THE H-PRINCIPLE, LECTURE 3: IMMERSION THEORY

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1. The questions at hand

The basic problem in immersion theory is the following.

Question 1.1. Classify immersions of M in N up to homotopy.

Here, a *homotopy between immersions* $f_0 \simeq f_1$ means a smooth map $F : M \times [0,1] \to N$ such that $F(-,t)$ is an immersion for every $t \in [0,1]$ with $f_0 = F(-,0)$ and $f_1 = F(-,1)$.

A more refined question, which we'll pursue in the next few lectures is

Question 1.2. What is the homotopy type of $\mathrm{Imm}(M,N)$, the space of immersions?

Notice that the basic question is a reduction of this refined question since

$$\pi_0 \,\mathrm{Imm}(M,N) = \{\text{immersions}\}/\sim \text{ of homotopy.}$$

Our basic approach to the refined question is to study the map

$$\mathrm{Imm}(M,N) \hookrightarrow \mathrm{Imm}^{\mathrm{f}}(M,N),$$

where the "formal immersions" $\mathrm{Imm}^{\mathrm{f}}(M,N)$ consists of the space of smooth bundle maps $F : TM \to TN$ that are injective on the fibers. We then ask

Question 1.3. What is the homotopy type of this map? When is it a homotopy equivalence?

We ask these questions because formal immersions are much easier to study. Observe that there is the forgetful map $\mathrm{Imm}^{\mathrm{f}}(M,N) \to \mathrm{Map}^{\mathrm{sm}}(M,N)$, sending a bundle map F to the map f on the base space. This forgetful map is a fibration, and it's easy to see that the fiber over a smooth map $f : M \to N$ is precisely $\mathrm{Hom}^{\mathrm{inj}}_{\mathrm{Vect}(M)}(TM, f^*TN)$, the space of injective bundle maps from TM to the pullback bundle f^*TN. The base space $\mathrm{Map}^{\mathrm{sm}}(M,N)$ is a comparatively easy space to study since it is homotopy equivalent to all continuous maps $\mathrm{Map}(M,N)$, which is well-studied in topology. (For example, if N is contractible, we know $\mathrm{Map}(M,N) \simeq \mathrm{pt}$.) And we discussed earlier how the fiber $\mathrm{Hom}^{\mathrm{inj}}_{\mathrm{Vect}(M)}(TM, f^*TN)$ is a pretty nice space and can be studied using bundle theory. So the upshot is that if the space of immersions is homotopy equivalent to the space of formal immersions, our life gets easier because formal immersions are easier to attack with homotopy theory.

Example 1.4. For $N = \mathbb{R}^n$, we have $\mathrm{Map}(M,N) \simeq \mathrm{pt}$, so there is an equivalence $\mathrm{Imm}^{\mathrm{f}}(M,N) \simeq \mathrm{Hom}^{\mathrm{inj}}_{\mathrm{Vect}(M)}(TM, \underline{\mathbb{R}^n})$, which is attackable with the theory of characteristic classes.

Example 1.5. Suppose as well that M has dimension n and is parallelizable, so that we can pick a bundle isomorphism $TM \cong \underline{\mathbb{R}^n}$. Then $\mathrm{Hom}^{\mathrm{inj}}_{\mathrm{Vect}(M)}(TM, \underline{\mathbb{R}^n}) \cong \mathrm{Map}(M, \mathrm{GL}_n\mathbb{R})$. For certain M, this is a well-studied space. For example, if $M = S^3$, we see that $\pi_0 \,\mathrm{Map}(S^3, \mathrm{GL}_3\mathbb{R}) = \pi_0\mathrm{GL}_3 \times \pi_3\mathrm{GL}_3 = \mathbb{Z}/2 \times \mathbb{Z}$. However, there are no immersions of S^3 into $\mathbb{R}^3$,[1] so we see that formal and actual immersions are not homotopy equivalent in this case.

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[1] In general, there are no immersions of a compact, closed n-manifold into $\mathbb{R}^n$. In this case, the argument goes as follows. Consider $U = S^3 - \mathrm{pt}$. Any immersion f restricted to U is just an embedding of U into $\mathbb{R}^3$, so its image $f(U)$ is an open set in $\mathbb{R}^3$. The boundary of this open $f(U)$ is either empty (if it's all of $\mathbb{R}^3$) or more than a point, so we can't extend f to all of S^3.

2. Working with sheaves

Recall the notion of a presheaf of spaces, which we introduced last time. A *presheaf* on M is a functor $F : \mathcal{U}_M^{op} \to \text{Spaces}$, where $\mathcal{U}_M$ is the category whose objects are the open subsets of M and whose morphisms are given by inclusion maps. Notice that $\mathrm{Imm}(-, N)$ and $\mathrm{Imm}^{\mathrm{f}}(-, N)$ define presheaves on M. In fact, their global sections are determined in a local-to-global fashion, so in fact they are sheaves. We will use the following definition of a sheaf.

Definition 2.1. A (classical) *sheaf* of spaces $\mathcal{F}$ on M is a presheaf such that for any finite cover $\{U_\alpha\}$ of an open set U, the natural map

$$\mathcal{F}(U) \longrightarrow \lim \left(\prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta) \right)$$

is an isomorphism.

Let $\mathrm{Imm}(-, N)$ and $\mathrm{Imm}^{\mathrm{f}}(-, N)$ be the presheaves of *bounded* immersions and formal immersions of open submanifolds of M into N (that is, immersions $g : U \to N$ for which the image $g(U)$ is contained in a compact subset of N). The property of a map being an immersion is local, as is the structure of a formal immersion, so they form sheaves on M. This fact suggests that maybe we can exploit the simple description of these sheaves on any open set diffeomorphic to $\mathbb{R}^m$.

Proposition 2.2. *The inclusion* $\mathrm{Imm}(D^m, N) \hookrightarrow \mathrm{Imm}^{\mathrm{f}}(D^m, N)$ *is a weak homotopy equivalence.*

Proof (Just a sketch ...) Let $V_m(TN)$ denote the space of m-frames in the tangent bundle TN over N. It is the total space of a bundle over N whose fiber is the Stiefel space $V_m(\mathbb{R}^n)$, with $n = \dim N$. There is a map $d_0 : \mathrm{Imm}(D^m, N) \to V_m(TN)$ sending an immersion g to its tangent map at the origin $d_0 g : T_0 M \to (g^* TN)_0$, which picks out an m-frame in $T_{g(0)}N$. Likewise, there is a map $d_0^{\mathrm{f}} : \mathrm{Imm}^{\mathrm{f}}(D^m, N) \to V_m(TN)$ sending a bundle map F to its behavior at the origin. Clearly, d_0 agrees with d_0^{f} following the inclusion $\mathrm{Imm} \hookrightarrow \mathrm{Imm}^{\mathrm{f}}$.

We want to show that both d_0 and d_0^{f} are weak homotopy equivalences and hence the inclusion must be as well.

First, note that $\mathrm{Imm}^{\mathrm{f}}(D^m, N) \cong \mathrm{Map}(\mathbb{R}^m, V_m(TN)) \simeq \mathrm{Map}(\mathrm{pt}, V_m(TN)) = V_m(TN)$, and d_0^{f} is compatible with this weak homotopy equivalence.

The case of d_0 is a little more tricky, so we'll explain why it's plausible. As we discussed in the first class, there is a section exp for d_0, which essentially constructs an immersion from an m-frame by using an exponential map on the tangent space $T_{f(0)}N$ to extend the linear map to an immersion. There is an analogous but simpler case. Consider the map $ev_0 : \mathrm{Map}(D^m, N) \to N$ sending a continuous map to its value at the origin. There is a section c sending a point $n \in N$ to the constant map to n. This is clearly a homotopy equivalence. Now observe that d_0 maps to ev_0 by forgetting the differential and the m-frame information. $\square$

Question 2.3. Since the map of sheaves $\mathrm{Imm}(-, N) \hookrightarrow \mathrm{Imm}^{\mathrm{f}}(-, N)$ is a weak homotopy equivalence on any open set diffeomorphic to $\mathbb{R}^m$, does this imply that the global sections are also homotopy equivalent? If not, what can we deduce from this local fact?

Earlier we showed that the answer to the first question is No, although it's an attractive thought. We need to explore how homotopy theory interacts with sheaf theory to get a handle on the questions above.

Remark 2.4. A good model to bear in mind while we discuss sheaves of spaces is the more familiar case of sheaves of chain complexes. Compare the three sheaves $\underline{\mathbb{R}} \to \Omega^*_{\mathrm{dR}} \to C^*_{\mathrm{Sing}}(-, \mathbb{R})$. It's familiar that all three agree (up to chain homotopy) on disks (the analogue of the proposition above). But they don't agree on global sections for interesting manifolds: the constant sheaf only counts the number of connected components, whereas de Rham and singular cohomology detect a lot more of the topology. De Rham's theorem tells us that de Rham and singular cohomology coincide, and his theorem follows from the fact that both functors behave nicely under restriction. Singular cochains are flabby (so restriction is surjective), and differential forms are soft (so that restriction of values on *closed* sets is surjective). We need to find the analogue of flabby/soft for sheaves of spaces.

Although we know that actual and formal immersions don't always agree, let's discover what goes wrong when we try to prove they *do* agree. This will help us focus elucidate the crucial features of the situation. We will try to prove they agree by induction on a cover of disks on M.

Pick a cover of M by disks: $M = \bigcup_{i=0}^{N} D_i^m$ (we assume M is compact), and so that the intersections $M_j \cap D_{j+1}^m$, where $M_j = \bigcup_{i=0}^{j} D_i^m$, are all products of a sphere with a disk.

Now assume, as an induction hypothesis, that $\mathrm{Imm}(M_j, N) \simeq \mathrm{Imm}^{\mathrm{f}}(M_j, N)$, and imagine that we have also shown the equivalence $\mathrm{Imm}(M_j \cap D_{j+1}^m, N) \simeq \mathrm{Imm}^{\mathrm{f}}(M_j \cap D_{j+1}^m, N)$.

We then have a map of pullback squares from

$$\begin{tikzcd}
\mathrm{Imm}(M_{j+1}, N) \arrow[r] \arrow[d] & \mathrm{Imm}(D_{j+1}^m, N) \arrow[d] \\
\mathrm{Imm}(M_j, N) \arrow[r] & \mathrm{Imm}(M_j \cap D_{j+1}^m, N)
\end{tikzcd}$$

to

$$\begin{tikzcd}
\mathrm{Imm}^{\mathrm{f}}(M_{j+1}, N) \arrow[r] \arrow[d] & \mathrm{Imm}^{\mathrm{f}}(D_{j+1}^m, N) \arrow[d] \\
\mathrm{Imm}^{\mathrm{f}}(M_j, N) \arrow[r] & \mathrm{Imm}^{\mathrm{f}}(M_j \cap D_{j+1}^m, N)
\end{tikzcd}$$

by functoriality. By our hypotheses, this map of squares is a weak homotopy equivalence for the entries other than (possibly) the upper left hand corner. We would like to deduce that the map $\mathrm{Imm}(M_{j+1}, N) \hookrightarrow \mathrm{Imm}^{\mathrm{f}}(M_{j+1}, N)$ is also a weak homotopy equivalence. This would give us the induction step.

In the category of spaces, however, we can't make that conclusion. For example, suppose we have a map $f : X \to Y$. Pick a point $\{0\} \in Y$ and let $F = f^{-1}\{0\}$ be the fiber. Pick a path extending the point $[0,1] \to Y$ and let $F' = f^{-1}([0,1])$ be its fiber. We get a map of diagrams

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & X \arrow[d] & \Longrightarrow & F' \arrow[r] \arrow[d] & X \arrow[d] \\
\{0\} \arrow[r] & Y & & {[0,1]} \arrow[r] & Y
\end{tikzcd}$$

and the maps from $\{0\} \to [0,1]$, $X \to X$, and $Y \to Y$ are homotopy equivalences. But this need not mean the map $F \to F'$ is a weak homotopy equivalence. We need $X \to Y$ to be a fibration to ensure this! (For instance, for $X \to Y$ is a fiber bundle, F' is homeomorphic to the a product $F \times [0,1]$.)

Definition 2.5. A *Serre fibration* is map $f : E \to B$ such that for any commuting square (with X a CW complex)

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & E \arrow[d] \\
X \times [0,1] \arrow[r] & B
\end{tikzcd}$$

there is a lift $X \times [0,1] \to E$.

This suggests we introduce an analogue of "flabbiness" for sheaves of spaces.

Definition 2.6 ([1])**.** A sheaf is *flexible* if every restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$, for any $U \subset V$ a pair of compact subspaces of M, is a Serre fibration.

The following proposition, which we will discuss later, shows that this is an extremely suitable notion for considering when a map of sheaves such as $\mathrm{Imm}(-, N) \to \mathrm{Imm}^{\mathrm{f}}(-N)$ is a weak homotopy equivalence.

Proposition 2.7. *Let $\mathcal{F} \to \mathcal{F}'$ be a map of flexible sheaves on M. If $\mathcal{F}(U) \xrightarrow{\sim} \mathcal{F}'(U)$ is a weak homotopy equivalence for every contractible $U \subset M$, then $\mathcal{F}(M) \to \mathcal{F}'(M)$ is a weak homotopy equivalence.*

References

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