THE H-PRINCIPLE, LECTURE 9: IMMERSIONS INTO EUCLIDEAN SPACE, FROM SMALE TO COHEN

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As we saw in the preceding lectures, the Hirsch-Smale theorem reduces the study of space of immersions, $\mathrm{Imm}(M, N)$, to a space of formal immersions, $\mathrm{Imm}^{\mathrm{f}}(M, N)$. This has the virtue that we can analyze this surrogate space $\mathrm{Imm}^{\mathrm{f}}(M, N)$ by a combination of the homotopy type of the space of all maps, $\mathrm{Map}(M, N)$, and of spaces of injective bundle maps, $\mathrm{Hom}^{\mathrm{inj}}(T_M, f^*T_N)$, as f varies over elements of $\mathrm{Map}(M, N)$. This is a major simplification, as both of these spaces are significantly more amenable to study than the space of immersions. However, despite the wealth of technical tools that topologists have developed to tackle the study such mapping spaces $\mathrm{Map}(M, N)$, calculating even the first few homotopy groups of $\mathrm{Map}(M, N)$ is still, in general, a potentially formidable task. But there is one case of N for which the study of $\mathrm{Map}(M, N)$ is never difficult (but for which the study of $\mathrm{Imm}(M, N)$ is still interesting): $N = \mathbb{R}^{n+k}$, Euclidean space.

In this case, the Hirsch-Smale theorem then reduces to the statement, using that $\mathrm{Map}(M, \mathbb{R}^{n+k})$ is contractible, that the space of immersions is homotopy equivalent to the space of injections of T_M into a trivial bundle. I.e., there is a weak homotopy equivalence

$$\mathrm{Imm}(M, \mathbb{R}^{n+k}) \simeq \mathrm{Hom}^{\mathrm{inj}}(T_M, \underline{\mathbb{R}}^{n+k})$$

provided that either k is strictly positive or M is open (this assumption will remain in place henceforth).

The most basic question to pose about the space $\mathrm{Imm}(M, \mathbb{R}^{n+k})$ is whether it contains a point, i.e., whether there exists an immersion of M into $\mathbb{R}^{n+k}$. Given an injective bundle map $F : T_M \to \underline{\mathbb{R}}^{n+k}$, we can choose a metric on $\underline{\mathbb{R}}^{n+k}$ and take the orthogonal complement of the image of F to obtain an isomorphism $\underline{\mathbb{R}}^{n+k} \cong T_M \oplus \mathrm{Coker}(F)$. This provides us with the following criterion:

Corollary 0.1 (Corollary of Hirsch-Smale). *There exists an immersion of M^n in $\mathbb{R}^{n+k}$ if and only if there exists a k-dimensional bundle V on M such that the sum of T_M with V is trivial:*

$$T_M \oplus V \cong \underline{\mathbb{R}}^{n+k}.$$

Let us see that this implies the weak version of Whitney's immersion theorem, just by using basic homotopy theory:

Theorem 0.2 (Whitney). *Any compact n-manifold M can be immersed in $\mathbb{R}^{2n}$.*

By the Corollary, it suffices to show the existence of an $\mathbb{R}^n$-bundle V such that the sum $T_M \oplus V$ is a trivial bundle. This can be quickly seen from two facts: (1) $K^0(M)$ is a group; (2) every $\mathbb{R}^{n+k}$-bundle on an n-dimensional CW complex is isomorphic to one of the form $V^n \oplus \underline{\mathbb{R}}^k$ for some $\mathbb{R}^n$-bundle V on M. We spell this out:

Proof. By the Corollary, it suffices to show the existence of an n-dimensional bundle V. By choosing a finite atlas of M, we can embed $\eta : M \to \mathbb{R}^k$ into a large-dimensional Euclidean space $\mathbb{R}^k$ and take the normal bundle N_η.

Now, recall the existence of a homotopy fiber sequence

$$S^r \to BO(r) \to BO(r+1),$$

Date: Lecture January 31, 2011. Last edited on January 31, 2011.

for each r, where $BO(n)$ is the classifying space for $\mathbb{R}^n$-bundles. From the long exact sequence of a fibration, this implies that the map on homotopy groups

$$\pi_i BO(n) \to \pi_i BO(k)$$

is an isomorphism for $i \leq n-1$ and is surjective for $i = n$, where $k \geq n$. In other words, the map $BO(n) \to BO(k)$, for is n-connective. As a consequence, for any map from an n-dimensional CW complex M to $BO(k)$, there exists a factorization up to homotopy:

$$\begin{tikzcd} & BO(n) \arrow[d] \\ M \arrow[r] \arrow[ur, dashed] & BO(k) \end{tikzcd}$$

Of course, the composition map on homotopy classes $[M, BO(n)] \to [M, BO(k)]$ corresponds to map on equivalence classes of vector bundles $\underline{\mathbb{R}}^{n-k} \oplus (-) : \mathrm{Vect}_n(M) \to \mathrm{Vect}_k(M)$ defined by taking a direct sum with a trivial bundle.

Applying the existence of the above lifting to the map g_η classifying the normal bundle N_η, and we obtain the existence of an $\mathbb{R}^n$-bundle V such that $T_M \oplus V \oplus \underline{\mathbb{R}}^{k-n} \cong T_M \oplus N_\eta \cong \underline{\mathbb{R}}^k$ is trivial, hence $T_M \oplus V$ is trivial.

□

Now, we can ask: Is this result optimal? Considering the instance of $n = 2$, we know independently from the classification of surfaces that every surface immerses into $\mathbb{R}^3$, so clearly the answer is not optimal.

Question 0.3. What is the function $\beta(n)$ defining the minimal dimension Euclidean space such that every n-manifold M immerses in $\mathbb{R}^{2n-\beta(n)}$?

So far, we have calculated the inequality $0 \leq \beta(n)$, for all n, and the example that $\beta(2) = 1$. We also have $\beta(n) < n$ for $n \geq 2$, since a compact and open n-manifold can be immersed in $\mathbb{R}^n$ if and only if its tangent bundle is trivial (Corollary 0.1). We can ask the following related question.

Question 0.4. For fixed k, what are the obstructions to finding a rank k vector bundle V on M such that $V \oplus T_M$ is trivial?

The Stiefel-Whitney classes $w_j(V) \in H^j(M, \mathbb{F}_2)$ provide such an obstruction. Recall that these characteristic classes satisfy $w_j(V) = 0$ if $j > \mathrm{rk}(V)$ and $w(V \oplus W) = w(V) \cup w(W)$. Thus if $V \oplus T_M$ is trivial, then $w(V) \cup w(M) = w(\underline{\mathbb{R}}^{n+k}) = 1$. This shows that the Stiefel–Whitney classes of such a V are determined by those of T_M, with recursively computable formulas

$$w_1(V) = w_1(M), w_2(V) = w_2(M) + w_1(M)^2, \text{etc.}$$

We'll use the unambiguous notation $\bar{w}_j(M) = w_j(V)$. Thus, for a vector bundle V to exist as in Question 0.4, or equivalently for M to be immerseable in $\mathbb{R}^{n+k}$, a necessary condition is for the classes $\bar{w}_j(M)$ to vanish, $\bar{w}_j(M) = 0$, for $j > k$.

Definition 0.5. Let M be a closed manifold of dimension n. The ith Wu class $v_i \in H^i(M, \mathbb{F}_2)$ is the unique class for which $v_i \cdot x = \mathrm{Sq}^i x$, where x is any element of $H^{n-i}(M, \mathbb{F}_2)$, and Sq^i is the ith Steenrod square, $\mathrm{Sq}^i \colon H^{n-i}(M, \mathbb{F}_2) \to H^n(M, \mathbb{F}_2)$.

Theorem 0.6 (Wu's formula)**.** *$w(M) = \mathrm{Sq}(v)$, i.e.,*

$$w_k(M) = \sum_{i+j=k} \mathrm{Sq}^i(v_j).$$

Proof. See [5]. □

Theorem 0.7 (Massey). *Let M be a compact manifold of dimension $n \geq 2$. Then $\bar{w}_j(M) = 0$ for $j > n - \alpha(n)$ where $\alpha(n)$ is the number of 1's in the dyadic expansion of n. Moreover, this is the optimal bound: If $n = 2^{h_1} + \cdots + 2^{h_{\alpha(n)}}$ is the dyadic expansion of n, then we have the nonvanishing $\bar{w}_{n-\alpha(n)}(\mathbb{R}P^{h_1} \times \cdots \times \mathbb{R}P^{h_{\alpha(n)}}) \neq 0$.*

The proof of the first part is an clever algebraic computation using Wu's formula. The second part of this theorem is a straightforward computation using $w(\mathbb{R}P^k) = (a+1)^{k+1}$ where $H^*(\mathbb{R}P^k) = \mathbb{F}_2[a]$, and it implies that $\mathbb{R}P^{h_1} \times \cdots \times \mathbb{R}P^{h_{\alpha(n)}}$ cannot be immersed in $\mathbb{R}^{2n-\alpha(n)-1}$, or in other words that $\beta(n) \leq \alpha(n)$.

It is still a fair ways to conclude from this that every $\alpha(n)$ and $\beta(n)$ are equal. After all, there are nontrivial bundles all of whose characteristic classes vanish. Nonetheless, the following was proven by applying still greater tools from topology to Hirsch-Smale:

Theorem 0.8 (Cohen). *If $n \geq 2$, $\beta(n) = \alpha(n)$. That is, every compact n-manifold can be immersed in $\mathbb{R}^{2n-\beta(n)}$.*

Remarks on the proof: Let I_n denote the ideal in $H^*(BO, \mathbb{F}_2)$ generated by all classes $\bar{w}_i(M^n)$. Brown-Peterson constructed a space BO/I_n mapping to BO whose cohomology realized the ring $H^*(BO, \mathbb{F}_2)/I_n$. Cohen completed Brown-Peterson's program, by showing: (1) the map classifying any normal bundle of a manifold factors through BO/I_n; (2) the map $BO \to BO/I_n$ factors through $BO(n - \alpha(n))$. See [2]. □

Remark 0.9. The best linear function $f(n)$ of n for which every M^n immerses in $\mathbb{R}^{f(n)}$ is still $f(n) = 2n - 1$, which is Whitney's (strong) immersion theorem, [7].

References

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