MATH 465, LECTURE 10: THE NORMAL FORM LEMMA, SECOND PART: HOMOLOGY LEMMA

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Throughout this lecture our handlebody W^n of dimension n will be obtained from $\partial_0 W \times [0,1]$ by attaching q-handles ϕ_i^q for i in the index set I_q, $q = 0, \dots, n$. That is

$$W = \partial_0 W \times [0,1] + \sum_{q=0}^{n} \sum_{i \in I_q} \phi_i^q$$

Note that attaching the handles in this order does not impose any restrictions by the reordering result of lecture 7. We also have a natural filtration

$$\partial_0 W \times [0,1] =: W_{-1} \subset W_0 \subset \cdots \subset W_n = W$$

where for $k \geq 0$, W_k is obtained from W_{k-1} by attaching handles of index k. That is

$$W_k := W_{k-1} + \sum_{i \in I_k} \phi_i^k$$

Our goal is to prove the following lemma which gives a homological condition under which the hypotheses of the cancellation lemma are satisfied.

Lemma 0.1. *Let W^n be as above with $n \geq 6$. Let $2 \leq q \leq n-3$, and let $f : S^q \to \partial_1 W_q$ be an embedding such that $[f] = \pm[\phi_1^q]$ in $C_q^{\mathrm{cell}}(W)$. Then f is isotopic to a map that intersects the transverse sphere of ϕ_1^q transversely in a single point and does not intersect the transverse sphere sphere of any other ϕ_i^q.*

The remainder of this lecture will reduce this lemma to the so-called Whitney trick (to be proven next time) by better understanding the $C_q^{\mathrm{cell}}(W)$. Recall that by definition $C_q^{\mathrm{cell}}(W) = H_q(W_q, W_{q-1})$. Hence by collapsing W_{q-1} we have

$$C_q^{\mathrm{cell}}(W) \cong \bigoplus_{i \in I_q} H_q(D^q \times D^{n-q}, S^{q-1} \times D^{n-q}).$$

Each term of the sum is itself isomorphic to $H_q(D^q, S^{q-1})$ and so it is one dimensional. Let $[\phi_i^q]$ denote a generator so that the collection $\{[\phi_i^q]\}_{i \in I_q}$ is a basis for $C_q^{\mathrm{cell}}(W)$.

In the statement of the lemma we view $f : S^q \to \partial_1 W_q$ as an element of $C_q^{\mathrm{cell}}(W)$. How can we do this? Well, we have maps

$$\pi_q(W_q) \to H_q(W_q) \to H_q(W_q, W_{q-1}) = C_q^{\mathrm{cell}}(W)$$

where the first map is the Hurewicz homomorphism (i.e. $[f] \mapsto f_*[S^n]$) and the second map is from the definition of relative homology.

The idea of the proof is to write $[f]$ in terms of the $[\phi_i^q]$. Consider the maps

$$\begin{tikzcd}
rS^q, \varnothing) \arrow[r, "f"] & (W_q, W_{q-1}) \arrow[r, "p"] \arrow[dr, "p_i"'] & (\bigvee_{I_q} D^q \times D^{n-q}, \bigvee_{I_q} S^{q-1} \times D^{n-q}) \\
& & (D^q \times D^{n-q}, S^{q-1} \times D^{n-q})
\end{tikzcd}$$

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where p is the collapse map and p_i projects onto the i^{th} factor.

Since p induces an isomorphism on homotopy and p_i induces projection onto the subgroup generated by $[\phi_i^q]$, $p_i \circ f$ determines an element of $\pi_q(D^q, S^{q-1}) \cong \mathbb{Z}$, that is an integer which we denote $[p_i \circ f]$. Then

$$[f] = \sum_{i \in I_q} [p_i \circ f] \cdot [\phi_i^q].$$

To compute the coefficients $[p_i \circ f]$ geometrically we look at the inverse image of a regular value. Namely let $0 \in (D^q, S^{q-1})$ be a regular value. Then

$$[p_i \circ f] = \sum_{x \in p_i^{-1}(0)} \epsilon_x$$

where $\epsilon_x = \pm 1$ is determined by the oriented intersection of $f(S^q)$ and ϕ_i^q in W. This is defined generally as follows.

Let V be any manifold and $f : M \to V$, $i : M' \to V$ smoooth maps of manifolds with $\dim M + \dim M' = \dim V$ and $f \pitchfork i$ and choices of orientation of TM and $N_i M'$. Then for $x \in M \cap M'$ we have the following diagram

$$\begin{tikzcd}
T_x M \arrow[r, "df"] \arrow[dr, "\cong"'] & T_x V \cong T_x M' \oplus N_i M'|_x \arrow[d, "proj"] \\
& N_i M'|_x
\end{tikzcd}$$

where the diagonal map is an isomorphism by transversality. That map is a map between oriented vector spaces. We define $\epsilon_x = \pm 1$ depending on whether the map is orientation preserving or reversing.

We apply this to $V = \partial_1 W_q$, $M = S^q$, $M' = \{0\} \times S^{n-q-1}$. The relevant bundles are orientable and we choose orientations. Then we have

$$[f] = \sum_{i \in I_q} \sum_{\substack{x \in f(S^q) \cap \\ \phi_i^q(\{0\} \times S^{n-q-1})}} \epsilon_x [\phi_i^q] \tag{1}$$

Note that changing the choice of orientation will only only change the coefficient of each basis vector $[\phi_i^q]$ up to a sign.

We now return to the proof of the homology lemma. We start by isotoping f to a map that is transverse to each of the transverse spheres of the q-handles (apply lecture 4, theorem 1.6 to $P = S^q$, $E = \partial_1 W_q$, $M = \sqcup_{I_q} S^{n-q-1}$). Then by hypothesis $[f] = [\phi_i^q]$ where we can now write $[f]$ using equation 1. Thus $f(S^q) \cap \phi_1^q(\{0\} \times S^{n-q-1})$ is non-empty.

The conclusion of the lemma is exactly that we can isotope f such that the sum in equation 1 has only one term. So if this is already the case, we are done. Otherwise there is cancellation in the sum and we will apply the Whitney trick to isotope f to eliminate the superfluous intersection points in pairs.

References

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