MATH 465, LECTURE 13: NORMAL FORM LEMMA, FINAL PART

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1. Recapitulation

Let us begin by recalling a couple of lemmas that we will need.

Lemma 1.1 (Homology Lemma)**.** *Let W be a handlebody of dimensions $n \geq 6$, and let $2 \leq q \leq n-3$. Let $f : S^q \to \partial_1 W_q$ be an embedding such that $[f] = \pm[\varphi_1^q]$, the class of the handle φ_1^q, inside $C_q^{\text{cell}}(W, \partial W)$. Then f is isotopic to an embedding that intersects the transverse sphere $\{0\} \times S_1^{n-q-1}$ of the handle φ_1^q transversally in one point and which does not intersect the transverse sphere of any other q-handle.*

Lemma 1.2 (Elimination Lemma)**.** *Let W be a handlebody with no handles of index less than q for $1 \leq q \leq n-3$. Recall that $\partial_1^\circ W_q$ denotes the closed complement of the images of the attaching maps of the $(q+1)$-handles inside $\partial_1 W_q$. If there exists an embedding $\psi^{q+1} : S^q \times D^{n-q-1} \to \partial_1^\circ W_q$ such that:*

- *$\psi^{q+1}|_{S^q \times \{0\}}$ intersects transversally the transverse sphere of φ_1^q in one point, and does not intersect the transverse spheres of the other q-handles, perhaps after a suitable isotopy in $\partial_1 W_q$;*
- *$\psi^{q+1}|_{S^q \times \{0\}}$ is isotopic trivial embedding inside $\partial_1 W_{q+1}$.*

. (Note the larger space we allow in which $\psi^{q+1}|_{S^q \times \{0\}}$ is a trivial embedding.) Then, we may add handles ψ^{q+1} and ψ^{q+2} so that

$$W \cong \partial_o W \times [0,1] + \sum_{I_q - \{1\}} \varphi_i^q + \sum_{I_{q+1}} \varphi_i^{q+1} + \psi^{q+2} + \sum_{I_{q+2}} \varphi_i^{q+2} + \dots$$

In other words, we may switch a q-handle φ_1^q for a $(q+2)$-handle ψ^{q+2} in the handle presentation of W.

Remark 1.3. Observe that the output of the Homology Lemma is very close to the first input condition for the Elimination Lemma. That is, the Homology Lemma provides a homological condition for when the core of the attaching map $f := \psi^{q+1}|_{S^q \times \{0\}}$ can be isotoped so as the satisfy the first elimination condition. Of course, the Homology Lemma does not make mention of the thicker embedding of $S^q \times D^{n-q-1}$, but this is not a problem. Namely, we have the following: If $\psi : S^q \times D^{n-q-1} \to M^{n-1}$ is an open embedding, and $g : S^q \to M^{n-1}$ is an embedding such that g is isotopic (or just homotopic) to $\psi|_{S^q \times \{0\}}$, then g extends to an embedding $\tilde{g} : S^q \times D^{n-q-1} \to M^{n-1}$. The argument is simple: By the tubular neighborhood theorem, the normal bundle of the map $\psi|_{S^q \times \{0\}}$ is identifiable with the interior of $S^q \times D^{n-q-1}$, and thus the normal bundle is trivial. Since g is homotopic, the normal bundle is isomorphic, thus the normal bundle is also trivial. Choosing a trivialization of the normal bundle thereby gives the desired fattened up embedding.

2. Preparation for the normal form lemma

2.1. **Universal covers.** We need two further results before we can prove the normal form lemma. We will recall the statement of the Normal Form lemma towards the end of the lecture (once we are well prepared).

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Recall we are working our way towards the h-cobordism and s-cobordism theorems. The fundamental group of our underlying manifold M will play a significant role in the s-cobordism theorem. However right now, we may essentially pass our arguments to the relevant universal covers.

Fix now a handlebody W with an ascending order of attached handles. We have the following filtration of spaces:

$$\partial_0 W \times [0,1] =: W_{-1} \subset \dots W_q \subset W_{q+1} \subset W_{q+2} \subset \dots \subset W_n = W$$

Let $\widetilde{W_i}$ denote the universal cover of W_i. Consider the cellular chain complex computing $H_*(\widetilde{W}, \widetilde{\partial_0 W})$ associated to the following filtration:

$$\dots \subset \widetilde{W}_q \subset \widetilde{W}_{q+1} \subset \dots$$

Let $\pi = \pi_1 W$. Once we assume the relevant homotopy equivalences (as in the assumptions of inclusions of boundaries for the h-cobordism theorem), π will be the only fundamental group of interest.

If we assume that $q \geq 2$, then a standard cellular argument gives that the inclusion $\widetilde{W}_q \subset \widetilde{W}$ is 1-connected. Moreover, π acts on $\widetilde{W}_q$ and $\widetilde{W}$ by deck transformations. This inclusion is π-equivariant.

We still assume W has no zero or 1-handles. Since $\widetilde{W}_{q-1} \subset \widetilde{W}_q$ is π-equivariant, $\mathbb{Z}[\pi]$ acts on the relative homology. Namely, the chain complex $(\oplus_q C_q^{\text{cell}}(\widetilde{W}_q, \widetilde{W}_{q-1}), \oplus d_q^{\text{cell}})$ has an action of $\mathbb{Z}[\pi]$, and the differential is linear with respect to this action.

Corollary 2.1 (Alternate condition for Homology Lemma)**.** *Instead of $f : S^q \to \partial_1 W_q$, we may pick an arbitrary lift $\tilde{f}$ to the universal cover (remember S^q is simply connected for $q \geq 2$). Write $[\tilde{f}] = \pm\gamma[\varphi_1^q] \in C_q^{\text{cell}}(\widetilde{W}_q, \widetilde{W}_{q-1})$ (note that the φ_i^q's form a free $\mathbb{Z}[\pi]$-basis).*

2.2. **The Modification Lemma.** The name "Homology Lemma" is not quite as apt as it could be, since it the condition is stated for a cellular cycle, rather than its associated homology class. We would like to improve the lemma so as not to depend on the choice of cellular cycle $[f] \in C_q^{\text{cell}}(W, \partial_0)$, but instead to only depend only on the homology class in $H_q(W, \partial_0 W)$. That is, we would like to be able to modify the the choice of cycle by adding a boundary. The following lemma enables us to do so.

Lemma 2.2 (Modification Lemma)**.** *Let W be a handlebody as before. Let $f : S^q \to \partial_1^\circ W_q$, which after picking a lift defines a class $[\tilde{f}] \in C_q^{\text{cell}}(\widetilde{W}_q, \widetilde{W}_{q-1})$. Let b be a boundary: $b \in \text{Image}(d_{q+1})$. Then we can find an embedding $g : S^q \to \partial_1^\circ W_q$ satisfying:*

- *f and g are isotopic in $\partial_1 W_{q+1}$ (again note they are not necessarily isotopic inside $\partial_1^\circ W_q$)*
- *there exists a lift $\tilde{g}$ such that $[\tilde{g}] = [\tilde{f}] + b$.*

Proof. It suffices to check for a basis element (over $\mathbb{Z}[\pi]$): $b = \pm\gamma d_{q+1}[\varphi_j^{q+1}]$, where γ is an arbitrary element of π. We construct g. Consider the two embeddings:

$$S^q \xrightarrow{f} \partial_1 W_q \xleftarrow{\varphi_j^{q+1}|_{S^q \times \{0\}}} S^q$$

These are just two embedded spheres inside our ambient space $\partial_1 W_q$. Choose any embedded path $\omega : [0,1] \to \partial_1 W_q$ connecting a point in the image sphere of f to a point in the image sphere of $\varphi_j^{q+1}|_{S^q \times \{0\}}$. Now the idea is to simply thicken this path to form the connect-sum of the two spheres we had originally. Explicitly, extend the path ω to an embedding $\tilde{\omega} : [0,1] \times D^q \to \partial_1 W_q$. The boundary of this gives us the connect sum $f \#_\omega (\varphi_j^{q+1}|_{S^q \times \{0\}}) : S^q \to \partial_1 W_q$. At the level of homology, $H_q(W_q, W_{q-1}) = C_q^{\text{cell}}(W, \partial_0 W)$, this class represents $[f] + d_{q+1}[\varphi_j^{q+1}]$.

Choosing a life $\tilde{f}$ determines a lift of $\tilde{g}$ of g agreeing with their common preimage in S^q. Therefore we obtain that, for some element $\gamma' \in \pi$, $[\tilde{g}] = [\tilde{f}] + \gamma d_{q+1}[\varphi_j^{q+1}]$. This is still not quite the correct result we need. However, recall we picked an arbitrary (embedded) path ω. We may now use our action of the fundamental group to alter this choice of path. Change ω by $\gamma(\gamma')^{-1}$ and follow the same procedure as was just outlined. Now defining $g := f\#_{\gamma(\gamma')^{-1}\omega}\varphi_j^{q+1}$, the corresponding lift $\tilde{g}$ will give the desired cycle class.

□

2.3. **Dual handlebodies.** Proceeding on, we introduce the simple idea of the Dual Handlebody decomposition. Let W be obtained from a cylinder by adding one q-handle φ^q:

$$W = \partial_0 W \times [0,1] + \varphi^q$$

Observe we picked the "∂_0" end of W. There is a dual decomposition of W such that:

$$W = \partial_1 W \times [0,1] + \eta^{n-q}$$

Recall that our formula for the boundary of the handlebody W was:

$$\partial_1 W = \partial_0 W \times \{1\} - \varphi^q(S^{q-1} \times \mathrm{int}(D^{n-q})) \cup_{S^{q-1}\times S^{n-q-1}} D^q \times S^{n-q-1}$$

Let η be the obvious inclusion of $S^{n-q-1} \times D^q$ (which equals $D^q \times S^{n-q-1}$) into $\partial_1 W$. Then indeed, $\partial_1(\partial_1 W \times [0,1] + \eta^{n-q}) = \partial_0 W$. When we come to our lectures on Morse theory (which will exhibit existence of handlebody decompositions), this dual decomposition corresponds to taking the associated handlebody decomposition of -1 of the Morse function.

For completeness, let us write down the dual handlebody decomposition of any handlebody W.

$$\begin{aligned} W &= \partial_0 W \times [0,1] + \sum_{I_0} \varphi_i^0 + \sum_{I_1} \varphi_i^1 + \cdots + \sum_{I_n} \varphi_i^n \\ &\cong \partial_1 W \times [0,1] + \sum_{I_n} \eta_i^0 + \sum_{I_{n-1}} \eta_i^1 + \cdots + \sum_{I_0} \eta_i^n \end{aligned}$$

3. Normal Form Lemma

Lemma 3.1 (Normal Form Lemma)**.** *Let W be an oriented h-cobordism handlebody of dimension $n \geq 6$. Then for any choice $2 \leq q \leq n-3$, there exists a diffeomorphism:*

$$W \cong \partial_0 W \times [0,1] + \sum_{I_q} \varphi_i^q + \sum_{I_{q+1}} \varphi_i^{q+1}$$

Proof. By induction. For our base case, we apply our previous lemma that if the map $\partial_0 W \to W$ is 1-connected, then W is diffeomorphic to handlebody with no 0- or 1-handles. Switching to the dual handlebody presentation, we can apply this lemma to also assume that there are no $(n-1)$- or n-handles. As our inductive step, we may assume $W_{-1} = W_{q-1}$: namely that there are no handles of index less than q. The basic idea is simple: we want to replace all the q-handles with $(q+2)$-handles. So we choose an arbitrary q-handle. The want to apply the Elimination Lemma to replace it with a $(q+2)$-handle: To do so, we first choose a trivial embedding; then we modify it so that it represents the same cellular cycle class; then we apply the Homology Lemma, which allows us to apply the Elimination Lemma. Proceeding, we will be left with a handlebody with handles all of index $n-2$ and $n-1$; we can then switch again to the dual handlebody decomposition and apply the same procedure to move our handles into an arbitrary pair of adjacent indices between 2 and $q-2$.

We now carry this out the inductive step, exchanging a q-handle for a $(q+2)$-handle: Let φ_1^q be a q-handle. Let $[\varphi_1^q]$ be the class of the handle inside $C_q^{\mathrm{cell}}(\widetilde{W}, \widetilde{\partial_0 W})$. The inductive assumption that there are no handles of index less than q ensures that the differential d_{q+1} is surjective. Thus, we may write it as $[\varphi_1^q] = \sum_{I_{q+1}} x_j d_{q+1}[\varphi_j^{q+1}]$.

We select an arbitary trivial embedding $\psi^{q+1} : S^q \times D^{n-1} \to \partial_1^\circ W_q$. Applying the Modification Lemma, we may isotope $\psi^{q+1}|_{S^q\times\{0\}}$ through $\partial_1 W_{q+1}$ to an embedding $g : S^q \to \partial_1^\circ W_q$ for which a lift of g represents the desired cellular class

$$[\tilde{g}] = [\psi^{q+1}] + \sum_{I_{q+1}} x_j d_{q+1}[\varphi_j^{q+1}].$$

Since $[\psi_1^q] = 0$, we obtain the equality $[\tilde{g}] = [\varphi_1^q]$.

Now, we are in a position to apply the Homology Lemma, which asserts that g can be isotoped to an embedding satisfying the first condition of the Elimination Lemma. Our assumption that ψ^{q+1} was a trivial embedding ensures the second condition of the Elimination Lemma. We apply the Elimination Lemma to replace φ_1^q with ψ^{q+2}. This completes the inductive step, hence the proof. $\square$

References

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