MATH 465, LECTURE 15: THE S-COBORDISM THEOREM, SIMPLE HOMOTOPY, AND WHITEHEAD TORSION

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Previously, given an h-cobordism W with a specified handlebody presentation, we had an associated contractible chain complex of free $\mathbb{Z}\pi$-modules: $C_*^{\mathrm{cell}}(\widetilde{W}, \widetilde{\partial_0 W})$. Here π denotes the group $\pi_1(\partial_0 W) \cong \pi_1 W$. Recall that this contractible complex came to us with a natural equivalence class of basis, given by lifts of the handles. That is, for each q-handle ϕ_i^q, we choose a lift of the characteristic map

$$\widetilde{\Phi}_i^q : (D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \longrightarrow (\widetilde{W}_q, \widetilde{W}_{q-1})$$

and then we pushforward a relative fundamental class of the q-disk to obtain a cellular class $[\phi_i^q] \in H_q(\widetilde{W}_q, \widetilde{W}_{q-1}) = C_q^{\mathrm{cell}}(\widetilde{W}, \widetilde{\partial_0 W})$. In selecting each basis element, we made an arbitrary choice of lift, which we could alter by acting by a deck transformation $\gamma_i \in \pi$, and an arbitrary choice of fundamental class, which we alter by multiplying by -1. This (unordered) basis is thus only well-defined up to multiplying by $\pm\gamma_i$, for each i.

It might be tempting to think that this complex $C_*^{\mathrm{cell}}(\widetilde{W}, \widetilde{\partial_0 W})$ might contain no information because it is contractible. But this would be a mistake. So let us now address the question: What invariant can extracted from a contractible finite complex of free $\mathbb{Z}\pi$-modules equipped with an equivalence class of basis?

1. The Whitehead group and Whitehead torsion

Let X be a contractible finite complex of free $\mathbb{Z}\pi$-modules with an unordered $\mathbb{Z}\pi$-basis of each X_q. Given such an X, we will produce an element of $K_1(\mathbb{Z}\pi)$, the first algebraic K-group of the ring $\mathbb{Z}\pi$.

Let us first recall the definition of $K_1(R)$. (This is originally due to Whitehead, who defined it for this exact purpose.) Let $\mathrm{GL}(R)$ denote the sequential colimit $\mathrm{colim}_{n\to\infty} \mathrm{GL}_n(R)$. Let δ_{ij} denote the matrix which is zero except for a single entry at the ith row and jth column. A invertible matrix is *elementary* if it equal to the sum of the identity matrix I with $r\delta_{ij}$, $r \in R$ and $i \neq j$, a matrix consisting of a single off-diagonal nonzero element. Let $E(R) \subset \mathrm{GL}(R)$ be the subgroup generated by all elementary matrices.

Lemma 1.1. *The subgroup $E(R)$ is equal to $[\mathrm{GL}(R), \mathrm{GL}(R)]$, the commutator subgroup of $\mathrm{GL}(R)$.*

Proof. Any elementary matrix $I + r\delta_{ik}$ is equivalent to the commutator

$$I + r\delta_{ik} = (I + r\delta_{ij})(I + \delta_{jk})(I - r\delta_{ij})(I - \delta_{jk}).$$

Conversely, show that for any $A, B \in \mathrm{GL}(R)$, the commutator $ABA^{-1}B^{-1}$ can be expressed a product of elementary matrices. See, e.g., [6]. □

Definition 1.2. $K_1(R)$ is $\mathrm{GL}(R)/E(R) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$, the abelianization of the group $\mathrm{GL}(R)$.

Observe the quotient $\mathrm{GL}(R)/E(R)$ can be expressed by imposing several equivalence relations:

- $A \sim B$ if B is obtained from A by adding r times the kth row of A to the lth row, for $r \in R$ and $k \neq l$;
- $A \sim B$ if B is obtained from A by switching a pair of rows or a pair of columns.

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We now return to the problem of our contractible complex X. Let X be a contractible finitely generated complex of free R-modules. Since X is contractible, we may choose a contracting homotopy h. That is, h is a sequence of R-module maps $h_i : X_i \to X_{i+1}$ such that $h_{i+1} \circ d_i + d_{i-1} \circ h_i = \mathrm{id}$ for each i. Let A denote the matrix defining the linear transformation $(d+h)_{\mathrm{odd}} : X_{\mathrm{odd}} \to X_{\mathrm{ev}}$. Likewise, let B denote the matrix of the linear transformation $(d+h)_{\mathrm{ev}} : X_{\mathrm{ev}} \to X_{\mathrm{odd}}$. Note that the matrices A and B both define classes, $[A]$ and $[B]$, in $K_1(R)$.

Now, the composition $B \circ A : X_{\mathrm{odd}} \to X_{\mathrm{ev}} \to X_{\mathrm{odd}}$ is the identity matrix I, exactly by the formula for h being a contracting homotopy. Thus, we obtain that the class $[A] + [B] = [B \circ A]$ is the identity element $[I] = 0 \in K_1(R)$, so $[A] = -[B]$.

Select the matrix A. This will be our invariant in $K_1(R)$ associated to such a contractible complex X. We will refer to this element as the *torsion* of X. Since the choice of $[A]$ versus B was somewhat arbitrary, we make the following definition.

Definition 1.3. Let X be a contractible finitely generated complex of free R-modules with a basis. The torsion $\tau(X)$ is class $[A] \in \widetilde{K}_1(R)$, where $\widetilde{K}_1(R) = K_1(R)/\mathbb{Z}^\times$ is the reduced K-group of R.

Remark 1.4. There is also a definition of torsion when X is not necessarily contractible, which is slightly more involved. See [6].

Our situation is slightly different, however, in that our basis for the contractible complex X was only well-defined up to a certain equivalence relation.

Since the determinant map $\det : \mathrm{GL}(R) \to R^\times$ is trivial when restricted to $E(R)$, there is thus a factorization:

$$\begin{tikzcd}
E(R) \arrow[r] \arrow[dr] & \mathrm{GL}(R) \arrow[r] \arrow[d] & K_1(R) \arrow[dl, dashed, "\det"] \\
& R^\times &
\end{tikzcd}$$

There is a section $R^\times \to K_1(R)$, defined by sending a unit $a \in R^\times$ to the class of the 1×1-matrix $[a] \in K_1(R)$. This defines a splitting $K_1(R) \cong R^\times \oplus \ker(\det)$.

Remark 1.5. Nothing in this construction depended on the ring $\mathbb{Z}\pi$ being a group ring. The same construction could be made for any ring R together with a choice of subgroup of the units of R, $G \subset R^\times$. In our case, this subgroup is $\mathbb{Z}^\times \times \pi \subset \mathbb{Z}[\pi]^\times$.

Definition 1.6. For a group π, the Whitehead group of π is $\mathrm{K}_1(\mathbb{Z}\pi)/\mathbb{Z}^\times \times \pi = \mathrm{Wh}(\pi)$.

We can also define the Whitehead torsion of a homotopy equivalence of finite CW complexes.

Definition 1.7. Let $f : K \to K'$ be a cellular homotopy equivalence of finite CW complexes. This induces $\tilde{f}_* : C_*^{\mathrm{cell}}(\widetilde{K}) \to C_*^{\mathrm{cell}}(\widetilde{K}')$. Then $\mathrm{cone}(\tilde{f}_*)$ is a contractible complex with a basis. The *Whitehead torsion* of f is $\tau(f) = \tau(\mathrm{cone}(\tilde{f}_*) \in \mathrm{Wh}(\pi_1 K')$.

Definition 1.8. An elementary expansion is a map $X \to X \bigcup_{D^n} D^{n+1}$ of the form

$$\begin{tikzcd}
X \arrow[r] & X \bigcup_{D^n} D^{n+1} \\
D^n \arrow[r] \arrow[u] & D^{n+1} \arrow[u]
\end{tikzcd}$$

where $D^n \to D^{n+1}$ factors through ∂D^{n+1}.

Definition 1.9. An elementary collapse is a collapse map $X \bigcup_{D^n} D^{n+1} \to X$ of the form where $D^n \to D^{n+1}$ factors through ∂D^{n+1}.

Definition 1.10. A map f is simple if it is a composition of elementary expansions and collapses.

Theorem 1.11. *A cellular homotopy equivalence of finite CW complexes f is homotopic to a simple homotopy equivalence if and only if $\tau(f) = 0$ in $\mathrm{Wh}(\pi_1 K')$.*

Lemma 1.12. *Whitehead torsion is a homotopy invariant.*

2. The s-cobordism theorem

We have the h-cobordism theorem to classify homotopy cobordisms with trivial fundamental group. We will now extend this to cases with non-trivial fundamental group.

Theorem 2.1 (The s-cobordism Theorem)**.** *Let M be a smooth compact manifold without boundary, of dimension $n \geq 5$. Then there is a isomorphism*

$$\mathrm{H-Cob}(M) \longrightarrow \mathrm{Wh}(\pi_1 M)$$

between the set of h-cobordisms with ingoing boundary M modulo diffeomorphisms relative to M and the Whitehead group of $\pi_1 M$. It is given by sending the class of a cobordism $\partial_0 M \to W$ to $\tau(\partial_0)$, the Whitehead torsion of ∂_0. in particular, an h-cobordism ∂_0 is trivial if and only if $\tau(\partial_0) = 0$.

Remark 2.2. $\mathrm{Wh}(1) = 0$, so if $\pi_1 M = 1$, $\mathrm{Wh}(\pi_1 \mathrm{M}) = 0$, which implies the h-cobordism theorem.

Proof. We will see that the s-cobordism follows from an argument very similar to that used in the proof of the h-cobordism theorem. So we will try to duplicate the argument for the h-cobordism theorem, but use the universal cover $\widetilde{W}$ of the cobordism W, whose homology has an action of the fundamental group of W.

Step 1. Apply the Normal Form Lemma to the h-cobordism W.

$$W \cong M \times [0,1] + \sum_{I_q} \varphi^q + \sum_{I_{q+1}} \varphi^{q+1}$$

Step 2. The differential d_{q+1} is an isomorphism given by a matrix A

$$A : H_{q+1}(\widetilde{W}_{q+1}/\widetilde{W}_q) \to H_q(\widetilde{W}_q/\widetilde{W}_{q-1})$$

where the homology groups have $\mathbb{Z}\pi$-bases given by lifts of the handles.

Step 3. Attempt to diagonalize A using handle operations. Reduce A to the 0 matrix from 0 to 0 by cancelling appropriate pairs of handles. We have a table comparing row reduction operations on a matrix to handle operations.

	matrix operations over $\mathbb{Z}\pi$	handle operations
1.	$A \sim \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$	Cancellation Lemma
2.	multiply kth row by $x \in \mathbb{Z}\pi$ and add to the lth row	Modification Lemma for $[\varphi_k^{q+1} \mid_{S^q}] + x d_{q+1}[\varphi_l^{q+1}]$
3.	interchange rows or columns	relabel handles
4.	multiply the ith row by $\pm\gamma$ for $\gamma \in \pi$	pre-compose by a map of degree -1 and/ or change the lift of φ

If the matrix corresponding to a cobordism W can be diagonalized by these steps, W is trivial. For a general ring R, not every matrix in $\mathrm{GL}(R)$ is diagonalizable. The s-cobordism theorem characterizes nontrivial h-cobordisms W by characterizing the matrices associated to them.

We must consider the matrix A corresponding to W modulo the above relations. A priori, we have $A \in \mathrm{GL}_n(\mathbb{Z}\pi_1 M)$ where n depends on the size of A. Imposing relation 1 corresponds to considering A as an element of $\mathrm{GL}(\mathbb{Z}\pi_1 M)$. Imposing relations 2 and 3 corresponds to considering

A as an element of $\mathrm{K}_1(\mathbb{Z}\pi_1 M)$. Imposing relation 4 corresponds to considering A as an element of $\mathrm{Wh}(\pi_1 M)$.

To show that the map in the theorem is well defined, we note that $[A] = \tau(C_*^{\mathrm{cell}}(\widetilde{W}, \partial_0 \widetilde{W}))$ in $\mathrm{Wh}(\pi_1 M)$. Then since Whitehead torsion is a homotopy invariant, $[A]$ is a diffeomorphism invariant.

To show that it is surjective, we note that given an element of $\mathrm{Wh}(\pi_1 M)$, we can choose a matrix representing that element and build a handlebody with that matrix as its cellular differential.

That it is injective follows from basic properties of Whitehead torsion (e.g., the sum and composition formulas, see [5]), or from the following section (though this is overkill). □

3. The topological invariance of Whitehead torsion

The Whitehead torsion of a map is a topological invariant of a CW complex. That is, if $f : X \to Y$ is a homeomorphism, then f is a simple homotopy equivalence: $\tau(f) = 0$ in $\mathrm{Wh}(\pi_1 Y)$. This was a longstanding conjecture proved by Chapman, [2], more than ten years after Barden, Mazur, and Stallings proved the s-cobordism theorem.

Chapman first proved the theorem using the theory of $\mathcal{Q}$-manifolds, where $\mathcal{Q} = \prod_{\mathbb{N}}[0,1]$ is the Hilbert cube, a countable product of intervals. A $\mathcal{Q}$-manifold is a separable metric space with a open cover of open subsets of $\mathcal{Q}$.

Theorem 3.1. *Let X be a finite CW complex. Then $X \times Q$ is a $\mathcal{Q}$-manifold. Likewise, let M be a compact $\mathcal{Q}$-manifold. Then there is a homeomorphism $M \cong X \times \mathcal{Q}$ for some finite CW complex X.*

Thus, there is a close relation between the homotopy theory of CW complexes and the homeomorphism theory of $\mathcal{Q}$-manifolds. Chapman, building on a result of West, proved the following very satisfying theorem:

Theorem 3.2. *A map $f : X \to Y$ of finite CW complexes is a simple homotopy equivalence if and only if the map of $\mathcal{Q}$-manifolds $f \times \mathrm{id} : X \times \mathcal{Q} \to Y \times \mathcal{Q}$ is homotopic to a homeomorphism.*

Corollary 3.3 (Topological invariance of Whitehead torsion)**.** *If f is a homeomorphism then $\tau(f)$ is zero in* $\mathrm{Wh}(\pi_1 Y)$.

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