MATH 465, LECTURE 17: MORSE THEORY AND MILNOR'S CONSTRUCTION OF EXOTIC 7-SPHERES, FIRST PART

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Today we'll finish up the main result in Morse theory and explain why it gives a handlebody decomposition to every cobordism. After that, we'll begin on the construction of exotic smooth structures on spheres.

1. Morse theory

Over the last few lectures we've discovered an intimate relationship between the nondegenerate critical points of a function and the process of attaching handles. What we'll now show is that the functions we want are plentiful, so that constructing handle decompositions is easy.

Definition 1.1. A *Morse function* $f : M \to \mathbb{R}$ is a smooth function all of whose critical points are nondegenerate.

Proposition 1.2. *For any smooth manifold M, there exists a Morse function.*

Proof. Our basic strategy, as in much of differential topology, is to modify the problem till we can apply Sard's lemma, which says that critical values of a function are a measure zero subset of the range. Our proof has two stages: first, we consider a manifold that is an open subset of $\mathbb{R}^n$, and then we reduce the general case to that embedded case.

<u>Case 1:</u>

Let M be an open subset of $\mathbb{R}^n$ and f a smooth function on all of $\mathbb{R}^n$. Consider the gradient function $g := \nabla f : M \to \mathbb{R}^n$. Notice that the Jacobian matrix dg is also the Hessian matrix $\mathrm{Hess}(f)$, and thus we have the following useful correspondence. The critical points of f are precisely the zero set of g, and $x \in M$ is a *degenerate* critical point of f if and only if $g(x) = 0$ and $dg(x)$ is noninvertible. In other words, the degenerate critical points of f are in bijection with the critical points of g on which g vanishes.

Consider the following family of functions $\{f_a := f + \langle a, -\rangle\}$, where we range over all $a \in \mathbb{R}^n$. We will show that almost every f_a is Morse.[1] Notice that $g_a = \nabla f_a = g + a$. As we saw above,

$$\{\text{degenerate critical points of } f_a\} = \{\text{critical points of } g_a\} \cap g_a^{-1}(0).$$

In other words, f_a has degenerate critical points if and only if $0 \in \mathbb{R}^n$ is a critical value of g_a. By Sard's lemma, we know that 0 will be a regular value for almost every choice of a.

<u>Case 2:</u>

We treat the case of a general manifold by piggybacking on our work above. Choose an embedding $M \hookrightarrow \mathbb{R}^N$ of our n-manifold into some Euclidean space. Pick an open cover $\{U_\alpha\}$ of M such that for every U_α there is a projection $\pi : U_\alpha \to \mathbb{R}^n \subset \mathbb{R}^N$ onto a coordinate subspace that defines a coordinate chart for U_α.[2] In other words, for each U_α, we get an embedding $U_\alpha \hookrightarrow \mathbb{R}^n$ by only paying attention to some subset of the coordinate variables in the full Euclidean space. Note that we restrict attention to manifolds such that we can make such a cover that is countable and locally finite.

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[1]Here "almost every" is in the sense of measure theory.

[2]Notice that, yet again, we depend on Sard's lemma here.

Given a smooth function $f : M \to \mathbb{R}$, we can again consider $f_a = f + \langle a, - \rangle$, for any $a \in \mathbb{R}^N$. By restricting our attention to U_α, we can apply Case 1 to determine the measure zero set in $\mathbb{R}^N$ of the vectors a for which $f_a|_{U_\alpha}$ is not Morse. Running over all opens in the cover, we construct a measure zero set of a for which f_a is not Morse on M (here we use that the cover is countable). Hence there is some value a for which f_a is Morse. □

Note that our proof suggests that Morse functions are dense in the space of all smooth functions.

Corollary 1.3. *Every cobordism W has a handlebody decomposition.*

Proof. With a little work, see [2], one can show that there exists a Morse function f such that $f(\partial_0 W) = 0$ and $f(\partial_1 W) = 1$. □

Remark 1.4. This was the last necessary ingredient in our proof of the h-cobordism theorem. Recall that our proof assumed the existence of a handlebody presentation for an h-cobordism W: The existence of such a handlebody presentation is now assured by existence of a Morse function on W and the result from the previous lecture that a Morse function begets a handle presentation.

2. Exotic spheres

By the h-cobordism theorem, there is a single *topological* manifold structure on the homotopy type of the sphere S^n, $n \geq 5$. This naturally leads us to ask the following.

Question 2.1. Is there also a unique *smooth* manifold structure? If not, what are the other smooth structures?

In this lecture we'll just try to construct an exotic sphere and see what we'll need to do to show they exist.

Observe that some spheres are obtained as total spaces of fibrations. For example, there are the Hopf fibrations $S^1 \hookrightarrow S^3 \to S^2$, $S^3 \hookrightarrow S^7 \to S^4$, and so on. Let's try to fiber them differently. In other words, we want to build a smooth fiber bundle

$$S^{n-1} \hookrightarrow M \to S^n$$

such that the connecting map $\delta : \pi_n S^n \to \pi_{n-1} S^{n-1}$ is an isomorphism because the manifold M will then be $(2n-2)$-connected. Note that this assertion about connectedness follows from looking at the homology Serre spectral sequence for this (putative) fibration: by the Hurewicz map from homotopy to homology, the connecting map δ is also the differential $d^n : H_n(S^n, H_0 S^{n-1}) \to H_0(S^n, H_{n-1} S^{n-1})$. Hence we know that M has the same homology as S^{2n-1} and M is simply connected, and so, by the h-cobordism theorem, M is homeomorphic to S^{2n-1}.

So we now have a strategy: classify the smooth fibrations of the form above. Let's try $S^1 \hookrightarrow M \to S^2$ first. Abstractly, we could construct such a smooth fibration by picking a map $S^2 \to B\mathrm{Diff}(S^1)$, but this is a hard space to work with. Instead, we will consider maps into $BSO(2) \subset B\mathrm{Diff}(S^2)$, since that space is much more tractable. Given a map $f : S^2 \to BSO(2)$, we pull back the tautological bundle to get an $SO(2)$-bundle on S^2. Since $SO(2)$ is the space S^1, we have our S^1-bundle. Since the map f induces a map of $SO(2)$-bundles (by construction), we get a map between the long exact sequences in homotopy. The boundary map $\delta : \pi_2 S^2 \to \pi_1 S^1$ is an isomorphism if and only if the map $f_* : \pi_2 S^2 \to \pi_2 BSO(2)$ is an isomorphism. But if f_* is an isomorphism, then M is precisely S^3 as f must then classify the Hopf bundle.

Let's try $S^3 \hookrightarrow M \to S^4$. Thus we consider maps $S^4 \to BSO(4)$. What is $\pi_4 BSO(4)$? Recall that we have a universal cover

$$Sp(1) \times Sp(1) \to SO(4) \cong \mathrm{Isom}^+(\mathbb{H}),$$
$$(u, v) \mapsto (x \to u \cdot x \cdot v^{-1}).$$

(Recall that $Sp(1) = SU(2)$ is the unit quaternions and hence the space S^3.) The kernel of this map is $\mathbb{Z}^\times = \{\pm 1\}$. Thus we have a fibration

$$\mathbb{Z}^\times \hookrightarrow Sp(1) \times Sp(1) \to SO(4),$$

and the long exact sequence in homotopy tells us $\pi_4 B(Sp(1)\times Sp(1)) \cong \mathbb{Z}\times\mathbb{Z} \cong \pi_4 BSO(4)$.

Now we need to address the question of what condition on $f : S^4 \to BSO(4)$ implies that the boundary map δ is an isomorphism? Recall that $S^3 = SO(4)/SO(3)$. Hence we can view M as the pullback along f of the bundle

$$S^3 \hookrightarrow ESO(4) \times_{SO(4)} S^3 \to BSO(4),$$

where the total space is the Borel construction

$$ESO(4) \times_{SO(4)} S^3 \cong (S^3)_{hSO(4)},$$

namely, the homotopy orbits of the $SO(4)$ action on S^3. Playing with notation, we see

$$S^3 = SO(4)/SO(3) \implies S^3/SO(4) = (SO(4)/SO(3))/SO(4) \simeq \mathrm{pt}/SO(3) = BSO(3).$$

Thus, f gives a map of fibrations

$$\begin{array}{ccccc} S^3 & \hookrightarrow & M & \to & S^4 \\ \downarrow & & \downarrow & & \downarrow \\ S^3 & \hookrightarrow & BSO(3) & \to & BSO(4) \end{array}$$

and hence a map between the long exact sequences in homotopy. In particular, we have a commuting square

$$\begin{tikzcd} \pi_4 S^4 \arrow[r, "\delta"] \arrow[d, "f_*"] & \pi_3 S^3 \arrow[d, "\cong"] \\ \pi_4 BSO(4) \arrow[r, "\delta'"] & \pi_3 S^3 \end{tikzcd}$$

and so we see that describing δ' will let us characterize which maps f_* make δ an isomorphism.

Understanding δ' is quite straightforward, thankfully. Notice that our S^3-fibration over $BSO(4)$ "rotates" to the fibration $SO(3) \hookrightarrow SO(4) \to S^3$ so that the map $\pi_3 SO(4) \to \pi_3 S^3$ is exactly the map $\delta' : \pi_4 BSO(4) \to \pi_3 S^3$. From our work above, we know we can lift this $SO(3)$-bundle to the bundle $Sp(1) \overset{\Delta}{\hookrightarrow} Sp(1) \times Sp(1) \to S^3$ (using the obvious universal cover stuff). But now we see

$$\pi_3(Sp(1)\times Sp(1)) \to \pi_3 S^3$$
$$(i, j) \mapsto i - j$$

and so δ' is precisely the "difference" map. Hence f_* must send a generator of $\pi_4 S^4$ to pair of integers $(i, j) \in \pi_4 BSO(4)$ that differ by ± 1.

References

[1] Guillemin, Victor; Pollack, Alan. Differential topology. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974. xvi+222 pp.

[2] Milnor, John. Lectures on the h-cobordism theorem. Notes by L. Siebenmann and J. Sondow. Princeton University Press, Princeton, N.J. 1965 v+116 pp. Available from `http://www.maths.ed.ac.uk/~aar/surgery/hcobord.pdf/`.