MATH 465, LECTURE 18: MILNOR'S CONSTRUCTION OF EXOTIC 7-SPHERES, SECOND PART

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1. Recapitulation

Recall we constructed manifolds M_{ij} by fibering S^3 over S^4. The regular 7-sphere is obtained by the Hopf fibration. M_{ij} is classified by a map $f_{ij} : S^4 \to BSO(4)$.

$$\begin{tikzcd} S^3 \arrow[r] & M_{ij} \arrow[d] & \\ & S^4 \arrow[r, "f_{ij}"] & BSO(4) \end{tikzcd}$$

Recall we computed $\pi_4 BSO(4) \cong \pi_4 B(Sp(1) \times Sp(1)) \cong \mathbb{Z} \times \mathbb{Z}$, indexed by (i, j). The connecting map in the long exact sequence of homotopy groups is an isomorphism if $i - j = 1$ and so M_{ij} is 6-connected. In particular, M_{ij} is homotopy equivalent to S^7.

2. Secondary Invariants

Therefore, we need to obtain an invariant which can distinguish homotopic 7-manifolds. Our current selection of invariants is insufficient for this purpose. For instance, Pontryagin classes only exist in $4k$-dimensions. Stiefel-Whitney numbers are also zero for our 7-manifolds. However, analogous to our construction of the Whitehead group, we can construct a secondary invariant. Heuristically, these exist typically when our primary invariant is zero: The secondary invariant describes in what manner the primary invariant is trivial.

The cobordism class is our primary invariant. By Thom's theory we will show that $\Omega_7^{\mathrm{SO}} = 0$, namely that every oriented 7-manifold is bounded by an oriented 8-manifold. Therefore, we may choose some 8-manifold B^8 with $\partial B^8 = M^7$ and compute an associated characteristic number of the 8-manifold. We will check to what degree this number is well defined.

Proposition 2.1. $\Omega_7^{\mathrm{SO}} = 0$

Proof. □

We now define the secondary invariant which is the main focus of this lecture.

Proposition 2.2. *Let M^7 be an oriented manifold with $H_3(M) = H_4(M) = 0$. Choose an oriented 8-manifold B^8 with boundary $\partial B^8 \cong M^7$. Then, the following is an invariant of M:*

$$\lambda(M) = \langle 2p_1^2(B), [B, M]\rangle - \mathrm{Sig}(B) \bmod 7.$$

Here, the signature of the manifold with boundary B is understood to mean the signature of the nondegenerate bilinear form on $H^4(B) \cong H^4(B, M)$ – this isomorphism makes use of the fact that the boundary M has middle cohomology equal zero.

We will need Hirzebruch's Signature Theorem, which gives the signature of a $4k$-dimensional manifold in terms of the Pontryagin numbers (with coefficients in terms of the L-genus of the formal power series $\frac{\sqrt{z}}{\tanh(\sqrt{z})}$). Signature is a cobordism invariant ([**?**], [**?**]). In particular, we will need the signature of an 8-manifold.

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However, to motivate the construction, we will compute the signature of a 4-manifold N^4.

$$\mathrm{Sig}(N^4) = \langle \frac{p_1}{3}, [N] \rangle$$

We check the scaling factor (here a third) since signature is determined by the first Pontryagin number. $\Omega_4^{\mathrm{SO}} \otimes \mathbb{Q} = \mathbb{Q}$. We only need to check on one 4-manifold: we will use $\mathbb{C}P^2$. Recall that in general the i-th Pontryagin class of a real vector bundle $E \to N$ is given in terms of the even Chern classes of the complexification of E:

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(N; \mathbb{Z})$$

For $\mathbb{C}P^2$, we compute the total Pontryagin character, $p(v) = \bar{c}(v \otimes \mathbb{C})$:

$$c(\mathbb{C}P^2) = (1+x)^3 \bmod x^3 \Rightarrow \bar{p}(\mathbb{C}P^2) = (1+x)^3(1-x)^3 \bmod x^3 = 1 - 3x^2 \Rightarrow p_1(\mathbb{C}P^2) = 3x^2$$

Finally, we use that the signature of $\mathbb{C}P^2$ is 1 to obtain the $1/3$ factor.

We are interested in the case of 8-manifolds. The signature of an arbitrary 8-manifold can be checked using two specific test cases, e.g., $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$. We obtain (exercise):

$$\mathrm{Sig}(N^8) = \langle \frac{7p_2 - p_1^2}{45}, [N] \rangle$$

We are now in a position to complete the proof of our proposition.

Proof. Choose two 8-manifolds B and B' which both bound M. $\partial B \cong \partial B' \cong M$. Consider $N = B \cup_M \bar{B}'$ as an oriented 8-manifold.

Remark 2.3. This proposition is closely related to the construction of "levels" of quantizations in quantum field theory, e.g., as in the Weiss-Zumino-Witten model for loop-groups.

By the long exact sequence of a cofibration, and our initial assumption of vanishing middle cohomology of M, $H_3(M) = H_4(M) = 0$, we observe that the middle cohomology of $B \cup_M \bar{B}'$ is obtained as a direct sum of that of B and $\bar{B}'$.

$$H^4(B \cup_M \bar{B}') = H^4(B) \oplus H^4(\bar{B}')$$

Using this direct sum splitting, we obtain that

$$\mathrm{Sig}(B \cup_M \bar{B}') = \mathrm{Sig}(B) + \mathrm{Sig}(\bar{B}') = \mathrm{Sig}(B) - \mathrm{Sig}(B')$$

The tangent bundle of $B \cup_M \bar{B}'$ is classified by a map $T : B \cup_M \bar{B}' \to BSO(8)$ which restricts to the inclusions of B and $\bar{B}'$ respectively. Again since $H_4(M) = 0$,

$$p_1(B \cup_M \bar{B}') = p_1(B) + p_1(\bar{B}') = p_1(B) - p_1(B')$$
$$\Rightarrow p_1^2(B \cup_M \bar{B}') = p_1^2(B) + p_1^2(B')$$

Let $\lambda(X) = \langle 2p_1^2(X), [X] \rangle - \mathrm{Sig}(X)$ be our putative invariant. We show that $\lambda(B) - \lambda(B')$ is an integer multiple of 7.

Note the sign change due to the orientation of B' as opposed to $B \cup_M \bar{B}'$ (recall we denote $N = B \cup_M \bar{B}'$):

$$\langle 2p_1^2(B), [B] \rangle) - \langle 2p_1^2(B'), [B'] \rangle = \langle 2p_1^2(N), [N] \rangle$$

Using that $\mathrm{Sig}(B) - \mathrm{Sig}(B') = \mathrm{Sig}(N)$, and dropping the pairing with the fundamental class $[N]$ in our notation (all Pontryagin classes appearing from now until the end of the proof will be those of N), we obtain that

$$\lambda(B) - \lambda(B') = 2p_1^2 - \mathrm{Sig}(N)$$

Using the signature theorem, $\mathrm{Sig}(N) = \frac{7p_2 - p_1^2}{45}$ (note 7 is coprime to 45), we are done. ◻

Exercise 2.4. Do the same procedure for 3-manifolds (every 3-manifold bounds a 4-manifold). Do you obtain an interesting invariant?

3. λ Invariant in action

Recall we constructed the following map of fibrations:

$$\begin{tikzcd}
S^3 \arrow[r] \arrow[d] & M^7_{ij} \arrow[r] \arrow[d] & S^4 \arrow[d, "f_{ij}"] \\
S^3 \arrow[r] & BSO(3) \arrow[r] & BSO(4)
\end{tikzcd}$$

We view the classifying map $f_{ij} \in \pi_4(BSO(4)) \cong \pi_3(SO(4)) \cong \pi_3(Sp(1) \times Sp(1))$ and thus as an element $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.

Let f_{ij} be such that $i - j = 1$, and so M^7 is a homology 7-sphere. Let $k = i + j$ be the free variable and denote $M^7_k = M^7_{ij}$.

We have constructed M^7 as a (3-)sphere bundle over a base manifold (S^4). Therefore, writing $M^7_k = \mathrm{Sph}(\xi_k)$, we naturally obtain M^7_k as the boundary of the 8-manifold $B = \mathrm{Disk}(\xi_k)$. In particular $\mathrm{Sig}(B) = 1$.

Lemma 3.1 (1)**.** *$p_1(\xi_{i,j}) = \pm 2(i + j)\iota$ Here ι denotes the standard generator of $H^4(S^4)$.*

We will use this lemma to prove:

Lemma 3.2 (2)**.** $\lambda(M^7_k) = k^2 - 1 \bmod 7$

Now, as long as k^2 is not congruent to 1 mod 7, M^7_k cannot be diffeomorphic to the usual 7-sphere.

Proof. We prove the first lemma. Consider reversing the orientation of our fiber S^3. The first Pontryagin class is invariant under this change. Observe that:

$$\xi(-1f_{i,j}) = \xi(f_{-j,-i})$$

Therefore, our formula must be symmetric in i and j. So, for some constant $c \in \mathbb{Z}$, $p_1 = c(i + j)\iota$ We just need to check in one non-zero example what this constant is. For $k = 1$ we have the usual Hopf fibration, and so $\mathrm{Disk}(\xi_1) \cong \mathbb{H}P^2 - D^8$. As a power series,

$$p(\mathbb{H}P^n) = \frac{(1 + x)^{2n+2}}{1 + 3x} \Rightarrow p_1(\mathbb{H}P^2) = 2x \Rightarrow c = \pm 2$$

$\square$