MATH 465, LECTURE 23: PLUMBING

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Our goal in this talk is to explain the construction known as "plumbing." The input is an even unimodular lattice Q and the output is a $4m$-manifold whose middle cohomology has intersection pairing described by Q.

Theorem 0.1 (Arf)**.** *The signature of an even unimodular lattice is a multiple of* 8.

Remark 0.2. Recall that for a lattice, *even* means that $\langle v, v\rangle$ is even for all $v \in Q$, and *unimodular* means $\det Q = 1$.

Example 0.3. The E_8 lattice is defined by the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Using Arf's theorem and plumbing, we will obtain a map

$$\mathbb{Z} \to bP_{4m} = \{ \text{ parallelizable manifolds bounding a } 4m\text{-manifold } \}$$

by starting with an even unimodular lattice Q and computing its $signature(Q)/8$ (this lives in $\mathbb{Z}$), and then using plumbing to construct a $4m$-manifold P_Q^{4m} and taking its boundary $\partial P_Q \in bP_{4m}$.

1. Plumbing

Plumbing goes as follows.

- Given Q and a basis indexed by I, for each index $i \in I$ choose a sphere $S_i := S_i^{2m}$ and take the disk bundle $Disk(TS_i) \to S_i$ of its tangent bundle.
- For each nonzero entry a_{ij} in Q with $i \neq j$, glue $Disk(TS_i)$ and $Disk(TS_j)$ around points in each as follows:
  1. Choose a disk in the base $D_i := D_i^{2m} \subset S_i$ and pick a splitting of the restriction of $Disk(TS_i)$ to D_i as $D_i \times D^{2m}$.
  2. Do likewise for S_j — pick a disk ...
  3. Glue the two product disks by "switching order": $D_i \times D^{2m} \overset{\cong}{\to} D^{2m} \times D_j$.

Remark 1.1. If $a_{ij} > 0$, pick a bunch of disjoint points in S_i and S_j and do the construction as above. If a_{ij} is negative, reverse orientations. Note, however, that one can pick a representing matrix Q so that all the off-diagonal entries are 0 or 1.

By following this construction we obtain a $4m$-manifold P_Q. The intersection number of the S_i and S_j is precisely a_{ij}. Hence, if Q is unimodular and has all 2's along the diagonal, we get an isomorphism

$$H_{2m}(P_Q) \to H_{2m}(P_Q, \partial P_Q)$$

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in the long exact sequence of the pair, where this map is given by the intersection matrix Q. Thus Q unimodular implies $H_{2m}(\partial P_Q) = 0$ and so ∂P_Q is $(4m-2)$-connected. By the h-cobordism theorem, we then see that ∂P_Q is then homotopy equivalent to S^{4m-1}. (Note that we don't need all 2's along the diagonal, just that we have a bundle V with $euler(V) = a_{ii}$ CANT READ MY NOTES HERE)

Theorem 1.2 (Kervaire-Milnor)**.** *Plumbing provides a group homomorphism*

$$(\mathbb{Z}, +) \to (bP_{4m}, \#)$$

that is surjective*. The kernel is* $\sigma_m \mathbb{Z}$*, with*

$$\sigma_m = a_m 2^{2m-2}(2^{2m-1} - 1) numerator(B_m/4m),$$

where $a_m = 1$ *or* 2 *and* B_m *denotes the* m^{th} *Bernoulli number.*

Remark 1.3. These particular numbers are a consequence of Adams, *J(X) IV*.

2. Example: $4m = 8$

Note that coker $J_7 = 0$ so $\Theta_7 = bP_8$. Earlier, we gave an example of an even unimodular lattice, E_8. We will now address the question: Is ∂P_{E_8} diffeomorphic to the standard 7-sphere? We know that

$$sig(P_{E_8}, \partial) = sig(E_8) = 8,$$

and we know that the tangent bundle TP_{E_8} restricted to its 4-skeleton is trivial, so $p_1(P_{E_8}) = 0$. Hence,

$$sig(E_8) = 7p_2/3^2 \cdot 5.$$

As P_{E_8} is 3-connected, it is a Spin-manifold. By the Atiyah-Singer index theorem, there thus exists a Dirac operator $\not{D}$ such that

$$\hat{A}_8(P_{E_8}) = ind\, \not{D} \in \mathbb{Z}.$$

(We don't need anything about the operator other than its existence.)

This integrality result has the following consequence. Recall that

$$\hat{A}_8(M) = \frac{7p_1^2 - 4p_2}{2^7 \cdot 3^2 \cdot 5}.$$

Now suppose that ∂P_{E_8} is the standard 7-sphere. Then it bounds the 8-disk, and so we can construct a boundaryless 8-manifold

$$X = P_{E_8} \cup_{S^7 = \partial P} D^8.$$

Then, because $p_1 = 0$,

$$\hat{A}(X) = \frac{-4p_2}{2^7 \cdot 3^2 \cdot 5} = -\frac{1}{28} \cdot \frac{1}{8}\left(\frac{7p_2}{3^2 \cdot 5}\right) = -\frac{1}{28} \cdot \frac{sig(E_8)}{8} = -\frac{1}{28},$$

since we saw earlier that $sig(E_8) = 8$. We thus have a contradiction! Hence ∂P is not diffeomorphic to S^7.

Note that by Kervaire-Milnor, we know that 28 connect-sums of P_{E_8} has the property

$$\hat{A}\left(P_{E_8}^{\#28} \cup_{\partial(P^{\#28})} D^8\right) = -1 \in \mathbb{Z}.$$

(Here $P^{\#28}$ means "iterate the connect-sum operation 28 times.") Thus we see that the obstruction/invariant we've constructed gives an isomorphism

$$bP_8 \cong \mathbb{Z}/28.$$

Remark 2.1. What we've just done is *not* a proof of Kervaire-Milnor. The goal was simply to show that these invariants we've been discussing are not exotic: they are consequences of the index theorem, one of the central theorems of mathematics. We simply wanted to explore the invariants in low dimensions.

3. Moving on from here

We just gave some techniques for exploring bP_{4m}. What about bP_{4m+2}? The important fact is that a framed $2k$-manifold has a quadratic refinement of the intersection pairing in the middle dimension. The Arf invariant of a quadratic form gives an invariant of P with $\partial P \in bP_{4m+2}$.

For example, if Q is in a symplectic basis, then $Arf(Q) = \sum_{i \in I} q(x_i)q(y_i)$ is a function taking values in $\mathbb{F}_2$.

By work of many topologists (notably Browder and recently Hill-Hopkins-Ravenel), we know that the Kervaire invariant detects bP.

Theorem 3.1.

$$bP_{4m+2} \cong \begin{cases} \mathbb{Z}/2 & 4m+2 \neq 2^k - 2 \\ 0 & \textit{else, or for } 4m+2 = 6, 14, 30, 62 \end{cases}$$

References