MATH 465, LECTURE 9: NORMAL FORM LEMMA, FIRST PART: HANDLE ELIMINATION

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Before proceeding, here is a brief reminder on notation for handlebodies:

Notation 0.1. A handlebody W of dimension n is constructed by inductively adding handles of increasing index to a trivial cobordism $\partial_0 W \times [0.1]$. W_q is the handlebody obtained by adding the handles of index $\leq q$: There is a sequence of inclusions

$$\partial_0 W \times [0,1] = W_{-1} \subset W_0 \subset \ldots W_{q-1} \subset W_q \subset W_{q+1} \ldots \subset W_n.$$

so $W_{-1} = \partial_0 W \times [0,1]$. Each subsequent manifold in the filtration is obtained from the previous by addition the addition of handles of the next higher index, $W_q = W_{q-1} + \sum_{I_q} \phi_i^q$, where I_q denotes the set indexing the q-handles. The outgoing boundary, $\partial_1 W_q$ is $\partial W_q - \partial_0 W \times \{0\}$, which is the side on which we attach additional handles.

1. Normal form lemma

Over the next several lectures, our goal is to prove the following technical lemma, which will be a key component in the proof of the s-cobordism theorem.

Lemma 1.1 (Normal Form Lemma)**.** *Let W be a handlebody of dimension $n \geq 6$ which is an oriented h-cobordism. Then for any choice of q with $2 \leq q \leq n-3$, W is diffeomorphic relative $\partial_0 W$ to*

$$\partial_0 W \times [0,1] + \sum_{I_q} \phi_i^q + \sum_{I_{q+1}} \phi_i^{q+1},$$

i.e., a handlebody whose handles are only of index q and $q+1$.

Before beginning the proof, I'll first outline how the s-cobordism will follow. Here's the basic idea. If W is an h-cobordism, then the relative homology $H_*(W, \partial_0 W)$ is zero. Additionally, if W has handles of index only q and $q+1$, then in order for these homology groups to vanish, the cellular differential d_{q+1} must be an isomorphism:

$$H_{q+1}(W_{q+1}/W_q) \xrightarrow{\cong} H_q(W_q/W_{q-1}) = H_q(W_q/W_0),$$

since all of the other groups in the cellular chain complex $C_k^{\text{cell}}(W, \partial_0 W) = H_k(W_k/W_{k-1})$ are zero unless $k = q, q+1$.

Now we know that these homology groups are free of rank $|I_{q+1}| = |I_q|$ and the matrix d_{q+1} has basis given by the handles ϕ_i^q. If this matrix was diagonal with entries ± 1, we could immediately apply the cancellation lemma $|I_q|$ times to each pair of a q-handle and a $(q+1)$-handle to show that W is diffeomorphic to the product $\partial_0 W \times [0,1]$. If the matrix d_{q+1} is not diagonal, we can try to diagonalize this matrix by moving handles, and whenever we apply the cancellation lemma, we use it to get rid of handles. However, we will not always be able to do so: This matrix determines an obstruction to killing off all the handles and making W diffeomorphic to a trivial cobordism. We'll describe this obstruction, which lives in what is known as the Whitehead group, which a quotient of the first algebraic K-group of the group ring of fundamental group, $K_1(\mathbb{Z}[\pi_1 W])$.

Remark 1.2. We still have yet to show that every cobordism has a handlebody structure. We will prove this later, in our discussion of Morse theory.

Date: Lecture April 19, 2010. Last edited April 28, 2010.

We now turn to the normal form lemma. The proof will require several subsidiary lemmas.

2. Eliminating a handle

We will need the following:

Notation 2.1. Let $\partial_1^\circ W_q \subset \partial_1 W_{q+1}$ denote the closed complement in $\partial_1 W_q$ to the images of the attaching maps of the $(q+1)$-handles:

$$\partial_1^\circ W_q = \partial_1 W_q - \coprod_{I_{q+1}} \phi_i^{q+1}(S^q \times \mathring{D}^{n-q-1}).$$

The next lemma provides a situation in which it is possible to swap a q-handle for a $(q+2)$-handle in a handlebody presentation of W, all the while without changing the diffeomorphism type of W.

Lemma 2.2 (Elimination Lemma)**.** *Let W be a handlebody with no handles of index less than q, where $1 \geq q \leq n-3$. Let ϕ_1^q be a specific q-handle we would like to eliminate from the presentation of W. If there exists an embedding $\psi^{q+1} : S^q \times D^{n-q-1} \longrightarrow \partial_1^\circ W_q$ such that*

- *the attaching sphere $\psi^{q+1}(S^q \times \{0\})$ intersects the transverse sphere $\{0\} \times S_1^{n-q-1}$ of the q-handle ϕ_1^q transversally and in a single point;*
- *the attaching sphere $\psi^{q+1}(S^q \times \{0\})$ does not intersect the transverse spheres of any of the other q-handle;*
- *ψ^{q+1} is isotopic to a trivial embedding in $\partial_1 W_{q+1}$,*

then there exists a diffeomorphism

$$W \cong \partial_0 W \times [0,1] + \sum_{I_q - \{1\}} \phi_i^q + \sum_{I_{q+1}} \tilde{\phi}_i^{q+1} + \psi^{q+2} + \sum_{I_{q+2}} \tilde{\phi}_i^{q+2} + \cdots$$

Remark 2.3. This formulation is something of a mouthful, but don't be discouraged: The proof is easier than the statement.

Proof. Note that we can isotope ψ^{q+1} so that it satisfies all of the conditions. We continue to denote this (possibly isotoped) embedding by ψ^{q+1}.

Case 1: Assume W has no handles of index greater than $q+1$. Now, add a handle along the map ψ^{q+1} to obtain $W + \psi^{q+1}$. Since ψ^{q+1} is a trivial embedding, by our cancellation lemma we can choose an attaching map $\psi^{q+2} : S^{q+1} \times D^{n-q-3} \to \partial_1 W$ so that the attached $(q+2)$-handle will cancel ψ^{q+1}. I.e., the map ψ^{q+2} avoids the image of other $q+1$-handles and intersects the transverse sphere of ψ^{q+1} in one point. By cancellation, we have diffeomorphisms

$$\begin{aligned} W &\cong W + \psi^{q+1} + \psi^{q+2} \\ &\cong \partial_0 W \times [0,1] + \sum_{I_q - \{1\}} \phi_i^q + \phi_1^q + \sum_{I_{q+1}} \phi_i^{q+1} + \psi^{q+1} + \psi^{q+2} \end{aligned}$$

and moving the ψ^{q+1} to cancel the ϕ_1^q

$$\cong \partial_0 W \times [0,1] + \sum_{I_q - 1} \phi_i^q + \sum_{I_{q+1}} \phi_i^{q+1} + \psi^{q+2}.$$

Which proves the lemma for this case.

General case: Apply the first case to W_{q+1}:

$$\begin{aligned} f : W_{q+1} &\overset{\cong}{\to} W_{q+1} + \psi^{q+1} + \psi^{q+2} \\ &\cong \partial_0 W \times [0,1] + \sum_{I_q - 1} \phi_i^q + \sum_{I_{q+1}} \phi_i^{q+1} + \psi^{q+2}. \end{aligned}$$

Now add the higher index handles of W along the diffeomorphism f. ∎

Lemma 2.4. *Let W be the handlebody of dimension $n \geq 6$ such that $\partial_0 W \to W$ is 1-connected. Then W is diffeomorphic to a handlebody with no 0-handles or 1-handles.*

Remark 2.5. What's a 0-handle? Applying the definition of k-handle, we see

$$\phi^0 : S^{-1} \times D^n = \emptyset \to \partial_1 W,$$

so ϕ^0 must be the empty map. Hence $W + \phi^0$ is thus $W \sqcup D^n$, and the outgoing boundary is $\partial_1(W + \phi^0) \cong \partial_1 W \sqcup S^{n-1}$.

Proof. First, we will see that the 1-connectedness hypothesis means that all the 0-handles have 1-handles attached when we construct W_1. Hence, the cancellation lemma assures us that there is a diffeomorphic handlebody with no 0-handles. Here's the argument more explicitly. To kill ϕ_i^0, we need a 1-handle ϕ_j^1 such that $\phi_j^1\big|_{S^0}$ sends one point in S^0 to $\partial_1 W_{-1}$ and one point to the 0-handle. Since the map $\partial_0 W \to W$ is connected, there exists for each ϕ_i^1 for each ϕ_i^0. Now apply the cancellation lemma.

The rest is done later $\cdots$ □

References

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