

# INTEGRALS ON SPIN MANIFOLDS AND THE $K$ -THEORY OF $K(\mathbb{Z}, 4)$

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ABSTRACT. We prove that certain integrals  $\int_M P(x)\widehat{A}$  take only integer values on spin manifolds. Our method of proof is to calculate the real connective theory  $ko_*K(\mathbb{Z}, 4)$  and  $ko_*BSpin$  through dimension 15 and determine an explicit set of spin manifolds whose image under the  $\widehat{A}$ -genus generate these groups.

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## 1. INTRODUCTION

This paper uses homotopy theory to prove that certain integrals take only integer values. These are integrals of the form

$$\int_M P(x)\widehat{A}$$

on a spin manifold  $M$ , in which:  $P$  is a polynomial with rational coefficients,  $x$  is an integral closed 4-form; and the  $\widehat{A}$ -genus is a multiplicative sequence of characteristic classes, defined by the curvature of the manifold. The particular

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integrality result we prove for spin 12-manifolds has relevance to mathematical physics, to the quantization of the action in M-theory, [DMW].

Integrality results of a similar form have had great conceptual and practical implications. We sketch a few: In geometry, for instance, Milnor applied an integrality theorem of Thom's for 8-manifolds in order to define an invariant of 7-manifolds with vanishing middle cohomology groups; he then used this invariant to distinguish certain 7-manifolds which were homotopic to the 7-sphere, i.e., to prove the existence of exotic 7-spheres. Later, in the 1980s, Freedman constructed a large number of topological 4-manifolds whose unsmoothability was a consequence of Rohklin's theorem, that the signature of a smooth spin 4-manifold is an integral multiple of 16. Both Thom's and Rohklin's integrality theorems were later subsumed by the Atiyah-Singer index theorem, which interpreted them in terms of the index theory of differential operators (for example, in terms of dimensions of spaces of holomorphic sections). In mathematical physics, integrality have also had deep significance: At the most basic level, a classical mechanical system modeled by a symplectic manifold may be quantized only if its symplectic form is integral. Then, relatedly, from the perspective of Chern-Weil theory, the integrality of Chern classes is related to the quantization of electromagnetic charge as interpreted in Yang-Mills gauge theory. Index theory has subsequently had further, related, applications, such as to anomaly cancellation.

The particular integrals which we consider in this work, however, do not arise straightforwardly in classical index theory, although Diaconescu, Moore, and Witten have interpreted them in terms of  $E_8$  gauge theory in [DMW]. Stable homotopy theory can also be applied to the study of these invariants, and the method is interesting in part because it can be applied quite generally, wherever one might want an integrality result. Below I overview the argument used in this paper.

The initial problem is roughly inverse to the original index theorem: determine the polynomials  $P$  such that  $\int_M P(x)\hat{A}$  is an integer, where  $M$  is a spin manifold,  $x$  is a differential 4-form representing an element of  $H^4(M; \mathbb{Z})$ . Were  $x$  an integral 2-form, the Atiyah-Singer index theorem theory would immediately provide the answer: Every element  $x \in H^2(M, \mathbb{Z})$  is the first Chern class of a complex line bundle  $L$ , and  $\text{Ch}(L) = \exp(x)$ , hence  $\int_M \exp(x)\hat{A}$  takes integer values. However, if one does not already know that the polynomial  $P$  is the Chern character of a bundle, the question of whether  $\int_M P(x)$  or  $\int_M P(x)\hat{A}$  is necessarily an integer appears opaque from the more rigid perspective of index theory.

The yoga of homotopy theory is to try to interpret the integral within the realm of stable homotopy theory and then apply the well-established machinery for computation available there. This is the outline of how Milnor produced exotic 7-spheres, Thom classified which manifolds are the boundaries of other manifolds, Hirzebruch proved the signature theorem, and how Adams determined the number of independent vector fields on spheres.

To begin this interpretation, note that the 4-cocycle  $x$  is equivalent to the data of a map to an Eilenberg-MacLane space,  $x : M^n \rightarrow K(\mathbb{Z}, 4)$ . The manifold together with the 4-class thus defines an element of the  $n$ th spin bordism group of  $K(\mathbb{Z}, 4)$ ,  $\text{MSpin}_n K(\mathbb{Z}, 4)$ . The  $\hat{A}$ -genus defines a ring homomorphism from the bordism ring of a point to  $\mathbb{Q}$ , and likewise any polynomial in a 4-class  $P$  defines a homomorphism

$$\pi_* \text{MSpin} \wedge K(\mathbb{Z}, 4) \rightarrow \mathbb{Q}$$

defined on an element of  $\mathbf{MSpin}_n \mathbf{K}(\mathbb{Z}, 4)$  by picking representative  $(M, x)$  of that element and evaluating the integral  $\int_M P(x) \widehat{A}$ . Fixing  $n$ , one can thereby prove that integral is always an integer by finding a set of spin  $n$ -manifolds together with 4-forms that generate the group  $\mathbf{MSpin}_n \mathbf{K}(\mathbb{Z}, 4)$ : The integrality result obtains just by checking that the integral takes only integer values on these generators. The requisite steps are therefore to compute the bordism group and to find explicit generators.

The work of Atiyah, Bott, and Shapiro constructs the  $\widehat{A}$ -genus as a map of ring spectra from  $\mathbf{MSpin}$  to the connective real  $K$ -theory spectrum,  $ko$ . Building on this work, Anderson, Brown, and Peterson gave a splitting of spin bordism in terms of connective covers of real  $K$ -theory. Standard tools of homotopy, such as the Adams spectral sequence, then allow one to begin the problem in earnest along these lines. The spectral sequence not only allows the computation of the group, but the relation between the group and the Adams filtration allow one to determine when a given set of manifolds are generators. Determining generators for a bordism ring is difficult in general; even for the spin bordism of a point complete generators are unknown. The existence of differentials in the spectral sequence computing the  $ko$ -theory of  $\mathbf{K}(\mathbb{Z}, 4)$  further complicates the analysis: The bulk of [Fr] was devoted to navigating these issues.

This paper circumvents some those technical intricacies by lifting the problem of 4-classes to one of spin bundles, i.e., by representing the 4-dimensional cohomology classes by spin bundles, a geometric result of independent interest. This motivation comes in part from the general issue of representing cohomology classes geometrically, e.g., every 2-dimensional cohomology classes is the first Chern classes of a line bundle. Naively, one might imagine every 4-class on a spin manifold to be representable as half the first Pontryagin class of a spin bundle. Unsurprisingly, this is false. However, through a range of dimensions the answer is true up to cobordism. The proof is via homotopy, proving that the spin bordism of  $\mathbf{BSpin}$ , the classifying space for spin bundles, surjects onto that of  $\mathbf{K}(\mathbb{Z}, 4)$ . (The integrals then become the indices of differential operators.) A consequence of the proof is a new proof theorem of Stong's computing the low dimensional spin bordism groups of  $\mathbf{K}(\mathbb{Z}, 4)$ , in particular that that  $\mathbf{MSpin}_{11} \mathbf{K}(\mathbb{Z}, 4)$  vanishes. This result is of independent result for M-theory, where the model of spacetime involves a spin 11-manifold equipped with a 4-class. The integrality result we prove has further relevance to M-theory, to the issue of quantization of the action. That is, we prove that if  $M$  is a spin 12-manifold with an integral closed 4-form  $x$ , then  $\int_M (\frac{1}{6}x^3 \pm x^2 + 30x) \widehat{A}$  is an integer.

We now give a more detailed overview of the contents of this paper: We use the Adams spectral sequence to compute the  $ko$ -theory of  $\mathbf{K}(\mathbb{Z}, 4)$  through 15 dimensions after completion at each prime  $p$ . This entails first computing the  $E_2$  term of the sequence, which requires knowing the structure of the cohomology of  $\mathbf{K}(\mathbb{Z}, 4)$  as a module over a certain subalgebra of the Steenrod algebra. Section 2 does this, based on Serre's original computation of the cohomology of  $\mathbf{K}(\mathbb{Z}, n)$  as a polynomial algebra. This computation alone is sufficient to reprove a certain integrality theorem for 8-manifolds, a proof which is then sketched. Further computation requires understanding several differentials in the spectral sequence, which is facilitated by analysis of the map  $\lambda : \mathbf{BSpin} \rightarrow \mathbf{K}(\mathbb{Z}, 4)$ , where  $\lambda$  corresponds to half of the first Pontryagin class. We prove that the spin bordism of  $\mathbf{BSpin}$  surjects onto that of

$K(\mathbb{Z}, 4)$  by working at each prime and for each spectrum in the Anderson-Brown-Peterson splitting of the spin bordism spectrum  $M\text{Spin}$ . This surjectivity allows certain ill-defined invariants for  $K(\mathbb{Z}, 4)$  to be lifted to well-defined invariants in the spin bordism of  $B\text{Spin}$ . This, coupled with the relative simplicity of the  $ko$ -theory of  $B\text{Spin}$ , resolves the differential for computing the 12th spin bordism group of  $K(\mathbb{Z}, 4)$ . We are simultaneously able to find explicit manifolds and prove they generate this group using the relation between values of invariants and the Adams filtration. A Massey product relates the differential in 13 to a lower differential, thereby reproving a theorem of Stong's that the 11th spin bordism group of  $K(\mathbb{Z}, 4)$  is zero.

Technically, the bulk of the text is devoted to proving that a specific set of manifolds with spin bundles generates the  $ko$ -theory of  $B\text{Spin}$  and that their images under the map  $\lambda$  generate the  $ko$ -theory of  $K(\mathbb{Z}, 4)$ . Once this set of manifolds is determined, all possible integrality results follow in the range of dimensions computed. For the reader's convenience, we briefly enumerate the various calculations and arguments contained in this paper:

- Description of  $H^*(K(\mathbb{Z}, 4), \mathbb{F}_p)$  as a module over the appropriate subalgebra  $\mathcal{A}(1)$  of the Steenrod algebra. Computation of the  $E_2$  term of the Adams spectral sequence converging to the  $p$ -adic  $ko$ -theory of  $K(\mathbb{Z}, 4)$ .
- Computation of the  $E_2$  term of the  $ko$ -theory spectral sequence from  $H^*(B\text{Spin}, \mathbb{F}_p)$  and evaluation of the maps  $H^*(\lambda)$  and  $\text{Ext}(\lambda)$  where  $\lambda : B\text{Spin} \rightarrow K(\mathbb{Z}, 4)$  classifies half the first Pontryagin class.
- Proof of the surjectivity of the  $ko$ -theory and spin bordism of  $B\text{Spin}$  onto that of  $K(\mathbb{Z}, 4)$  through 15 dimensions at each prime, and thus integrally.
- Determination of generators of  $\pi_{4n} ko \wedge K(\mathbb{Z}, 4)$  and  $\pi_{4n} ko \wedge B\text{Spin}$  through 14 dimensions. Determination of the groups.
- Relation of values of integrals on spin manifolds to the Adams filtration in the Adams spectral sequence for  $ko_*K(\mathbb{Z}, 4)$ . Determination of all integrality results for integrals of the form  $\int_M P(x_4) \hat{A}$  for spin manifolds of dimension less than 16.
- Massey product argument connecting the differential coming from dimension 13 to that determining the 11-stem. Alternative proof of Stong's theorem, that  $\Omega_{11}^{\text{Spin}} K(\mathbb{Z}, 4) = 0$ . Computation of Bockstein of Pontryagin square determining the  $d_2$  differential in the spectral sequence.

**Remark 1.** *Throughout, the homology theories will be reduced, i.e., the quotient of actual homology theory by the homology of a point.*

**1.1. Acknowledgements.** This work is a slight adaptation of my undergraduate thesis, [Fr]. I am enormously grateful to my advisor, Mike Hopkins, for guiding me in this work and for all his support and encouragement.

## 2. 2-LOCAL SPIN BORDISM

The cohomology of  $B\text{Spin}$  can be computed from that of  $BSO$  or  $BO$  by standard arguments. A short exact sequence of topological groups gives rise to a fibration after apply the classifying space functor  $B$ , which is in this case  $B\mathbb{Z}/2 \hookrightarrow B\text{Spin} \rightarrow BSO$ . The Serre spectral sequence then applies, and one can shortly see that the mod 2 cohomology of  $B\text{Spin}$  is exactly the cohomology of  $BSO$  quotiented out by the ideal generated by the Stiefel Whitney class  $w_2$ . All this is done explicitly in [St2].

The result is a polynomial algebra in the Stiefel-Whitney classes:  $H^*(\mathrm{BSpin}; \mathbb{F}_2) \cong \mathbb{F}_2[w_n : n \neq 2^i + 1, n \geq 4]$ . The action of the Steenrod algebra is induced from that of  $\mathrm{BSO}$ , given by the Wu relations.

$\mathrm{BSpin}$  is evidently 3-connected, and the cohomology class  $w_4$  gives a map to the Eilenberg-MacLane space  $\mathrm{K}(\mathbb{Z}/2, 4)$ . This class has a lift to integral cohomology. We will denote this integral lift by  $\lambda$ , since it is half of the pullback of the first Pontryagin class  $p_1$  to  $\mathrm{BSpin}$ . A main focus of this paper will be the effect of this map  $\lambda$  in spin bordism. Our approach will be to use the cohomology of these two spaces to compute their spin bordism by the Adams spectral sequence, and to relate the effect of the map in cohomology to the map in spin bordism.

Serre first computed the mod  $p$  cohomology of Eilenberg-MacLane spaces inductively using the results of Borel's thesis. The result, at 2, is a polynomial algebra in terms of Steenrod operations on a single generator. The induced map in mod 2 cohomology,  $H^*(\lambda)$ , is a map of modules of the Steenrod algebra, and so is completely determined by the action of the Steenrod operations in  $H^*(\mathrm{BSpin}; \mathbb{F}_2)$  and in  $H^*(\mathrm{K}(\mathbb{Z}, 4); \mathbb{F}_2)$ .

Thus, we have some command of this situation in classical cohomology. The next step is to pass to spin bordism. Spin bordism is a difficult homology theory to manage. At odd primes it is identical to oriented bordism, since  $\mathrm{Spin}$  is a double cover of  $\mathrm{SO}$  the map is an odd prime homotopy equivalence. At the prime 2, Anderson, Brown, and Peterson [ABP] constructed a splitting of the homology theory as a sum of copies of connective real  $K$ -theories,  $ko\langle n \rangle$ . That is, there is a map

$$\mathrm{MSpin} \longrightarrow ko \vee ko\langle 8 \rangle \vee ko\langle 10 \rangle \vee ko\langle 12 \rangle \vee \dots$$

where the maps are given by  $KO$ -theory characteristic numbers. The first map is a lifting to spectra of the classical Atiyah-Bott-Shapiro orientation. This map of ring spectra,  $\mathrm{MSpin} \rightarrow ko$ , we will denote  $\widehat{A}$  since its effect in homotopy (modulo torsion) is exactly the classical  $\widehat{A}$ -genus. There is more information in the map than the genus, however, so the map should be thought of as a rigidified, homotopy-theoretic improvement of the genus. For instance, given a map from a spin manifold  $f : M \rightarrow X$  and a cohomology class in  $x \in H^*(X)$  we may think of the integral  $\int_M f^*(x) \widehat{A}$  as factoring

$$\begin{array}{ccc} \pi_* \mathrm{MSpin} \wedge X_+ & & \\ \downarrow & \searrow \widehat{A} & \\ \int_M f^*(x) \widehat{A} & & \pi_* ko \wedge X_+ \\ \downarrow & \swarrow \text{---} & \\ \mathbb{Z} & & \end{array}$$

and, in particular, think of  $X$  as  $\mathrm{BSpin}$  or  $\mathrm{K}(\mathbb{Z}, 4)$ .

The Bott periodicity of real  $K$ -theory implies that  $\Sigma^8 ko \simeq ko\langle 8 \rangle$ . Therefore, to understand spin bordism in this range of dimensions at the prime 2, it completely suffices to look at  $ko$ -theory.

The tool of choice is the Adams spectral sequence. The Adams spectral sequence takes as inputs the mod  $p$  cohomologies of  $ko$  and  $X$  as modules over the Steenrod algebra and outputs, if computed correctly, the  $p$ -adic completion of  $\pi_* ko \wedge X$ . The

first term of this spectral sequence is  $\text{Ext}_{\mathcal{A}}^{s,t}(H^*(ko \wedge X; \mathbb{F}_p); \mathbb{F}_p)$ . From [Ad1] and [St1], the spectrum cohomology of  $ko$  can be given by the quotient of the Steenrod algebra by the ideal generated by  $Sq^1$  and  $Sq^2$ , so  $H^*(ko; \mathbb{F}_2) \cong \mathcal{A}/\mathcal{A}(1)$ . At odd primes,  $ko$  splits as a wedge of spaces with simpler cohomology.

Computing Ext over the Steenrod algebra is generally daunting. However, in the specific case of  $ko$ -theory, a hom-tensor interchange massively simplifies this task.

First, the Künneth formula implies that

$$H^*(ko \wedge X) \cong H^*(ko) \otimes H^*(X) \cong \mathcal{A}/\mathcal{A}(1) \otimes H^*(X)$$

However,  $\mathcal{A}/\mathcal{A}(1) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{F}_2$ . So substituting this into the first term of our spectral sequence gives:

$$\begin{aligned} \text{Ext}_{\mathcal{A}}^{s,t}(H^*(ko \wedge X), \mathbb{F}_2) &\cong \text{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{F}_2 \otimes H^*(X), \mathbb{F}_2) \\ &\cong \text{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A} \otimes_{\mathcal{A}(1)} H^*(X), \mathbb{F}_2) \end{aligned}$$

The final step in simplifying our Ext term is the derived hom-tensor interchange, so

$$\text{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A} \otimes_{\mathcal{A}(1)} H^*(X), \mathbb{F}_2) \cong \text{Ext}_{\mathcal{A}(1)}^{s,t}(H^*(X), \mathbb{F}_2).$$

Now, instead of computing Ext over the whole infinite-dimensional Steenrod algebra, we can instead work over the finite subalgebra  $\mathcal{A}(1)$  generated by  $Sq^1$  and  $Sq^2$ . We will now do this for BSpin and  $K(\mathbb{Z}, 4)$ .

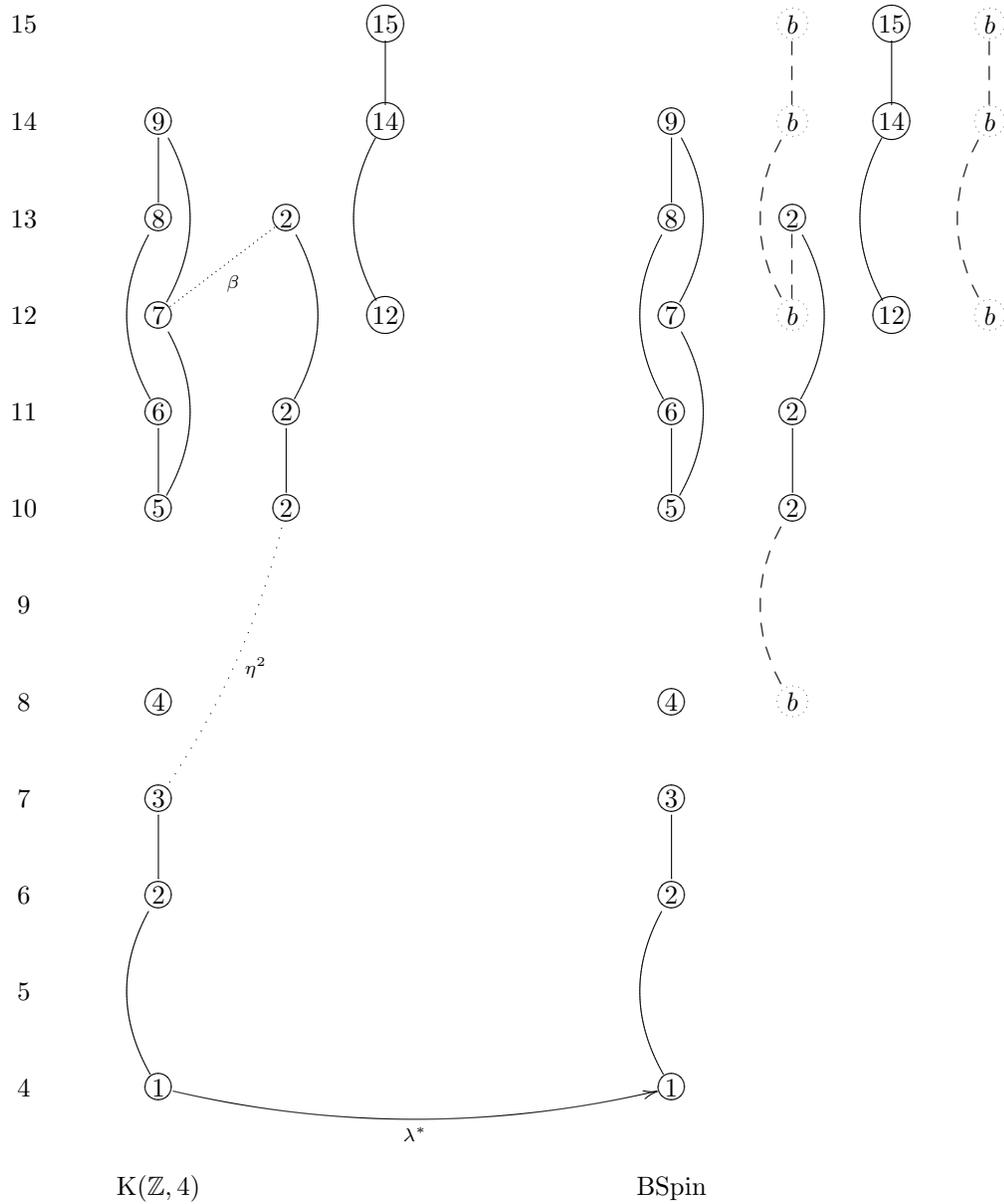
The cohomology of BSpin is a polynomial algebra, as stated before. Its structure as a module over  $\mathcal{A}$  is given by the Wu relations.  $H^*$ BSpin becomes a module over  $\mathcal{A}(1)$  by restriction. The explicit computation is easy to do, and the result through the range of dimensions is given in the diagram at the end of the paper.

Likewise, Serre has expressed  $H^*K(\mathbb{Z}, 4)$  as a polynomial algebra on admissible sequences of Steenrod operations on a generator in degree 4. Its structure as an  $\mathcal{A}(1)$  module, i.e., the action of  $Sq^1$  and  $Sq^2$ , is given by the Adem relations. The outcome of this routine computation is given in an upcoming diagram.

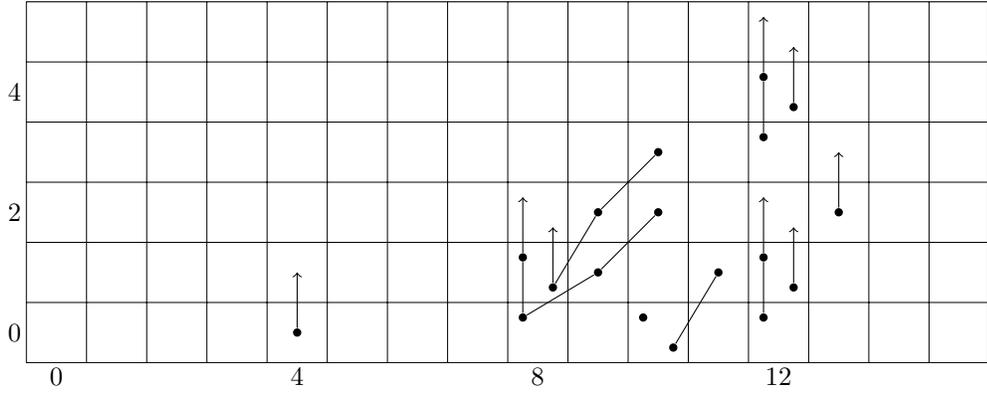
The induced map is also evident, since the map commutes with Steenrod operations. The cohomology of  $K(\mathbb{Z}, 4)$  injects into that of BSpin. As  $\mathcal{A}(1)$ -modules, through 15 dimensions,  $K(\mathbb{Z}, 4)$  is a direct sum of five smaller  $\mathcal{A}(1)$ -modules. Four of these modules are also direct summands in the cohomology of BSpin, but the image of one of them sits in a larger module of which it is not a summand. This has an enormous effect in homotopy.

From the effect of  $\lambda$  in cohomology, we can immediately see the effect in Ext, the first term of the Adams spectral sequence. The computation of Ext for these types of modules is standard in stable homotopy theory, and one can check, e.g., [AP].

**2.1.  $K(\mathbb{Z}, 4)_{(2)}$  and  $B\text{Spin}_{(2)}$ .** The following picture shows the structure of the  $\mathbb{F}_2$ -cohomology of BSpin and of  $K(\mathbb{Z}, 4)$  as an  $\mathcal{A}(1)$ -module, i.e., the action of the Steenrod squares  $Sq^1$  and  $Sq^2$ , and the effect of the map  $\lambda$  in cohomology. The elements (and operations) of the cohomology of BSpin that are not in the image of  $K(\mathbb{Z}, 4)$  are dotted. The map is an injection. (Dually, one may think of the diagrams as describing the cellular structures of the spaces.)



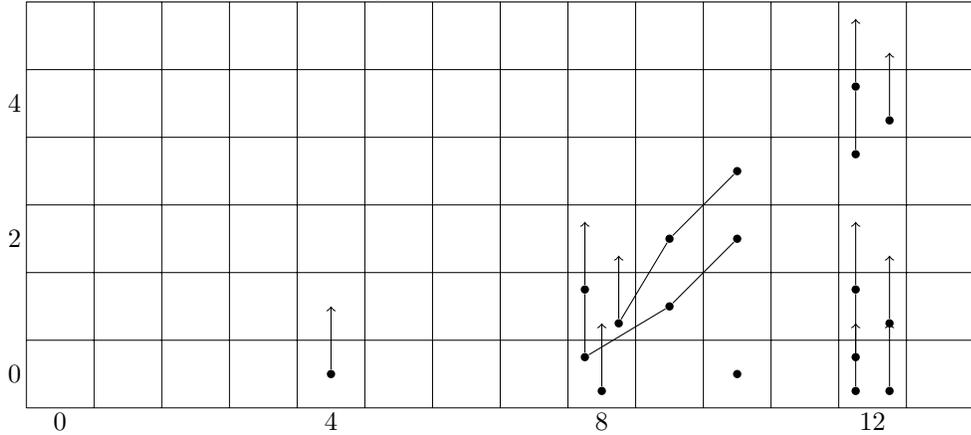
2.2.  $\text{Ext}_{\mathcal{A}(1)}^{s,t}(H^*K(\mathbb{Z}, 4); \mathbb{F}_2)$  and  $\text{Ext}_{\mathcal{A}(1)}^{s,t}(H^*B\text{Spin}; \mathbb{F}_2)$ . The decomposition of the mod 2 cohomology of  $K(\mathbb{Z}, 4)$  developed in the previous section reduces the computation of the  $E_2$  term of the Adams spectral sequence to a routine computation of the bigraded Ext using minimal resolutions. The result of this computation looks as follows:



The upward arrow indicate an infinite “ $\mathbb{Z}$ -tower.” In the target of the spectral sequence this  $\mathbb{Z}$ -tower represents the 2-adically complete integers, with the action of  $h_0$  corresponding to multiplication by 2, and with associated graded in each horizontal degree being the group  $\mathbb{Z}/2$ . The vertical Adams filtration thus relates to what values a homomorphism from the bordism group to  $\mathbb{Z}$  may take. This fact will be exploited.

One should note that there is the possibility of exactly two differentials, one supported by the  $\mathbb{Z}$ -tower in dimension  $t - s = 13$  and one of degree 2 supported on the pair of  $\mathbb{Z}/2$ s related by an  $h_1$  in  $t - s = 10$  and 11. These differentials do exist and are both of degree 2, as will be described later.

The decomposition of the mod 2 cohomology of  $B\text{Spin}$  developed in the previous section reduces the computation of the  $E_2$  term of the Adams spectral sequence to a routine computation of  $\text{Ext}$  using minimal resolutions. This result is as follows.



There are no differentials in this spectral sequence, because the differentials are derivations with respect to the action of the  $h_i$ . Therefore one can immediately read off the low dimensional spin bordism groups of  $B\text{Spin}$ .

**2.3. The map  $\text{Ext}(\lambda)$ .** The map on  $\text{Ext}$  induced by the map  $\lambda$  is a consequence of the effect of  $\lambda$  on cohomology. From the decomposition of the cohomologies

of  $B\text{Spin}$  and  $K(\mathbb{Z}, 4)$  as  $\mathcal{A}(1)$ -modules, it is evident that the  $\text{Ext}$  term of  $B\text{Spin}$  surjects onto that of  $K(\mathbb{Z}, 4)$  in degrees which are a multiple of four.

The issue is to show surjectivity on  $ko$ -theory from this: in this range of dimensions,  $\pi_* ko \wedge K(\mathbb{Z}, 4)$  is a quotient of a direct summand of  $\pi_* ko \wedge B\text{Spin}$ , that is, that  $\pi_* ko \wedge B\text{Spin}$  is isomorphic to  $N \oplus P$  and  $\pi_* ko \wedge K(\mathbb{Z}, 4)$  is isomorphic to  $(\lambda_* N)/d_* M$ , where the  $d_*$  are the differentials in the Adams spectral sequence.

We establish this surjectivity by using the first term of the Adams spectral sequence: in this range of dimensions we can fairly obviously express  $\pi_* ko \wedge K(\mathbb{Z}, 4)$  as a quotient  $N/(d_* M)$ , where  $A \oplus B$  is a splitting of  $\text{Ext}$  into the part which supports a nontrivial differential (i.e.,  $M$ ) and the part that does not (i.e.,  $N$ ). We will then see that the  $E_2$  map composition below is a surjection:

$$\text{Ext}_{\mathcal{A}}(H^* B\text{Spin} \otimes H^* ko; \mathbb{F}_p) \xrightarrow{E_2(\lambda)} \text{Ext}_{\mathcal{A}}(H^* K(\mathbb{Z}, 4) \otimes H^* ko; \mathbb{F}_p) \longrightarrow N$$

where the first map the  $\text{Ext}$  functor of the map on spaces,  $\lambda : B\text{Spin} \rightarrow K(\mathbb{Z}, 4)$ , and the second map  $s$  is projection onto the first term of the splitting defined above. That is,  $s \circ (E_2(\lambda))$  is a surjection. Since there are no differentials in the Adams spectral sequence for  $\pi_* ko \wedge B\text{Spin}$ , therefore  $\pi_* ko \wedge B\text{Spin}$  surjects onto  $N$ , and we know  $N$  surjects onto  $N/(d_* M) = \pi_* ko \wedge K(\mathbb{Z}, 4)$ . This will imply that  $\pi_* ko \wedge B\text{Spin}$  surjects onto  $\pi_* ko \wedge K(\mathbb{Z}, 4)$ , once worked out at each prime.

The induced map  $\lambda^* : H^*(K(\mathbb{Z}, 4); \mathbb{F}_2) \rightarrow H^*(B\text{Spin}; \mathbb{F}_2)$  is an injection on mod 2 cohomology, sending  $\iota$  to  $w_4$ . Since the cohomology of  $K(\mathbb{Z}, 4)$  is a polynomial algebra with generators given by Steenrod operations on  $\iota$ , therefore the homomorphism  $\lambda^*$  is determined by the value  $\lambda^* \iota$ , which can be computed with knowledge of the ring structure of  $H^*(B\text{Spin}; \mathbb{F}_2)$  and the action of Steenrod operations.

The induced map  $\text{Ext}(\lambda)$  is easy to compute.  $\text{Ext}(K(\mathbb{Z}, 4))$  is a direct sum of  $\text{Ext}(M)$  and  $\text{Ext}(N)$  where  $N$  supports a differential. Our result will follow from showing that  $\text{Ext}(\lambda)$  surjects onto  $\text{Ext}(M)$  and that there are no differentials in the spectral sequence computing  $\pi_* ko \wedge B\text{Spin}$ .

In fact, the map is surjective on 2-torsion spin bordism as a consequence of the differentials in the spectral sequence for  $K(\mathbb{Z}, 4)$ . First, the  $\mathbb{Z}$ -tower in dimension 13 cannot survive the spectral sequence, because if it did it would imply that the 13th spin bordism group of  $K(\mathbb{Z}, 4)$  is nonzero rationally. This can be easily seen to be false, by because rational homotopy of spectra is equal rational homology, which is nonzero only in degrees  $4n$  by the Künneth formula. This has the upshot that the  $\mathbb{Z}$ -tower in 13 supports a differential. A more precise description will follow in section 3.

### 3. MANIFOLD GENERATORS

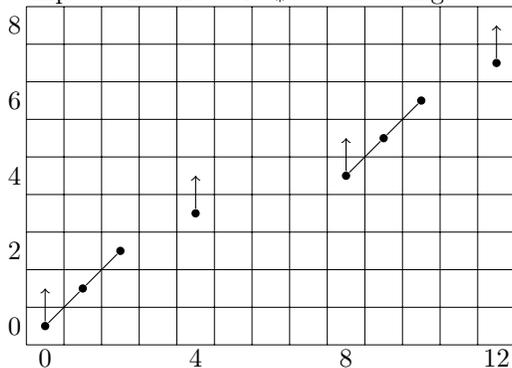
In this section, we will determine an explicit set of manifolds with bundles that generate the spin bordism of  $B\text{Spin}$  through dimension 15 and whose image generates the spin bordism of  $K(\mathbb{Z}, 4)$ , all after localization at 2. Because of the connectivity of the maps  $\widehat{A} : M\text{Spin} \rightarrow ko$  and  $\lambda : B\text{Spin} \rightarrow K(\mathbb{Z}, 4)$ , therefore  $ko$ -theory may serve as proxy for spin bordism throughout. The proof that these manifolds generate at all primes, hence over  $\mathbb{Z}$ , will be completed in the odd primes section.

To better explain the interplay between the algebra and the geometry at work in these arguments, we start with two warm up cases: the  $ko$ -theory of a point, and then  $ko_4$  for  $B\text{Spin}$ . The second case is identical for any other 3-connected space  $X$  with  $\pi_4 X \cong \mathbb{Z}$ .

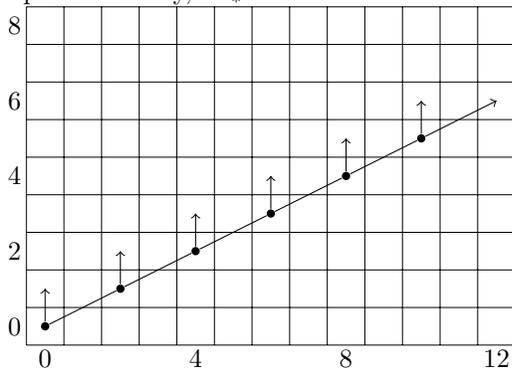
This interplay between manifold theory and algebraic topology is first realized in the following coincidence. First, for any spin 4-manifold, the  $\widehat{A}$ -genus is an even integer. Second, the map on spectra defined by complexification,  $ko \rightarrow ku$ , induces multiplication by 2 on the homotopy group  $\pi_4$ . These two factors of 2 are identical in a fundamental way.

This analogy is a consequence of the  $ko$ -orientability of spin bundles, which one may think of as a lift of  $\widehat{A}$  from being valued in complex  $K$ -theory to being real  $K$ -theory valued. This subsumes Rohklin's theorem for spin 4-manifolds, and further proves that the  $\widehat{A}$ -genus is even for all spin  $8n + 4$ -manifolds, and takes all integer values for spin manifolds of dimension  $8n$ . One vehicle for expressing this is the Adams spectral sequence, which allows a more general relation between the Adams filtration of bordism groups and the invariants used to measure the groups.

As described in the previous section, the spectrum cohomology of  $ko$  allows one to compute the value of  $\pi_* ko \wedge$  as being:



We have a map induced on Adams spectral sequences to that of connective complex  $K$ -theory,  $ku_*$ :



This map is an isomorphism in dimensions divisible by 8, and multiplication by 2 in dimensions equivalent to 4 modulo 8 since, e.g., the  $\mathbb{Z}$ -tower in dimension 4 for  $ko$  starts in Adams filtration 3 and maps to  $h_0$  applied to the generator of the  $\mathbb{Z}$ -tower in  $ku$  starting in Adams filtration 3. (Note:  $h_0$  is an operation induced by  $Sq^1$  and it corresponds to multiplication by 2 in the target of the spectral sequence.) Thus we can see that the  $\widehat{A}$ -map has 'slope'  $1/2$  as realized in the Adams spectral sequence. This allows the interplay of certain geometric versus algebraic facts.

Consequently, to represent elements generating  $ko_{4n}$  as manifolds in the preimage of the  $\widehat{A}$ -map, we need only pick a spin 4-manifold  $M$  such that  $\int_M \widehat{A}_4 = \pm 2$  and a spin 8-manifold  $B$  such that  $\int_B \widehat{A}_8 = \pm 1$ . An example of the first manifold is a  $K_3$  manifold, a degree 4 hypersurface in  $\mathbb{C}P^3$ . An example of the second can be constructed as follows: plumb together sphere bundles according to the Dynkin diagram of  $E_8$ ; take connect sums until the boundary is no longer an exotic sphere; finally, glue on a disk smoothly along the boundary  $S^7$  to obtain a compact closed 8-manifold. The resulting manifold is called the Bott manifold, since taking products with it represents real Bott periodicity in the image of the  $\widehat{A}$ -map.

Now, having discussed this intersection of the geometric and algebraic perspectives in the case of a point, we turn to the second case, of  $\pi_4 ko \wedge BSpin \cong \pi_4 \mathbf{MSpin} \wedge BSpin$ . As computed by the Adams spectral sequence,  $\pi_4 \mathbf{MSpin} \wedge BSpin \cong \mathbb{Z}$ . The mod 2 cohomology of  $BSpin$  is generated by  $w_4$ , and thus evaluating the Kronecker pairing  $\langle [M], w_4(V) \rangle$  gives an isomorphism between  $\mathbb{Z}/2$  and the mod 2 spin bordism of  $BSpin$ . To explicitly lift this invariant to an integral giving an the isomorphism  $\pi_4 \mathbf{MSpin} \wedge BSpin \rightarrow \mathbb{Z}$  is equivalent to lifting the mod 2 cohomology class  $w_4$  to an integral cohomology class. It is easy to check that  $\lambda$  is an integral lift of  $w_4$ .

Therefore, the isomorphism  $\pi_4 \mathbf{MSpin} \wedge BSpin \rightarrow \mathbb{Z}$  is given explicitly by taking the integral  $\int_M \lambda(V)$  over the spin 4-manifold  $M$ . Thus, a pair  $(M, V)$  generates  $\pi_4 \mathbf{MSpin} \wedge BSpin$  if and only if  $\lambda(V)$  generates  $H^4(M; \mathbb{Z})$ . It suffices to take the manifold  $M$  to be  $S^4$ , and the bundle  $V$  to be pulled back from  $BSpin$  by picking a map generating  $\pi_4 BSpin \cong \mathbb{Z}$ . Alternatively, this bundle can be constructed in terms of the tautological bundle over  $\mathbb{H}P^1 \cong S^4$ .

3.1.  $\pi_8 \mathbf{MSpin} \wedge BSpin$ . Again,  $\pi_{12} ko \wedge BSpin \cong \mathbb{Z}^3$ . The invariants in  $\text{Ext}$  are induced by the classes  $w_4^2$ ,  $\frac{1}{4}w_4v_1^2$ , and  $w_8$ . These invariants must now be interpreted geometrically as integrals on manifolds. Our first step is to relate these invariants to the  $\mathbb{Z}$ -towers in the spectral sequence: This is crucial, because then the Adams filtration in which the towers begin will then bound the values that the resulting integrals.

In the absence of differentials, as is the case here, there is a straightforward relation between the  $Q_0$ -homologies in  $H^*(BSpin; \mathbb{F}_2)$  and the homomorphisms giving an isomorphism  $(\pi_* ko \wedge BSpin)_{(2)} \rightarrow \mathbb{Z}_{(2)}^n$ . We need only two ingredients, the Hurewicz homomorphism and the Künneth formula.

Take a minimal CW complex structure for the space  $BSpin$ . Denote the  $n$ -skeleton  $BSpin^{(n)}$ , and consider the cofiber sequence  $BSpin^{(7)} \hookrightarrow BSpin^{(8)} \rightarrow BSpin^{(8)}/BSpin^{(7)}$ .  $BSpin^{(8)}/BSpin^{(7)}$  is 7-connected, and on  $H^8(BSpin^{(8)}/BSpin^{(7)})$  injects into  $H^8 BSpin^{(8)}$ . Identify  $w_8$  and  $w_4^2$  in  $H^8(BSpin^{(8)}/BSpin^{(7)})$  by their images in  $H^8 BSpin^{(8)}$ .

$$\begin{array}{ccc}
\pi_8 ko \wedge BSpin & \cong & \pi_8 ko \wedge BSpin^{(8)} \twoheadrightarrow \pi_8 ko \wedge BSpin^{(8)}/BSpin^{(7)} \\
\langle w_8, [M] \rangle \downarrow & & \downarrow h \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 & \xleftarrow[u \otimes w_4^2]{} & H_8(ko \wedge BSpin^{(8)}/BSpin^{(7)})
\end{array}$$

where  $u$  is a Thom class in  $H^0 ko$  and  $u \otimes w$  is a homomorphism of  $H_8(ko \wedge BSpin^{(8)}/BSpin^{(7)})$  identified with the class in  $H^8(ko \wedge BSpin^{(8)}/BSpin^{(7)}) \cong H^0 ko \otimes H^8(BSpin^{(8)}/BSpin^{(7)})$ .

Essentially, this uses the fact that the  $\mathbb{Z}$ -towers represented by these  $Q_0$ -homologies start in Adams filtration 0, and thus are elements of  $\text{Ext}^0 = \text{Hom}$ . The conclusion is that these homomorphisms of the bordism groups are expressed in terms of cohomology in such a way as to be comparable to the Adams spectral sequence. Further refinement of these invariants obtains by lifting  $w_8$  and  $w_4^2$  to the integral cohomology of  $BSpin$ . Integrating these integer cohomology classes will then give a homomorphism to  $\mathbb{Z} \oplus \mathbb{Z}$  that is surjective after localization at 2.

The class  $\lambda$  lifts  $w_4^2$ , and we can pick the invariant  $\int_M \lambda(V)^2$  as an integral lift of  $\langle w_4(V)^2, [M] \rangle$ . Lifting  $w_8$  requires a minor trick, which is to consider the map  $Bi : BSU \rightarrow BSpin$ . We know exactly how lift Stiefel-Whitney classes in the cohomology of  $BSU$ . The formula for the total Pontryagin class,  $p = \overline{c\overline{c}}$ , implies that  $Bi^* \lambda$  equals  $-c_2$  and that  $Bi^* p_2$  equals  $c_2^2 + 2c_4$ . Therefore  $Bi^* \frac{1}{2}(p_2 - (\lambda)^2) = c_4$ . Since  $c_4$  reduces to  $w_8$  and  $Bi^*$  is a monomorphism on the 8th cohomology group with both integral and  $\mathbb{F}_2$  coefficients, therefore  $\frac{1}{2}(p_2 - (\lambda)^2)$  is an integral lift of  $w_8$  in the cohomology of  $BSpin$ , and the integral  $\int_M p_2(V) - \lambda(v)^2$  lifts  $\langle w_8(V), [M] \rangle$ .

Finally, we analyze the invariant in dimension 8 coming from the class  $w_4$ , or rather  $h_0 v_1^2[w_4]$ , where  $h_0 v_1^2$  is an element of  $\text{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2; \mathbb{F}_2) \cong \pi_* ko$ . which [ABS] shows is lifted integrally by  $\widehat{A}_4$ . Thus, the lift of the second invariant is given by  $\frac{1}{4} \int_M \lambda(V) \widehat{A}$ .

Now (after inverting odd primes, but this will prove unnecessary), we have shown that the three integrals listed above geometrically define the isomorphism  $\pi_8 \mathbf{MSpin} \wedge BSpin \rightarrow \mathbb{Z}^3$ .

Three manifolds together with bundles generate this group if manifolds together with their invariants can be written in a 3x3 matrix which, over the integers, can be made upper diagonal such the values of the diagonal entries are those dictated by the Adams filtration that in which the relevant  $\mathbb{Z}$ -towers.

Now, consider the manifolds  $\mathbb{H}P^2$ ,  $\mathbb{C}P^1 \times \mathbb{C}P^3$ , and  $S^8$  together with the bundles specified below.

To  $\mathbb{H}P^2$ , we assign the tautological bundle  $\gamma$ , oriented so that  $\lambda(\gamma) = u$  the canonical generator of  $H^4(\mathbb{H}P^2; \mathbb{Z})$ . Together with the ring structure of  $\mathbb{H}P^2$ , this gives the values of our three invariants:  $\int \lambda^2 = 1$ ,  $\int \lambda \widehat{A} = \dots$

To  $\mathbb{C}P^1 \times \mathbb{C}P^3$ , we assign the bundle  $LL' - L - L'$ , where  $L$  and  $L'$  are the canonical line bundles over each of projective spaces, and their first Chern classes are minus the canonical generators of  $H^2$  for each of the spaces. Thus, its total Chern class is  $(1 - a - b)(1 - a)^{-1}(1 - b)^{-1}$ . Its total Pontryagin class is then given by the product of the Pontryagin classes for each of the bundles,

$$p(LL' - L - L') = (1 + a^2 + 2ab + b^2)(1 + a^2)^{-1}(1 + b^2)^{-1}$$

and since  $a^2 = 0$  and  $b^4 = 0$

$$p(LL' - L - L') = (1 + 2ab + b^2)(1 - b^2) = 1 + 2ab - 2ab^3.$$

We can now compute the relevant invariants easily:  $\int \lambda^2 = 0$ ,  $\int \lambda \widehat{A} = \int ab(-4b^2/24) = -1/6$ , and  $\int \frac{1}{2}(p_2 - \lambda^2) = -1$ .

On  $S^8$ , we consider the bundle  $V$  pulled back from  $BO$  by the generator of  $\pi_8 BSO \cong \mathbb{Z}$  compatible with a chosen orientation of  $S^8$ . This can also be thought

of as the canonical bundle over the octonionic projective space  $\mathbb{O}P^1 \cong S^8$ . Then  $p(V) = 1 + 2z$ , where  $z$  is the preferred generator of  $H^8(S^8; \mathbb{Z})$ . One sees this by looking at BString, the 7-connected cover of BSO or BSpin.  $w_8$  pulls back to a generator of  $H^8(\text{BString}; \mathbb{F}_2)$ . Then, since  $\frac{1}{2}(p_2 - \lambda^2)$  reduces to  $w_8$  in the cohomology of BSpin, therefore their respective pull backs are so related in the cohomology of BString. The final piece of the argument is to use the Hurewicz homomorphism: since BString is 7-connected,  $\pi_8 \text{BString}$  is isomorphic to  $H_8 \text{BString}$ . This computes the characteristic classes of the bundle over  $S^8$  pulled back by this map: the image of  $w_8$  generates  $H^8(S^8; \mathbb{F}_2)$  and its lift generates over  $\mathbb{Z}$ .

These manifolds together with their invariants are summarized in the table below, which shows that they do indeed generate the bordism group in question.

	$\int \widetilde{w}_8$	$\int \lambda^2$	$\int \lambda \widehat{A}_4$
$S^8$	1	0	0
$\mathbb{H}P^2$	-4	1	-1/12
$\mathbb{C}P^1 \times \mathbb{C}P^3$	-1	0	-1/6

Since these elements generate the group  $\pi_8 \text{MSpin} \wedge \text{BSpin}$ , this gives a new proof of the previously known result that the integral

$$\int_M \lambda^2 \pm 12\lambda \widehat{A}_4 \in 2\mathbb{Z}$$

takes only even integer values for any spin 8-manifold and spin bundle  $V$  with  $\lambda = \frac{1}{2}p_1(V)$ .

Together with the surjectivity of the spin bordism of BSpin on to that of  $K(\mathbb{Z}, 4)$  in this dimension, this proves the also well known fact that for any spin 8-manifold  $M$  and any 4-dimensional cohomology class  $x$ , then

$$\int_M x^2 \pm 12x \widehat{A}_4 \in 2\mathbb{Z}$$

or in other words, that for any spin 8-manifold  $M$ ,  $\lambda(TM)$  is a characteristic element of the bilinear form module  $H^4(M; \mathbb{Z})$ . In the next section these methods extend to deal with spin 12-manifolds.

**3.2.  $\pi_{12} \text{MSpin} \wedge \text{BSpin}$ .** The Adams spectral sequence in the last section gave the computation that  $\pi_{12} ko \wedge \text{BSpin} \cong \mathbb{Z}^6$ . From the construction of spectral sequence coming from the direct sum decomposition of the cohomology of BSpin as  $\mathcal{A}(1)$ -modules, we can associate  $\mathbb{Z}/2$ -invariants to each  $\mathbb{Z}$ -tower that measure the position of a manifold (representing a class in the target of spectral sequence) in the respective  $\mathbb{Z}$ -towers.

The invariants defined, coming from the  $Q_0$ -homologies  $w_4, w_4^2, w_8, w_4^3, w_6^2, w_4 w_8$  in the cohomology of BSpin, can be lifted to integer invariants exactly by the considerations of the previous section. These may be expressed by the integrals  $\int \lambda \widehat{A}_8, \int \lambda^2 \widehat{A}_4, \int \widetilde{w}_8 \widehat{A}_4, \int \lambda^2, \int p_3, \int \lambda \widetilde{w}_8$ . We should note that  $p_3$  is an integral lift of  $w_6^2$ , a standard fact about the cohomology of BO that pulls back to BSpin. The fact that  $Q_0$ -homologies for BSpin lift to integer invariants, in contrast to  $K(\mathbb{Z}, 4)$ , facilitates these sorts of arguments in the spectral sequence. An additional convenience is the last of differentials in the Adams spectral sequence computing  $\pi_* ko \wedge \text{BSpin}$ . The differentials for  $K(\mathbb{Z}, 4)$  significantly complicate the process of relating the Adams filtration to bounding values of integral; nonetheless, this is doable in cases, and was the approach in [Fr].

The manifolds listed in the following table will generate the 2-local spin bordism of  $K(\mathbb{Z}, 4)$ . As a consequence, their images under the map  $\lambda$  generate the 12th spin bordism group of  $K(\mathbb{Z}, 4)$ , 2-locally. In the last section of this paper, we will prove that they also generate  $p$ -local spin bordism group of  $K(\mathbb{Z}, 4)$  for odd  $p$ , and hence they generate over this group over  $\mathbb{Z}$ .

	$\int \lambda \widetilde{w}_8$	$\int \widetilde{w}_8 \widehat{A}_4$	$\int \lambda^3$	$\int p_3$	$\int \lambda^2 \widehat{A}_4$	$\int \lambda \widehat{A}_8$
$(\mathbb{HP}^1 \times S^8, \gamma \times f)$	1	0	0	2	0	0
$(\mathbb{CP}^1 \times \mathbb{CP}^5)$	0	-1/4	0	2	0	$9/2^4 \cdot 5$
$(\mathbb{HP}^3, \gamma)$	0	0	-1	0	-1/6	$-1/2 \cdot 3^2 \cdot 5$
$(\mathbb{CP}^3 \times \mathbb{CP}^3)$	-1	2/3	1	-2	0	$1/2^3 \cdot 3 \cdot 5$
$(\mathbb{HP}^2 \times K_3, \gamma)$	0	6	0	0	2	-1/6
$(\mathbb{HP}^1 \times B^8, \gamma)$	0	0	0	0	0	1
$(V, \mathcal{L})$	3	-1/3	1	-4	-1/6	-1/18

Here, the manifold  $V$  is a Milnor manifold, a hypersurface of bidegree  $(2, 1)$  in the product of projective spaces  $\mathbb{CP}^3 \times \mathbb{CP}^4$ . The bundle  $\mathcal{L}$  on  $V$  is the restriction of the bundle  $LL' - L - L'$  on  $\mathbb{CP}^3 \times \mathbb{CP}^4$ .

The invariants above are listed in order of ascending Adams filtration: 0, 0, 0, 1, 3, 4. In order for a list of manifolds to generate the group, lower filtration invariants should vanish on the manifolds generating the  $\mathbb{Z}$ -towers starting higher filtration. More concretely, one should be able to write the values of the manifolds as an upper triangular matrix such that the diagonal values are exactly those dictated by the Adams filtration of the associated  $\mathbb{Z}$ -tower. Again, these values can be determined by comparison of the behavior of  $\widehat{A}$  on  $ko_*$ . Thus, the invariant  $\int \lambda^3$  must take value 1 on the manifold generating its associated  $\mathbb{Z}$ -tower, since that  $\mathbb{Z}$ -tower starts in Adams filtration 1. Likewise,  $\int \lambda \widetilde{w}_8$  must take value 1/4 on the manifold generating its associated  $\mathbb{Z}$ -tower.

In order to verify that a subset of the manifolds above generate  $(\pi_{12} \mathbf{MSpin} \wedge \mathbf{BSpin})_{(2)}$ , we take the following linear combinations. One can easily verify that the diagonal values are exactly those proscribed the filtrations in the Adams spectral sequence.

	$\int \lambda \widetilde{w}_8$	$\int \widetilde{w}_8 \widehat{A}_4$	$\int \lambda^3$	$\int p_3$	$\int \lambda^2 \widehat{A}_4$	$\int \lambda \widehat{A}_8$
$\mathbb{HP}^1 \times S^8$	1	0	0	2	0	0
$\mathbb{CP}^1 \times \mathbb{CP}^5$	0	-1/4	0	2	0	$9/2^4 \cdot 5$
$\mathbb{HP}^3$	0	0	-1	0	-1/6	$-1/2 \cdot 3^2 \cdot 5$
$V - 3\mathbb{HP}^1 \times S^8 - \frac{4}{3}\mathbb{CP}^1 \times \mathbb{CP}^5 + \mathbb{HP}^3$	0	0	0	$-2 \cdot 19/3$	-1/3	$-13/2^2 \cdot 4 \cdot 5$
$\mathbb{HP}^2 \times K_3$	0	0	0	0	2	-1/6
$\mathbb{HP}^1 \times B^8$	0	0	0	0	0	1

Together with the results of the final section, we have the following:

**Theorem 1.** *The manifolds with bundles listed above generate the 2-local  $ko$ -homology of  $\mathbf{BSpin}$ . Under the image of the map  $\widehat{A} \wedge \lambda$ , the manifolds  $S^4 \times B^8$ ,  $\mathbb{HP}^2 \times K(\mathbb{Z}, 4)_3$ ,  $V$ ,  $\mathbb{CP}^3 \times \mathbb{CP}^3$ , and  $\mathbb{HP}^3$  generates  $\pi_{12} ko \wedge K(\mathbb{Z}, 4)$  integrally.*

#### 4. SURJECTIVITY AND INTEGRALITY

We can put the results above to work to understand several things about spin manifolds equipped with 4-dimensional cohomology classes. Firstly, we have used algebra to give conditions sufficient for a set of manifolds with 4-classes to generate the  $ko$ -theory of  $K(\mathbb{Z}, 4)$ . However, we did not resolve the issue of differentials in this spectral sequence. By reapplying some geometry, coupled to standard homological algebra, we can resolve the exact images of the differentials in terms of manifolds, providing a concrete hold on the low dimensional  $ko$ -theory of  $K(\mathbb{Z}, 4)$ .

A consequence of knowing explicit generators, as explained in the introduction, is that it allows the proof of integrality theorems simply by checking them on that generating set. Thus, another corollary of the work of the last section is the computation of all integrality theorems for integrals of the form  $\int P(x)\widehat{A}$  previously mentioned. One may ask for more, however, by wanting the integral to not just be an integer, but also the index of a differential operator. The surjectivity of the spin bordism of  $BSpin$  on to the spin bordism of  $K(\mathbb{Z}, 4)$  allows for such a result. We show using standard homotopy that when the 4-class  $x$  is equal  $\lambda$  of a spin bundle, then the integer-valued integral expresses the index of an operator. Thereby, a spin manifold with a 4-class may be moved through a cobordism, preserving the 4-class, to a spin cobordant manifold whose associated 4-class is  $\lambda$  of a spin bundle, and for whom the integrals express indices.

**4.1. The differentials for  $ko_{n < 16} K(\mathbb{Z}, 4)$  and Stong's theorem.** Rational stable homotopy theory provides a quick argument showing that the  $\mathbb{Z}$ -tower in dimension  $t - s = 13$  necessarily supports a differential. As a functor on spectra, rational homotopy is canonically identical to rational homology, i.e.,  $\pi\mathbb{Q}_* \cong H\mathbb{Q}_*$ , which can be proved just by checking on spheres. (The argument proves more generally that,  $\mathbb{Q}$ -locally, all spectra split as a wedge sum of  $\vee \Sigma^i H\mathbb{Q}$ .) Since rational homology satisfies a Kunneth formula, which facilitates a quick computation for our case:

$$(\pi_* ko \wedge K(\mathbb{Z}, 4)) \otimes \mathbb{Q} \cong H\mathbb{Q}_*(ko \wedge K(\mathbb{Z}, 4)) \cong H\mathbb{Q}_*(ko) \otimes H\mathbb{Q}_*(K(\mathbb{Z}, 4))$$

and therefore in the  $ko$ -theory of  $K(\mathbb{Z}, 4)$ ,  $\mathbb{Z}$ 's occur only in degrees  $4n$ .

Thus the  $\mathbb{Z}$ -tower in  $t - s = 13$  cannot survive the spectral sequence. Additionally, it must support a differential, since there is no  $\mathbb{Z}$ -tower in dimension 14 to map into it. While rational stably homotopy does not help in determining the order of the differential, together with knowledge of spin manifolds and their rational invariants, it can tell us what the image is exactly up to multiplication by a power of 2.

By our computations with  $BSpin$ , we found a list of manifolds that generate  $\pi_{12} ko \wedge BSpin$  and whose image under the map  $\lambda$  generated  $\pi_{12} ko \wedge K(\mathbb{Z}, 4)$ . In fact, the subset of these associated to the invariants involving  $\lambda$  suffice to generate  $\pi_{12} ko \wedge K(\mathbb{Z}, 4)$ . We can thus choose  $\mathbb{H}P^3$ ,  $\mathbb{C}P^3 \times \mathbb{C}P^3$ ,  $\mathbb{H}P^2 \times K_3$ , and  $S^4 \times B^8$ , which generate the group 2-locally. (Since this differential occurs at 2, we can ignore odd prime factors.)

Our rational Kunneth formula proved that fact that  $\pi_{12} ko \wedge K(\mathbb{Z}, 4) \otimes \mathbb{Q} \cong \mathbb{Q}^3$ . We can think of this isomorphism as being defined by evaluating the integrals  $\int x^3$ ,  $\int x^2 \widehat{A}_4$ ,  $\int x \widehat{A}_8$  on an element of  $\pi_{12} ko \wedge K(\mathbb{Z}, 4)$  represented by an element  $(M, x)$ . By evaluating these rational invariants on the four chosen elements of our generating set, we may find some linear combination of these four elements such all of the rational invariants vanish. This linear combination would therefore represent

a torsion class in  $\pi_{12} ko \wedge K(\mathbb{Z}, 4)$ , and therefore some  $2^i$  multiple of it is the image of the differential supported by the  $\mathbb{Z}$ -tower in dimension 13.

We have already computed the values of these invariants on these manifolds, and therefore can check that

$$2^3 \cdot 3^2 \cdot 5(\mathbb{H}\mathbb{P}^3 - \mathbb{C}\mathbb{P}^3 \times \mathbb{C}\mathbb{P}^3) + 2 \cdot 3 \cdot 5(\mathbb{H}\mathbb{P}^2 \times K_3) + 11(S^4 \times B^8)$$

evaluates to zero under all three of the rational invariants. The factors of 2 involved imply that this combination represents an element in Adams filtration 4, and it contains the generator for the  $\mathbb{Z}$ -tower beginning in Adams filtration zero. Next we will show by algebraic means that the differential must be a  $d_2$ , of order 2. This will imply, for instance, that  $\pi_{12} ko \wedge K(\mathbb{Z}, 4)$  is isomorphic to  $\mathbb{Z}^3$ , i.e., it is torsion free.

The order of the differential is determined by the order of the Bockstein  $\beta_{2^i}$  such that  $\beta_{2^i}(Sq^2\iota)^2$  is nonzero. Here the Bockstein is defined as the connecting homomorphism in the long exact sequence of cohomology groups induced by the short exact sequence of coefficients:  $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2^i \rightarrow \mathbb{Z}/2^{i-1}$ . Thus, if  $b$  and  $c$  are integer singular cochains such that  $\delta(c) = 2^i b$ , where  $\delta$  is the coboundary differential, then the Bockstein has the property that  $\beta_{2^i}[c] = [b]$ . Note that  $\beta_2 = Sq^1$ .

By our rational homotopy argument  $(Sq^2\iota)^2$  cannot lift to a nontorsion class, and therefore there exists a Bockstein of some order on which it is nonzero. This can be determined as follows:

Since  $Sq^1(Sq^2\iota)^2$  is equal zero, this implies that the class lifts to a class in mod 4 cohomology. The mod 4 lift of this class may be represented by the Pontryagin square  $\mathcal{P}(Sq^2\iota)$ , where here  $\mathcal{P}$  denotes the family of nonadditive cohomology operations of type  $(n, \mathbb{Z}/2; 2n, \mathbb{Z}/4)$  defined on integer cochains in terms of the cup- $i$  product,

$$\mathcal{P}(x) := x \smile x + \delta x \smile_1 x$$

where this cup-1 product  $x \smile_1 y$  is expressed on integer cochains by a choice of cochain homotopy between  $x \smile y$  and  $y \smile x$ .

One can compute which order Bockstein is nonzero on  $\mathcal{P}$  by checking the power of 2 in the evaluation of the coboundary  $\delta$  on  $\mathcal{P}(x)$  where  $x$  is an integer cocycle whose reduction mod 2 represents the cohomology class  $Sq^2\iota$ . We can further pick an integer cocycle lift  $y$  for  $Sq^3\iota$  by  $2y = \delta x$ . Therefore

$$\begin{aligned} \delta\mathcal{P}(x) &:= \delta(x \smile x) + \delta(\delta x \smile_1 x) \\ &= 2\delta x \smile x - \delta x \smile_1 \delta x = 4y \smile x - 4y \smile_1 y \end{aligned}$$

and thus the cocycle  $y \smile x - y \smile_1 y$  represents the class  $\beta_4\mathcal{P}(Sq^2\iota)$ . Since generally  $Sq^{2n}y_{2n+1}$  can be defined by the cup- $i$  product  $y_{2n+1} \smile_1 y_{2n+1}$ , which proves that:

**Lemma 1.** *For  $x$  a  $2n$ -dimensional integer cocycle, then  $\beta_4\mathcal{P}(x) = Sq^1(x) \smile x + Sq^{2n}Sq^1x$*

And in the specific case of  $x = Sq^2\iota$ , therefore

$$\begin{aligned} \beta_4\mathcal{P}(Sq^2\iota) &= Sq^1(x) \smile x + Sq^6Sq^1x \\ &= Sq^3\iota \smile Sq^2\iota + Sq^6Sq^3\iota \end{aligned}$$

and this allows the determination of the order of the differential in the Adams spectral sequence.

This determination happens by a general relation between differentials in the Bockstein spectral sequence and those in Adams spectral sequence: the composition  $\mathcal{P}Sq^2$  defines a map of Eilenberg-MacLane spaces  $K(\mathbb{Z}, 4) \rightarrow K(\mathbb{Z}/4, 12)$  and thus a natural map of the  $ko$ -theory Adams spectral sequences,  $\text{Ext}_{\mathcal{A}(1)}(H^*K(\mathbb{Z}, 4); \mathbb{F}_2) \rightarrow \text{Ext}_{\mathcal{A}(1)}(H^*K(\mathbb{Z}/4, 12); \mathbb{F}_2)$ , commuting with the differentials.

The beginning of the  $E_2$  term of this spectral sequence for  $K(\mathbb{Z}/4, 12)$  is given by the same techniques used earlier for  $K(\mathbb{Z}, 4)$ . Through dimension 14, the mod 2 cohomology of  $K(\mathbb{Z}/4, 12)$  breaks up as an  $\mathcal{A}(1)$ -module direct sum as  $A \oplus S$  where  $S$  is a 1-dimensional module in dimension 13, and  $A$  consists of the generator in degree 12 and the nonzero element  $Sq^2$  on it. The generalized Hurewicz homomorphism gives an isomorphism  $\pi_{12} koK(\mathbb{Z}/4, 12) \cong \pi_{12}K(\mathbb{Z}/4, 12) \cong \mathbb{Z}/4$ , so this dictates the degree of the differential supported on the  $\mathbb{Z}$ -tower in dimension 13: it is a  $d_2$ .

By naturality, the  $d_2$  differential computing the 12-stem for  $ko \wedge K(\mathbb{Z}/4, 12)$  commutes with that for  $ko$  for  $K(\mathbb{Z}, 4)$ . Therefore that differential is also a  $d_2$ . By the manifold theory used earlier, the image of that differential can therefore be represented by the  $\hat{A}$ -image of that linear combination manifolds. This proves:

**Theorem 2.** *The 12th  $ko$ -homology group of  $K(\mathbb{Z}, 4)$  is torsion free, and thus there is an isomorphism  $\pi_{12} ko \wedge K(\mathbb{Z}, 4) \cong \mathbb{Z}^3$  given by the preceding integrals.*

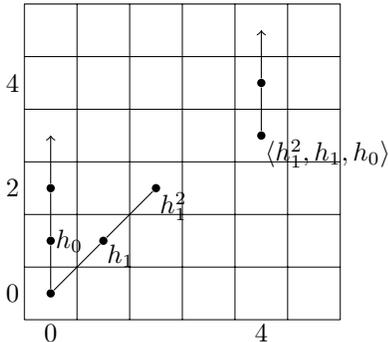
This differential implies a lower-dimensional differential that completes the computation of the  $ko$ -theory of  $K(\mathbb{Z}, 4)$  in this range, originally done by Stong. Together with the Anderson, Brown, Peterson splitting of  $M\text{Spin}$ , this gives that  $M\text{Spin}_{10}K(\mathbb{Z}, 4) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$  and  $M\text{Spin}_{11}K(\mathbb{Z}, 4) \cong 0$ .

Here, we relate the differential computing the 12-stem to a lower differential by a Massey product. Intuitively, one should think that the relation between the differentials comes from the multiplicative structure on the spectral sequence. However, because of the nature of the ‘question mark’ as an  $\mathcal{A}(1)$ -module, the ring structure has been ‘forgotten’ in low degrees. However, there is a Massey product that ‘remembers’ this information, and the differentials are derivations with respect to these Massey products.

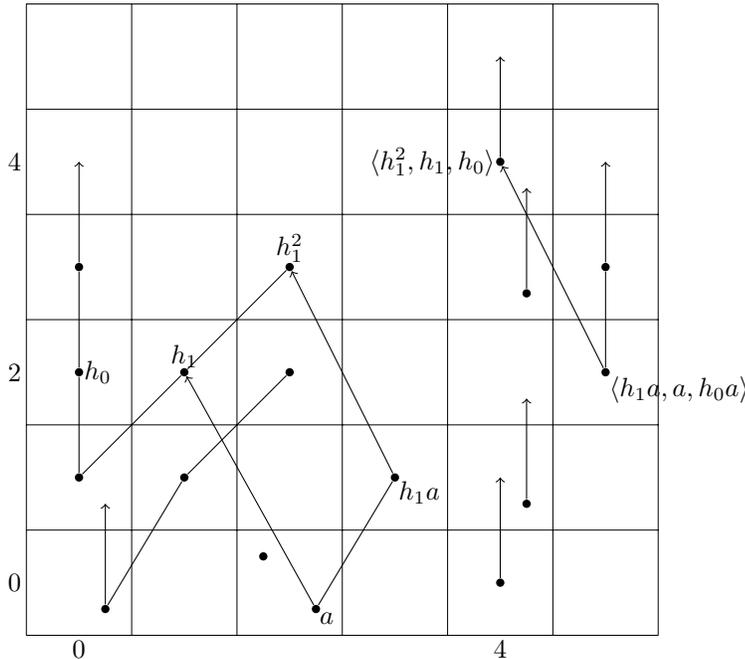
More accurately, we have the following purely homological fact, which can be checked manually in the bar complex:

**Lemma 2.** *In  $\text{Ext}_{\mathcal{A}(1)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ ,  $\eta = \langle h_1^2, h_1, h_0 \rangle$ .*

We obtain the following picture.



This allows us to use that the differentials in the Adams spectral sequence are derivations with respect to the action of  $\text{Ext}_{\mathcal{A}(1)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ : the computation with the Bockstein in dimension  $t - s = 13$  then implies the lower differential, as pictured below.



**4.2. Integrality theorems.** The Massey product argument of the last section proves the 2-local surjectivity of  $ko_*\lambda$  in this range of dimensions. As a result of [ABP], the map  $\widehat{A} : \mathbf{MSpin} \rightarrow ko$  is 8-connected. From basic cohomology computations, the map  $\lambda : \mathbf{BSpin} \rightarrow \mathbf{K}(\mathbb{Z}, 4)$  is also 8-connected. Thus, in conjunction with the results on the surjectivity of  $ko$ -homology at odd primes proved in the final section, this provides the following theorem:

**Theorem 3.** *For any  $M$  be a spin manifold of dimension less than 16, there exists a spin manifold  $N$  and a spin bundle  $V \rightarrow N$  with  $\lambda(V) = y$  and such that  $(N, y)$  represents the same element as  $(M, x)$  in  $\pi_* \mathbf{MSpin} \wedge \mathbf{K}(\mathbb{Z}, 4)$ .*

Equivalently, one could say that for any spin manifold  $M$ , dim less than 16, equipped with a 4-class, one can move  $M$  through a spin cobordism preserving the 4-class to another manifold  $N$  such that the map  $x$  lifts to  $\mathbf{BSpin}$ . In other words, if we work only up to cobordism, every four dimensional cohomology class of a spin manifold is half the first Pontryagin class for a spin bundle over that manifold.

Using our knowledge of explicit generators, we may now determine all polynomials  $P$  which satisfy an integrality result of the form: for any spin  $n$ -manifold  $M$  equipped with an integral 4-class  $x$ , the integral

$$\int_M P(x) \widehat{A}$$

is an integer. The following table of manifolds provides these results

*	$\int_M x^3$	$\int_M x^2 \widehat{A}_4$	$\int_M x \widehat{A}_8$
$\mathbb{H}\mathbb{P}^3$	1	$\frac{-1}{2 \cdot 3}$	$\frac{1}{2 \cdot 3^2 \cdot 5}$
$\mathbb{H}\mathbb{P}^2 \times K_3$	0	2	-1/6
$S^4 \times B^8$	0	0	-1
$V_{2,1}$	1	-1/6	-1/18
$\mathbb{C}\mathbb{P}^3 \times \mathbb{C}\mathbb{P}^3$	1	0	$\frac{1}{2^2 \cdot 3^2}$

This provides the following formulas simply by checking on generators:

**Theorem 4.** *For  $M$  any spin 12-manifold and  $x$  an integral closed 4-form, then the following integrals*

$$\begin{aligned} & \int_M \frac{1}{3!} x^3 \pm 3x^2 \widehat{A}_4 + 30x \widehat{A}_8 \\ 180 \int_M x \widehat{A}_8 &= \frac{1}{2^3 \cdot 3} \int_M x(7p_1^2 - 4p_2) \\ 6 \int_M x^2 \widehat{A}_4 &= \frac{-1}{4} \int_M x^2 p_1 \end{aligned}$$

take only integer values. Here  $p_i$  denote the Pontryagin classes of the tangent bundle of  $M$ .

Several other results for spin manifolds along these lines may be found in [Fr].

## 5. SPIN BORDISM AT ODD PRIMES

To complete the proofs of the results stated in the last section, it remains to show that through the given range the spin bordism of  $B\text{Spin}$  still surjects onto that of  $K(\mathbb{Z}, 4)$  after completion at odd primes. The reason this happens is somewhat different than the reason at the prime 2. At 2, the cohomology of  $K(\mathbb{Z}, 4)$  injects into that of  $B\text{Spin}$ . At odd primes  $p$ , the cohomology of  $K(\mathbb{Z}, 4)$  never injects into that of  $B\text{Spin}$ , since  $B\text{Spin}$  has no odd torsion.

Nonetheless, our previous approach still applies. We use the Adams spectral sequence with input the  $\mathbb{F}_p$  cohomology of  $M\text{Spin}$  and  $K(\mathbb{Z}, 4)$  or  $B\text{Spin}$ . The map induced by  $\lambda$  on  $\text{Ext}$  is again computable by the map  $H^* \lambda$ , likewise a consequence of the action of the Steenrod operations in the  $\mathbb{F}_p$  cohomology of  $B\text{Spin}$ . To understand spin bordism through this range of dimensions, it again suffices to understand a simple summand in a splitting of  $M\text{Spin}$ : For each  $p$  odd, there is a map  $ko \rightarrow \bigvee_{0 \leq k \leq \frac{1}{2}(p-3)} \Sigma^{4k} BP\langle 1 \rangle$  that is a homotopy equivalence after localizing at  $p$ . Here  $BP\langle 1 \rangle$  is the Johnson-Wilson spectrum, constructed from the Brown-Peterson spectrum  $BP$  by killing the higher  $v_i$ . The spectrum cohomology  $H^*(BP\langle 1 \rangle; \mathbb{F}_p)$  is equal  $\mathcal{A}_p // E[Q_0, Q_1]$ , so using the Adams spectral sequence to compute  $ko$  we again have a hom-tensor interchange to simplify the  $E_2$  term:

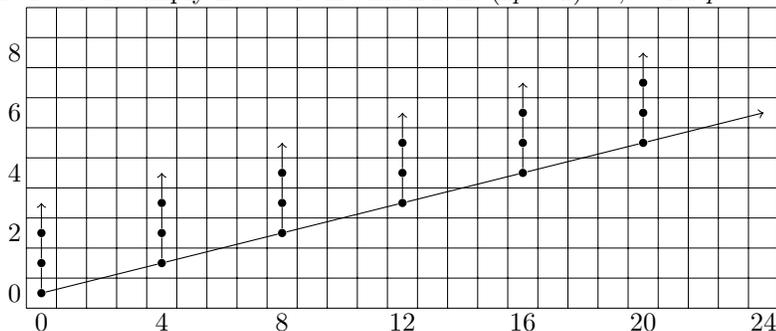
$$\begin{aligned} & \text{Ext}_{\mathcal{A}_p}(\mathcal{A} // E[Q_0, Q_1] \otimes H^*(K(\mathbb{Z}, 4); \mathbb{F}_p), \mathbb{F}_p) \\ & \cong \text{Ext}_{\mathcal{A}_p}((\mathcal{A} \otimes_{E[Q_0, Q_1]} \mathbb{F}_p) \otimes H^*(K(\mathbb{Z}, 4); \mathbb{F}_p), \mathbb{F}_p) \cong \text{Ext}_{E[Q_0, Q_1]}(H^*(K(\mathbb{Z}, 4); \mathbb{F}_p), \mathbb{F}_p) \end{aligned}$$

and the relative simplicity of this exterior algebra relative to  $\mathcal{A}(1)$  greatly eases the necessary computations.

The  $\mathbb{F}_p$ -cohomology of both spaces is well known, the  $E[Q_0, Q_1]$  module structure easily computed. After completion at an odd prime,  $B\text{Spin}$  is homotopy equivalent to  $BSO$  after localization at any odd prime, and therefore its  $\mathbb{F}_p$ -cohomology is a

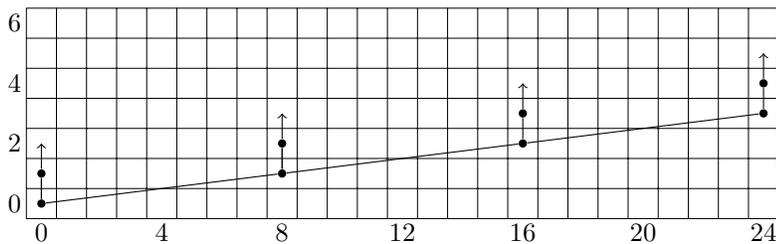
polynomial algebra on the mod  $p$  reductions of the Pontryagin classes. Since all cohomology is in even degrees, and the degree of the primitives is odd, therefore the module structure is trivial. The  $\mathbb{F}_p$  cohomology of  $K(\mathbb{Z}, 4)$  was computed by Serre and is well known, if complicated. The action of the Milnor primitives is a computable consequence of the Adem relations.

5.1.  $\text{Ext}_{E[Q_0, Q_1]}^{s,t}$  for  $\text{BSpin}$  and  $K(\mathbb{Z}, 4)$ . In this section we compute  $p$ -local  $ko$ -theory for  $\text{BSpin}$  and  $K(\mathbb{Z}, 4)$ , through dimension 14. The computation is quite simple, once the basic facts have been cited. Simply by writing out a minimal resolution of  $\mathbb{F}_2$  by free  $E[Q_0, Q_1]$ , it is easy to compute by hand  $\text{Ext}_{E[Q_0, Q_1]}^{s,t}(\mathbb{F}_p; \mathbb{F}_p)$ . The result is simply  $\mathbb{Z}$ -towers in dimensions  $(2p - 2) \cdot n$ , so for  $p = 3$  we have:

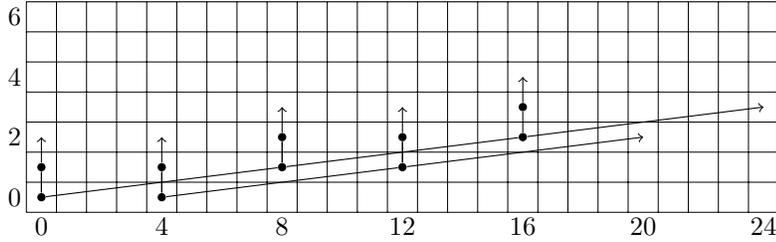


This is the  $E_2$  term of the Adams spectral sequence computing the 3-completion of  $BP\langle 1 \rangle_*$  the  $BP\langle 1 \rangle$ -theory of a point, and the spectral sequence. Since the map  $ko \rightarrow BP\langle 1 \rangle$  is a 3-local homotopy equivalence, this spectral sequence also computes the 3-completion of  $ko_*$ . Since the spectral sequence can have no differentials, this unsurprisingly implies that  $(ko_*)_3^\wedge$  consists of copies of  $\mathbb{Z}_3$  in dimensions divisible by 4.

For  $p = 5$ , we can similarly compute  $BP\langle 1 \rangle$  by hand resolving  $\mathbb{F}_p$  by a minimal or bar resolution of projective (i.e., free)  $E[Q_0, Q_1]$ -modules. At  $p = 5$ , the Milnor primitive  $Q_1$  has degree 9, which results in the  $\mathbb{Z}$ -towers being spaced in every 8 dimensions. We have:

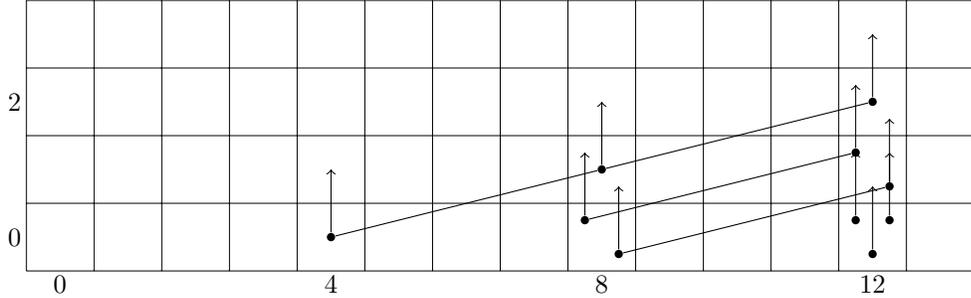


Since there is a 5-local homotopy equivalence  $ko \rightarrow BP\langle 1 \rangle \vee \Sigma^4 BP\langle 1 \rangle$ , the spectral sequence computing  $ko$  of a point at 5 consists of a direct sum of the term above with another copy whose dimension has been shifted by 4. This looks like:



Again, there are no differentials in this spectral sequence, so it verifies what we already knew, that  $(ko_*)_5^\wedge \cong \bigoplus \Sigma^{4i} \mathbb{Z}_5$ .

To return to the issue of computing the  $p$ -complete  $ko$ -theory of  $B\text{Spin}$ : since the  $Q_i$  act as zero on the cohomology  $H^*(B\text{Spin}; \mathbb{F}_p)$ , as an  $E[Q_0, Q_1]$ -module it is a direct sum of suspensions of trivial modules, one for each cohomology class  $p_i^n$ . Ext for these modules was exactly what was computed above, so we have thereby computed  $(\pi_* ko \wedge B\text{Spin})_p^\wedge$  for  $p$  odd. For  $p = 3$ , and through 15 dimensions, this is:



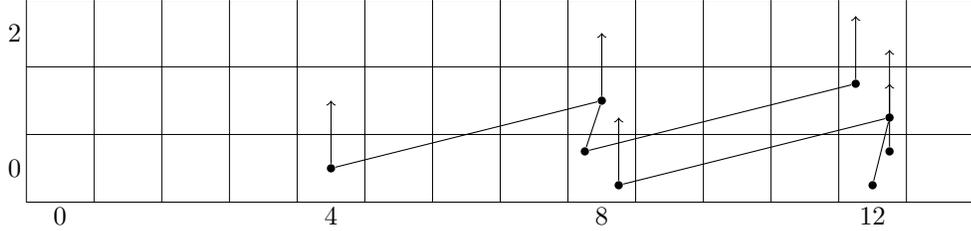
The computation for  $(\pi_* ko \wedge B\text{Spin})_5^\wedge$ , which we omit for the sake of brevity, is virtually identical, except easier.

Considering primes greater than 5, the  $\mathbb{Z}$ -towers through dimension 15 all lie in Adams filtration zero. More precisely, at the prime  $p$  the first  $\mathbb{Z}$ -tower in Adams filtration greater than zero occurs in  $t - s = 2p + 2$ , which is same degree as if we were considering a 4-sphere  $S^4$  instead of  $B\text{Spin}$ . This has the consequence that the  $p$ -local 14-skeleton of  $K(\mathbb{Z}, 4)$  is summand of the  $p$ -local 14-skeleton of  $B\text{Spin}$  for  $p$  greater than 5. Therefore, the spin bordism of  $K(\mathbb{Z}, 4)$  is a  $p$ -local summand of the spin bordism of  $B\text{Spin}$  through dimension 15 for  $p$  greater than 5. Thus, to prove surjectivity at 3 and 5 is the only remaining part of the proof.

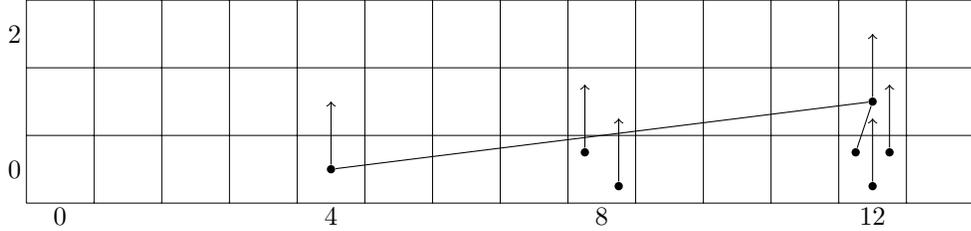
We can now turn our attention to the slightly more intricate case of  $\pi_* ko \wedge K(\mathbb{Z}, 4)$  at 3 and 5. The cohomology of  $B\text{Spin}$  has a trivial  $E[Q_0, Q_1]$ -module structure, but this is not at all the case for  $K(\mathbb{Z}, 4)$ . Using the usual formula for the cohomology of  $K(\mathbb{Z}, 4)$  in terms of admissible sequences given by Serre, the  $E[Q_0, Q_1]$  action can be compute using the odd prime Adem relations, using that  $Q_0 = \beta$  and  $Q_{i+1} = [Q_i, P^{i+1}]$ . In low dimensions,  $H^*(K(\mathbb{Z}, 4); \mathbb{F}_3)$  has the structure:



The picture above shows the decomposition of the cohomology as a module, showing the modules starting in dimensions less or equal 13. The computation of Ext is then equivalent to the direct sum of Ext for each of the submodules:



Likewise, doing the computations at the prime 5 give the spectral sequence:



Before dealing with the issue of the map between the induced map on the spectral sequences, we verify that the  $\widehat{A}$ -images of the spin manifolds listed in section 2 do indeed generate the  $ko$ -homology of  $K(\mathbb{Z}, 4)$  not only after inverting odd primes, but also integrally. Due to the lack of differentials in the odd prime spectral sequence in this range of dimensions, the verification is purely algebraic: one need only check the degree of Adams filtrations and the value of the associated invariant on the candidate manifold. At the prime 3, for instance, the  $\mathbb{Z}$ -tower in  $t - s = 8$  coming from the  $Q_0$ -homology  $\iota$  sits in Adams filtration zero, and a spin manifold  $(M, x)$  sent to the bottom of that tower must take value  $\int_M x \widehat{A}_4 = 1/3$ , modulo factors of other primes. However, since  $\int_{\mathbb{C}P^3 \times S^2} ab \widehat{A}_4 = -1/6$ , our generating set of manifolds for the prime 2 also works after completion at 3, giving the following:

**Proposition 1.** *The elements  $(\mathbb{C}P^3 \times S^2, ab)$ ,  $(\mathbb{H}P^2, y)$  generate the group  $(\pi_8 ko \wedge K(\mathbb{Z}, 4)) \otimes \mathbb{Z}_3$ .*

Likewise, examining the Adams filtrations in which the  $\mathbb{Z}$ -towers in  $t - s = 12$  begin, a spin manifold sent to the bottom of the  $\mathbb{Z}$ -tower starting in Adams filtration 1 should take value  $1/3$  on the invariant  $\int_M x \widehat{A}_8$  (modulo factors of other primes) and value 0 on the two other invariants,  $\int_M x^2 \widehat{A}_4$  and  $\int_M x^3$ . This is precisely the case of  $(\mathbb{H}P^3, u) - (V_{2,1}, ab)$ . Similarly,  $(\mathbb{H}P^3, u)$  and  $(\mathbb{C}P^3 \times \mathbb{C}P^3, ab) - (\mathbb{H}P^3, u)$  generate the  $\mathbb{Z}$ -towers starting in filtration 0. (The cohomology classes are those specified in section 3.)

**Proposition 2.** *The elements  $(\mathbb{C}P^3 \times \mathbb{C}P^3)$ ,  $(\mathbb{H}P^3)$ ,  $(V_{2,1})$  generate the group  $(\pi_{12} ko \wedge K(\mathbb{Z}, 4)) \otimes \mathbb{Z}_3$ .*

Similar considerations for the  $\mathbb{Z}$ -towers at the prime 5 give:

**Proposition 3.** *After completion at the prime 5, the elements  $(V_{2,1}, vw)$ ,  $(\mathbb{H}P^3)$ ,  $(\mathbb{C}P^3 \times \mathbb{C}P^3)$  generate the homotopy group  $\pi_{12} ko \wedge K(\mathbb{Z}, 4)$ .*

Now, to turn to the issue of surjectivity from  $BSpin$  to  $K(\mathbb{Z}, 4)$ . For  $BSpin$  the prime 3, the  $\mathbb{Z}$ -tower in  $t - s = 8$  starting in Adams filtration 1 maps to  $h_0$  of the corresponding  $\mathbb{Z}$ -tower in the spectral sequence for  $K(\mathbb{Z}, 4)$ , which starts in Adams filtration 0. Thus, surjectivity for this group is determined by where the other  $\mathbb{Z}$ -towers in  $t - s = 8$  in the  $BSpin$  spectral sequence map to in the  $K(\mathbb{Z}, 4)$  spectral sequence. A similar question determines the issue of surjectivity in dimension 12, as well as the issue of surjectivity at the prime 5 in dimension 12.

The behavior of the map  $H^*\lambda$  determines the value of  $\text{Ext}(\lambda)$  on these ‘extra’  $\mathbb{Z}$ -towers, and we compute both of these in the next section, completing the proof of surjectivity.

**5.2. The map  $\text{Ext}_{E[Q_0, Q_1]}(\lambda)$  and surjectivity.** The action of the Steenrod operations on  $H^*(BSpin; \mathbb{F}_p)$  completely determines  $H^*\lambda$ , as is the case with all maps whose target is an Eilenberg-MacLane space. At the prime 2, this is easily computed by the Wu formula. At odd primes, an expression like the Wu formula is not common knowledge, so we will explicitly compute the Steenrod operations using basic combinatorics.

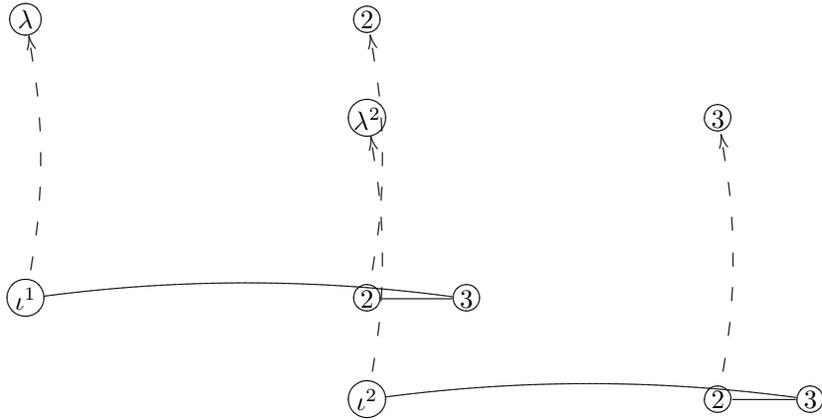
First, consider the inclusion of a maximal torus  $i : T^n \rightarrow Spin(2n)$ ,  $n$  sufficiently large. Modulo 2-torsion, the induced map on the cohomology of the classifying spaces,  $Bi^*$ , pulls the Pontryagin classes back to the elementary symmetric functions of the squares of the generators  $t_i$  in  $H^2(BT; \mathbb{Z})$ . These symmetric functions are the invariants under the action of the Weyl group on the cohomology of  $BT$ . We compute the action of the Steenrod operations in  $BT$  and then work backward to get the action on  $BSpin$ . This is equivalent, e.g., to complexifying and then using the splitting principle.

To compute a Steenrod reduced power  $P^i$  on  $H^*BT$  is purely an issue of combinatorics.  $H^*BT$  is a polynomial algebra on generators in degree 2,  $t_n$ . Directly from the axioms for cohomology operations, we have that  $P^1 t_i$  is equal to  $t_n^p$ , where  $p$  is the prime we are working at, and  $P^i t_n$  is zero for  $i$  greater than 1. Since all the nonzero cohomology lives in even degrees, all the Milnor primitives, the  $Q_i$ , act as zero.

The  $\mathbb{F}_p$  cohomology of  $BSpin$  is a polynomial algebra generated by the mod  $p$  reductions of the Pontryagin classes. Pulling these classes back to the cohomology of  $BT$ ,  $Bi^* p_1 = e_1 = \sum t_n^2$  the first elementary symmetric function in the squares of the  $t_n$ . Therefore, we have:  $Bi^* P^1 p_1 = P^1 Bi^* p_1 = P^1 e_1 = P^1 \sum t_n^2 = 2 \sum t_n^{p+1}$ , where the last step is by the Cartan formula. Thus, for  $p = 3$ ,  $P^1 p_1$  is the inverse image of  $2 \sum t_n^4$ . Determining this is a purely combinatorial problem, to rewrite the Newton polynomial  $s_2(t_1^2, \dots)$  in terms of elementary symmetric functions  $e_i(t_1^2, \dots)$ . Since  $s_2 = e_1^2 - 2e_2$ , we have that  $P^1 p_1 = 2p_1^2 + 2p_2$  at the prime 3. To compute  $P^1 p_n$  at the prime  $p$  is likewise the purely combinatorial question of rewriting the Newton polynomial  $2s_{1, \dots, 1, p+1/2}$  (where there are  $n - 1$  1s) in terms of elementary symmetric functions  $e_i$ . Thus,  $P^1 p_2$  is the inverse image of  $2s_{1,2}$ . Since  $s_{1,2} = e_1 e_2 \text{ mod } 3$ , therefore  $P^1 p_2 = 2p_1 p_2$  at the prime 3.

At the prime 5,  $P^1$  has degree  $2(5-1) = 8$ , and  $P^1 t_n^2 = 2t_n^6$ . Therefore computing  $P^1 p_1$  is equivalent to rewriting  $2s_3$  in terms of the  $e_i$ . The answer, mod 5, is that  $P^1 p_1 = 2(p_1^3 + p_1 p_2 - p_3)$ .

This is all the knowledge required to compute the map  $\text{Ext}(\lambda)$  between  $\text{BSpin}$  and  $\text{K}(\mathbb{Z}, 4)$  at all primes through 12 dimensions. Standard homological algebra supplies the relation between the map on cohomology and the map on  $\text{Ext}$ . That is, given a map of graded modules over the exterior algebra  $E[Q_0, Q_1]$ , from a module  $M$  to a direct sum of trivial 1-dimensional modules  $\Sigma^i S$ , there is a certain automatic surjectivity. In the example of the mod 3 cohomologies below:



The maps between the modules as given above (where the subset of element in the top module, the cohomology of  $\text{BSpin}$ , are those computed to be the image of  $\text{K}(\mathbb{Z}, 4)$ ),  $\text{Ext}$  of the direct sum of the 1-dimensional modules surjects onto  $\text{Ext}$  of the module below (the cohomology of  $\text{K}(\mathbb{Z}, 4)$ ), simply by looking at the long exact sequence on  $\text{Ext}$  coming from the associated short exact sequence of modules. This completes the theorem at the prime 3, and the result at 5 is identical. Thus, the action of the reduced powers on the cohomology of  $\text{BSpin}$  show that the extra  $\mathbb{Z}$ -towers in the  $\text{Ext}$  terms in degrees 8 and 12 pick up the extensions in Adams filtration zero for  $\text{K}(\mathbb{Z}, 4)$ .

## 6. NOTES

The methods used here are not at all special to spin manifolds,  $ko$ -theory, or 4-dimensional cohomology classes. One could imagine them having application to prove integrality results for oriented manifolds, using the Laures-McClure  $L$ -theory lift of the signature  $\text{Sig} : \mathbf{MO} \rightarrow L$ , or string manifolds, using the  $\text{tmf}$  orientation of String bordism,  $\sigma : \mathbf{MString} \rightarrow \text{tmf}$ .

The formulas in this paper are quite reminiscent of those in the index theorem, since the leading coefficient of  $x^n$  is  $(n!)^{-1}$ . So it would be very interesting if there were a similar multiplicative family of polynomials, like the Chern character of line bundles, taking integer values on spin manifolds. The integral lifts of the spin Wu characteristic classes satisfy a product formula, which is highly suggestive. The method used here, however, becomes more difficult in dimension 16, in the determination of spin manifold generators of  $ko_{16}\text{K}(\mathbb{Z}, 4)$ .

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