On the connective real K-theory of $\mathrm{K}(\mathbb{Z}, 4)$: an application of stable homotopy to integrals on spin manifolds

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Abstract

This thesis studies integrals on $4n$-dimensional spin manifolds M given by integrating a closed integral differential 4-form against the $\widehat{A}$-genus: that is, integrals of the form $\int_M x^k \widehat{A}$. The longer range goal, in analogy with the Atiyah-Singer index theorem, is to show that $\int_M P(x)\widehat{A}$ is an integer, for a specific $P(x)$ a polynomial with leading term $\frac{1}{n!}x^n$. A spin manifold M together with an integral 4-form can be interpreted as an element of the spin bordism of the Eilenberg-MacLane space $\mathrm{K}(\mathbb{Z}, 4)$. The $\widehat{A}$-genus is induced by Atiyah-Bott-Shapiro orientation, $\widehat{A} : \mathbf{M}\mathrm{Spin} \to bo$. By the Anderson, Brown, Peterson splitting of $\mathbf{M}$Spin, this integral then factors through the connective real K-theory of $\mathrm{K}(\mathbb{Z}, 4)$. Using an Adams spectral sequence, the thesis offers new proofs for such integrality in 8 and 12 dimensions.

1 Introduction

The goal of the work presented here is to use methods of stable homotopy theory to show that on spin manifolds certain integrals take only integer values. I relate certain analytic invariants of a manifold equipped with a specific 4-form to objects in the realm of stable homotopy. Construing the $\widehat{A}$-genus both as the index of a Dirac operator and as a map from spin bordism to connective K-theory allows the application of a fairly computable Adams spectral sequence to these analytic invariants.

Historically, the notion of spin arose in two fundamental ways. First, in studying representations of the Lie algebra of the special orthogonal group, Elie Cartan discovered that there were certain "spin" representations of this Lie algebra that were not induced by SO_n but came rather from its double cover, the spin group. Ten years later, in search of a relativistic theory of the electron, Paul Dirac discovered the most basic case of a Dirac operator in order to squareroot a Laplacian. Physicists then developed the notion of the spin (i.e., intrinsic angular momentum) of subatomic particles and found that of two basic types, fermions and bosons, bosons could be modeled on a vector bundle whose structure comes from a representation of SO_n, but fermions can only be modeled by a spin structure on a space, i.e., a vector bundle whose structure comes from a spin representation.

Mathematicians found this remarkable intersection of ideas even more engaging for two reasons: first, a large class of operators could be realized as Dirac operators on spinor bundles; and second, a large class of manifolds ("spin manifolds") naturally admitted a spin structure (that is, the structure group of their tangent bundle could be lifted from the orthogonal group to the spin group). These spaces had already long been studied for purely geometric interest, and the property of a manifold being spin has intuitive geometric meaning as a second order notion of orientability. The old picture of orientability is that one can say whether an n-dimensional person living on your space is right- or left-handed. That is, a local orientation cannot be reversed by traveling through your space. (If on a Möbius strip, for instance, a 2-dimensional man with a flag in what seems to be his right hand moves through the strip once and returns to where he has started, the flag will then appear to be in his left hand.) Technically, a space is orientable if the restriction of its tangent bundle to any embedded loop is not twisted. Analogously, a space is spin if additionally the bundle on any embedded surface induced by the tangent bundle is not twisted (i.e., is a product of the tangent space and the surface). This is true, for instance, of any orientable space for which every embedded circle and 2-sphere can be contracted to a point. Examples include higher dimensional spheres, Lie groups, projective spaces, hypersurfaces, and homogeneous spaces.

Atiyah and Singer, in studying the fundamental invariant of a Dirac operator (the index), found that it had a topological rigidity. Their result, the Atiyah-Singer index theorem, widely regarded as one of the great mathematical achievements of the past century, asserts that the index may be computed from purely topological data by integrating the $\widehat{A}$-genus associated to a spinor bundle against a specific polynomial in a differential 2-form (the Chern character). Furthermore, it was found that the natural home of this index theory and the $\widehat{A}$-genus was in K-theory.

One facet of this result is that, for a $4n$-dimensional spin manifold, integrating the $\widehat{A}$-genus against this polynomial yields an integer (as opposed to just a rational number). This integrality is a deep topological fact about spin manifolds.

One may ask the question whether there is not a similar integrality result for differential 4-forms over a spin manifold. In analogy with the result for 2-forms, one might like to produce polynomial functions with rational coefficients λ_i in an integral 4-form x on a $4n$-dimensional spin manifold, $P(x) = (\frac{1}{n!}x^n + \lambda_1\, x^{n-1} + \cdots + \lambda_{n-1}x)$ such that when integrated over the manifold against the total $\widehat{A}$-class, $\widehat{A} = \sum_k \widehat{A}_{4k}$, the result is an integer: $\int_M P(x)\widehat{A} \in \mathbb{Z}$.

There exists a classical result of this form for 8-dimensional spin manifolds. Recently, an expression for 12-manifolds was shown by Diaconescu, Moore, and Witten using a particular low-dimensional coincidence. The exceptional Lie group E_8 has only one nontrivial homotopy group through 13 dimensions, which is $\pi_3 E_8 = \mathbb{Z}$. This coincidence ends up meaning that an integral 4-form on a space of dimension less than 13 classifies a vector bundle with structure group E_8 over that space. Using index theory particular to E_8, in [6] and [16] Diaconescu, Moore, and Witten prove an expression for 12-manifolds. This method, however, cannot generalize to higher dimensional manifolds because E_8 has higher homotopy groups.

The purpose of the present work is to formulate a homotopy-theoretic method that will apply to higher dimensional manifolds, and using this method re-prove these results for 8- and 12-dimensional spin manifolds.

A sequence of steps is required to reinterpret the problem in terms of homotopy. First, for a 4-form x on a $4n$-dimensional spin manifold M, the integrals we are interested in, $\int_M x^i \widehat{A}_{n-i}$, depend not on x specifically but only on the cohomology class of x. Secondly, there is homotopy realization of cohomology as homotopy classes of maps into Eilenberg-MacLane spaces. Thus, a spin manifold equipped with a 4-form gives an element in the spin bordism of the Eilenberg-MacLane space that classifies 4-dimensional cohomology classes, $\mathrm{K}(\mathbb{Z},4)$ (where spin bordism is the relation given by saying that two spin n-manifolds are equivalent if their disjoint union is the boundary of a spin $(n+1)$-manifold). Further, this integral is an invariant of that spin bordism class. The spin bordism group $\Omega_*^{\mathrm{Spin}}(\mathrm{K}(\mathbb{Z},4))$ has an interpretation as a homotopy group, and the $\widehat{A}$-genus can be realized by a map from the spectrum representing spin bordism to the real K-theory spectrum or the connective real K-theory spectrum. This allows the application of powerful, but fairly technical, tools of homotopy theory, namely the Steenrod algebra and the Adams spectral sequence.

It is surprising that these methods should be applicable to this problem. Topology and homotopy first arose to attack problems in classical analysis: Poincaré devised the foundations of modern topology not because he was especially interested in higher dimensional spaces themselves, but out of interest in problems in celestial mechanics, where questions of the stability and periodicity of planetary orbits can be answered by considering the topology of the higher dimensional phase space of the mechanical system. Homotopy, likewise, was born from the calculus of variations, when Lagrange considered maximization problems over spaces of paths. Later generations of mathematicians, however, decided that higher dimensional spaces and spaces of maps between spaces (and their homotopy groups) were not just tools for analytic problems, but were intrinsically interesting.

Over the past century topological notions have been essential in studying analysis and particularly differential operators, but the technical tools of homotopy theory rarely come into play in addressing these sorts of questions. Homotopy theory applies most effectively to stable phenomena, meaning something that can occur in any dimension or in large enough dimension. Most geometric and analytic questions, in contrast, appear completely unstable. However, there exists a history of approaching unstable problems by finding a relation to some stable phenomenon, and then applying the tools of stable homotopy to solve the equivalent stable problem. This is how Thom classified which manifolds are the boundary of another manifold, and how Adams solved the question of vector fields on spheres. The work of this thesis takes a similar form.

2 Preliminaries

The volume of background material that supports this thesis is considerable and precludes complete treatment here. In particular, singular cohomology, which assigns algebraic objects and homomorphisms to spaces and maps between spaces, is fundamental to the study. It is developed in the standard textbooks. The reader's knowledge of singular cohomology and topology will be assumed throughout, but I will review the necessary homotopy theory and properties of the $\widehat{A}$-genus.

The mathematical machinery fundamental to the purpose and methods at hand lies entirely in the mainstream of homotopy theory: particularly, the Steenrod algebra and the Adams spectral sequence. The proofs of the fundamental results concerning these tools of stable homotopy are found in the standard texts. I will try to give a sense of how stable homotopy and its algebraic tools work, omitting many of the technical details.

2.1 Vector bundles and spin structures

The special orthogonal group SO_n of n-by-n self-adjoint matrices with determinant one has fundamental group $\mathbb{Z}/2$ (for $n > 2$). From the theory of covering spaces, there then exists a simply-connected cover. Since SO_n is connected, this cover comes naturally equipped with a group structure of loops on SO_n. This group is Spin_n. A vector bundle is orientable if the structure group can be reduced from GL_n to SO_n. It is spin if there then exists a lift to Spin_n.

A great asset of the theory of vector bundles is that the spaces that arise naturally from the homotopic viewpoint have natural geometric structure. Unlike most Eilenberg-MacLane spaces, which classify singular cohomology groups, the spaces that classify vector bundle structures over a given base, Grassmann manifolds, were of interest long before it was known that they possessed this powerful homotopy aspect. I will elaborate.

Given a particular fiber bundle $p : E \to B$ with fiber F and a map f from another space X into B, the pullback construction canonically constructs a fiber bundle over X with the same fiber and structure group as that given over B, given by the fiber product.

The total space of the bundle, $f^{-1}E = E \times_B X$, is defined as $\{(e, x) : p(e) = f(x)\}$, and the map $f^{-1}p$ projects each point (e, x) onto its second coordinate, x.

Homotopy enters into this world because of the following fact: up to isomorphism, the pullback bundle over X depends only on the homotopy class of the map f in $[X, B]$, not on the specific choice of the map. Further, maps which are not homotopic lead to distinct fiber bundle structures on X. This intuitively leads to the idea that for a given structure group and fiber, there may exist a 'universal' bundle by which we can construct all such fiber bundles as pullbacks.

For the case of vector bundles, the classifying space is the infinite Grassmannian of k-planes in $\mathbb{R}^\infty$. The Grassmann manifold $G_k(\mathbb{R}^{n+k})$ is defined to be the space of k-planes in $\mathbb{R}^{n+k}$, topologized appropriately. Thus, homotopy classes of maps $[X, G_k(\mathbb{R}^\infty)]$ are in one-to-one correspondence with isomorphism classes of vector bundles over X (say, with a choice of inner product on the fibers so as to reduce the structure group to O_n). By this correspondence, the problem of classifying all vector bundles over a space is equated with a homotopy problem, that of determining the homotopy classes of maps into a space.

A space BG is the classifying space for principal G-bundles if there exists a fibration $G \hookrightarrow EG \to BG$ with EG contractible: we can show that the infinite Grassmannian satisfies this property because of the following principal O_k-bundle: $\mathrm{O}_k \hookrightarrow V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$, where $V_k(\mathbb{R}^n)$ is the Stiefel manifold consisting of orthonormal k-frames in $\mathbb{R}^n$. There is a canonical map from the Stiefel manifold to the Grassmann manifold defined by sending each k-frame to the k-dimensional subspace it defines. The fiber is exactly O_k, which is also the structure group. By the inclusion of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$, we derive maps $V_k(\mathbb{R}^n) \to V_k(\mathbb{R}^{n+1})$ and $G_k(\mathbb{R}^n) \to G_k(\mathbb{R}^{n+1})$ which preserve the fiber O_k. Taking the limit of this sequence produces a bundle $\mathrm{O}_k \hookrightarrow V_k(R^\infty) \to G_k(\mathbb{R}^\infty)$. By a spectral sequence argument, the total space is contractible, which thus implies that the infinite Grassmannian is the classifying space for the orthogonal group. An analogous argument works for the special orthogonal group by considering the oriented Grassmannian.

This notion of lifting the structure group to Spin_n may not seem geometric. Also, the equivalent notion via characteristic classes that we will deal with (that the second Stiefel-Whitney class, w_2, is zero), probably does not clarify matters, despite its computational convenience. However, having a spin structure is very geometric, and essentially constitutes a second order notion of orientability. Where a bundle is orientable if the restriction of the bundle to any embedded circle S^1 is trivial, it has a spin structure if any map of a compact surface has trivial pullback bundle. In the case that the base space is simply-connected, then it is sufficient for the restriction of the bundle to an embedded 2-sphere to be trivial. See [8].

While this is a nice geometric characterization, the helpful computational perspective is that provided by characteristic classes, which will henceforth be central to the work here.

The most direct ways to get at Stiefel-Whitney classes is as follows: the mod 2 cohomology of $B\mathrm{O}_k$ (i.e., $G_k(\mathbb{R}^\infty)$) is a polynomial algebra on k elements. Call them $w_1, \ldots, w_k$, where w_i has degree i, and with no polynomial relations between them. A $\mathbb{R}^k$-bundle ξ over a space X is classified by a map $f : X \to B\mathrm{O}_k$, and thus induces a map on mod 2 cohomology going in the opposite direction $f^* : H^*(B\mathrm{O}_k; \mathbb{Z}/2) \to H^*(X; \mathbb{Z}/2)$. Then the Stiefel-Whitney classes $w_i(\xi)$ of ξ are the images in $H^*(X; \mathbb{Z}/2)$ of the generators w_i of the classifying space under the map on cohomology f^* induced by the map f classifying the vector bundle ξ. (References to the characteristic classes of a manifold refer to the characteristic classes of the tangent bundle of the manifold.)

The Steenrod squares provide two other ways of getting at Stiefel-Whitney classes: first, in terms of the Thom isomorphism and the action of the Steenrod squares on the Thom class of the vector bundle; second, as the total Steenrod square of the total Wu class. There is, further, an Euler class characterization, which has the advantage of making clear the relation between Stiefel-Whitney classes and the Chern and Pontryagin classes: for instance, for a complex vector bundle the Stiefel-Whitney classes are the mod 2 reductions of Chern classes. All of these approaches are done in [12].

For each vector bundle ξ over a base space X, there exist unique cohomology classes $w_i(\xi) \in H^i(X; \mathbb{Z}/2)$ that satisfy:

(1) Naturality: Given ξ and η $\mathbb{R}^n$-bundles and a commutative diagram:

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\widetilde{f}} & E(\eta) \\ {\scriptstyle p_\xi}\downarrow & & \downarrow{\scriptstyle p_\eta} \\ B(\xi) & \xrightarrow[f]{} & B(\eta) \end{array}$$

where $\widetilde{f}$ restricted to each fiber is an isomorphism. (In other words, $\widetilde{f}$ is a bundle map.) Then $f^*(w_i(\eta)) = w_i(\xi)$.

(2) For η and ξ vector bundles over the same base space X, the Stiefel-Whitney classes of their Whitney sum are given by the expression: $w_k(\eta \oplus \xi) = \sum_{i+j=k} w_i(\eta) \smile w_j(\xi)$.

(3) For w the total Stiefel-Whitney class of the tangent bundle of the circle, then $w = 1$. Also, if ξ is an $\mathbb{R}^n$-bundle, then $w_i(\xi) = 0$ for $i > n$.

It is further the case, and will be important for our purposes, for Stiefel-Whitney classes (and also for Chern and Pontryagin classes) that $w(\eta \times \xi) = w(\eta)w(\xi)$.

Definition 1. *A vector bundle possesses a spin structure if its classifying map can be lifted to $BSpin$. More geometrically, this is the case if the transition functions can be defined by an action of the Spin group on the fibers compatible with the action of the special orthogonal group.*

It is a fact that a vector bundle ξ is orientable if and only if $w_1(\xi) = 0$. ξ admits a spin structure if and only if it is orientable and $w_2(\xi) = 0$. For a proof see [8].

These axioms characterize the Stiefel-Whitney classes uniquely. As mentioned, one way of realizing them is as the pullbacks of the generators of the classifying space for vector bundles, but there exist other ways to construct them: by the inverse of the Thom isomorphism on the action of the Steenrod squares on the Thom class; by the total Steenrod square of the Wu class; and as an Euler class.

2.2 Spin bordism, spectra, and Thom spaces

The machinery of algebraic topology seems most powerful and well suited to the computation of stable homotopy groups. Stable homotopy, while an intricate and fascinating world by itself, is perhaps not one into which a differential geometer or others interested in spin manifolds alone normally venture. However, bordism theories are generalized homology theories, and thus admit techniques of stable homotopy. This is an example of an important avenue of algebraic topology, that of relating geometric phenomena (which is ostensibly unstable) to stable phenomena.

A cornerstone of homotopy theory is the notion of a spectrum, due to Lima and Whitehead. The usefulness of spectra comes from two fundamental facts about homotopy classes between maps. First, for two spaces X and Y such that Y is n-connected and the dimension of X is less than $2n-1$, the set of homotopy classes of maps $[X, Y]$ is stable, that is, naturally equivalent to $[\Sigma^k X, \Sigma^k Y]$. (Here, Σ is the suspension operation, which takes the product of a space with the unit interval and then collapses the boundary to points.) This implies that for any two spaces, regardless of connectivity and dimension, $[\Sigma^k X, \Sigma^k Y]$ is stable for large k. The second fact is that looping and suspension satisfy an adjoint relation : $[\Sigma X, Y] \cong [X, \Omega Y]$.

Furthermore, the set homotopy classes of maps between spaces, $[X, Y]$, can come by a group structure in two ways: X may be the suspension of some other space (this is why maps from spheres have a group structure), or Y can have some continuous multiplicative structure (such as a topological group or loops on some space). Where both of these conditions are met, the group structures coincide. For this reason, stable homotopy has a group structure, whereas the unstable homotopy classes of maps between arbitrary spaces need only be a set.

This leads to two different conceptions of a spectrum:

Definition 2. *An Ω-spectrum E is a sequence of spaces with basepoint, E_i, and based maps $\tau_i : E_{i-1} \to \Omega E_i$ that are homotopy equivalences.*

Definition 3. *A suspension spectrum K is a sequence of spaces with basepoint, K_i, and based maps $\kappa_i : \Sigma K_i \to K_{i+1}$ that are homotopy equivalences.*

These definitions are not equivalent. For example, the spheres obviously form a suspension spectrum, but they are not an Ω-spectrum.

The original, and perhaps most important, example of an Ω-spectrum is that made up of the Eilenberg-MacLane spaces $\mathrm{K}(\mathbb{Z}, n)$. A space X is a $\mathrm{K}(G, n)$ if, by definition, $\pi_* X = \begin{cases} G & \text{if } * = n \\ 0 & \text{otherwise} \end{cases}$

A fibration (e.g., a fiber bundle) $F \hookrightarrow E \to X$ induces a long exact sequence on homotopy groups. One of the most important examples of fibrations in homotopy theory is the path space fibration, where the total space is the contractible space of all paths in a space X (maps of the interval into X) and the fiber is the space of all loops with fixed basepoint, $\Omega X \hookrightarrow PX \to X$ that if X is a $\mathrm{K}(\mathbb{Z}, n)$. Since PX is contractible and its homotopy groups are zero, then $\pi_* X \cong \pi_{*-1} \Omega X$. By considering the path space fibration for $X = \mathrm{K}(\mathbb{Z}, n)$ we see that ΩX is a $\mathrm{K}(\mathbb{Z}, n-1)$.

Eilenberg-MacLane spaces do not generally admit nice geometric representatives. That is, nice geometric spaces typically have many and complicated higher homotopy groups. There are several exceptions in low dimensions: the circle S^1 is a $\mathrm{K}(\mathbb{Z}, 1)$; infinite real projective space $\mathbb{R}\mathrm{P}^\infty$ is a $\mathrm{K}(\mathbb{Z}/2, 1)$; and $\mathbb{C}\mathrm{P}^\infty$ is a $\mathrm{K}(\mathbb{Z}, 2)$.

These results are not hard to see. For instance the universal covering space of S^1 is $\mathbb{R}^1$, and it is a classical result of covering space theory that a map $X \to Y$ lifts to the universal cover of Y if X is simply connected. Therefore, all maps from higher dimensional spheres into S^1 factor through $\mathbb{R}^1$, which is contractible. Therefore the higher homotopy groups of S^1 are zero.

Following an argument similar to that we used to show that the infinite Grassmannian is the classifying space for the orthogonal group, we can look at the fibration $\mathbb{Z}/2 \hookrightarrow S^n \xrightarrow{f} \mathbb{R}\mathrm{P}^n$, where f identifies antipodal points. There is an obvious action of $\mathbb{Z}/2$ on the fiber, which consists of 2 points and hence may be identified with $\mathbb{Z}/2$. We can then include S^n into S^{n+1} as the equator, and consider the limit of the sequence. S^∞ is contractible, so from the long exact sequence on homotopy we see that $\pi_1\mathbb{R}\mathrm{P}^\infty \cong \pi_0\mathbb{Z}/2 \cong \mathbb{Z}/2$, and the other homotopy groups of $\mathbb{Z}/2$ vanish, implying that they also do for $\mathbb{R}\mathrm{P}^\infty$. The same argument works for $\mathbb{C}\mathrm{P}^\infty$. (This does not, however, work for $\mathbb{H}\mathrm{P}^\infty$, which has higher homotopy and is therefore not a $\mathrm{K}(\mathbb{Z}, 4)$.)

An important property of these spaces is that they classify singular cohomology. That is:

Theorem 1. *There exist canonical isomorphisms* $H^n(X; G) \cong [X, \mathrm{K}(G, n)]$ *and* $H_n(X; G) \cong \mathrm{colim}_k \pi_{n+k}\mathrm{K}(G, k) \wedge X_+$.

Stable homotopy is a generalized homology theory, and we can see that this definition is what we want because $S^k \wedge X_+ = \Sigma^k X_+$, that is, smashing with the k-sphere is exactly the same as taking the kth suspension. The definition of a $\mathrm{K}(G, n)$ defines the space up to homotopy.

Similarly, a generalized cohomology theory E^* is a functor which satisfies all the Eilenberg-Steenrod axioms except the dimension axiom specifying $E^*(\mathrm{pt.})$.

The Brown representability theorem states that for any such theory there exists a spectrum which classifies it. Generalized (co)homology theories come up in a variety of places, often very geometrically, and this result allows the application of the tools of stable homotopy theory to be applied to their study. For instance, complex K-theory can be thought of geometrically as stable isomorphism classes of complex vector bundles over a given base space, together with Whitney sums and tensor products. Formulated this way the theory has great geometric significance in a variety of problems concerning manifolds (such as vector fields on spheres). The application of the spectrum approach to this geometric theory is a deep combination of geometry and homotopy.

In the case of bordism theories, these spaces can be constructed quite explicitly from the classifying spaces of the (B, f)-structure as Thom spaces. [15] is the standard reference for all of this material.

The Thom space of a vector bundle is the total space E with all vectors of length greater than, say, 1 contracted to a point. Alternatively, it is the disk bundle quotiented out by the sphere bundle, $D(E)/S(E)$. If the base space is compact, the Thom space is just the one-point compactification of the total space.

The spectrum that represents spin bordism can be constructed this way: there is a universal bundle with base space $B\mathrm{Spin}$ classifying spin bundles. By taking the Thom space of the contractible total space $E\mathrm{Spin}$, we get our Thom space $M\mathrm{Spin}$. The Thom spectrum can then be constructed based on this term, and work of Thom shows that the generalized (co)homology theory defined by this spectrum coincides with the original geometric bordism theory defined by a (B, f)-structure.

2.3 The $\widehat{A}$-genus

A primary focus of topology is finding invariants of spaces that are reasonably computable and that behave nicely under basic operations on spaces (e.g., products). A genus is a particularly nice type of invariant: it is an invariant not just of a manifold, but of the oriented bordism class of the manifold, and it is additive under disjoint union and multiplicative under products. Every genus can be given by as a multiplicative sequence. The $\widehat{A}$-genus is defined by that coming from the formal power series of $\frac{\sqrt{x}}{\sinh(\sqrt{x}/2)}$ in terms of Pontryagin classes. In dimensions less than 16, $\widehat{A}$ can be computed from the Pontryagin classes of manifold by the following formulas.

$$\widehat{A}_4 = \frac{-p_1}{24}$$

$$\widehat{A}_8 = \frac{7p_1^2 - 4p_2}{2^7 \cdot 3^2 \cdot 5}$$

$$\widehat{A}_{12} = \frac{-31p_1^3 + 44p_1p_2 - 16p_3}{2^{10} \cdot 3^3 \cdot 5 \cdot 7}$$

and the value of the $\widehat{A}$-genus on a $4k$-dimensional manifold M is

$$\widehat{A}(M) = \int_M \widehat{A}_{4k}.$$

For any real vector bundle ξ, we can consider its complexification, i.e., we can tensor each fiber with $\mathbb{C}$ to get a complex vector bundle over the same base space. The complexification of a real bundle is its own conjugate bundle. This complex bundle then has Chern classes $c_i(\xi \otimes \mathbb{C})$. However, for any complex bundle ζ, the Chern classes of the conjugate bundle $\overline{\zeta}$ satisfy: $c_i(\overline{\zeta}) = (-1)^i c_i(\zeta)$. Thus for $\xi \otimes \mathbb{C}$, which is its own conjugate bundle, $c_i(\xi \otimes \mathbb{C}) = c_i(\overline{\xi \otimes \mathbb{C}}) = (-1)^i(\xi \otimes \mathbb{C})$. For i odd, $c_i(\xi \otimes \mathbb{C}) = -c_i(\xi \otimes \mathbb{C})$ implies that $2c_i(\xi \otimes \mathbb{C}) = 0$. That is, $c_i(\xi \otimes \mathbb{C})$ is 2-torsion for i odd. The total Pontryagin class of the real vector bundle ξ is then given by $p = \overline{c\overline{c}}$.

This then gives the relation to the Stiefel-Whitney classes that $p_i \equiv w_{2i}^2 \pmod 2$. If a manifold is spin then its first Pontryagin class is even.

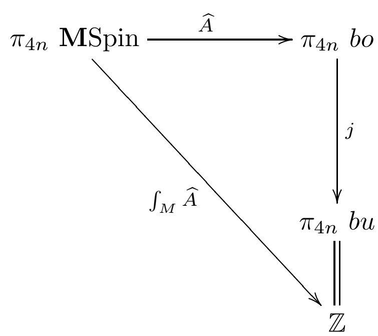
For most of the manifolds of interest in this work, computing the Pontryagin classes is fairly straightforward. For $\mathbb{C}\mathrm{P}^n$, $p(\mathbb{C}\mathrm{P}^n) = (1 + u^2)^{n+1}$ where u is a generator of $H^2(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$, the first Pontryagin class of the tautological line bundle. For quaternionic projective space, $p(\mathbb{H}\mathrm{P}^n) = (1 + y)^{2n+2}(1 + 4y)^{-1}$, and where y is the first Pontryagin class of the tautological SU_2 bundle, a generator of $H^4(\mathbb{H}\mathrm{P}^n; \mathbb{Z})$. These results are done in [12].

For a hypersurface V in $\mathbb{C}\mathrm{P}^n$ of degree d, the results follow directly from that of $\mathbb{C}\mathrm{P}^n$. The Lefschetz hyperplane theorem asserts that the embedding of $i : V \hookrightarrow \mathbb{C}\mathrm{P}^n$ induces an isomorphism in cohomology in dimension less than n. Stably, the Whitney sum of the normal bundle of the embedding and tangent bundle of V is equivalent to the restriction of the tangent bundle of $\mathbb{C}\mathrm{P}^n$, that is,

$N_i \oplus TV \equiv T\mathbb{C}\mathrm{P}^n|_V$. From this the Pontryagin classes can be directly computed, and the value of the $\widehat{A}$-genus can be obtained. See [8]. From these calculations, $\mathbb{C}\mathrm{P}^n$ is a spin manifold whenever n is odd. $V_d \subset \mathbb{C}\mathrm{P}^n$ is spin if $|d-n|$ is odd.

One can also obtain a spin 8-manifold B^8 with $\widehat{A}(B^8) = -1$. It is constructed by first plumbing together the boundaries of eight copies of the tangent disk bundle over S^4 according to the Dynkin diagram of the exceptional Lie group E_8. This makes an 8-dimensional manifold with boundary a homotopy sphere. After taking the connect sum of 28 copies, the boundary is then no longer exotic. B^8 is the closed manifold obtained by additionally gluing together this boundary with the boundary of the disk D^8. The tangent bundle is trivial away from a point, therefore its first Pontryagin class is zero. Details of the construction can be found in [7].

By [5], the $\widehat{A}$-genus is induced by a K-theory orientation of the spin bordism spectrum, that is, as a map $\widehat{A} : \mathbf{M}\mathrm{Spin} \to bo$. There is an injection $j : bo \hookrightarrow bu$ such that a spin manifold $M \in \Omega^{\mathrm{Spin}}$, the following diagram commutes:



where $\pi_{4n}\ bu \cong \mathbb{Z}$, and j is an isomorphism when n is even and corresponds to multiplication by 2 when n is odd.

2.4 Stable cohomology operations

Definition 4. *A cohomology operation ψ of type $(G, n; \pi, m)$ is a function $H^n(\ ; G) \to H^m(\ ; \pi)$ changing coefficients and degree, and defined canonically on spaces. That is, for any map $f : M \to N$ and any ψ, then $\psi \circ H^n(f; G) = H^m(f; \pi) \circ \psi$.*

Definition 5. *A cohomology operation ψ of type $(G, n; \pi, m)$ is stable if there exists a family of cohomology operations ψ_k of type $(G, n+k; \pi, m+k)$ extending ψ and commuting with the suspension operation Σ on spaces.*

Unpacking this somewhat, for any space X, suspension induces an isomorphism on reduced cohomology (with any coefficients): $\widetilde{H}^{n+1}(\Sigma X) \to \widetilde{H}^n(X)$. (This isomorphism can be understood as the connecting homomorphism in a Mayer-Vietoris sequence.) Then the ψ_k are stable if the following diagram commutes:

$$\begin{array}{ccc} H^{n+k+1}(\Sigma X;G) & \xrightarrow{\psi_{k+1}} & H^{m+k+1}(\Sigma X;\pi) \\ \downarrow & & \downarrow \\ H^{n+k}(X;G) & \xrightarrow[\psi_k]{} & H^{m+k}(X;\pi) \end{array}$$

(In this case ψ is usually just referred to as a single stable cohomology operation, and the subscripts are omitted.) The set of stable cohomology operations of type $(\mathbb{Z}/p, n; \mathbb{Z}/p, m)$ fit together in a near miraculous fashion, called the Steenrod algebra, which are uniquely characterized by the these additional properties:

(1) $Sq^n x = \begin{cases} x^2 & \text{if } |x| = n \\ 0 & \text{if } |x| < n \end{cases}$

(2) Coproduct structure: $Sq^n(x \smile y) = \sum_{i=0}^{n} Sq^i x \smile Sq^{n-i} y$.

(3) Adem relations: if $a < 2b$ then $Sq^a Sq^b = \sum_{i=0}^{[a/2]} \binom{b-1-i}{a-2i} Sq^{a+b-i} Sq^i$.

The Steenrod algebra has a basis, as an algebra, of $\{Sq^i \,:\, i \text{ is a power of } 2\}$. From this, and the Adem relations, we can see easily that it has a basis as a vector space consisting of so-called admissible monomials $Sq^{\{a_n,\ldots,a_1\}} := Sq^{a_n} \ldots Sq^{a_1}$ such that $a_i \geq 2a_{i-1}$ (that is, the $a_n, \ldots, a_1$ form an admissible sequence). The Adem relations give a rule for expressing inadmissable monomials as linear combinations of admissible monomials, so these elements clearly generate the entire Steenrod algebra. The admissible monomials then form a basis since our axioms provide no additive relations between them. The excess $e(A)$ of an admissible sequence $a_n, \ldots, a_1$ is defined to be $e(A) = a_n - a_{n+1} - \ldots - a_1$.

The existence of operations that satisfy these relations has powerful and immediate consequences about the possible structures of the cohomology ring of a space. For instance consider the truncated polynomial ring with generator y in dimension $2k$ such that k is not a power of 2. One may ask whether this ring can be the cohomology ring of a space. Since the Sq^{2^i} are an algebra basis for the Steenrod algebra, therefore Sq^{2k} can be expressed as a sum of compositions of Sq^{2^i}. By our axioms, $Sq^{2k} y = y^2$, but this Sq^{2k} operation can be factored through terms in intermediate degrees. By assumption on the structure of our cohomology, these are zero, however, which gives a contradiction and implies that this ring structure cannot be the cohomology ring of any space.

Another geometric question that the existence of these Steenrod operations answers is whether the suspensions of $\mathbb{C}\mathrm{P}^2$ and $S^2 \vee S^4$ can be homotopy equivalent. The ring structure of a cohomology ring cannot carry over to the cohomology ring of the suspension of the space (for obvious reasons of degree), so while it distinguishes these spaces, it does not obviously distinguish their suspensions. However, the action of Sq^2 on the third cohomology class will give the generator of H^5 for $\Sigma\mathbb{C}\mathrm{P}^2$ but zero for $\Sigma(S^2 \vee S^4)$ because of the stability condition.

For M a connected n-dimensional manifold, then there exist classes $v_i \in H^i(M;\mathbb{Z}/2)$ such that $Sq^i y = v_i \smile y$ and for any class $y \in H^{n-i}(M;\mathbb{Z}/2)$. This is clear, since $H^n(M;\mathbb{Z}/2) \cong \mathbb{Z}/2$ and Sq^i, in this narrow context, is then an element of $\mathrm{Hom}_{\mathbb{Z}/2}(H^{n-i}(M;\mathbb{Z}/2),\mathbb{Z}/2)$. By Poincaré duality, we then get the existence of v_i representing this element. The total Stiefel-Whitney class is then the total Steenrod square of the sum of the v_i (called the total Wu class), $w = Sq(v)$.

There exists another way to realize the Stiefel-Whitney classes in terms of the Steenrod squares and Thom isomorphism ϕ. In this case for ξ an $\mathbb{R}^n$-bundle over X, $\phi : H^i(X;\mathbb{Z}/2) \xrightarrow{\sim} H^{i+n}(E,E_0;\mathbb{Z}/2)$. Then $w_i(\xi) = \phi^{-1}(Sq^i u)$, where u is the Thom class.

Considering stable cohomology operations with $\mathbb{Z}/p$ coefficients, p odd, yields an analogous algebra satisfying similar axioms, with slight variation. For one, the degree of P^n is $2n(p-1)$. Also:

(1) $P^n x = \begin{cases} x^p & \text{if } |x| = 2n \\ 0 & \text{if } |x| < 2n \end{cases}$

(3) $P^a P^b = \sum_{j=0}^{[a/p]} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^j$.

A sequence $I = (\varepsilon_0, s_1, \varepsilon_1, s_2, \ldots s_n, \varepsilon_n)$ is called admissible if $s_{i+1} \geq ps_i + \varepsilon_{i+1}$. The excess $e(I)$ is the defined by $e(I) = 2s_1 + \varepsilon_n - \varepsilon_{n-1} - \ldots - \varepsilon_0$.

For a module M over the Steenrod algebra, one can define the Q_i-homology $H(M;Q_i) = \mathrm{Ker}\ Q_i / \mathrm{Im}\ Q_i$. The Q_i are defined inductively by successive commutators (e.g., Q_1 is the commutator of Q_0 and P^1) and act as differentials on $\mathcal{A}$-modules. Q_i-homologies are known as the Margolis homologies, and they play an important role in the stable structure theory of modules over $\mathcal{A}(1)$ and $E[Q_0, Q_1]$, as in [3], particularly in computations involving the Adams spectral sequence.

2.5 Classical Adams spectral sequences

Spectral sequences were invented by Jean Leray in a German concentration camp during World War II. They are formal way of organizing information, and have proved an essential tool for tackling topological problems over the past 40 years. One can construct a spectral sequence via an exact couple or a filtered complex to make a computation whose steps are the refinement of an initial over-approximation of an object by sequence of differentials. For example: the Serre spectral sequence relates the cohomologies of the base, fiber, and total space of fibration; the Bockstein spectral sequence computes rational cohomology from mod p cohomology; there exists a spectral sequence generalizing the Mayer-Vietoris short exact sequence.

There is also a spectral sequence due to Adams which addresses, as he puts it, the basic question of what can be known about stable homotopy classes of maps $[\Sigma^t Y, X]$ given knowledge of $H^*(Y;\mathbb{Z}/p)$ and $H^*(X;\mathbb{Z}/p)$.

Theorem 2. *There exists a spectral sequence converging to the p-completion of the stable homotopy group $[\Sigma^{t-s}Y, X]$, i.e., $[\Sigma^{t-s}Y, X] \otimes \mathbb{Z}_p$. The spectral sequence has first term $E_2^{t-s,s} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X;\mathbb{Z}/p), H^*(Y;\mathbb{Z}/p))$ and has differentials $d_n^{-1,n}$.*

The spectral sequence is bigraded, where s (the Adams filtration) is the homological degree, and t is the internal degree of a homomorphism based on the gradation (lowering degree by t).

Actually, it converges to the stable homotopy group once it has been localized with respect to the Eilenberg-MacLane spectrum, but this localization does not do anything, i.e., it does not result in any loss of information. Novikov later proved a similar theorem for complex cobordism, and there exists a so-called Adams-Novikov spectral sequence for a broad range of generalized cohomology theories.

The spectral sequence is constructed by filtering a map through maps to spheres that are zero on the level of homology. The Ext functor is then applied, giving a filtration to the first term of the spectral sequence. This construction can be found in [10] or [2]. The idea of cohomology is to provide a partial dictionary between spaces and algebra, so that certain statements about spaces and maps between them can be translated into statements about algebraic objects and homomorphisms (which will frequently be more tractable). Cohomology is an obviously useful tool for showing the nonexistence of maps with specific properties. It is not obviously useful for showing the existence of maps. Supplementing cohomology with cohomology operations, however, gives an algebraic structure sufficient to detect the existence of maps, which is precisely what the Adams spectral sequence is designed to do.

3 The Adams spectral sequence

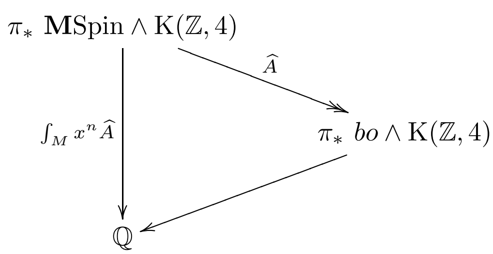
I want to apply this homotopy theory to understand certain maps on spin manifolds given by integrating powers of integral 4-forms against the $\widehat{A}$-genus. The first step is to interpret this phenomenon in homotopy-theoretic terms. For a manifold M, de Rham's theorem gives an isomorphism between the de Rham cohomology of differential forms and singular cohomology with real coefficients, $H^*_{DR}(M;\mathbb{R}) \cong H^*(M;\mathbb{R})$. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces a monomorphism $H^*(M;\mathbb{Z})/T \hookrightarrow H^*(M;\mathbb{R})$ of integer singular cohomology modulo torsion into singular cohomology with real coefficients. The images in de Rham cohomology $H^*_{DR}(M;\mathbb{R})$ of this map are called *integral*.

There exists yet another characterization of singular cohomology as homotopy classes of maps into Eilenberg-MacLane spaces, so that $H^n(M;\mathbb{Z}) \cong [M, \mathrm{K}(\mathbb{Z}, n)]$. Thus, a closed integral n-form x on M yields a map $x : M \to [M, \mathrm{K}(\mathbb{Z}, n)]$ (modulo torsion) representing that same cohomology class. (I will use the same symbol to represent the differential form and the map, with the understanding that all subsequent constructions will use only properties of the homotopy class of the map, not the map itself.)

So a k-dimensional manifold M together with a fixed integral n-form, by considering the n-form as a map into $\mathrm{K}(\mathbb{Z}, n)$, represents an element of the unoriented bordism ring of $\mathrm{K}(\mathbb{Z}, n)$ (because this depends only the homotopy class of the map into $\mathrm{K}(\mathbb{Z}, n)$). If M is a spin manifold, (M, x) can also be thought of as representing a unique element of the spin bordism ring of $\mathrm{K}(\mathbb{Z}, n)$, i.e., $(M, x) \in \Omega_k^{\text{Spin}}(\mathrm{K}(\mathbb{Z}, n))$. Hereafter, the concern is with 4-forms on a $4n$-dimensional spin manifold, elements of $\Omega_{4n}^{\text{Spin}}(\mathrm{K}(\mathbb{Z}, 4))$.

Spin bordism is a generalized homology theory represented by the Thom spectrum $\mathbf{M}\text{Spin}$, realizing the spin bordism of a space M as a stable homotopy group, $\Omega_*^{\text{Spin}}(M) \cong \pi_* \, \mathbf{M}\text{Spin} \wedge M_+$. (Note: the disjoint basepoint notation will be typically omitted.)

The Atiyah-Bott-Shapiro orientation gives a map $\widehat{A} : \mathbf{M}\text{Spin} \to bo$, where bo is the connective K-theory spectrum, obtained from the usual real K-theory spectrum by killing its negative homotopy groups, see [2]. So the $\mathbb{Q}$-valued maps given by integrating an integral 4-form against the total $\widehat{A}$-class factor through the connective K-theory of $\mathrm{K}(\mathbb{Z}, 4)$, giving a commutative diagram:



The purpose of this chapter is to partially compute the connective K-theory of $\mathrm{K}(\mathbb{Z}, 4)$, the stable homotopy group $\pi_n \, bo \wedge \mathrm{K}(\mathbb{Z}, 4)_+ = [\Sigma^n S^0, bo \wedge \mathrm{K}(\mathbb{Z}, 4)_+]$, after completion at the prime 2. This will be the most important case, due to the complexity of bo at 2 and the fact that spin bordism has no odd torsion. In the next chapter, these results will be applied to integrals on spin manifolds.

The tool for this computation is an Adams spectral sequence that is particularly effective for connective K-theory. The first term of this spectral sequence is $\text{Ext}_{\mathcal{A}}^{s,t}(H^*(bo \wedge \mathrm{K}(\mathbb{Z}, 4); \mathbb{Z}/p)$. From [1] and [14], the spectrum cohomology of bo can be given by the quotient of the Steenrod algebra by the subalgebra generated by Sq^1 and Sq^2, so $H^*(bo; \mathbb{Z}/2) = \mathcal{A}//\mathcal{A}(1)$. At odd primes, bo splits as a wedge of spaces with simpler cohomology.

Computing Ext over the Steenrod algebra is generally daunting. However, in the specific case of computing connective real K-theory, we have a Hom-tensor interchange for simplifying this term. (Cohomology for the rest of this chapter will be mod 2, so this is frequently omitted from the notation.)

First, we have a Künneth formula so that

$$\begin{aligned} H^*(bo \wedge \mathrm{K}(\mathbb{Z}, 4)) &= H^*(bo) \otimes_{\mathbb{Z}/2} H^*(\mathrm{K}(\mathbb{Z}, 4)) \\ &= \mathcal{A}//\mathcal{A}(1) \otimes_{\mathbb{Z}/2} H^*(\mathrm{K}(\mathbb{Z}, 4)) \end{aligned}$$

However, $\mathcal{A}//\mathcal{A}(1) = \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2$. So substituting this into the first term of our spectral sequence gives:

$$\text{Ext}_{\mathcal{A}}^{s,t}(H^*(bo \wedge \mathrm{K}(\mathbb{Z}, 4)) = \text{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2 \otimes_{\mathbb{Z}/2} H^*(\mathrm{K}(\mathbb{Z}, 4)), \mathbb{Z}/2)$$

$$= \operatorname{Ext}^{s,t}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{A}(1)} H^*(\mathrm{K}(\mathbb{Z},4)), \mathbb{Z}/2)$$

The final step in simplifying our Ext term is the Hom-tensor interchange (in this case an Ext-tensor interchange), so

$$\operatorname{Ext}^{s,t}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{A}(1)} H^*(\mathrm{K}(\mathbb{Z},4)), \mathbb{Z}/2) = \operatorname{Ext}^{s,t}_{\mathcal{A}(1)}(H^*(\mathrm{K}(\mathbb{Z},4)), \mathbb{Z}/2).$$

Now, instead of computing Ext over the whole infinite-dimensional Steenrod algebra, we can instead work over the finite subalgebra $\mathcal{A}(1)$ generated by Sq^1 and Sq^2.

3.1 $H^*(\mathrm{K}(\mathbb{Z},4);\mathbb{Z}/2)$ as an $\mathcal{A}(1)$-module

The mod p cohomology of Eilenberg-MacLane spaces can be computed inductively using the Serre spectral sequence for the pathspace fibration relating $\mathrm{K}(\mathbb{Z},n)$ and $\mathrm{K}(\mathbb{Z},n-1)$. That is, the based loopspace $\Omega\mathrm{K}(\mathbb{Z},n)$ is a $\mathrm{K}(\mathbb{Z},n-1)$, and so one has the pathspace fibration

$$\begin{array}{ccc} \Omega\mathrm{K}(\mathbb{Z},n) & \longrightarrow & P\mathrm{K}(\mathbb{Z},n) \\ & & \downarrow \\ & & \mathrm{K}(\mathbb{Z},n) \end{array}$$

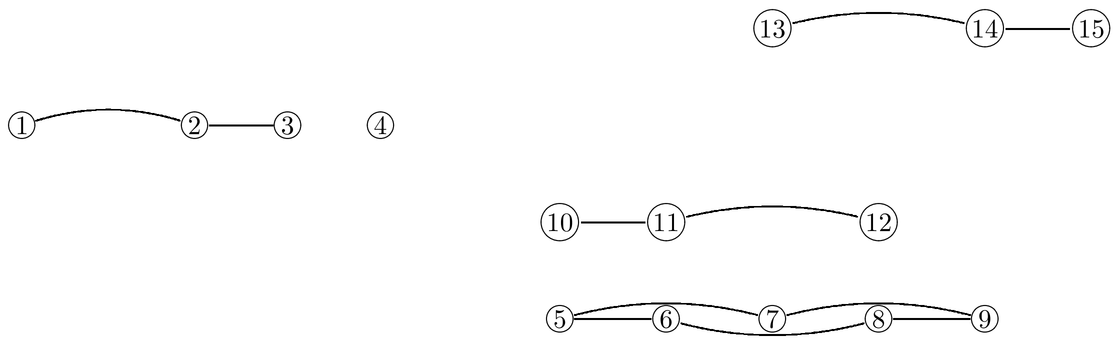
where $P\mathrm{K}(\mathbb{Z},n)$ is the space of paths in $\mathrm{K}(\mathbb{Z},n)$ and thus is contractible. Serre first made this computation shortly after Steenrod's introduction of the Steenrod algebra in the early 60's, expressing the mod p cohomology of $\mathrm{K}(\mathbb{Z},n)$ as a freely generated polynomial algebra with generators given by Steenrod operations on a single element in degree n. In particular, the mod 2 cohomology of $\mathrm{K}(\mathbb{Z},4)$ is:

$$H^*(\mathrm{K}(\mathbb{Z},4);\mathbb{Z}/2) = \mathbb{F}_2[Sq^I\iota \ : \ 1 < e[I] < 4]/(Sq^1\iota = 0)$$

where Sq^I is an element of the Cartan basis, i.e., I is an admissible sequence. These computations can be found in [10].

Although one can then understand the action of Sq^1 and Sq^2 just by using the Adem formula, decomposing this cohomology, which is a left $\mathcal{A}(1)$-module, into a direct sum of simpler $\mathcal{A}(1)$-modules may be complicated. However, the present purpose for applications to 8- ands 12-manifolds does not require much. A general understanding of this cohomology and higher-dimensional manifolds entails significant further work, which will be addressed in the epilogue.

This gives us, with only moderate effort, the structure of the action of Sq^1 and Sq^2 over a small range of dimensions.



The digits represent the following elements of the cohomology:

$1 - \iota$	$6 - \iota(Sq^3\iota)$	$11 - Sq^5Sq^2\iota$
$2 - Sq^2\iota$	$7 - (Sq^2\iota)^2$	$12 - Sq^6Sq^3\iota$
$3 - Sq^3\iota$	$8 - (Sq^3\iota)(Sq^2\iota)$	$13 - \iota^3$
$4 - \iota^2$	$9 - (Sq^3\iota)^2$	$14 - \iota^2(Sq^2\iota)$
$5 - \iota(Sq^2\iota)$	$10 - Sq^4Sq^2\iota$	$15 - \iota^2(Sq^3\iota)$

3.2 $\mathbf{Ext}_{\mathcal{A}(1)}$ of stably invertible $\mathcal{A}(1)$-modules

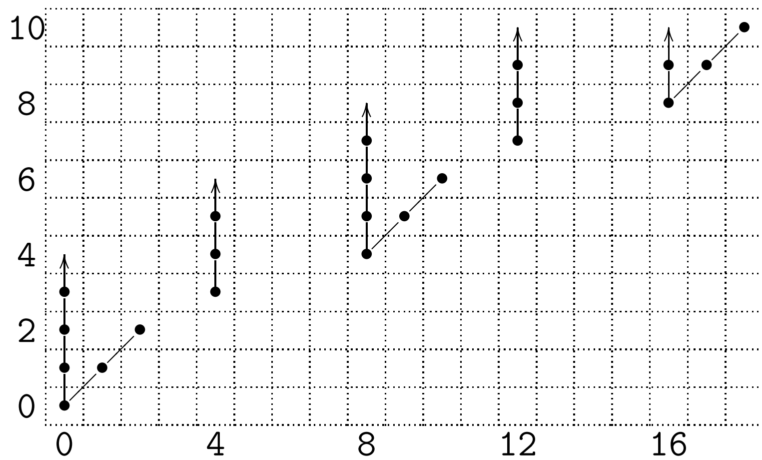
The $\mathcal{A}(1)$-modules comprising the mod 2 cohomology of $\mathrm{K}(\mathbb{Z}, 4)$ through dimension 15 (neglecting an unimportant free module) all share the property of being *stably invertible*. That is:

Definition 6. *$\mathcal{A}(1)$-module X is stably invertible if there exists an $\mathcal{A}(1)$-module Y such that as $\mathcal{A}(1)$-modules $X \otimes_{\mathbb{Z}/2} Y = \mathbb{Z}/2 \oplus F$ where F is a free $\mathcal{A}(1)$-module.*

The stable inverses for the above modules are not hard to find. For instance, for $X := \{x_0, x_1, x_3 : Sq^1x_0 = x_1, Sq^2x_1 = x_3, \text{ and otherwise } Sq^ix_j = 0\}$, then we can define $Y := \{y_{-3}, y_{-1}, y_0 : Sq^2y_{-3} = y_{-1}, Sq^1y_{-1} = y_0\}$, and then the tensor product $X \otimes Y \cong \mathbb{Z}/2 \oplus F$.

These stably invertible modules have a well developed structure theory due to Margolis, Adams, and Priddy. For instance, it is not hard to prove that a finite $\mathcal{A}(1)$-module is stably invertible if and only if both its Q_0-homology and its Q_1-homology are 1-dimensional. Using this structure theory as in [3], or otherwise [11], it is straightforward to compute $\mathrm{Ext}_{\mathcal{A}(1)}$ for them.

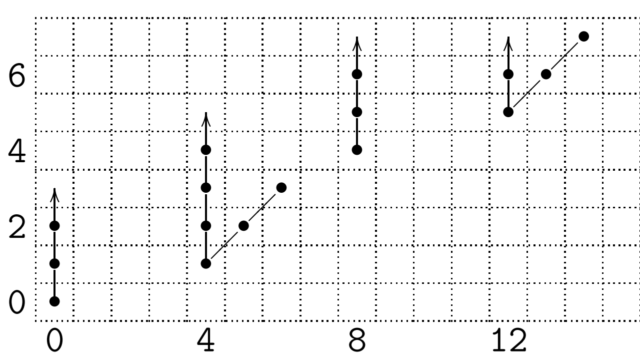
The mod 2 cohomology of a point has the structure of an $\mathcal{A}(1)$-module consisting of a single copy of $\mathbb{Z}/2$ in degree 0. In this case, Ext looks as follows:



As is customary in drawing Adams spectral sequences, s is measured along the vertical axis, and $t-s$ is measured along the horizontal axis. This Ext term is easy to compute just by writing down a projective (i.e., free) resolution of $\mathbb{Z}/2$. This resolution is periodic, reflected by the periodicity in the Ext chart, and the relation of this Ext chart to Ext for other stably invertible modules can be seen by the fact that they occur as kernels in each others projective resolutions. Thus, their Ext charts are translates of each other.

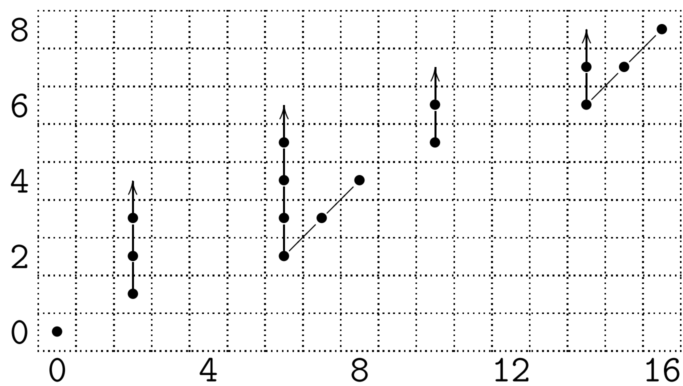
The ring structure and the action of $h_0 \in \mathrm{Ext}^{1,1}$ and $h_1 \in \mathrm{Ext}^{2,1}$ come from the action of Sq^1 and Sq^2 in the kernels of the resolution. The action of h_0 on $h_0 v_1^2$ generates the $\mathbb{Z}$-tower in $t-s=8$. These computations can be found in [11] as well as [9]. The action of h_0 in each $\mathbb{Z}$-tower corresponds to multiplication by 2 in the limit of the spectral sequence. (The $\mathbb{Z}$-towers converge to the 2-adic integers.) The differentials in the spectral sequence are derivations with respect to this multiplication, therefore there can be no differentials (because only $\mathbb{Z}$-towers can map into $\mathbb{Z}$-towers, and this is impossible because there are no adjacent $\mathbb{Z}$-towers).

For the module $\Sigma^3 Y$, obtained by shifting the degree of the elements in Y (as defined above) by 3, the Ext chart is:

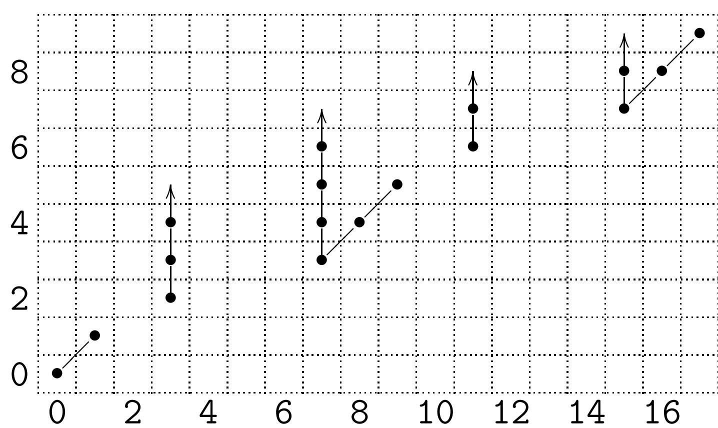


For the joker (as Adams termed it), $J := \{j_0, j_1, j_2, j_3, j_4 : Sq^1 j_0 = j_1, Sq^1 j_3 =$

$j_4, Sq^2Sq^2j_0 = Sq^2j_2 = j_4, Sq^2j_1 = j_3\}$, a translate of which starts in dimension 10 in the mod 2 cohomology of $\mathrm{K}(\mathbb{Z}, 4)$, the Ext term is:



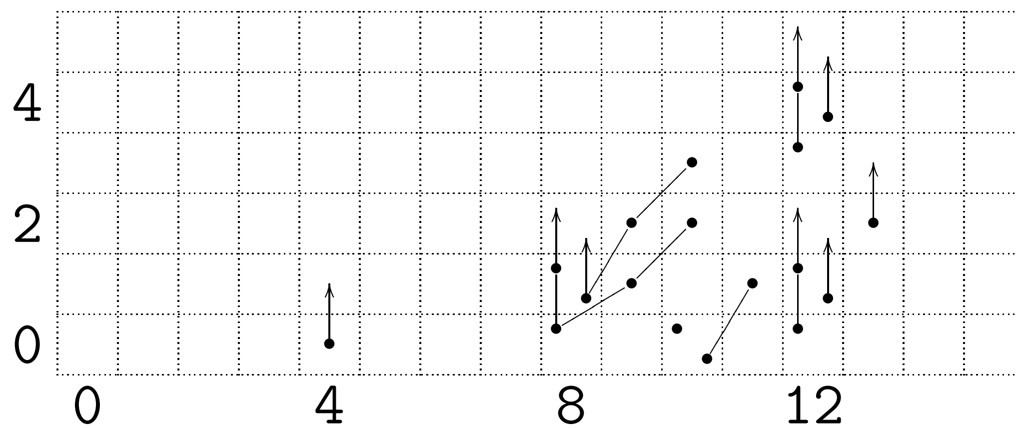
And for X as above, Ext is:



All of these computations are stable. Namely, shifting the degree of the $\mathcal{A}(1)$-modules results only in a translation of the Ext chart. Thus, the beginning of the Ext term for $H^*(\mathrm{K}(\mathbb{Z}, 4); \mathbb{Z}/2)$ can be obtained from the above computations simply by observing the degrees of the $\mathcal{A}(1)$-submodules in the decomposition.

3.3 $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(bo \wedge \mathrm{K}(\mathbb{Z}, 4); \mathbb{Z}/2), \mathbb{Z}/2)$

The Ext term for the Adams spectral sequence converging to the connective K-theory of $\mathrm{K}(\mathbb{Z}, 4)$ through 15 dimensions is a direct sum of the Ext terms just computed:



Additionally, there must exist a differential from the $\mathbb{Z}$-tower in $t-s=13$ to $t-s=12$. We know this in two ways: first, from rational homotopy, we know that once we tensor this homotopy group with $\mathbb{Q}$, nonzero homotopy groups occur only in dimensions $4n$ by Bott periodicity. Secondly, the computations of the Ext term at any odd prime (see chapter 5) will have no $\mathbb{Z}$-tower in $t-s=13$. Therefore, $\pi_{13}\ bo \wedge \mathrm{K}(\mathbb{Z},4)$, modulo torsion, is zero. This means that the $\mathbb{Z}$-tower in $t-s=13$ cannot survive the sequence of differentials. Generally, a $\mathbb{Z}$-tower may get killed either by supporting a nontrivial differential or by being in the image of a differential. Since there is no $\mathbb{Z}$-tower in $t-s=14$, the $\mathbb{Z}$-tower in 13 cannot be in the image of differential. Thus, there exists a differential coming from the bottom of the $\mathbb{Z}$-tower in $t-s=13$ and hitting some linear combination of the $\mathbb{Z}$-towers in $t-s=12$.

4 Applications to spin manifolds

Our purpose is to understand the relation between integrals of the form $\int_M x^n \widehat{A}$ on a spin manifold. By [5], the $\widehat{A}$-genus has a realization as a map on spectra from the spin bordism spectrum $\mathbf{M}$Spin to the real K-theory spectrum BO given by the difference bundle construction. By the splitting of the spin bordism spectrum due to Anderson, Brown, and Peterson [4], on the level of homotopy this map $\widehat{A} : \mathbf{M}\mathrm{Spin} \to bo$ is surjective after completion at 2.

In [4] it is proved that at the prime 2 the spin bordism spectrum splits as product of connective covers of the real K-theory spectrum BO and Eilenberg-MacLane spaces $\mathrm{K}(\mathbb{Z}/2,n)$. That is, there exists a map from $\mathbf{M}$Spin to such a product, and it induces an isomorphism on mod 2 cohomology. Additionally, spin bordism has no odd torsion, so this determines the structure of the spin bordism spectrum.

Using this work, one can prove certain results about all spin manifolds equipped with a 4-form by finding a set of spin manifolds with 4-forms whose image under the map $\widehat{A}$ generates the connective K-theory of $\mathrm{K}(\mathbb{Z},4)$ and then verifying the result for this generating set. This is the goal of this chapter.

In dimension 8, this is relatively straightforward. However, in higher dimensions there are differentials in the spectral sequence, and this complicates the procedure. Ideally, one could prove these results by working directly with the Ext chart and not the final term of the spectral sequence (which will be a complicated quotient of Ext).

One way to get around this problem in dimension 12 is by working with filtrations of $\mathrm{K}(\mathbb{Z},4)$ given by its mod 2 cell structure. Using quotients of this filtration, one can work with more manageable spaces whose Ext terms nonetheless encode the same information as $\mathrm{K}(\mathbb{Z},4)$ after completing at 2.

The prospect for 16 and higher dimensions is briefly discussed in the epilogue.

4.1 $\int_{M_{16n+8}} x^{4n+2} \pm 12\ x^{4n+1}\widehat{A}_4 \in 2\mathbb{Z}$

Before doing the hard work of defining analytic invariants of the spin bordism of $\mathrm{K}(\mathbb{Z},4)$ and determining a set of manifolds that generate its connective K-theory, we can get some facts from this spectral sequence relatively easily. We will need only a little information from this spectral sequence to prove the following classical formula for spin 8-manifolds.

Theorem 3. *M an 8-dimensional spin manifold, and x an integral 4-form, then*

$$\int_M x^2 \ \pm\ 12x\widehat{A}_4 \ \equiv\ 0 \ (\mathrm{mod}\ 2).$$

Proof: First, for any spin manifold M, $w_4(M) \equiv \frac{1}{2}p_1(M)$ mod 2. This can be shown simply by looking at the cohomology of the classifying space $B\mathrm{Spin}$, see [15], and observing that $\frac{1}{2}p_1$ generates $H^4(B\mathrm{Spin};\mathbb{Z})$ while w_4 generates $H^4(B\mathrm{Spin};\mathbb{Z}/2)$, so that w_4 must be the mod 2 reduction of $\frac{1}{2}p_1$. (This actually shows something stronger: that $w_4(V) \equiv \frac{1}{2}p_1(V)$ mod 2, for any spin bundle V.)

Therefore, for a spin 8-manifold M and an integral 4-form x,

$$\langle w_4\ x, [M]_{\mathbb{Z}/2}\rangle \equiv \int_M \frac{p_1}{2}\ x \ \ (\mathrm{mod}\ 2).$$

Consider the map $\pi_8\ bo \wedge \mathrm{K}(\mathbb{Z},4) \to \mathbb{Z}/2$ given by $(M,x) \to \int_M x^2$ (mod 2). This sits in filtration 0, thus is an element of Hom by definition of Ext as a derived functor of Hom. The following diagram then commutes.

$$\begin{array}{ccc} \pi_8\ bo \wedge \mathrm{K}(\mathbb{Z},4) & \longrightarrow & \mathrm{Hom}^8_{\mathcal{A}}(\mathcal{A}//\mathcal{A}(1) \otimes H^*(\mathrm{K}(\mathbb{Z},4))) \\ & \searrow^{\int_M x^2} & \downarrow \\ & & \mathbb{Z}/2 \end{array}$$

By definition, a homomorphism over the Steenrod algebra ξ must commute with the Steenrod operations, i.e., $\xi \circ Sq^n = Sq^n \circ \xi$. Consider $u \in H^0(bo)$ and $x \in H^4(\mathrm{K}(\mathbb{Z},4))$.

Then from the coproduct structure of $\mathcal{A}$ and the Thom class definition of the Stiefel-Whitney classes, $Sq^4(u \otimes x) = Sq^4 u \otimes x + u \otimes Sq^4 x = w_4 u \otimes x + u \otimes x^4$. So for ξ as above, $\xi(w_4 u \otimes x + u \otimes x^4) = \xi \circ Sq^4(u \otimes x) = Sq^4 \xi(u \otimes x) = 0$.

Therefore for an orientable manifold M equipped with an integral 4-form x (whose mod 2 reduction will also be denoted x), it is true that

$$\langle w_4 x + x^2, [M]_{\mathbb{Z}/2} \rangle \equiv 0 \pmod 2.$$

This is the mod 2 reduction of $\int_M x^2 \pm \frac{1}{2} p_1 x$ for M a spin manifold, so $\int_M x^2 \pm \frac{1}{2} p_1 x$ is even. □

One can interpret this result in the language of lattices: the middle cohomology class of an even-dimensional manifold, equipped with the intersection form, has the structure of a lattice, and the result above means that $p_1/2$ is a characteristic vector of that lattice if the 8-manifold is spin. It also means that if $p_1/2$ is even, then the intersection form of the manifold must be even.

This is promising, because it gives us an interesting analytic result while seeming to require little from our spectral sequence. One would like to generate further integrality results by first using the observation that certain elements lie in the image of Steenrod operations, and must therefore be sent to zero under some element of Hom, and secondly, by lifting this argument to talk about integers instead of invariants defined only mod 2. The difficulty, however, is that generally when a Steenrod operation is applied to an element of $H^*bo \otimes H^*\mathrm{K}(\mathbb{Z}, 4)$ the result will not be defined only in terms of powers of x and Stiefel-Whitney classes, but also in terms of Steenrod operations $Sq^n x$. This means that our invariant will only be defined mod 2, and cannot be obviously lifted to an interesting integer invariant. We do get a slew of mod 2 invariants, and the above argument deduces similar results for these mod 2 invariants. However, these are not the present subject of interest.

The exception is in this dimension: Again consider $u \in H^0 bo$ and x as before. Then $Sq^4(ux^{4n+1}) = w_4 u x^{4n+1} + u\binom{4n+1}{2}(Sq^2 x)^2 x^{4n-1} + u\binom{4n+1}{1} x^{4n+2} = w_4 u x^{4n+1} + u x^{4n+2}$. An identical argument gives:

Theorem 4. *For M a $(16n+8)$-dimensional spin manifold with a x a 4-form, then $\int_M \ x^{4n+2} \pm 12 \ x^{4n+1} \widehat{A}_4 \equiv 0$ mod 2.*

This method is a nice way of interpreting elements in Adams filtration zero but does not access the properties of elements in higher Adams filtration.

4.2 A second proof that $\int_{M_8} \frac{1}{2} x^2 \ \pm \ 6x\widehat{A}_4 \in \mathbb{Z}$

This proof will be a basic application of the method of finding a manifold basis for our invariants. The method is fruitful in dimensions 12 and 16, so we apply it here in $t - s = 8$ as an introduction.

Again, a d-degree element η of the cohomology of $\mathrm{K}(\mathbb{Z}, 4)$ with coefficients in $\mathbb{Z}/p$ produces a $\mathbb{Z}/p$-invariant of $(M^d, x) \in \Omega_d^{\mathrm{Spin}}(\mathrm{K}(\mathbb{Z}, 4))$ given a lifting of the class to a cohomology class defined rationally. For a map $x : M^d \to \mathrm{K}(\mathbb{Z}, 4)$, the induced map on cohomology $x^* : H^*(\mathrm{K}(\mathbb{Z}, 4)) \to H^*(M^d)$ is completely determined by what it does to the generator $\iota \in H^4(\mathrm{K}(\mathbb{Z},4))$, namely, the value $x^*(\iota) \in H^4(M^d)$. But $x^*(\iota) = [x]$. Since an element η in $H^d(\mathrm{K}(\mathbb{Z},4))$ can be written in terms of the algebra basis (discussed earlier) as a polynomial in Steenrod operations on ι, $\eta = P(\iota)$. Then $x^*(\eta) \in H^d(M^d)$, and since Steenrod operations commute with induced maps on cohomology (that is, x^* in this case), therefore $x^*(\eta) = x^*(P(\iota)) = P(x^*(\iota)) = P(x)$. Then the Kronecker pairing $\langle x^*(\eta), [M]\rangle \in G$ (with the fundamental class $[M]$) produces a spin bordism invariant of the element (M^d, x) coinciding with the mod p reduction of integral of lift, $\in_M x^*(\eta)$.

This is a spin bordism invariant (in fact, a bordism invariant for any bordism theory with structure at least that of oriented) by a basic application of Stokes theorem. (First let there exist another $N^d \xrightarrow{f} \mathrm{K}(\mathbb{Z},4)$ and an oriented manifold $L^{d+1} \xrightarrow{g} \mathrm{K}(\mathbb{Z},4)$ such that the boundary $\partial L = M \coprod -N$ and g restricts to our other maps $g|_M = x$ and $g|_N = f$.)

With the Adams spectral sequence, we can work with each $\mathbb{Z}/p$. However, our main interest is in invariants that are defined rationally, namely as integrals of 4-forms. By using the surjectivity of the map on the homotopy groups induced by the natural map from **M**Spin to *bo*, we can pullback to find a basis for Ext in terms of elements of the spin bordism of $\mathrm{K}(\mathbb{Z},4)$, i.e., spin manifolds equipped with four-dimensional cohomology classes. Our Ext term is a right $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{Z}/2,\mathbb{Z}/2)$-module, and the right action of h_0 corresponds to multiplication by 2. In terms of the structure of the spin bordism ring, this multiplication by 2 is just what we would expect, the disjoint union of a manifold with itself. Our Ext term also has a ring structure because $\mathrm{K}(\mathbb{Z},4)$ is an H-space (coming from the fact that it can be seen as the space of loops on a $\mathrm{K}(\mathbb{Z},5)$). This ring structure is given in terms of manifolds as follows: if (M,x) and (N,y) are manifolds with four-dimensional mod 2 cohomology classes, then their product is given by $(M\times N, x+y)$. If they sit in $\mathrm{Ext}^{s,t}$ and $\mathrm{Ext}^{s',t'}$ respectively, then their product lies in $\mathrm{Ext}^{s+s'+k,t+t'+k}$ where $k \geq 0$. (Typically k is zero, but irregular things can happen with the Adams filtration s.)

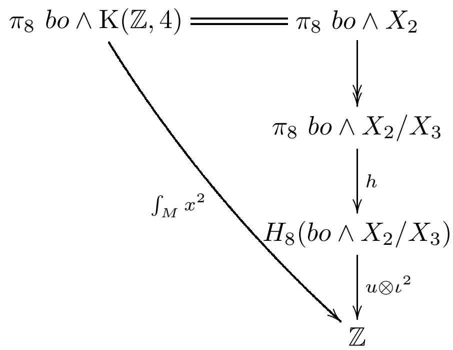
Theorem 5. *M an 8-dimensional spin manifold, and x an integral 4-form, then*

$$\int_M x^2 \ \pm\ 12x\widehat{A}_4 \ \equiv\ 0 \text{ (mod 2)}$$

Proof: We have two spin bordism invariants of a spin manifold M^8 equipped with $x \in H^4M$ that are defined rationally in $t-s=8$. These are $I_1 = \int_M x^2$ and $I_2 = \int_M x\widehat{A}_4$.

The idea of the proof is that these two invariants measure the placement of a manifold in the two $\mathbb{Z}$-towers in $t-s=8$, and from this information we can find manifolds that lie in the bottoms of the $\mathbb{Z}$-towers and therefore generate the homotopy group.

It is easy to see that I_1 measures the $\mathbb{Z}$-tower starting in Adams filtration zero. Considering the 2-completion of the space $\mathrm{K}(\mathbb{Z},4)$, it has a minimal skeleton (given by the same picture as that of its mod 2 cohomology, where the cells are now cones on 2-complete spheres instead of mod 2 cohomology classes). Call the 8-skeleton X_2 and the 7-skeleton X_3. Then there is an inclusion map $X_2 \hookrightarrow \mathrm{K}(\mathbb{Z}, 4)$, and this map is an isomorphism on homotopy in dimension 8 and lower. Further, the map that $i : X_3 \hookrightarrow X_2$ then induces on homotopy $\pi_8\ bo \wedge X_2 \to \pi_8\ bo \wedge X_2/X_3$ is a surjection. X_2/X_3 is 7-connected, so the generalized Hurewicz homomorphism $h : \pi_8\ bo \wedge X_2/X_3 \to H_8(bo \wedge X_2/X_3) = \mathbb{Z}$ is an isomorphism. Here $H_8(bo \wedge X_2/X_3) = H_0(bo) \otimes H_8(X_2/X_3)$, and this isomorphism is just given by the generator of the dual of H_8, which is $u \otimes \iota^2 \in H^8(bo \wedge X_2/X_3)$. (Here $\iota^2 \in H^8(X_2/X_3)$ can be identified with the image of $\iota^2 \in H^8(\mathrm{K}(\mathbb{Z}, 4))$.) This produces the commutative diagram. (Note: for the rest of the chapter, all groups have been 2-adically completed.)



Thus, the value of the invariant $\int_M x^2$ determines the Adams filtration of the image of the element (M, x) in the $\mathbb{Z}$-tower coming from the $\mathcal{A}(1)$-module $\{\iota^2\}$ (that is, the $\mathbb{Z}$-tower in $t - s = 8$ starting in Adams filtration $s = 0$).

Now, we can analogously make the same determination for the invariant $\int_M x\widehat{A}_4$, by considering the embedding $X_3 \hookrightarrow \mathrm{K}(\mathbb{Z}, 4)$. From the Adams spectral sequence it is the case that $\pi_8\ bo \wedge X_3 = \mathbb{Z}$. Working rationally, $\pi_8\ bo \wedge X_3 \otimes \mathbb{Q} = \mathbb{Q}$. The rational stable homotopy of a spectrum E coincides with its rational spectrum homology, $\pi_* E \otimes \mathbb{Q} \cong H_*(E; \mathbb{Q})$. Denote $e \in \pi_4\ \mathrm{K}(\mathbb{Z}, 4)$ whose image under the Hurewicz isomorphism is dual to ι. The inclusion $bo \hookrightarrow bu$ induces a map given by multiplication by 2 for $\pi_4\ bo \to \pi_4\ bu$. Thus the generator of $\pi_4\ bo$, represented by $h_0 v_1^2$ in the Ext term of the Adams spectral sequence, maps to twice the Bott generator of $\pi_4\ bu$. Working rationally (which in this context is equivalent to inverting h_0), $v_1^2 \in \pi_4\ bo \otimes \mathbb{Q}$ maps to the generator of $\pi_4\ bu \subset \pi_4\ bu \otimes \mathbb{Q}$. By the Künneth formula, $\pi_8\ bo \wedge X_3 \otimes \mathbb{Q} \cong \pi_4\ bo \otimes \pi_4\ X_3 \otimes \mathbb{Q}$, and so has a basis element $v_1^2 \otimes e$. Since the $\mathbb{Z}$-tower starts one Adams filtration lower than $v_1^2 \otimes e$, the generator of $\pi_8\ bo \wedge X_3$ is $\frac{1}{2} v_1^2 \otimes e$.

Any element of $H^8(bo \wedge X_3; \mathbb{Q})$ gives an isomorphism $\pi_8\ bo \wedge X_3 \otimes \mathbb{Q} \to \mathbb{Q}$. Rationally, $\widehat{A}_4$ is induced by a map

$$\begin{array}{ccc}
\mathbf{M}\mathrm{Spin} & \longrightarrow & bo \\
 & & \big\downarrow \\
 & & bu \longrightarrow \mathrm{K}(\mathbb{Q},4)
\end{array}$$

So this composition is given by an element $u\widehat{A}_4 \in H^4(\mathbf{M}\mathrm{Spin};\mathbb{Q})$. In addition, the element of $H^8(bo \wedge X_3;\mathbb{Q})$ that maps $v_1^2 \otimes e \to 1$ is in the image of $u\widehat{A}_4 x \in H^8(\mathbf{M}\mathrm{Spin} \wedge X_3;\mathbb{Q})$. Therefore the image of $(M,x) \in \pi_8\ \mathbf{M}\mathrm{Spin} \wedge X_3$ under the map $\widehat{A}$ generates $\pi_8\ bo \wedge X_3$ if and only if $\int_M xA_4 = 1/2$.

Two manifolds with 4-forms generate the homotopy group $\pi_8\ bo \wedge \mathrm{K}(\mathbb{Z},4)$ if one of each is sent to the bottom of these two $\mathbb{Z}$-towers (and the values the invariants take on them are linearly independent). We can then see that $(\mathbb{H}\mathrm{P}^2, y)$ and $(\mathbb{C}\mathrm{P}^3 \times S^2, ab)$ form a basis for this homotopy group, where y is a generator of $H^4(\mathbb{H}\mathrm{P}^2;\mathbb{Z})$, a is a generator of $H^2(\mathbb{C}\mathrm{P}^3;\mathbb{Z})$ and b is a generator of $H^2(S^2;\mathbb{Z})$. $\int_{\mathbb{H}\mathrm{P}^2} y^2 = 1$ and $\int_{\mathbb{C}\mathrm{P}^3\times S^2} ab\widehat{A}_4 = \int_{\mathbb{C}\mathrm{P}^3\times S^2} ab(-p_1(\mathbb{C}\mathrm{P}^3)/24)$. Since $p_1(\mathbb{C}\mathrm{P}^3) = 4a^2$, this integral becomes $\int_{\mathbb{C}\mathrm{P}^3\times S^2} ab(-4a^2/24) = -1/6$. And because I_1 is zero on $(\mathbb{C}\mathrm{P}^3 \times S^2, ab)$, these two elements form a mod 2 basis.

Any element of the spin bordism of $\mathrm{K}(\mathbb{Z},4)$ is then sent to the same element in this Ext term as a linear combination of $(\mathbb{H}\mathrm{P}^2, y)$ and $(\mathbb{C}\mathrm{P}^3 \times S^2, ab)$. Thus, a linear formula concerning these invariants I_1 and I_2 can be proved simply by verifying it for a linear combination of $(\mathbb{H}\mathrm{P}^2, y)$ and $(\mathbb{C}\mathrm{P}^3 \times S^2, ab)$. Since $p_1(\mathbb{H}\mathrm{P}^2) = 2y$, we can compute I_2 on $(\mathbb{H}\mathrm{P}^2, y)$ and obtain the following table of values:

*	$\int x^2$	$\int x\widehat{A}_4$
$(\mathbb{H}\mathrm{P}^2, y)$	1	-1/12
$(\mathbb{C}\mathrm{P}^3 \times S^2, ab)$	0	-1/6

Even though the spin bordism ring has no odd torsion, here we have not shown that these manifolds form a basis integrally, only after completion at 2. They do, however, and this will be proved in the next chapter, where we compute this Ext term at the necessary odd primes. These results will be assumed here.

Thus, we can now see that $\int_{M_8} x^2 \pm 12x\widehat{A}_4 \equiv 0 \pmod 2$, since it is clearly the case for a linear combination of our two basis elements. That is, for any (M^8, x) as before, there exist integers n and m so that:

$$\int_{M_8} x^2 \pm 12x\widehat{A}_4 = \int_{n\mathbb{H}\mathrm{P}^2 \coprod m\mathbb{C}\mathrm{P}^2\times S^2} \left(\sum_{i=1}^{n} y_i + \sum_{j=1}^{m} a_j b_j\right)^2 \pm 12 \left(\sum_{i=1}^{n} y_i + \sum_{j=1}^{m} a_j b_j\right)\widehat{A}_4$$

$$= n\left(\int_{\mathbb{H}\mathrm{P}^2} y^2 \pm 12y\widehat{A}_4\right) + m\left(\int_{\mathbb{C}\mathrm{P}^2\times S^2} (ab)^2 \pm 12ab\widehat{A}_4\right)$$

$$= n \pm -n \pm 2m \equiv 0 \pmod 2$$

Thus our formula holds for our basis, and once the manifold basis is checked after completion at odd primes in chapter 5, that will complete the proof for any (M^8, x) an 8-dimensional spin manifold equipped with an integral 4-form.

4.3 $\int_{M_{12}} \frac{1}{3!}x^3 \pm 3x^2\widehat{A}_4 + 30x\widehat{A}_8 \in \mathbb{Z}$

Although the first proof of the integrality result for spin 8-manifolds may have seemed easier than the second, as though we were getting more out of our spectral sequence while doing less, this method has limited use in higher dimensions for two reasons: as opposed to the simplicity of an element lying in Adams filtration zero, it is unclear how to interpret elements in higher Adams filtration; secondly, even working just in filtration zero we typically get invariants that are only defined mod 2 and cannot be lifted to $\mathbb{Z}$. Our second method of proof for 8-manifolds, while comparatively more difficult in dimension 8, extends to 12-manifolds whereas the first proof does not.

However, the invariants produced by most elements of $H^*(\mathrm{K}(\mathbb{Z}, 4))$ are defined in terms of Steenrod operations, and developing the previous relation between the values of the invariants and the Adams filtration requires more work.

An additional complication is that there is a differential from the $\mathbb{Z}$-tower in $t-s = 13$, so that the homotopy group π_{12} is not actually the $t-s = 12$ part of the Ext term, but rather a quotient of it by the image of the differential. Thus, we cannot directly write down a basis for the Ext term in terms of manifolds, because part of that Ext term exists only algebraically. The most obvious way to avert this problem is to work with the skeleton filtration of the space $\mathrm{K}(\mathbb{Z}, 4)$ and consider a subcomplex with the cells of the module with Q_0-homology in 13 deleted. By this method, we get a modified sequence of spaces by which we can isolate the effect of this differential, and can eventually write down a generating set of manifolds with which to prove relations between our analytic invariants.

Again, each Q_0-homology in dimension d produces a spin bordism invariant for a d-dimensional spin manifold equipped with a map into $\mathrm{K}(\mathbb{Z}, 4)$ (because of the surjectivity of the induced map on homotopy groups). If this invariant can be defined rationally (e.g., as an integral over a power of the associated 4-form $\int x^n$) then we further get a spin bordism invariant for a $d + 4k$-dimensional element of the spin bordism of $\mathrm{K}(\mathbb{Z}, 4)$ (i.e., a $d+4k$-dimensional spin manifold equipped with an integral 4-form) by integrating against $\widehat{A}_{4k}$.

Given a spin manifold M^{d+4k} equipped with x and a cohomology class η in $H^d(\mathrm{K}(\mathbb{Z}, 4); G)$, let us assume $x^*(\eta)$ can be canonically lifted. Then we can further define a canonical rational invariant of (M, x), that is $\langle x^*(\eta)\widehat{A}_{4k}, [M]\rangle$, which equals $\int_M x^*(\eta)\widehat{A}_{4k}$.

Now turning our attention back to spin 12-manifolds, the $\mathbb{Z}$-towers in $t-s = 12$ come from Q_0-homologies $\iota, \iota^2, \iota^3, (Sq^2\iota)^2$ to which we can associate invariants $\int_M x^3$, $\int_M x^2\widehat{A}_4$, $\int_M x\widehat{A}_8$. For the fourth Q_0-homology, the invariant will only be mod 2, and to define it we will need the following lemma.

Lemma 1. *For any additive mod 2 cohomology operation C, if $Sq^1 \circ C = 0$,* *then C lifts to a cohomology operation C' of type $(\mathbb{Z}/2, *; \mathbb{Z}/4, *)$ such that C is the mod 2 reduction of C'.*

Proof: Sq^1, the Bockstein homomorphism, is by definition the connecting homomorphism of the long exact sequence that results from the cohomology functor applied to the short exact sequence $\mathbb{Z}/2 \rightarrowtail Z/4 \twoheadrightarrow Z/2$. If $Sq^1 \circ C = 0$ then the image of C is in the kernel of Sq^1, which is exactly the image of the map induced on cohomology by the mod 2 reduction of $\mathbb{Z}/4$. Thus, the map lifts. $\square$

This basic fact about lifting cohomology operations to different coefficients will be key in defining the invariants that are defined in terms of Steenrod operations.

For the cohomology operation $C : z_6 \to (z_6)^2$, clearly $Sq^1 \circ C = 0$ from the coproduct structure, and therefore C lifts to an operation of type $(\mathbb{Z}/2, 6; \mathbb{Z}/4, 12)$. By the Adem relations, $Sq^6 = Sq^1 Sq^4 Sq^1 + Sq^2 Sq^4$. Thus, for $x \in H^4(M^{12}; \mathbb{Z}/2)$,

$$(Sq^2 x)^2 = Sq^6 Sq^2 x = Sq^1 Sq^4 Sq^1 Sq^2 x + Sq^2 Sq^4 Sq^2 x.$$

Since this element lies in the top cohomology class of the manifold,

$$Sq^1 Sq^4 Sq^1 Sq^2 x + Sq^2 Sq^4 Sq^2 x = v_1 Sq^4 Sq^1 Sq^2 x + v_2 Sq^4 Sq^2 x$$

by definition of the Wu classes v_1 and v_2. By the Wu class characterization of the Stiefel-Whitney classes (that $w = Sq(v)$), $w_1 = v_1$ and $w_2 = Sq^1 v_1 + v_2$. Now for an orientable manifold $w_1 = 0$ and thus $v_1 = 0$. If M^{12} is also spin, then $w_2 = 0$, and by working backward so $v_2 = 0$. Therefore $(Sq^2 x)^2 = 0$, meaning that the operation C' takes values 0 and 2 in $\mathbb{Z}/4$. The invariant associated to the Q_0-homology $(Sq^2 x)^2$ is thereby $\langle (Sq^2 x)^2, [M]_{\mathbb{Z}/4} \rangle$, the Kronecker pairing with the $[M]_{\mathbb{Z}/4}$, the orientation class $\in H_{12}(M^{12}; \mathbb{Z}/4)$. For $Sq^2 x$ the mod 2 reduction of an integral class(as it is $\mathbb{C}\mathrm{P}^n$) then this last invariant equals $\int_M (Sq^2 x)^2$. This is denoted with the Pontryagin square, $\mathcal{P}$, so that the invariant is $\int_M \mathcal{P}(Sq^2 x)$

The Adams filtrations in which these $\mathbb{Z}$-towers start imply the values (modulo odd primes) of each invariant on a manifold (M, x) generating the associated $\mathbb{Z}$-tower. (If there is no differential, this works just as in dimension 8. The existence of the differential will complicate this procedure.)

The following are examples of spin 12-manifolds with specified elements of H^4: $(\mathbb{H}\mathrm{P}^3, y)$, $(\mathbb{H}\mathrm{P}^2 \times K_3, y)$, $(V_{2,1}, vw)$, $(S^4 \times B^8, x)$ and $(\mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^3, vw)$.

12-dimensional quaternionic projective space, $\mathbb{H}\mathrm{P}^3$, is a spin manifold because since it is 2-connected therefore the Stiefel-Whitney class obstructions to being spin must vanish. Let y denote the first Pontryagin class of the tautological SU_2-bundle. Then y is a generator of $H^4(\mathbb{H}\mathrm{P}^3; \mathbb{Z})$, and the pair $(\mathbb{H}\mathrm{P}^3, y)$ is an element of the spin bordism of $\mathrm{K}(\mathbb{Z}, 4)$.

$(\mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^3, vw)$, where v and w are the first Chern classes of each tautological complex line bundle over each $\mathbb{C}\mathrm{P}^3$, is another such element.

Hypersurfaces of degree d in $\mathbb{C}\mathrm{P}^n$ are spin manifolds if the absolute value $|n - d|$ is odd. The K_3 surface, a degree 4 hypersurface in $\mathbb{C}\mathrm{P}^3$, is thus a 4-dimensional spin manifold, and can be equipped with the trivial class $0 \in$

$H^4(K_3;\mathbb{Z})$. For y the first Pontryagin class of tautological bundle over $\mathbb{H}\mathrm{P}^2$, one has the product given by taking the product of the manifolds and the direct sum of their classes, $(\mathbb{H}\mathrm{P}^2 \times K_3, y)$.

Hypersurfaces of bidegree a, b in $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^m$ are spin if both $|n-a|$ and $|m-b|$ are odd. For instance, $V_{2,1} \subset \mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^4$ is spin 12-manifold. For v and w again the first Chern classes of each tautological line bundle over $\mathbb{C}\mathrm{P}^3$ and $\mathbb{C}\mathrm{P}^4$, respectively, then $vw \in H^4(\mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^4;\mathbb{Z})$. The inclusion map $i : V_{2,1} \hookrightarrow \mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^4$ induces a contravariant map on cohomology, so that $i^*(vw) \in H^4(V_{2,1};\mathbb{Z})$.

Let B^8 be the spin 8-manifold with $\widehat{A}(B^8) = -1$ as mentioned in the section on the $\widehat{A}$-genus. Then equip S^4 with a class $x \in H^4(S^4;\mathbb{Z})$ dual to the orientation class, and $(S^4 \times B, x)$ constitutes another element of $\pi_{12}\ \mathbf{M}\mathrm{Spin} \wedge \mathrm{K}(\mathbb{Z},4)$.

The purpose of the rest of this section is to show that the image of these elements under the map $\widehat{A} : \pi_{12}\ \mathbf{M}\mathrm{Spin} \wedge \mathrm{K}(\mathbb{Z},4) \to \pi_{12}\ bo \wedge \mathrm{K}(\mathbb{Z},4)$ generates the group $\pi_{12}\ bo \wedge \mathrm{K}(\mathbb{Z},4)$ after completion at the prime 2. (For the rest of the section $\mathbb{Z}$ will mean the 2-adic integers, and, to exclude issues of odd torsion, the cells in our skeleton filtration are actually cones on 2-complete spheres.)

In dimension 8, this was a straightforward procedure due to the absence of differentials. There exists a differential coming from $t - s = 13$, so we cannot work directly with the Ext term in $t - s = 12$ because the homotopy group is a quotient of Ext, not Ext itself.

The basic idea is to use the obvious skeleton filtration of $\mathrm{K}(\mathbb{Z},4)$ (after completing at 2). First I define a cofibration sequence $X_3 \hookrightarrow X_2 \hookrightarrow X_1 \hookrightarrow X \to \mathrm{K}(\mathbb{Z},4)$, and show that the image of these manifolds under $\widehat{A}$ generates each of the groups $\pi_{12}\ bo \wedge X_i/X_j$. This will then imply that the manifolds generate the group $\pi_{12}\ bo \wedge \mathrm{K}(\mathbb{Z},4)$.

Let X be the 17-skeleton of $\mathrm{K}(\mathbb{Z},4)$ (given by same diagram as its cohomology). Taking the cube of ι, the generator of $H^4(\mathrm{K}(\mathbb{Z},4);\mathbb{Z})$, defines a map $\mathrm{K}(\mathbb{Z},4) \xrightarrow{\iota^3} \mathrm{K}(\mathbb{Z},12)$. Let X_1 be the cofiber of this sequence. X_1 therefore has a cell structure given by the cells of X excluding the cells associated to ι^3, $\iota^2 Sq^2 \iota$, and $\iota^2 Sq^3 \iota$. Finally, X_2 and X_3 are as in the previous section; they are given by the 8-skeleton and 7-skeleton of $\mathrm{K}(\mathbb{Z},4)$, respectively.

Proposition 1. *The image of* $(\mathbb{H}\mathrm{P}^3, u)$ *under* $\widehat{A}$ *generates the group* $\pi_{12}\ bo \wedge X/X_1$.

Proof: X/X_1 is 11-connected, therefore we have the following commutative diagram.

$$\begin{array}{ccc}
\pi_{12}\ bo \wedge X & \xrightarrow{p} & \pi_{12}\ bo \wedge X/X_1 \\
 & \searrow & \downarrow h \\
 & & H_{12}(bo \wedge X/X_1;\mathbb{Z}) \\
 & & \downarrow u\otimes\iota^3 \\
 & & \mathbb{Z}
\end{array}$$

where p is the map induced by $X \to X/X_1$, and h is the Hurewicz homomorphism, which is an isomorphism in this case since π_{12} is the first nontrivial homotopy group. The map $H_{12}(bo \wedge X/X_1; \mathbb{Z}) \to \mathbb{Z}$ is given by the generator of $H^{12}(bo \wedge X/X_1; \mathbb{Z})$, which is $u \otimes \iota^3$, where u is the Thom class in $H^0(bo; \mathbb{Z})$. There are no other terms because X/X_1 is 11-connected, and $H^{12}(bo \wedge X/X_1; \mathbb{Z}) \cong H^0(bo; \mathbb{Z}) \otimes H^{12}(X/X_1; \mathbb{Z})$.

The composition of the maps is therefore an isomorphism given by

$$(u \otimes \iota^3) \circ h : (M, x) \longrightarrow \int_M x^3$$

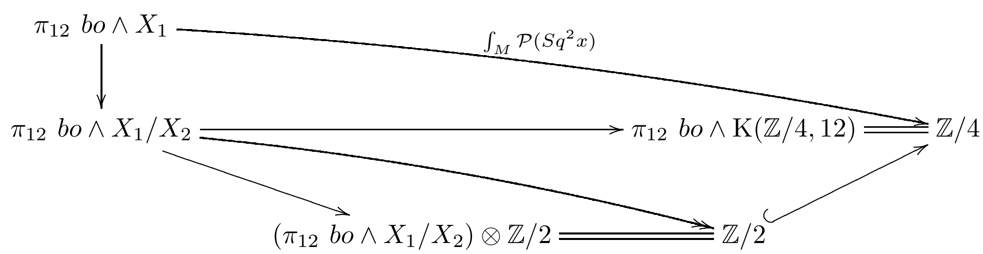
and under this map $(\mathbb{H}\mathrm{P}^3, y)$ is sent to the generator $1 \in \mathbb{Z}$. Therefore the image of $(\mathbb{H}\mathrm{P}^3, y)$ under $\widehat{A}$ generates $\pi_{12}\ bo \wedge X/X_1$. $\square$

We therefore have a short exact sequence of homotopy groups induced by the fibration $X_1 \to X \to X/X_1$,

$$\pi_{12}\ bo \wedge X_1 \hookrightarrow \pi_{12}\ bo \wedge X \xrightarrow{p} \pi_{12}\ bo \wedge X/X_1$$

p is surjective since $p(\mathbb{H}\mathrm{P}^3, y)$ generates $\pi_{12}\ bo \wedge X/X_1$.

Proposition 2. *$\pi_{12}\ bo \wedge X_1/X_2$ is cyclic and nonzero, and is generated by $(\mathbb{H}\mathrm{P}^3, y) - (\mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^3, vw)$. In addition, the following composition commutes,*



where $\mathcal{P}$ is the Pontryagin square, so that

$$\mathrm{K}(\mathbb{Z}, 4) \xrightarrow{Sq^2} \mathrm{K}(\mathbb{Z}/2, 6) \xrightarrow{\mathcal{P}} \mathrm{K}(\mathbb{Z}/4, 12)$$

Proof: The group is cyclic because there are $\mathbb{Z}$-towers in 12 and 13, and understanding the homotopy group rationally shows that the group $\mathbb{Q} \otimes \pi_{13}$ is trivial. Therefore there exists a differential d_n, $n \geq 2$, and, by simple inspection of the Adams filtrations, the 2-component of $\pi_{12}\ bo \wedge X_1/X_2$ is $\mathbb{Z}/2^{n+1}$.

In dimension 12 and 13, the minimal cell-complex for $\mathrm{K}(\mathbb{Z}/4, 12)$ has a sphere in dimension 12 and a sphere in dimension 13, so that in its Ext chart there is a $\mathbb{Z}$-tower in both dimensions 12 and 13.

The map $\mathcal{P} \circ Sq^2$ induces a map on the $\mathbb{Z}$-towers in $t - s = 12$ which must be an injection by the following lemma. Since it is an injection, an element (M, x) generates the group $\pi_{12}\ bo \wedge X_1/X_2$ if and only if it is sent to an element of Adams filtration 1 in the $\mathbb{Z}$-tower for the Ext chart of $K(\mathbb{Z}/4, 12)$. This is determined by the value of the map $\int_M \mathcal{P}(Sq^2 x)$. On $(\mathbb{H}P^3, y) - (\mathbb{C}P^3 \times \mathbb{C}P^3, vw)$, the map takes value 2 (so the manifold is sent to Adams filtration 1), and thus it generates the group $\pi_{12}\ bo \wedge X_1/X_2$. $\square$

Lemma 2.

$$h_0^{-1}\mathrm{Ext}_{\mathcal{A}(1)}(M, \mathbb{Z}/2)/(h_0 v_1^2) \cong h_0^{-1}\mathrm{Ext}_{\mathcal{A}(0)}(M, \mathbb{Z}/2)$$

Alternatively, define $R := h_0^{-1}\mathrm{Ext}_{\mathcal{A}(0)}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{F}_2[h_0^{\pm 1}, h_0 v_1^2]$ *and* $S := R/(h_0 v_1^2)$. *Then*

$$h_0^{-1}\mathrm{Ext}_{\mathcal{A}(1)}(M, \mathbb{Z}/2) \otimes_R S \cong h_0^{-1}\mathrm{Ext}_{\mathcal{A}(0)}(M, \mathbb{Z}/2)$$

Proof: Consider the short exact sequence of $\mathcal{A}(1)$-modules $S \overset{i}{\hookrightarrow} Y \overset{j}{\twoheadrightarrow} Z$, where $S = \{s_2, s_3 : Sq^1 s_2 = s_3\}$, $Y = \{y_0, y_2, y_3 : Sq^2 y_0 = y_2, Sq^1 y_2 = y_3\}$, $Z = \{z_0\}$, the maps are defined by $i(s_2) = y_2$ and $j(y_0) = z_0$.

For M any finite $\mathcal{A}(1)$-module, $h_0^{-1}\mathrm{Ext}_{\mathcal{A}(1)}(M \otimes S; \mathbb{Z}/2) = 0$, since by the stable structure theory of [3], the Q_0-homology of $M \otimes S$ will be 0-dimensional, and thus h_0 will not act freely on any element of $\mathrm{Ext}_{\mathcal{A}(1)}(M \otimes S; \mathbb{Z}/2) = 0$. Therefore, after inverting h_0 every element will be in the image of the action of h_0^{-1} of 0, and the new module is 0.

The $\mathcal{A}(1)$-module Z is the identity under tensor multiplication, so that $M \otimes Z \cong M$ for any M. The short exact sequence $M \otimes S \hookrightarrow M \otimes Y \twoheadrightarrow M \otimes Z$ induces a long exact sequence in Ext. After inverting h_0, the term involving S becomes 0, and this implies the following isomorphism:

$$h_0^{-1}\mathrm{Ext}_{\mathcal{A}(1)}(M \otimes Y, \mathbb{Z}/2) \cong h_0^{-1}\mathrm{Ext}_{\mathcal{A}(1)}(M, \mathbb{Z}/2).$$

Therefore, the map $\mathrm{Ext}_{\mathcal{A}(1)}(M, \mathbb{Z}/2) \otimes_E \mathrm{Ext}_{\mathcal{A}(1)}(Y, \mathbb{Z}/2) \to \mathrm{Ext}_{\mathcal{A}(1)}(M \otimes Y, \mathbb{Z}/2)$, where $E := \mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{Z}/2, \mathbb{Z}/2)$, is an isomorphism after inverting h_0.

Using this trick once more involving $T := \{t_0, t_2, t_3, t_5 : Sq^2 t_0 = t_2, Sq^1 t_2 = t_3, Sq^2 t_3 = t_5\}$, and then using that $T \cong \mathcal{A}(1)//\mathcal{A}(0)$, proves the lemma.

That is, there exists a short exact sequence $M \otimes Z' \hookrightarrow M \otimes T \twoheadrightarrow M \otimes Y$, where $Z' := \{z_5\}$, induced by the maps sending z_5 to t_5, and t_0 to y_0. This produces two long exact sequences:

$$\begin{array}{ccccccc}
\mathrm{Ext}(M) \otimes_E \mathrm{Ext}(Z') & \longleftarrow & \mathrm{Ext}(M) \otimes_E \mathrm{Ext}(T) & \longleftarrow & \mathrm{Ext}(M) \otimes_E \mathrm{Ext}(Y) & \longleftarrow & \cdots \\
\downarrow p & & \downarrow q & & \downarrow r & & \\
\mathrm{Ext}(M \otimes Z') & \longleftarrow & \mathrm{Ext}(M \otimes T) & \longleftarrow & \mathrm{Ext}(M \otimes Y) & \longleftarrow & \cdots
\end{array}$$

where $\mathrm{Ext}(*)$ denotes $\mathrm{Ext}_{\mathcal{A}(1)}(*, \mathbb{Z}/2)$. After inverting h_0, p and r are isomorphisms as determined earlier. By the five-lemma, q is therefore an isomorphism. Using the Hom-tensor interchange on $T = \mathcal{A}(1)//\mathcal{A}(0)$ completes the proof. $\square$

Proposition 3. *The image of* $(\mathbb{H}P^2 \times K_3, y)$ *under* $\widehat{A}$ *generates the group* $\pi_{12}\ bo \wedge X_2/X_3$.

Proof: X_2/X_3 is 7-connected and in cohomology has only one Q_0-homology in dimension 8. Thus, $\pi_{12}\ bo \wedge X_2/X_3 \cong (\pi_4\ bo) \otimes (\pi_8\ bo \wedge X_2/X_3)$.

$$\begin{array}{ccc}
\pi_{4k}\ \mathbf{M}\mathrm{Spin} & \xrightarrow{\widehat{A}} & \pi_{4k}\ bo \\
 & \searrow^{\int_M \widehat{A}} & \big\downarrow j \\
 & & \pi_{4k}\ bu \\
 & & \| \\
 & & \mathbb{Z}
\end{array}$$

where j is an isomorphism when k is even, and corresponds to multiplication by 2 when k is odd. The value of the $\widehat{A}$-genus on a spin manifold M^{4k}, representing an element of $\pi_{4k}\ \mathbf{M}\mathrm{Spin}$, is given by the integer value $(j \circ \widehat{A})[M^{4k}] = \int_M \widehat{A}$. For $k = 1$, j corresponds to multiplication by 2. Since for the K_3 surface, $\int_{K_3} \widehat{A} = 2$, therefore $\widehat{A}$ maps $[K_3] \in \pi_4\ \mathbf{M}\mathrm{Spin}$ to the generator of $\pi_4\ bo$.

Exactly the same argument that shows that the image of $(\mathbb{H}\mathrm{P}^3, y)$ generates $\pi_{12}\ bo \wedge X/X_1$ applies to show that the image of $(\mathbb{H}\mathrm{P}^2, u)$ generates $\pi_8\ bo \wedge X_2/X_3$:

$$\begin{array}{ccccc}
\pi_8\ \mathbf{M}\mathrm{Spin} \wedge \mathrm{K}(\mathbb{Z},4) & & & & \\
\big\downarrow \widehat{A} & & & & \\
\pi_8\ bo \wedge \mathrm{K}(\mathbb{Z},4) & \xrightarrow{\sim} & \pi_8\ bo \wedge X_2 & \xrightarrow{p} & \pi_8\ bo \wedge X_2/X_3 \\
 & & & & \big\downarrow h \\
 & & & & H_8(bo \wedge X_2/X_3; \mathbb{Z}) \\
 & & & & \big\downarrow u \otimes \iota^2 \\
 & & & & \mathbb{Z}
\end{array}$$

and the composition sends $(M^8, x) \to \int_M x^2$. Since $\int_{\mathbb{H}\mathrm{P}^2} u^2 = 1$, and $(\mathbb{H}\mathrm{P}^2, y)$ generates $\pi_8\ bo \wedge X_2/X_3$. Therefore the product $(K_3, 0)(\mathbb{H}\mathrm{P}^2, u) = (\mathbb{H}\mathrm{P}^2 \times K_3, u)$ maps to a generator of $(\pi_4\ bo) \otimes (\pi_8\ bo \wedge X_2/X_3)$. $\square$

Proposition 4. *The image of* $(S^4 \times B^8, u)$ *under* $\widehat{A}$ *generates the group* $\pi_{12}\ bo \wedge X_3$.

Proof: From the Adams spectral sequence, the $\pi_{12}\ bo \wedge X_3 \cong \mathbb{Z}$. Rationally then, it is the case that $\pi_{12}\ bo \wedge X_3 \otimes \mathbb{Q} \cong (\pi_8 bo) \otimes (\pi_4 X_3) \otimes \mathbb{Q}$. By the argument that if (M^8, x) generates the group $\pi_8\ bo \wedge X_3$ only if $\int_M x\widehat{A}_4 = \pm\frac{1}{2}$, so we see that (M^{12}, x) generates $\pi_{12}\ bo \wedge X_3$ only if $\int_M x\widehat{A}_8 = \pm 1$. Considering the mapping $\widehat{A} : \pi_8\ \mathbf{M}\mathrm{Spin} \to \pi_8\ bo$, the image of B^8 generates $\pi_8\ bo$ (can see this by looking at the Hurewicz homomorphism, just as in Lemma 2). Likewise, the image under the $\widehat{A}$ map of (S^4, x) is an element of $\pi_4\ bo \wedge X_3$. Their product $(S^4, x) \times (B^8, 0) = (S^4 \times B^8, x + 0)$ is not only in $\pi_{12}\ bo \wedge \mathrm{K}(\mathbb{Z}, 4)$ but generates $\pi_{12}\ bo \wedge X_3$, by evaluating the integral. $\square$

Combining propositions 1 through 4 proves:

Theorem 6. *After completion at 2, the images under the $\widehat{A}$ map of $(\mathbb{H}\mathrm{P}^3, y)$, $(\mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^3, vw)$, $(\mathbb{H}\mathrm{P}^2 \times K_3, y)$, $(S^4 \times B^8, x)$ generate $\pi_{12}\ bo \wedge \mathrm{K}(\mathbb{Z}, 4)$.*

Now, these manifolds plus 4-forms generate the 2-completed connective real K-theory of $\mathrm{K}(\mathbb{Z}, 4)$, so a linear relation between our invariants based on the values they take on these elements will hold for any 12-dimensional spin manifold equipped with a specified four dimensional cohomology class, after inverting odd primes. We will then write down a relation, which we can then verify easily. In addition to element $(S^4 \times B^8, x)$, the hypersurface $(V_{2,1}, vw)$ is also included in our table. This is not because it is needed to generate the 2-completed homotopy group, but it is used in the consideration of odd primes in the next chapter, which will show that the addition is sufficient to then have a set that generates $\pi_{12}\ bo \wedge \mathrm{K}(\mathbb{Z}, 4)$ integrally.

*	$\int_M x^3$	$\int_M x^2 \widehat{A}_4$	$\int_M x \widehat{A}_8$	$\langle \frac{1}{2}(Sq^2 x)^2, [M] \rangle$
$\mathbb{H}\mathrm{P}^3$	1	$\frac{-1}{2\cdot 3}$	$\frac{1}{2\cdot 3^2\cdot 5}$	0
$\mathbb{H}\mathrm{P}^2 \times K_3$	0	2	-1/6	0
$S^4 \times B^8$	0	0	-1	0
$V_{2,1}$	1	-1/6	-1/18	0
$\mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^3$	1	0	$\frac{1}{2^2\cdot 3^2}$	1

We can rewrite this, since these invariants are additive:

*	$\int_M x^3$	$\langle \frac{1}{2}(Sq^2 x)^2, [M] \rangle$	$\int_M x^2 \widehat{A}_4$	$\int_M x \widehat{A}_8$
$\mathbb{H}\mathrm{P}^3$	1	0	-1/6	$\frac{1}{2\cdot 3^2\cdot 5}$
$\mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^3 - \mathbb{H}\mathrm{P}^3$	0	1	1/6	$\frac{1}{2^2\cdot 3\cdot 5}$
$\mathbb{H}\mathrm{P}^2 \times K_3$	0	0	2	-1/6
$S^4 \times B^8$	0	0	0	-1
$-V_{2,1} + \mathbb{H}\mathrm{P}^3$	0	0	0	1/15

With the manifolds above, the four dimensional cohomology classes are understood to be as previously defined. (The plus and minus come from the structure of the spin bordism ring, namely disjoint union and disjoint union and reversing orientation.) These manifolds plus classes x form a basis for the connective K-theory of $\mathrm{K}(\mathbb{Z}, 4)$ in dimension 12. Thus, by finding a linear formula for the values of these invariants on these manifolds, and using the surjectivity of the map from **M**Spin, we can find a rational polynomial of the type sketched in the introduction, and prove that it must take integer values on spin 12-manifolds. This will re-prove the result of Diaconescu, Moore, and Witten.

That is, the values that our invariants take on a spin 12-manifold together with 4-form (M, x) are determined by the image of (M, x) under the map from the spin bordism of $\mathrm{K}(\mathbb{Z},4)$ to the connective K-theory of $\mathrm{K}(\mathbb{Z},4)$. The manifolds above form a basis for the Ext term converging to that stable homotopy group, thus they generate that final term. (M,x) is then sent to the same element of $\pi_*\ bo\wedge\mathrm{K}(\mathbb{Z},4)$ as a linear combination of our basis elements. Proving a linear formula (mod 2) on a spin 12-manifold then constitutes checking it on our manifold basis above.

This procedure easily produces such results as the following (modulo factors of odd primes):

Proposition 5. *(M^{12},x) as above, then $\int_M 6x^2\widehat{A}_4\in\mathbb{Z}$.*

Proposition 6. *(M^{12},x) again as above, then $\langle\frac{1}{2}(Sq^2x)^2,[M]\rangle\equiv 180\int_M x\widehat{A}_8\ \ (mod\, 2)$.*

Likewise, we can easily evaluate for each of these elements of $\Omega_{12}^{\mathrm{Spin}}\mathrm{K}(\mathbb{Z},4)$ that $\int(x^3/6)\pm 3x^2\widehat{A}_4+30x\widehat{A}_8$ is an integer. Therefore, modulo factors of odd primes (which we will deal with in the next chapter) we have proved:

Theorem 7. *M^{12} an 12-dimensional spin manifold, and x an integral 4-form, then*

$$\int_{M_{12}}\frac{x^3}{6}\ \pm\ 3x^2\widehat{A}_4\ +30x\widehat{A}_8\in\mathbb{Z}$$

5 At odd primes

The purpose of this section is to check that the mod 2 bases used in the previous chapter are actually bases integrally, completing the proofs of the formulas given.

The mod p cohomology of *bo* looks significantly different from the even prime case. After completion at any odd prime p, *bo* splits as a wedge of suspensions of Johnson-Wilson spectra $BP\langle 1\rangle$. The cohomology of each is similar, but simpler, to the mod 2 cohomology of *bo*. Where $H^*(bo;\mathbb{Z}/2)=\mathcal{A}//\mathcal{A}(1)$, it is the case that $H^*(BP\langle 1\rangle;\mathbb{Z}/p)=\mathcal{A}_p//E[Q_0,Q_1]$ the mod p Steenrod algebra quotiented out by an exterior algebra of elements which act as differentials. See [13]. This is easier for two reasons. First, computing Ext over an exterior algebra is cleaner than computing it over $\mathcal{A}(1)$. Second, through a range of dimensions $\mathcal{A}_p//E[Q_0,Q_1]$ as a direct sum of $E[Q_0,Q_1]$-submodules is fairly simple, while as a $\{Q_0,\mathcal{P}^1\}$-module it is complicated (and not at all the direct sum of stably invertible modules).

5.1 $H^*(\mathrm{K}(\mathbb{Z},4);\mathbb{Z}/p)$ as an $E[Q_0,Q_1]$-module

At odd primes, *bo* splits as a sum of Johnson-Wilson spectra. More precisely, there exists a map: $bo\to\bigvee_{k=0}^{\frac{1}{2}(p-3)}\Sigma^{4k}BP\langle 1\rangle$ that induces an isomorphism on cohomology mod p, i.e., is a $\mathcal{C}_p$-homotopy equivalence.

Taking Ext over the wedge of spaces is the same as the direct sum of the Ext terms for each individual space. Further, there can be no differentials in the spectral sequence relating the Ext terms of the wedged spaces.

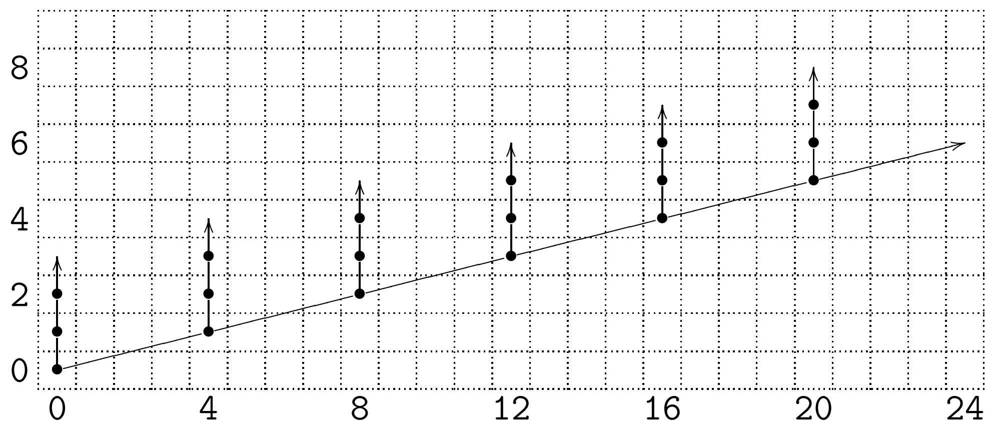
The trick that we used to simplify the Ext term for *bo* in the mod 2 case works verbatim:

$$\mathrm{Ext}_{\mathcal{A}_p}(\mathcal{A}//E[Q_0,Q_1]\otimes_{\mathbb{Z}/p}H^*(\mathrm{K}(\mathbb{Z},4);\mathbb{Z}/p),\mathbb{Z}/p)$$

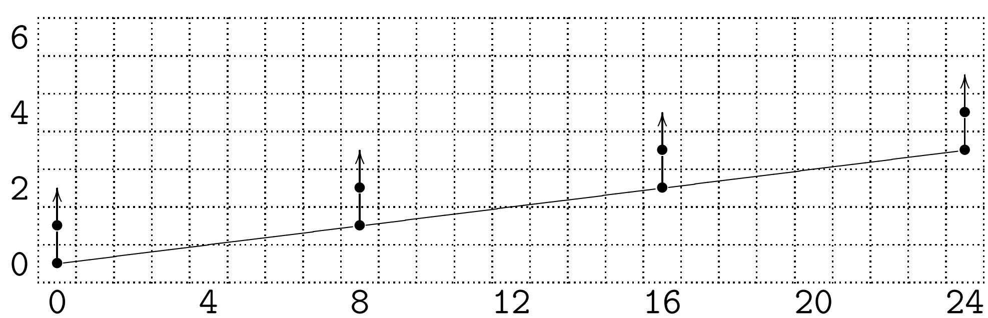
$$=\mathrm{Ext}_{\mathcal{A}_p}((\mathcal{A}\otimes_{E[Q_0,Q_1]}\mathbb{Z}/p)\otimes_{\mathbb{Z}/p}H^*(\mathrm{K}(\mathbb{Z},4);\mathbb{Z}/p),\mathbb{Z}/p)$$

$$=\mathrm{Ext}_{E[Q_0,Q_1]}(H^*(\mathrm{K}(\mathbb{Z},4);\mathbb{Z}/p),\mathbb{Z}/p)$$

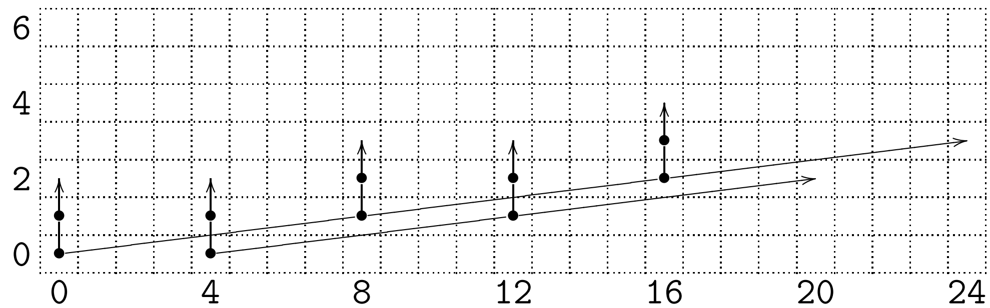
Additionally, the structure of this Ext term for a p-complete sphere is simpler in the odd prime case. The Ext chart for $\mathrm{Ext}_{E[Q_0,Q_1]}(\mathbb{Z}/3,\mathbb{Z}/3)$ is as follows.



The task of computation is then to determine the structure of the mod p cohomology of $\mathrm{K}(\mathbb{Z},4)$ as a direct sum of $E[Q_0,Q_1]$-modules. Taking suspensions of $BP\langle 1\rangle$ translate the resulting structure. The arguments at 2 relating the Adams filtration of the image of $(M,x)\in\pi_*\mathbf{M}\mathrm{Spin}\wedge\mathrm{K}(\mathbb{Z},4)$ to the values it takes under the given invariants carry over exactly. The Ext term $\mathrm{Ext}_{E[Q_0,Q_1]}(\mathbb{Z}/5,\mathbb{Z}/5)$ is then:



Since then connective K-theory will be given by the wedge of two Johnson-Wilson spectra, the second of which suspended 4 times, so:



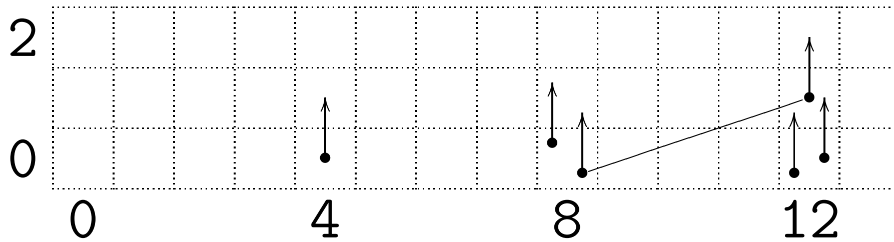
5.2 Verifying the manifold basis by the mod p spectral sequences

The purpose of this section is to determine that the particular spin manifolds that generated the connective K-theory of $\mathrm{K}(\mathbb{Z},4)$ in dimensions 8 and 12 after completion at the prime 2 still generate the connective K-theory after completion at any odd prime p. This will prove that those manifolds generate the connective K-theory integrally (that is, without completing at any prime), giving the final piece of the proof for the linear relations between our analytic invariants $\int_M x^n \widehat{A}$. This will be much easier than the prime 2 case, because in this range of dimensions there are no differentials at odd primes and all the invariants measuring the Adams filtration are defined rationally (as opposed to the invariant coming from the Q_0-homology $(Sq^2\iota)^2$ at the prime 2). Thus, all that is required is to show that our previous manifolds map to the bottom of each $\mathbb{Z}$-tower at each prime p, which can be done just by noting the power of p in the value that the associated invariant takes on that manifold.

Of course, first we must decompose the mod p cohomology of $\mathrm{K}(\mathbb{Z},4)$ into $E[Q_0,Q_1]$-modules. Just as at the prime 2, at each odd prime p the mod p cohomology of $\mathrm{K}(\mathbb{Z},4)$ is a free polynomial algebra on admissible monomials (of excess less than 4 and not ending in Q_0). The cell chart for $\mathrm{K}(\mathbb{Z},4)$ after completion at $p=3$ is given below, showing the action of Q_0 and Q_1:

$1 - \iota$	$4 - \iota^3$	$7 - 2\iota(Q_1\iota)$
$2 - P^1\iota$	$5 - \iota^2$	$8 - (P^1\iota)^2$
$3 - Q_1\iota$	$6 - \iota(P^1\iota)$	$13 - 2(P^1\iota)(Q_1\iota)$

which gives the spectral sequence:



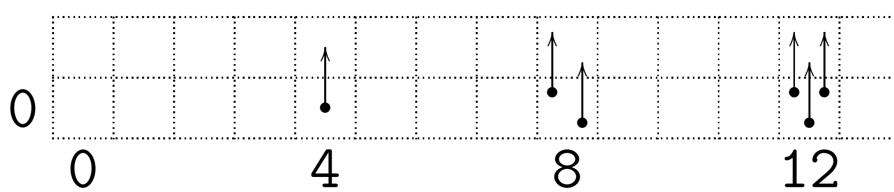
There can be no differentials in this range, because there are no adjacent $\mathbb{Z}$-towers. Since the $\mathbb{Z}$-tower in $t-s=8$ coming from the Q_0-homology ι sits in Adams filtration zero, a spin manifold (M,x) sent to the bottom of that tower must take value $\int_M x\widehat{A}_4 = 1/3$, modulo factors of other primes. However, since $\int_{\mathbb{C}\mathrm{P}^3\times S^2} ab\widehat{A}_4 = -1/6$, our generating set of manifolds for the prime 2 also works after completion at 3, giving the following:

Proposition 7. *The elements* $(\mathbb{C}\mathrm{P}^3 \times S^2, ab)$*,* $(\mathbb{H}\mathrm{P}^2, y)$ *generate the group* $(\pi_8\ bo \wedge \mathrm{K}(\mathbb{Z},4)) \otimes \mathbb{Z}_3$*.*

Likewise, examining the Adams filtrations in which the $\mathbb{Z}$-towers in $t-s=12$ begin, a spin manifold sent to the bottom of the $\mathbb{Z}$-tower starting in Adams filtration 1 should take value 1/3 on the invariant $\int_M x\widehat{A}_8$ (modulo factors of other primes) and value 0 on the two other invariants, $\int_M x^2\widehat{A}_4$ and $\int_M x^3$. This is precisely the case of $(\mathbb{H}\mathrm{P}^3, y) - (V_{2,1}, vw)$. Similarly, $(\mathbb{H}\mathrm{P}^3, y)$ and $(\mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^3, vw) - (\mathbb{H}\mathrm{P}^3, y)$ generate the $\mathbb{Z}$-towers starting in filtration 0.

Proposition 8. *The elements* $(\mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^3, vw)$*,* $(\mathbb{H}\mathrm{P}^3, y)$*,* $(V_{2,1}, vw)$ *generate the group* $(\pi_{12}\ bo \wedge \mathrm{K}(\mathbb{Z},4)) \otimes \mathbb{Z}_3$*.*

At the prime 5 there is a similar picture for the structure of the mod 5 cohomology of $\mathrm{K}(\mathbb{Z},4)$, again showing the action of Q_0 and Q_1, which gives:



This is also what the spectral sequence looks like through $t-s=13$ for all prime p greater than 5.

Also, we do not need to do any significant work at the prime 7 because the slope of $\widehat{A}$ is $2(p^1-1) = 2(7-1) = 12$, which means that the first place that we may conceivably need a 7 in the denominator of the value that our invariant takes on the basis element associated to it is in dimension $4+12=16$, outside the range of dimensions for which we are presently working. Were the present goal to understand these spectral sequences globally, we would have to deal with all primes, but for the range of dimensions we are concerned with, we do not need to look more than cursorily at $p>5$.

Proposition 9. *After completion at the prime 5, the elements* $(V_{2,1}, vw)$, $(\mathbb{H}\mathrm{P}^3, y)$, $(\mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^3, vw)$ *generate the homotopy group* $\pi_{12}\ bo \wedge \mathrm{K}(\mathbb{Z}, 4)$.

This completes the proofs of Theorems 5, 6, and 7, that the previous set of manifolds generate the connective real K-theory of $\mathrm{K}(\mathbb{Z}, 4)$ at 8 and 12, and that the given integrals are then always integers.

6 Epilogue

This work is conceived as the first step in a deeper investigation with three goals: first, to find a general polynomial $P(x) = (\frac{1}{n!}x^n + \ldots)$ in a 4-form x for a $4n$-dimensional spin manifold M such that $\int_M P(x)\widehat{A}$ is an integer for all M; second, to find a set of spin manifolds equipped with 4-forms that generate the connective K-theory $\pi_{4n}\ bo \wedge \mathrm{K}(\mathbb{Z}, 4)$, compute this group exactly, and thereby obtain all such integrality results simply by checking formulas on the set of manifolds; and third, to find some meaning for the polynomial P related to index theory.

At present, these three things are far from accomplished. The obvious prescription for their achievement would be to put the infinite dimensional structure of $\mathrm{K}(\mathbb{Z}, 4)$ into workable order at each prime and determine some regularity or periodicity to move through each Ext term mapping manifolds to the bottom of $\mathbb{Z}$-towers in the families coming from each invertible module. This approach encounters the immediate difficulty that the structure for $\mathrm{K}(\mathbb{Z}, 4)$, past even a small range of dimensions, is opaque. This can be dealt with using the Milnor basis for Steenrod algebra, but it happens that at the prime 2, for instance, the cohomology of $\mathrm{K}(\mathbb{Z}, 4)$ is a direct sum of stably invertible modules only through 48 dimensions, after which noninvertible modules appear. This is likely the case at all primes. There is the additional problem of even defining the invariants coming from Q_0-homologies that do not exist integrally. Further, there are differentials at every prime p, and especially many at 2.

For spin 16-manifolds, these difficulties are manageable. Addressing these issues, and explicitly dealing with 16-manifolds, will be the focus of future work.

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