Trapping Regions for Discontinuously Coupled Systems of Evolution Variational Inequalities and Application

S. Carl

Fachbereich Mathematik und Informatik, Institut für Analysis
Martin-Luther-Universität Halle-Wittenberg
06099 Halle, Germany. *E-mail: carl@mathematik.uni-halle.de*

S. Heikkilä

Department of Mathematical Sciences, University of Oulu
Box 3000, FIN-90014 University of Oulu, Finland.
E-mail: Seppo.Heikkila@oulu.fi

J.W. Jerome

Department of Mathematics, Northwestern University
Evanston, IL 60208-2730, USA. *E-mail: jwj@math.northwestern.edu*

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Address for manuscript correspondence:
Siegfried Carl
Fachbereich Mathematik und Informatik, Institut für Analysis
Martin-Luther-Universität Halle-Wittenberg
D-06099 Halle, Germany
E-mail: carl@mathematik.uni-halle.de
Tel: +49 345 5524639
Fax: +49 345 5527001

Abstract

We consider initial-boundary value problems for weakly coupled systems of parabolic equations under coupled nonlinear flux boundary condition. Both coupling vector fields $f : Q \times \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \Gamma \times \mathbb{R}^2 \to \mathbb{R}^2$ are assumed to be either of competitive or cooperative type, but may otherwise be discontinuous with respect to all their arguments. The main goal is to provide conditions for the vector fields f and g that allow the identification of regions of existence of solutions (so called trapping regions). To this end the problem is transformed to a discontinuously coupled system of evolution variational inequalities. Assuming a generalized outward pointing vector field on the boundary of a rectangle of the dependent variable space, the system of evolution variational inequalities is solved via a fixed point problem for some increasing operator in an appropriate ordered Banach space. The main tools used in the proof are evolution variational inequalities, comparison techniques, and fixed point results in ordered Banach spaces.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be a bounded domain with C^1-boundary $\partial\Omega$, $Q = \Omega \times (0,T)$ and $\Gamma = \partial\Omega \times (0,T)$, with $T > 0$. We consider the following initial-boundary value problem (IBVP for short): $k = 1, 2$.

$$\frac{\partial u_k}{\partial t} - \nabla \cdot [a_k \nabla u_k] + f_k(u_1, u_2) = 0 \qquad \text{in} \;\; Q \tag{1.1}$$

$$u_k = 0 \;\; \text{on} \; \Omega \times \{0\}, \;\; \frac{\partial u_k}{\partial \nu_k} + g_k(u_1, u_2) = 0 \qquad \text{on} \;\; \Gamma, \tag{1.2}$$

where $a_k \in L^\infty(Q)$ with $a_k(x,t) \geq \mu_k > 0$ in Q, and $\partial/\partial\nu_k$ denotes the outward conormal derivative at Γ related to the corresponding elliptic operator. It should be noted that the theory we are going to develop in this paper is applicable to more complicated systems, in which the elliptic operators may be the sum of a monotone divergence type operator and lower order convection terms, the vector fields f, and g may depend, in addition, on the space-time variables (x,t) and the initial condition may be nonhomogeneous, i.e., of the form $u_k(x,0) = \psi_k(x)$ with $\psi_k \in L^2(\Omega)$. Even mixed Dirichlet-Robin type boundary conditions can be treated. Only for the sake of simplifying our presentation and in order to emphasize the main idea we consider here problem (1.1), (1.2) as a model problem. Existence results for discontinuously coupled elliptic systems have been obtained by the authors in [4, 6].

The novelty of the IBVP (1.1), (1.2) is that the vector fields f and g may be discontinuous in all their arguments. The minimum regularity requirement for these vector fields is their superpositional measurability, i.e., whenever $u, v : Q \to \mathbb{R}$ (resp. $u, v : \Gamma \to \mathbb{R}$) are measurable, then also the superposition $(x,t) \mapsto f_k(u(x,t), v(x,t))$ (resp. $(x,t) \mapsto g_k(u(x,t), v(x,t))$) is measurable in Q (resp. Γ). In order to formulate the conditions imposed on the vector fields let us introduce the following notion.

Definition 1.1. A vector field $h = (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be of *competitive type* if the component functions $h_1(s_1, s_2)$ and $h_2(s_1, s_2)$ are both separately increasing in s_1, s_2. The argument s_k of $h_k(s_1, s_2)$ is called the *principal argument*. A vector field $h = (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be of *cooperative type* if $h_k(s_1, s_2)$ is increasing in its principal argument and decreasing in its nonprincipal argument.

From Definition 1.1 it follows that IBVP (1.1), (1.2) with f and g of cooperative type can be transformed into a system of competitive type, and vice versa. To this end one only needs to perform the transformation $(w_1, w_2) := (u_1, -u_2)$, and define $\hat{f}_1(w_1, w_2) = f_1(w_1, -w_2)$, $\hat{f}_2(w_1, w_2) = -f_2(w_1, -w_2)$, as well as $\hat{g}_1(w_1, w_2) = g_1(w_1, -w_2)$, $\hat{g}_2(w_1, w_2) = -g_2(w_1, -w_2)$ to get a IBVP in (w_1, w_2) of competitive type. Thus cooperative and competitive systems are qualitatively equivalent.

Throughout the rest of this paper we assume the following hypotheses on the vector fields f and g:

(H1) The component functions $f_k : \mathbb{R}^2 \to \mathbb{R}$ and $g_k : \mathbb{R}^2 \to \mathbb{R}$ are Baire-measurable and satisfy a growth condition of the form

$$\begin{aligned} |f_k(s_1, s_2)| &\leq c\,(1 + |s_1| + |s_2|),\ \forall\ (s_1, s_2) \in \mathbb{R}^2, \\ |g_k(s_1, s_2)| &\leq c\,(1 + |s_1| + |s_2|),\ \forall\ (s_1, s_2) \in \mathbb{R}^2, \end{aligned}$$

where c is some positive generic constant.

(H2) The vector fields f and g are assumed to be of competitive type.

2 Notations and Preliminaries

Let $W^{1,2}(\Omega)$ denote the usual Sobolev space of square integrable functions and let $(W^{1,2}(\Omega))^*$ denote its dual space. Then by identifying $L^2(\Omega)$ with its dual space, $W^{1,2}(\Omega) \subset L^2(\Omega) \subset (W^{1,2}(\Omega))^*$ forms an evolution triple with all the embeddings being continuous, dense and compact, cf. [8]. We let $V = L^2(0, T; W^{1,2}(\Omega))$, denote its dual space by $V^* = L^2(0, T; (W^{1,2}(\Omega))^*)$, and define a function space W by

$$W = \{w \in V \mid \frac{\partial w}{\partial t} \in V^*\},$$

where the derivative $\partial/\partial t$ is understood in the sense of vector-valued distributions and characterized by (cf. [8])

$$\int_0^T u'(t)\phi(t)\, dt = -\int_0^T u(t)\phi'(t)\, dt, \quad \text{for all } \phi \in C_0^\infty(0, T).$$

The space W endowed with the norm

$$\|w\|_W = \|w\|_V + \|\partial w/\partial t\|_{V^*}$$

is a Banach space which is separable and reflexive due to the separability and reflexivity of V and V^*, respectively. (Note that for any Banach space B the space $L^2(0, T; B)$ of vector-valued functions consists of all measurable functions $u : (0, T) \to B$ for which $\|u\| = (\int_0^T \|u(t)\|_B^2\, dt)^{1/2}$ is finite.) Furthermore it is well known that the embedding $W \subset C([0, T]; L^2(\Omega))$ is continuous, cf. [8]. Finally, because $W^{1,2}(\Omega) \subset L^2(\Omega)$ is compactly embedded, we have a compact embedding of $W \subset L^2(Q) \equiv L^2(0, T; L^2(\Omega))$ due to Aubin's lemma, cf. e.g. [7]. We introduce the natural partial ordering in $L^2(Q)$ by $u \leq w$ if and only if $w - u$ belongs to the cone $L^2_+(Q)$ of all nonnegative elements of $L^2(Q)$. This induces a corresponding partial ordering also in the subspaces V and W of $L^2(Q)$, and if $u \le w$ then $[u, w] := \{v \mid u \le v \le w\}$ denotes the order interval formed by u and w. Further, if $(B, \le)$ is any ordered Banach space, then we furnish the Cartesian product $B \times B$ with the componentwise partial ordering, i.e., $x = (x_1, x_2) \le (y_1, y_2) = y$ iff $x_k \le y_k,\ k = 1, 2$. Thus the order interval $[x, y] \subset B \times B$ corresponds to the rectangle $\mathcal{R} = [x_1, y_1] \times [x_2, y_2] \subset B \times B$. In what follows we will make use of the following Cartesian products: $X := V \times V$, $Y := L^2(Q) \times L^2(Q)$, and $Z := L^2(\Gamma) \times L^2(\Gamma)$. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V^* and V, and by $\gamma : V \to L^2(\Gamma)$ the trace operator which is linear and continuous, and if considered as a mapping $\gamma : W \to L^2(\Gamma)$ it is even compact, see, e.g., [2, Lemma 3.1]. In order to apply functional analytic methods to the IBVP (1.1), (1.2) we introduce operators A_k generated by the elliptic operators $-\nabla \cdot [a_k \nabla w]$, and F_k and G_k related to the vector fields f and g, respectively, as follows: Let $k = 1, 2$, and $\varphi \in V$

$$\begin{aligned}
A_k : V \to V^*;\ \ \langle A_k w, \varphi \rangle &:= \int_Q a_k \nabla w \nabla \varphi \, dxdt \\
F_k : Y \to V^*;\ \ \langle F_k(u_1, u_2), \varphi \rangle &:= \int_Q f_k(u_1, u_2) \varphi \, dxdt \\
G_k : Z \to V^*;\ \ \langle G_k(z_1, z_2), \varphi \rangle &:= \int_\Gamma g_k(z_1, z_2) \gamma\varphi \, dxdt.
\end{aligned}$$

One easily verifies that $A_k : V \to V^*$ is linear and monotone. By (H1) the operators $F_k : X \to V^*$ and $G_k \circ \gamma : X \to V^*$ are well defined and bounded, but not necessarily continuous. The time derivative $\partial/\partial t : V \to V^*$ is given by

$$\langle \partial u/\partial t, \varphi \rangle = \int_0^T < \frac{\partial u(\cdot, t)}{\partial t}, \varphi(\cdot, t) > dt, \quad \forall\ \varphi \in V,$$

with $< \cdot, \cdot >$ denoting the duality pairing between $(W^{1,2}(\Omega))^*$ and $W^{1,2}(\Omega)$, and we denote its restriction to the subspace of functions having homogeneous initial data by L, i.e., let $L := \partial/\partial t$ and its domain $D(L)$ of definition is given by

$$D(L) = \{u \in W \mid u(x, 0) = 0 \quad \text{in} \quad \Omega\}.$$

The linear operator $L : D(L) \subset V \to V^*$ can be shown to be closed, densely defined and maximal monotone, e.g., cf. [8, Chapter 32]. With $u = (u_1, u_2)$, $Au = (A_1 u_1, A_2 u_2)$, $\gamma u = (\gamma u_1, \gamma u_2)$, $F(u) = (F_1(u), F_2(u))$, $G(u) = (G_1(u), G_2(u))$, and $Lu = (Lu_1, Lu_2)$ the weak formulation of a solution of system (1.1), (1.2) reads as follows: Find $u \in [D(L)]^2 \subset X$ such that the following vector equation holds:

$$Lu + Au + F(u) + G \circ \gamma(u) = 0 \quad \text{in} \quad X^*. \tag{2.1}$$

In view of the discontinuous behaviour of the operators F and G this notion is too restrictive for establishing a solution theory as can be seen by the following simple example:

Example 2.1. Consider the system

$$\frac{\partial u_1}{\partial t} - \Delta u_1 = 0, \quad u_1(x,0) = 0 \text{ in } \Omega, \qquad \frac{\partial u_1}{\partial \nu} + g_1(u_1, u_2) = 0 \text{ on } \Gamma, \tag{2.2}$$

$$\frac{\partial u_2}{\partial t} - \Delta u_2 = 0, \quad u_2(x,0) = 0 \text{ in } \Omega, \qquad \frac{\partial u_2}{\partial \nu} = 0 \text{ on } \Gamma, \tag{2.3}$$

where

$$g_1(s_1, s_2) = \begin{cases} -1, & s_1 \leq 0, \\ 0, & s_1 > 0. \end{cases}$$

Obviously the system (2.2), (2.3) is of competitive type, however, it has no solution in the sense of equation (2.1) as will be shown by contradiction as follows. Assume $u = (u_1, u_2)$ is a solution. From (2.3) it readily follows that $u_2 = 0$. Thus the component u_1 must satisfy the IBVP (2.2) with $g_1(u_1, 0)$, whose weak formulation is

$$\langle \frac{\partial u_1}{\partial t}, \varphi \rangle + \int_Q \nabla u_1 \nabla \varphi \, dx dt + \int_\Gamma g_1(\gamma u_1, 0) \gamma \varphi \, d\Gamma = 0, \ \forall \ \varphi \in V. \tag{2.4}$$

Taking in (2.4) the special test function $\varphi = u_1$ we get in view of

$$\int_\Gamma g_1(\gamma u_1, 0) \, \gamma u_1 \, d\Gamma \geq 0,$$

the following inequality

$$\frac{1}{2} \|u_1(\cdot, T)\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(Q)} \leq 0, \tag{2.5}$$

which implies $u_1 = 0$. Therefore, if (u_1, u_2) is a solution it must be the trivial one. This, however, is a contradiction to the boundary condition in (2.2), since $\frac{\partial u_1}{\partial \nu} + g_1(u_1, u_2)|_{(0,0)} = -1 \neq 0$.

To establish a consistent solution theory for the discontinuous system (1.1), (1.2) we extend the notion of its solution by introducing multivalued vector fields $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ associated with f and g, respectively, as follows:

$$\begin{aligned} \alpha_1(s_1, s_2) &= [f_1(s_1-, s_2), f_1(s_1+, s_2)], \ \ \alpha_2(s_1, s_2) = [f_2(s_1, s_2-), f_2(s_1, s_2+)], \\ \beta_1(s_1, s_2) &= [g_1(s_1-, s_2), g_1(s_1+, s_2)], \ \ \beta_2(s_1, s_2) = [g_2(s_1, s_2-), g_2(s_1, s_2+)], \end{aligned} \tag{2.6}$$

where $f_1(s_1\pm, s_2)$, $f_2(s_1, s_2\pm)$ and $g_1(s_1\pm, s_2)$, $g_2(s_1, s_2\pm)$ denote the one-sided limits. Thus $\alpha_k : \mathbb{R}^2 \to 2^{\mathbb{R}} \setminus \emptyset$ and $\beta_k : \mathbb{R}^2 \to 2^{\mathbb{R}} \setminus \emptyset$ are the maximal monotone graphs of f_k and g_k with respect to their principal arguments s_k. In what follows we consider instead of the IBVP (1.1), (1.2) the following multivalued version of it:

$$\frac{\partial u_k}{\partial t} + A_k u_k + \alpha_k(u_1, u_2) \ni 0 \quad \text{in} \quad Q, \tag{2.7}$$

$$u_k = 0 \quad \text{on } \Omega \times \{0\}, \quad \frac{\partial u_k}{\partial \nu_k} + \beta_k(u_1, u_2) \ni 0 \quad \text{on} \quad \Gamma. \tag{2.8}$$

Next we develop the concept of a *trapping region* for systems which is an appropriate extension of the notion of super- and subsolutions in the scalar case. To this end let $\mathcal{R} = [\underline{u}, \bar{u}]$ be the rectangle formed by the ordered pair $\underline{u} = (\underline{u}_1, \underline{u}_2)$ and $\bar{u} = (\bar{u}_1, \bar{u}_2)$, where $\underline{u}$, $\bar{u} \in W \times W$.

Definition 2.1. The vector field $\partial u/\partial t + Au + F(u) + G \circ \gamma(u)$ is called a *generalized outward pointing vector* on the boundary $\partial\mathcal{R}$ of the rectangle $\mathcal{R}$ if the following inequalities hold for all $\varphi \in V \cap L^2_+(Q)$:

$$\begin{aligned}
&\langle \partial \underline{u}_1/\partial t + A_1\underline{u}_1 + F_1(\underline{u}_1, v) + G_1 \circ \gamma(\underline{u}_1, v), \varphi\rangle \le 0, \quad \forall\, v \in [\underline{u}_2, \bar{u}_2];\\
&\langle \partial \underline{u}_2/\partial t + A_2\underline{u}_2 + F_2(v, \underline{u}_2) + G_2 \circ \gamma(v, \underline{u}_2), \varphi\rangle \le 0, \quad \forall\, v \in [\underline{u}_1, \bar{u}_1];\\
&\langle \partial \bar{u}_1/\partial t + A_1\bar{u}_1 + F_1(\bar{u}_1, v) + G_1 \circ \gamma(\bar{u}_1, v), \varphi\rangle \ge 0, \quad \forall\, v \in [\underline{u}_2, \bar{u}_2];\\
&\langle \partial \bar{u}_2/\partial t + A_2\bar{u}_2 + F_2(v, \bar{u}_2) + G_2 \circ \gamma(v, \bar{u}_2), \varphi\rangle \ge 0, \quad \forall\, v \in [\underline{u}_1, \bar{u}_1].
\end{aligned}$$

Using the notion of the generalized outward pointing vector we define the trapping region.

Definition 2.2. Let $\underline{u}$, $\bar{u} \in W \times W$ satisfy $\underline{u} \le \bar{u}$, and $\underline{u}(x, 0) \le 0 \le \bar{u}(x, 0)$. Then $\mathcal{R} = [\underline{u}, \bar{u}]$ is called a *trapping region* for the system (1.1), (1.2) if $\partial u/\partial t + Au + F(u) + G \circ \gamma(u)$ is a generalized outward pointing vector on $\partial\mathcal{R}$.

Our main goal is to show that each trapping region for the system (1.1), (1.2) contains a solution of its multivalued version (2.7), (2.8) in the following sense.

Definition 2.3. The vector $u \in D(L) \times D(L) \subset X$ is a *solution* of the IBVP (2.7), (2.8) if there is an $\xi \in Y$ and an $\eta \in Z$ such that for $k = 1, 2$ the following holds:

(i) $\xi_k(x, t) \in \alpha_k(u_1(x, t), u_2(x, t))$, for a.e. $(x, t) \in Q$,

(ii) $\eta_k(x, t) \in \beta_k(\gamma u_1(x, t), \gamma u_2(x, t))$, for a.e. $(x, t) \in \Gamma$,

(iii) $\langle L u_k + A_k u_k + \xi_k + \gamma^* \eta_k, \varphi\rangle = 0, \quad \forall\, \varphi \in V$, where $\gamma^* : L^2(\Gamma) \to V^*$ denotes the adjoint operator to the trace operator γ with

$$\langle \xi_k, \varphi \rangle = \int_Q \xi_k(x,t)\,\varphi(x,t)\,dxdt, \quad \langle \gamma^* \eta_k, \varphi \rangle = \int_\Gamma \eta_k(x,t)\,\gamma\varphi(x,t)\,d\Gamma.$$

For the analysis of the multivalued system (2.7), (2.8) it will be convenient to use its equivalent formulation in terms of a discontinuously coupled system of evolution variational inequalities of the form: Find $u_k \in D(L)$ such that for all $\varphi \in V$ we have

$$\begin{aligned} &\langle Lu_1 + A_1 u_1, \varphi - u_1 \rangle + J_1(\varphi, u_2) - J_1(u_1, u_2) \\ &\qquad\qquad + \Phi_1 \circ \gamma(\varphi, u_2) - \Phi_1 \circ \gamma(u_1, u_2) \quad \geq 0, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\langle Lu_2 + A_2 u_2, \varphi - u_2 \rangle + J_2(u_1, \varphi) - J_2(u_1, u_2) \\ &\qquad\qquad + \Phi_2 \circ \gamma(u_1, \varphi) - \Phi_2 \circ \gamma(u_1, u_2) \quad \geq 0, \end{aligned} \tag{2.10}$$

where the functionals J_k and Φ_k are defined by

$$J_1(u_1, u_2) := \int_Q \Big(\int_0^{u_1(x,t)} f_1(s, u_2(x,t))\,ds \Big)\,dxdt, \quad u \in Y;$$

$$J_2(u_1, u_2) := \int_Q \Big(\int_0^{u_2(x,t)} f_2(u_1(x,t), s)\,ds \Big)\,dxdt, \quad u \in Y;$$

$$\Phi_1(v_1, v_2) := \int_\Gamma \Big(\int_0^{v_1(x,t)} g_1(s, v_2(x,t))\,ds \Big)\,d\Gamma, \quad v \in Z;$$

$$\Phi_2(v_1, v_2) := \int_\Gamma \Big(\int_0^{v_2(x,t)} g_2(v_1(x,t), s)\,ds \Big)\,d\Gamma, \quad v \in Z.$$

By hypotheses (H1) and (H2) the functionals $J_k : Y \to \mathbb{R}$ and $\Phi_k : Z \to \mathbb{R}$ are well defined, convex and locally Lipschitz continuous with respect to their principal argument. Moreover, since X is dense in Y and $\gamma(X)$ is dense in Z we obtain by applying [5, Theorem 2.2, Theorem 2.3] along with the chain rule for subdifferentials (see, e.g., [9, p.403]) the following formula for the subdifferentials $\partial_k J_k$ and $\partial_k(\Phi_k \circ \gamma)$ of the functionals J_k and $\Phi_k \circ \gamma$ with respect to their principal argument: If $u \in X$, then

$$\begin{aligned} \partial_k J_k(u_1, u_2) &= \alpha_k(u_1, u_2), \\ \partial_k(\Phi_k \circ \gamma)(u_1, u_2) &= \gamma^* \circ \partial_k \Phi_k(\gamma u_1, \gamma u_2) = \beta_k(\gamma u_1, \gamma u_2). \end{aligned}$$

Using the sum rule for subgradients (see, e.g., [9, Theorem 47.B]) we finally get

$$\begin{aligned} &\partial_k \Big(J_k(u_1, u_2) + \Phi_k \circ \gamma(u_1, u_2) \Big) = \partial_k J_k(u_1, u_2) + \partial_k(\Phi_k \circ \gamma)(u_1, u_2) \\ &= \alpha_k(u_1, u_2) + \beta_k(\gamma u_1, \gamma u_2). \end{aligned} \tag{2.11}$$

By definition of the subgradient and using (2.11) one easily verifies the equivalence of (2.7), (2.8) and (2.9), (2.10).

3 Scalar Equation

This section examines the individual equations of our system (2.7), (2.8) and its associated equivalent evolution variational inequality. Let us consider the scalar IBVP

$$\frac{\partial u}{\partial t} - \nabla \cdot [a\nabla u] + f(x,t,u) = 0 \qquad \text{in } \ Q, \tag{3.1}$$

$$u = 0 \ \text{ on } \Omega \times \{0\}, \ \ \frac{\partial u}{\partial \nu} + g(x,t,u) = 0 \qquad \text{on } \ \Gamma, \tag{3.2}$$

with $a \in L^\infty(Q)$ and $a(x,t) \geq \mu > 0$, and where the nonlinearities $f : Q \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ are assumed to be Baire-measurable in their respective domains of definitions, and satisfy the following growth and monotonicity conditions: For some $p \in L^2_+(Q)$ and $q \in L^2_+(\Gamma)$ we suppose

(S1) $|f(x,t,s)| \leq p(x,t) + c\,|s|, \ \ \forall (x,t,s) \in Q \times \mathbb{R}$,

(S2) $|g(x,t,s)| \leq q(x,t) + c\,|s|, \ \ \forall (x,t,s) \in \Gamma \times \mathbb{R}$, and

(S3) $s \mapsto f(x,t,s)$ and $s \mapsto g(x,t,s)$ are increasing.

The weak formulation of (3.1), (3.2) reads as follows: Find $u \in D(L)$ such that

$$\langle Lu + Au + F(u) + G \circ \gamma(u), \varphi = 0, \ \ \forall\, \varphi \in V, \tag{3.3}$$

where the operators in (3.3) are defined similarly as in section 2, i.e.,

$$\langle Au, \varphi\rangle = \int_Q a\nabla u\,\nabla\varphi\,dxdt, \qquad \langle F(u), \varphi\rangle = \int_Q f(x,t,u)\varphi\,dxdt,$$

$$\langle G \circ \gamma(u), \varphi\rangle = \int_\Gamma g(x,t,\gamma u)\gamma\varphi\,d\Gamma.$$

Let us recall for convenience the notion of (weak) super- and subsolution for the IBVP (3.1), (3.2).

Definition 3.1. The function $w \in W$ is called a *supersolution (subsolution)* of (3.1), (3.2) if $w(x,0) \geq 0$ ($w(x,0) \leq 0$), and for all $\varphi \in V \cap L^2_+(Q)$ the following inequality holds:

$$\langle \partial w/\partial t + Aw + F(w) + G \circ \gamma(w), \varphi\rangle \geq 0 \ (\leq 0). \tag{3.4}$$

Let $\alpha : Q \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ and $\beta : \Gamma \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ be the maximal monotone graphs associated with f and g, respectively, then the multivalued version of the IBVP (3.1), (3.2) reads as

$$\frac{\partial u}{\partial t} + Au + \alpha(x,t,u) \ni 0 \ \text{ in } \ Q, \tag{3.5}$$

$$u = 0 \ \text{ on } \Omega \times \{0\}, \ \ \frac{\partial u}{\partial \nu} + \beta(x,t,u) \ni 0 \ \text{ on } \ \Gamma, \tag{3.6}$$

which is equivalent to the evolution variational inequality: Find $u \in D(L)$ such that for all $\varphi \in V$

$$\langle Lu + Au, \varphi - u\rangle + J(\varphi) - J(u) + \Phi \circ \gamma(\varphi) - \Phi \circ \gamma(u) \geq 0. \tag{3.7}$$

The functionals that appear in (3.7) are given by

$$\begin{aligned} J(u) &:= \int_Q \Big(\int_0^{u(x,t)} f(x,t,s))\, ds \Big)\, dxdt, \\ \Phi(v) &:= \int_\Gamma \Big(\int_0^{v(x,t)} g(x,t,s)\, ds \Big)\, d\Gamma, \end{aligned}$$

and by similar arguments from convex analysis as in section 2 we have that $J : V \to \mathbb{R}$ and $\Phi \circ \gamma : V \to \mathbb{R}$ are convex and continuous, and satisfy

$$\partial(J(u) + \Phi \circ \gamma(u)) = \alpha(\cdot,\cdot,u) + \beta(\cdot,\cdot,\gamma u)). \tag{3.8}$$

The following uniqueness and enclosure result will be used in the analysis of the multi-valued system or its equivalent system of evolution variational inequalities, respectively.

Lemma 3.1. *Assume hypotheses (S1)–(S3) and let $\bar u$, $\underline{u} \in W$ be super- and subsolution of (3.1), (3.2). Then $\underline{u} \leq \bar u$, and there exists a unique solution u of the multivalued IBVP (3.5), (3.6) (resp. of (3.7)) which is contained in the order interval $[\underline{u}, \bar u]$.*

Proof. The proof of the lemma will be done in steps (a), (b), and (c).

(a) $\underline{u} \leq \bar u$

Subtracting the corresponding inequalities for super- and subsolution (3.4) and taking the special nonnegative test function $\varphi = (\underline{u} - \bar u)^+ := \max(\underline{u} - \bar u, 0) \in V \cap L^2_+(Q)$ we obtain

$$\begin{aligned} &\langle \frac{\partial(\underline{u} - \bar u)}{\partial t}, (\underline{u} - \bar u)^+\rangle + \int_Q a\, \nabla(\underline{u} - \bar u)\nabla(\underline{u} - \bar u)^+\, dxdt \\ &+ \int_Q (f(x,t,\underline{u}) - f(x,t,\bar u)(\underline{u} - \bar u)^+\, dxdt \\ &+ \int_\Gamma (g(x,t,\gamma\underline{u}) - g(x,t,\gamma\bar u)\gamma(\underline{u} - \bar u)^+\, d\Gamma \leq 0. \end{aligned} \tag{3.9}$$

By the monotonicity assumptions on f and g the integral terms in (3.9) related to them are nonnegative. Since $\underline{u}(x,0) \leq 0 \leq \bar u(x,t)$, it follows $(\underline{u} - \bar u)^+(x,0) = 0$ which implies for the first term of (3.9)

$$\langle \frac{\partial(\underline{u} - \bar u)}{\partial t}, (\underline{u} - \bar u)^+\rangle = \frac{1}{2}\int_\Omega \Big((\underline{u} - \bar u)^+(x,T) \Big)^2\, dx,$$

and therefore from (3.9) we obtain the estimate

$$\frac{1}{2}\|(\underline{u}-\bar{u})^+(\cdot,T)\|^2_{L^2(\Omega)}+\mu\|\nabla(\underline{u}-\bar{u})^+\|^2_{L^2(Q)}\le 0, \tag{3.10}$$

which implies $(\underline{u}-\bar{u})^+=0$, and thus $\underline{u}\le\bar{u}$.

(b) *Existence and Uniqueness*
For the existence proof we first transform the IBVP (3.5), (3.6) by the exponential shift

$$u(x,t)=e^{\lambda t}w(x,t),\quad \lambda>0, \tag{3.11}$$

with λ to be specified later, into an equivalent IBVP in w of the form

$$\frac{\partial w}{\partial t}+Aw+\lambda w+\tilde{\alpha}(x,t,w)\ni 0 \ \text{ in } \ Q, \tag{3.12}$$

$$w=0 \ \text{ on } \Omega\times\{0\},\quad \frac{\partial w}{\partial\nu}+\tilde{\beta}(x,t,w)\ni 0 \ \text{ on } \ \Gamma, \tag{3.13}$$

where the transformed maximal monotone graphs $\tilde{\alpha}$ and $\tilde{\beta}$ preserve the structure of the original α and β, respectively. The associated equivalent evolution variational inequality reads then: Find $w\in D(L)$ such that for all $\varphi\in V$

$$\langle Lw+Aw+\lambda w,\varphi-w\rangle+\tilde{J}(\varphi)-\tilde{J}(w)+\tilde{\Phi}\circ\gamma(\varphi)-\tilde{\Phi}\circ\gamma(w)\ge 0. \tag{3.14}$$

The maximal monotonicity of $L: D(L)\subset V\to V^*$, the monotonicity (even strong monotonicity) of $A+\lambda I: V\to V^*$ and the continuity of the convex functional

$$\Psi:=\tilde{J}+\tilde{\Phi}\circ\gamma: V\to\mathbb{R},$$

permit the application of [8, Theorem 32.J], which ensures the existence of a solution of the evolution variational inequality (3.14) provided that the multivalued operator

$$L+A+\lambda I+\partial\Psi: V\to 2^{V^*}$$

is coercive with respect to 0. The latter means that we need to show the existence of an $r>0$ and a $w_0\in D(L)\cap D(A)\cap D(\Psi)$ such that

$$\langle w^*,w-w_0\rangle>0,\quad \forall\ (w,w^*)\in L+A+\lambda I+\partial\Psi,\quad \text{with } \ \|w\|>r. \tag{3.15}$$

To verify (3.15) we set $w_0=0$ and show that for $r>0$ sufficiently large the following inequality holds:

$$\langle Lw+Aw+\lambda w+\xi+\eta,w\rangle>0,\quad \forall w:\ \|w\|>r, \tag{3.16}$$

where $\xi \in \partial \tilde{J}(w)$ and $\eta \in \partial(\tilde{\Phi} \circ \gamma)(w)$. Let $\xi_0 \in \partial \tilde{J}(0)$, and $\eta_0 \in \partial(\tilde{\Phi} \circ \gamma)(0)$. Then by the maximal monotonicity of the subgradients we get for any $\xi \in \partial \tilde{J}(w)$ and any $\eta \in \partial(\tilde{\Phi} \circ \gamma)(w)$ the following inequality

$$\langle \xi + \eta, w \rangle \geq \langle \xi_0 + \eta_0, w \rangle \geq -\|\xi_0\|_{V^*}\|w\|_V - \|\eta_0\|_{V^*}\|w\|_V. \tag{3.17}$$

Selecting $\lambda > \mu$ we obtain for any $w \in D(L)$ the estimate

$$\langle Lw + Aw + \lambda w, w \rangle \geq \frac{1}{2}\|w(\cdot, T)\|^2_{L^2(\Omega)} + \mu\|w\|^2_V. \tag{3.18}$$

Applying Young's inequality to the right-hand side of (3.17) we get from (3.17) and (3.18) for any $\varepsilon > 0$ the estimate

$$\langle Lw + Aw + \lambda w + \xi + \eta, w \rangle \geq (\mu - \varepsilon)\|w\|^2_V - C(\varepsilon)(\|\xi_0\|^2_{V^*} + \|\eta_0\|^2_{V^*}), \tag{3.19}$$

where $C(\varepsilon)$ denotes some constant only depending on ε. Finally, by selecting $\varepsilon < \mu$ the right-hand side of (3.19) becomes positive for all w with $\|w\|_V > r$ if r is large enough, and hence the coercivity follows which completes the existence proof. The uniqueness of the solution is an immediate consequence of the the strong monotonicity of $A + \lambda I$ along with the maximal monotonicity of L and $\partial\Psi$. Let u be the corresponding unique solution of the original IBVP (3.5), (3.6) (resp. of 3.7) given via the exponential shift (3.11). To complete the proof of the lemma we show that u is enclosed by the super- and subsolutions $\bar{u}$ and $\underline{u}$, respectively.

(c) $\underline{u} \leq u \leq \bar{u}$

We only show that $u \leq \bar{u}$, since the proof of $\underline{u} \leq u$ follows by similar arguments. By definition we have $\bar{u}(x, 0) \geq 0$ and

$$\langle \partial\bar{u}/\partial t + A\bar{u} + F(\bar{u}) + G \circ \gamma(\bar{u}), \varphi \rangle \geq 0, \quad \forall\, \varphi \in V \cap L^2_+(Q), \tag{3.20}$$

and the unique solution u satisfies $u \in D(L)$ and

$$\langle \partial u/\partial t + Au + \xi + \eta, \varphi \rangle = 0, \quad \forall\, \varphi \in V, \tag{3.21}$$

where $\xi(x, t) \in \alpha(x, t, u(x, t))$ a.e. in Q and $\eta(x, t) \in \beta(x, t, \gamma u(x, t))$ a.e. in Γ. Using the special test function $\varphi = (u - \bar{u})^+$ we obtain by subtracting (3.20) from (3.21) the inequality

$$\langle \partial(u - \bar{u})/\partial t + A(u - \bar{u}) + \xi - F(\bar{u}) + \eta - G \circ \gamma(\bar{u}), (u - \bar{u})^+ \rangle \leq 0. \tag{3.22}$$

If $\bar{\xi}(x, t) = f(x, t, \bar{u}(x, t))$ then $\bar{\xi}(x, t) \in \alpha(x, t, \bar{u}(x, t))$, and thus we have

$$\begin{aligned} &\langle \xi - F(\bar{u}), (u - \bar{u})^+ \rangle = \int_Q (\xi - \bar{\xi})(u - \bar{u})^+ \, dxdt \\ &= \int_{\{u > \bar{u}\}} (\xi - \bar{\xi})(u - \bar{u}) \, dxdt \geq 0, \end{aligned} \tag{3.23}$$

and similarly we get

$$\langle \eta - G \circ \gamma(\bar{u}), (u-\bar{u})^+ \rangle \ge 0. \tag{3.24}$$

Thus from (3.22), (3.23) and (3.24) we obtain the estimate

$$\frac{1}{2}\|(u-\bar{u})^+(\cdot,T)\|^2_{L^2(\Omega)} + \mu\,\|\nabla(u-\bar{u})^+\|^2_{L^2(Q)} \le 0,$$

which implies $(u-\bar{u})^+ = 0$, i.e., $u \le \bar{u}$. This completes the proof of the lemma. □

4 Main Result

The proof of our main result is based on Lemma 3.1 and the following fixed point theorem in ordered normed spaces, see [3, Proposition 1.1.1].

Lemma 4.1. *Let $[\underline{u},\bar{u}]$ be a nonempty order interval in an ordered normed space $(N,\le)$, and let $\mathcal{P} : [\underline{u},\bar{u}] \to [\underline{u},\bar{u}]$ be an increasing mapping, i.e., $v \le w$ implies $\mathcal{P}v \le \mathcal{P}w$. If monotone sequences of $\mathcal{P}([\underline{u},\bar{u}])$ converge weakly or strongly in N, then $\mathcal{P}$ has the least fixed point u_* and the greatest fixed point u^* in $[\underline{u},\bar{u}]$.*

Our main result is given by the next theorem.

Theorem 4.1. *Let $\mathcal{R} = [\underline{u},\bar{u}]$ be a trapping region in the sense of Definition 2.2. Then the multivalued system (2.7), (2.8) and its equivalent system of evolution variational inequalities (2.9), (2.10) possesses solutions within $\mathcal{R}$.*

Proof. For convenience we recall the system (2.9), (2.10) of evolution variational inequalities: Find $u_k \in D(L)$ such that for all $\varphi \in V$ we have

$$\begin{aligned} \langle Lu_1 + A_1u_1, \varphi - u_1\rangle + J_1(\varphi,u_2) - J_1(u_1,u_2) & \\ +\Phi_1\circ\gamma(\varphi,u_2) - \Phi_1\circ\gamma(u_1,u_2) &\ge 0, \qquad (4.1)\\ \langle Lu_2 + A_2u_2, \varphi - u_2\rangle + J_2(u_1,\varphi) - J_2(u_1,u_2) & \\ +\Phi_2\circ\gamma(u_1,\varphi) - \Phi_2\circ\gamma(u_1,u_2) &\ge 0, \qquad (4.2) \end{aligned}$$

Let $\underline{u} = [\underline{u}_1,\underline{u}_2]$ and $\bar{u} = [\bar{u}_1,\bar{u}_2]$. Then we define first a mapping $\mathcal{T}$ as follows: $[\underline{u}_1,\bar{u}_1] \ni v_1 \mapsto \mathcal{T}v_1 = z$, where z is a solution of the evolution variational inequality (4.2) with $u_1 := v_1$ fixed. Applying the property of the trapping region one readily observes that for any $v_1 \in [\underline{u}_1,\bar{u}_1]$ the functions $\bar{u}_2$ and $\underline{u}_2$ are super- and subsolutions for the corresponding equation related to (4.2), und thus we can apply Lemma 3.1 to ensure the existence of a unique solution $z = \mathcal{T}v_1$ of (4.2) satisfying $z \in [\underline{u}_2,\bar{u}_2]$. Moreover, by means of the monotonicity assumptions on f and g we will show that $\mathcal{T} : [\underline{u}_1,\bar{u}_1] \subset V \to [\underline{u}_2,\bar{u}_2]$ is decreasing. To this end let $v_1,\ \hat{v}_1 \in [\underline{u}_1, \bar{u}_1]$ with $v_1 \le \hat{v}_1$ be given, and denote $\mathcal{T}v_1 = z$ and $\mathcal{T}\hat{v}_1 = \hat{z}$. Since z and $\hat{z}$ are the unique solutions of 4.2 with $u_1 = v_1$ and $u_1 = \hat{v}_1$, respectively, we get by adding the corresponding inequalities and using the test function $\varphi = z + (\hat{z} - z)^+$ for the inequality in z and $\varphi = \hat{z} - (\hat{z} - z)^+$ for the inequality in $\hat{z}$ the following:

$$
\begin{aligned}
&\Big[J_2(v_1, z + (\hat{z} - z)^+) - J_2(v_1, z) + J_2(\hat{v}_1, \hat{z} - (\hat{z} - z)^+) - J_2(\hat{v}_1, \hat{z})\Big]\\
&+\Big[\Phi_2 \circ \gamma(v_1, z + (\hat{z} - z)^+) - \Phi_2 \circ \gamma(v_1, z)\\
&+\Phi_2 \circ \gamma(\hat{v}_1, \hat{z} - (\hat{z} - z)^+) - \Phi_2 \circ \gamma(\hat{v}_1, \hat{z})\Big]\\
&\ge \langle L(\hat{z} - z) + A_2(\hat{z} - z), (\hat{z} - z)^+\rangle. \qquad (4.3)
\end{aligned}
$$

Using the definition of the functionals and applying the monotonicity of the vector fields f and g we are going to show that the brackets on the left-hand side of (4.3) are nonpositive. Let us consider the first bracket on the left-hand side of (4.3), and discuss its value in the subsets of Ω according to the partition $\Omega = \{\hat{z} < z\} \cup \{\hat{z} \ge z\}$. For the set $\{\hat{z} < z\}$ we have $(\hat{z} - z)^+ = 0$ and thus the bracket is zero. For $\{\hat{z} \ge z\}$ we get $(\hat{z} - z)^+ = \hat{z} - z$ and thus this bracket becomes

$$
\begin{aligned}
& J_2(v_1, \hat{z}) - J_2(v_1, z) + J_2(\hat{v}_1, z) - J_2(\hat{v}_1, \hat{z})\\
= & \int_{\{\hat{z} \ge z\}} \Big(\int_z^{\hat{z}} f_2(v_1, s)\, ds\Big) dxdt + \int_{\{\hat{z} \ge z\}} \Big(\int_{\hat{z}}^{z} f_2(\hat{v}_1, s)\, ds\Big) dxdt\\
= & \int_{\{\hat{z} \ge z\}} \Big(\int_z^{\hat{z}} (f_2(v_1, s) - f_2(\hat{v}_1, s))\, ds\Big) dxdt \le 0,
\end{aligned}
$$

because $v_1 \le \hat{v}_1$ and f_2 is increasing in its first argument. In a similar way one shows that the second bracket on the left-hand side of (4.3) is nonpositive. Thus from (4.3) we obtain

$$
\langle L(\hat{z} - z) + A_2(\hat{z} - z), (\hat{z} - z)^+\rangle \le 0,
$$

which yields

$$
\|(\hat{z} - z)^+(\cdot, T)\|^2_{L^2(\Omega)} + \mu_2 \|\nabla(\hat{z} - z)^+\|^2_{L^2(Q)} \le 0,
$$

and thus $(\hat{z} - z)^+ = 0$, i.e., $\hat{z} \le z$, which shows that $\mathcal{T} : [\underline{u}_1, \bar{u}_1] \to [\underline{u}_2, \bar{u}_2]$ is decreasing. By means of the evolution variational inequality (4.1) we define a mapping $\mathcal{S}$ on the range of $\mathcal{T}$ in the following way: Let $z = \mathcal{T}v_1$, then $z \mapsto \mathcal{S}z := u_1$, where due to Lemma 3.1 u_1 is the uniquely defined solution of the evolution variational inequality (4.1) with $u_2 = z$ fixed, which in view of the property of the trapping region satisfies $u_1 \in [\underline{u}_1, \bar{u}_1]$. Moreover, in an analogous way as for $\mathcal{T}$ one can show that $\mathcal{S} : \mathcal{T}([\underline{u}_1, \bar{u}_1]) \to [\underline{u}_1, \bar{u}_1]$ is also decreasing. Hence, it follows that the composed operator $\mathcal{P} = \mathcal{S} \circ \mathcal{T} : [\underline{u}_1, \bar{u}_1] \to [\underline{u}_1, \bar{u}_1]$ is an increasing operator from the interval $[\underline{u}_1, \bar{u}_1] \subset V$ to itself. To apply the abstract fixed point result given in Lemma 4.1 we need to show that any monotone sequence of the image $\mathcal{P}([\underline{u}_1, \bar{u}_1])$ converges weakly or strongly in V. So let $(u_{1,n}) \subset \mathcal{P}([\underline{u}_1, \bar{u}_1])$ be a monotone sequence. Because $u_{1,n} \in [\underline{u}_1, \bar{u}_1]$, the $(u_{1,n})$ and the sequence $(\gamma u_{1,n})$ are bounded in $L^2(Q)$ and $L^2(\Gamma)$, respectively, and satisfy the evolution variational inequality

$$\begin{aligned}\langle Lu_{1,n} + A_1 u_{1,n}, \varphi - u_{1,n}\rangle + J_1(\varphi, z_n) - J_1(u_{1,n}, z_n) \\ +\Phi_1 \circ \gamma(\varphi, z_n) - \Phi_1 \circ \gamma(u_{1,n}, z_n) \geq 0,\end{aligned} \tag{4.4}$$

for some $z_n \in [\underline{u}_2, \bar{u}_2]$. Since (z_n) and (γz_n) are also bounded in $L^2(Q)$ and $L^2(\Gamma)$, respectively, we obtain from (4.4) with $\varphi = 0$ and by applying the growth condition (H1) the following estimate

$$\|u_{1,n}(\cdot, T)\|^2_{L^2(\Omega)} + \mu_1 \|\nabla u_{1,n}\|^2_{L^2(Q)} \leq \langle Lu_{1,n} + A_1 u_{1,n}, u_{1,n}\rangle \leq C, \tag{4.5}$$

which due to the boundedness of $(u_{1,n})$ in $L^2(Q)$ yields

$$\|u_{1,n}\|_V \leq C. \tag{4.6}$$

Since $(u_{1,n}) \subset [\underline{u}_1, \bar{u}_1]$ is monotone, it must be convergent in $L^2(Q)$ by Lebesgue's dominated convergence theorem. In view of (4.6) there exists a subsequence of $(u_{1,n})$ which is weakly convergent in V, and because all weakly convergent subsequences have the same limit, the entire sequence must be weakly convergent. Thus Lemma 4.1 can be applied which ensures the existence of extremal fixed points of $\mathcal{P}$ in $[\underline{u}_1, \bar{u}_1]$. Finally, let u_1 be any fixed point of $\mathcal{P}$, i.e., $u_1 = \mathcal{P}u_1 = \mathcal{S}(\mathcal{T}u_1)$. Then if $u_2 := \mathcal{T}u_1$, it follows that $u = (u_1, u_2)$ is a solution of the system (4.1), (4.2), which completes the proof. □

Corollary 4.1. *Let u_1^* and $u_{1,*}$ be the greatest and least fixed point of $\mathcal{P}$, respectively, and let $u_2^* = \mathcal{T}u_1^*$ and $u_{2,*} = \mathcal{T}u_{1,*}$. Then any solution (u_1, u_2) of (2.7), (2.8) (resp. (2.9), (2.10)) within the trapping region $\mathcal{R}$ satisfies: $u_{1,*} \leq u_1 \leq u_1^*$, $u_2^* \leq u_2 \leq u_{2,*}$.*

Proof. Obviously any solution (u_1, u_2) of the system of evolution variational inequality (2.9), (2.10) within $\mathcal{R}$ satisfies $u_1 = \mathcal{P}u_1 = \mathcal{S}(\mathcal{T}u_1) = \mathcal{S}u_2$, where $u_2 = \mathcal{T}u_1$. Since u_1^* and $u_{1,*}$ are the extremal fixed points of $\mathcal{P}$, we have $u_{1,*} \leq u_1 \leq u_1^*$, and because $\mathcal{T}$ is decreasing it follows $\mathcal{T}u_{1,*} \geq \mathcal{T}u_1 \geq \mathcal{T}u_1^*$, i.e., $u_{2,*} \geq u_2 \geq u_2^*$. □

5 Application

In this application, we illustrate the extension of the preceding theory to the case where first order convection terms are present in the system and where inhomogeneous initial conditions are selected. Thus, we consider a non-isothermal model of fluid contaminant transfer, with passive advection induced by a divergence free velocity field, associated with an incompressible fluid. The template is a river, flowing with velocity $\vec{v}$, into which a heated contaminant is released by one or more service facilities, situated along the river embankments. The facilities must discharge material when temperature and concentration exceed threshold values. This constitutes a set of discontinuous flux boundary conditions. Environmental probes are positioned in such a way that control sinks can be activated for certain temperature and concentration ranges, so as to reduce both the fluid temperature Θ and the contaminant concentration ρ. These control activations depend discontinuously upon Θ and ρ, and act as competitive species vector fields. Since $\vec{v}$ is divergence free, we have the equations (see [1, Sections 1.4–1.5]):

$$
\begin{aligned}
\frac{\partial \Theta}{\partial t} - \nabla\cdot [\kappa \nabla \Theta] + \vec{v}\cdot \nabla \Theta + f(\Theta, \rho) = 0, \\
\frac{\partial \rho}{\partial t} - \nabla\cdot [K \nabla \rho] + \vec{v}\cdot \nabla \rho + g(\Theta, \rho) = 0.
\end{aligned}
\tag{5.1}
$$

These equations arise from conservation principles. The second equation is an expression of conservation of mass, with the use of Fick's law for the concentration flux. The first equation is derived from a conservation of energy principle with certain simplifying assumptions, including constant values for the heat capacity and density of the fluid. In particular, we are assuming that variations in ρ do not essentially affect the fluid density. Fourier's law for the heat flux is also employed. In these equations, κ and K are the corresponding thermal and contaminant species diffusivities, assumed constant. The control source terms are designated by f, g. The boundary conditions are specified as follows.

$$
-\frac{\partial \Theta}{\partial \nu_\Theta} = a\, \chi_{[\Theta_0,\infty)}\, \rho\Theta, \quad -\frac{\partial \rho}{\partial \nu_\rho} = b\, \chi_{[\rho_0,\infty)}\, \rho. \tag{5.2}
$$

Since the inward heat flux beyond the threshold Θ_0 varies according to the product of concentration and temperature, and the concentration flux beyond the threshold ρ_0 varies according to the concentration, these boundary conditions are monotone in both arguments. Here, a and b are smooth nonnegative functions of position on the boundary, are nonzero at the locations of the facilities, and smoothly decrease to zero away from facility locations. Note that the boundary condition for the heat flux is simply the statement that the flux is proportional to the internal energy of the contaminant. The control mechanism is not activated until Θ, ρ reach threshold values, Θ_1, ρ_1, resp., and are defined by

$$
f(\Theta, \rho) = d\, \chi_{[\Theta_1,\infty)}\, \rho\Theta, \quad g(\Theta, \rho) = m\, \chi_{[\rho_1,\infty)}\, \rho.
$$

Here, d and m are smooth nonnegative functions of position in Ω, which are nonzero at the probe locations, and smoothly decrease to zero away from these locations. With the initial conditions for Θ and ρ arbitrary nonnegative functions $\hat{\Theta}$ and $\hat{\rho}$ which are mathematically and physically consistent, we see that this example fits within an extension of the theory which allows for the inclusion of convective terms and inhomogeneous initial conditions. Assuming that $\hat{\Theta},\ \hat{\rho} \in L^\infty(\Omega)$, one easily verifies that the constant vectors $\underline{u} = (0,0)$ and $\bar{u} = (\sup\hat{\Theta}, \sup\hat{\rho})$ form a trapping region of the IBVP for the coupled system (5.1) with the boundary condition (5.2).

References

[1] Allen, M., Herrera, I. and Pinder, G. *Numerical Modeling in Science and Engineering.* Wiley Interscience, New York, 1988.

[2] S. Carl and S. Heikkilä, Discontinuous reaction-diffusion equations under discontinuous and nonlocal flux conditions, Mathematical and Computer Modelling 32 (2000), 1333–1344.

[3] S. Carl and S.Heikkilä, *Nonlinear Differential Equations in Ordered Spaces*, Chapman & Hall/CRC, London, 2000.

[4] S. Carl and J.W. Jerome, Trapping region for discontinuous quasilinear elliptic systems of mixed monotone type, Nonlinear Analysis (to appear).

[5] K.C. Chang, Variational methods for non-differentiable functionals and their applications to partial differential equations, J. Math. Anal. Appl. 80 (1981), 102–129.

[6] J.W. Jerome, A trapping principle for discontinuous elliptic systems of mixed monotone type, J. Math. Anal. Appl. 262 (2001), 700–721.

[7] R.E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Vol. 49, AMS, Providence, 1997.

[8] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Vol. II A/B: Monotone Operators*, Springer-Verlag, New York, 1990.

[9] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Vol. III: Variational Methods*, Springer-Verlag, New York, 1985.