Compressible Euler-Maxwell Equations

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Abstract

The Euler-Maxwell equations as a hydrodynamic model of charge transport of semiconductors in an electromagnetic field are studied. The global approximate solutions to the initial-boundary value problem are constructed by the fractional Godunov scheme. The uniform bound and H^{-1} compactness are proved. The approximate solutions are shown convergent by weak convergence methods. Then, with some new estimates due to the presence of electromagnetic fields, the existence of a global weak solution to the initial-boundary value problem is established for arbitrarily large initial data in L^∞.

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1 Introduction

In this paper, we consider the hydrodynamic model of charge transport of semiconductors in an electromagnetic field. The analysis of physical models of charge transport in semiconductor devices is important in order to predict and understand device behavior. The hydrodynamic model treats the propagation of electrons in a semiconductor device as the flow of a compressible charged fluid. One of the important models is the Euler-Poisson system for a charged fluid in an electric field, which consists of the Euler

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equations for the conservation of density, momentum and energy, coupled to Poisson's equation for the electrostatic potential. See [3, 17, 11, 12, 16, 6, 8] and the references therein for discussion and analysis of the Euler-Poisson equations. When semiconductor devices are operated under high frequency conditions (in such technologies as photoconductive switches, electro-optics, semiconductor lasers, high-speed computers), magnetic fields are generated by moving charges inside the device, and the charge transport interacts with the propagating electromagnetic waves. In this case, the electromagnetic field obeys Maxwell's equations. Therefore, the hydrodynamic model for high-frequency charge transport in semiconductors consists of the Euler equations for conservation laws, coupled to Maxwell's equations for the electric and magnetic fields, instead of Poisson's equation for the electric field only. This Euler-Maxwell system in the isentropic case assumes the following form ([1, 2, 16]):

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = q\rho(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \frac{\rho \mathbf{u}}{\tau}, \\ \mu \mathbf{H}_t + \nabla \times \mathbf{E} = 0, \\ \varepsilon \mathbf{E}_t - \nabla \times \mathbf{H} + \mathbf{J} = 0, \\ -\varepsilon \nabla \cdot \mathbf{E} = q\rho - D(\vec{x}), \quad \nabla \cdot \mathbf{H} = 0, \\ \mathbf{B} = \mu \mathbf{H}, \quad \mathbf{J} = -q\rho \mathbf{u}, \quad \vec{x} \in \mathbf{R}^3, \quad t > 0, \end{cases} \tag{1.1}$$

where ρ is the electron density, $\mathbf{u} \in \mathbf{R}^3$ is the electron velocity, $p = p(\rho) = \rho^\gamma/\gamma$ is the pressure of the flow, $\gamma > 1$ is the adiabatic exponent, $\mathbf{E} \in \mathbf{R}^3$ is the electric field, $\mathbf{H} \in \mathbf{R}^3$ is the magnetic field, $\mathbf{J} \in \mathbf{R}^3$ is the current density, $\mathbf{B} \in \mathbf{R}^3$ is the magnetic induction, $\mathbf{E} + \mathbf{u} \times \mathbf{B}$ is the Lorentz force, $D \in L^\infty$ is the doping profile, τ is the momentum relaxation time, q is the electronic charge, μ is the permeability of the medium and ε is the permittivity of the medium. There is no electrostatic potential for the electric field in this model.

There have been studies of the Euler-Poisson equations by many authors, including ourselves (see [8, 6, 12] and the references therein). The Euler-Maxwell equations are much more complicated than the Euler-Poisson equations, not only because of Maxwell's equations, but also because of the complicated coupling of the Lorentz force. There have been some numerical simulations ([1, 2]) but no mathematical studies of this model. In this paper, we study the global solution to the Euler-Maxwell flow with special structures:

the flow depends only on one space variable $x \in \mathbf{R}$, but the velocity, the electric field and the magnetic filed still have three components. That is, this is a three-dimensional flow which does not change in the transverse directions. For general large initial data, the hyperbolic mode of this system supports the formation of shocks in the solutions. Therefore, we should seek global weak solutions, including shock waves with large initial data. The discontinuities in the solutions cause considerable difficulty in the study of this flow. To solve this problem, we employ a finite difference method. The basic strategy is to construct approximate solutions by a certain finite difference scheme, and then to show that the approximate solutions converge to the solution of the system in a certain compactness framework.

To construct the approximate solutions, we will use Godunov's method with the fractional step procedure. The Riemann solutions will be used as the natural fundamental building blocks. The convergence of the approximate solutions will be studied by the weak convergence method and by the compensated compactness framework developed for isentropic gas dynamics by DiPerna [10], Chen [4], Lions-Perthame-Souganidis [13] and Lions-Perthame-Tadmor [14]. (See [7] for Chen-LeFloch's recent work on the isentropic Euler equations with general pressure law.) For this purpose, we first have to make a uniform estimate on the approximate solutions. The high nonlinearity in the Lorentz force causes extreme difficulty in solving this system, and the Lorentz force is usually removed from the model in current literature. We resolve this problem with this term linearized. The approximate solutions will be shown to be uniformly bounded, and therefore are convergent in the weak-star topology of L^∞. To show that the limit of the approximate solutions is a weak solution, we need the strong convergence of the approximate density ρ^h and velocity u^h. To this end, we need to show the H^{-1} compactness of the entropy dissipation measures of the approximate solutions. This can be achieved by certain energy-type estimates and embedding theorems. For $1 < \gamma < 2$, the H^{-1} compactness can be proved by using the strict convexity of the mechanical energy. For $\gamma > 2$, however, the mechanical energy is no longer strictly convex, and then the H^{-1} compactness can be obtained by an estimate similar to the proof in [9]. Due to the presence of electromagnetic fields, some new estimates on the approximate solutions are also needed to show the consistency.

In Section 2, we will formulate our problem in one space variable and state the initial-boundary value problem. Section 3 is for the review of the Riemann solutions, and some important facts are given in this section. In Section 4, we will construct the global approximate solutions to the initial-boundary value problem by the Godunov scheme, with the fractional step procedure. Then we will prove, in Section 5, the L^∞ stability to obtain the uniform estimate on the approximate solutions. In Section 6, we will show the H^{-1} compactness of the entropy dissipation measures and some energy-type estimates will also be made. Finally, in Section 7, we will use the weak convergence method and the compensated compactness framework to show weak or strong convergence of the approximate solutions and the property that the limit function is the weak solution of the initial-boundary value problem.

2 Reformulation of the Problem

The Euler-Maxwell flow (1.1) in one space variable $x \in \mathbf{R}$:

$$\begin{aligned}
&\rho = \rho(x,t),\\
&\mathbf{u} = (u_1, u_2, u_3)(x,t) = (u, \mathbf{w})(x,t), \quad u = u_1,\ \mathbf{w} = (u_2, u_3),\\
&\mathbf{E} = (E_1, E_2, E_3)(x,t) = (E, \mathbf{e})(x,t), \quad E = E_1,\ \mathbf{e} = (E_2, E_3),\\
&\mathbf{H} = (H_1, H_2, H_3)(x,t) = (H, \mathbf{h})(x,t), \quad H = H_1,\ \mathbf{h} = (H_2, H_3),
\end{aligned} \tag{2.1}$$

with $\mathbf{w}$, $\mathbf{e}$, $\mathbf{h} \in \mathbf{R}^2$ the transverse velocity, transverse electric field, and transverse magnetic field, respectively (the velocity, the electric field and the magnetic filed still have three components), satisfies the following equations:

$$\begin{cases}
\rho_t + (\rho u)_x = 0,\\
(\rho u)_t + (\rho u^2 + p)_x = \rho(E + f_1(\mathbf{w}, \mathbf{h})) - \frac{\rho u}{\tau},\\
(\rho \mathbf{w})_t + (\rho u \mathbf{w})_x = \rho(\mathbf{e} + A\mathbf{w} + f_2(u, \mathbf{h})) - \frac{\rho \mathbf{w}}{\tau},\\
\mathbf{e}_t + A\mathbf{h}_x = f_3(\rho, \mathbf{w}),\\
\mathbf{h}_t - A\mathbf{e}_x = 0,\\
-E_x = \rho - D(x),
\end{cases} \tag{2.2}$$

where $f_1 = \mathbf{w} \cdot A\mathbf{h}$, $f_2 = -uA\mathbf{h}$, $f_3 = \rho\mathbf{w}$, and A is the 2×2 constant matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

In (2.2), we take all the parameters to be 1, and the constant $H = 1$, without loss of generality.

E can be solved as

$$E = E_0(t) + \int_0^x (D(\xi) - \rho(\xi, t))d\xi, \tag{2.3}$$

where $E_0(t) \in L^\infty$ is the boundary value:

$$E|_{x=0} = E_0(t). \tag{2.4}$$

Set

$$\mathbf{a} = \mathbf{e} + A\mathbf{h}, \quad \mathbf{b} = \mathbf{e} - A\mathbf{h};$$

then

$$\mathbf{e} = (\mathbf{a} + \mathbf{b})/2, \quad \mathbf{h} = A(\mathbf{b} - \mathbf{a})/2, \tag{2.5}$$

and $\mathbf{a}$ and $\mathbf{b}$ satisfy the following equations:

$$\mathbf{a}_t + \mathbf{a}_x = f_3(\rho, \mathbf{w}), \quad \mathbf{b}_t - \mathbf{b}_x = f_3(\rho, \mathbf{w}). \tag{2.6}$$

We consider the following initial-boundary value problem:

$$\begin{cases} \rho_t + m_x = 0, \qquad 0 < x < 1, \quad t > 0, \\ m_t + \left(\frac{m^2}{\rho} + p\right)_x = \rho(E + f_1(\mathbf{w}, \mathbf{h})) - \frac{m}{\tau}, \\ \mathbf{q}_t + \left(\frac{m\mathbf{q}}{\rho}\right)_x = \rho(\mathbf{e} + A\mathbf{w} + f_2(u, \mathbf{h})) - \frac{\mathbf{q}}{\tau}, \\ (2.6), \end{cases} \tag{2.7}$$

with the initial-boundary conditions

$$\begin{cases} (\rho, m, \mathbf{q}, \mathbf{a}, \mathbf{b})(x, 0) = (\rho_0(x), m_0(x), \mathbf{q}_0(x), \mathbf{a}_0(x), \mathbf{b}_0(x)), \\ m|_{x=0} = m|_{x=1} = 0, \\ \mathbf{a}|_{x=0} = \bar{\mathbf{a}}(t), \quad \mathbf{b}|_{x=1} = \bar{\mathbf{b}}(t), \end{cases} \tag{2.8}$$

where

$$m = \rho u, \quad \mathbf{q} = \rho \mathbf{w},$$

and the initial-boundary values satisfy the following condition:

$$\begin{aligned} &0 \le \rho_0(x) \le C, \quad |u_0(x)| = \left|\frac{m_0(x)}{\rho_0(x)}\right| \le C, \\ &|\mathbf{w}_0(x)| = \left|\frac{\mathbf{q}_0(x)}{\rho_0(x)}\right| \le C, \quad |\mathbf{a}_0(x)| \le C, \quad |\mathbf{b}_0(x)| \le C, \\ &|\bar{\mathbf{a}}(t)| \le C, \quad |\bar{\mathbf{b}}(t)| \le C, \end{aligned} \tag{2.9}$$

for some positive constant C. For the general large initial data in L^∞, the solutions of (2.7) will develop singularities or shocks in finite time. Therefore, there are only global weak solutions, including shock waves, for general large initial data. We will construct the global solution of (2.7)-(2.8) with linearized f_1, f_2, f_3, and show that if the initial-boundary conditions are bounded in L^∞, the global solution

$$V(x,t) = (\rho, m, \mathbf{q}, \mathbf{a}, \mathbf{b})(x,t)$$

to (2.7)-(2.8) will be bounded globally. The global approximate solutions

$$V^h = (\rho^h, m^h, \mathbf{q}^h, \mathbf{a}^h, \mathbf{b}^h)(x,t)$$

will be constructed by the Godunov method with the fractional step procedure, and then shown to be convergent to the global weak solution.

3 Riemann Solutions

We first recall some basic properties about the Riemann solutions to the homogeneous equations ([4, 5]):

$$\rho_t + m_x = 0, \qquad 0 < x < 1, \quad t > 0, \tag{3.1}$$

$$m_t + \left(\frac{m^2}{\rho} + p\right)_x = 0, \tag{3.2}$$

$$\mathbf{q}_t + \left(\frac{m\mathbf{q}}{\rho}\right)_x = 0, \tag{3.3}$$

or in compact form:

$$\tilde{v}_t + f(\tilde{v})_x = 0, \tag{3.4}$$

where

$$\tilde{v} = (\rho, m, \mathbf{q})^\top, \qquad f(\tilde{v}) = \left(m, \frac{m^2}{\rho} + p(\rho), \frac{m\mathbf{q}}{\rho}\right)^\top,$$

with $p(\rho) = \rho^\gamma/\gamma$, $\gamma > 1$. The eigenvalues of (3.4) are

$$\lambda_1 = \frac{m}{\rho} - \rho^\theta, \quad \lambda_2 = \frac{m}{\rho} + \rho^\theta, \quad \lambda_3 = \lambda_4 = \frac{m}{\rho},$$

where $\theta = (\gamma-1)/2$. The first two characteristic fields of (3.4) are genuinely nonlinear. The third and the forth characteristic field are linearly degenerate. The Riemann invariants are

$$w = \frac{m}{\rho} + \frac{\rho^\theta}{\theta}, \quad z = \frac{m}{\rho} - \frac{\rho^\theta}{\theta}, \quad \mathbf{w} = \frac{\mathbf{q}}{\rho}.$$

The discontinuity in the weak solutions of (3.4) satisfies the Rankine-Hugoniot condition:

$$\sigma(\tilde{v} - \tilde{v}_0) = f(\tilde{v}) - f(\tilde{v}_0),$$

where σ is the propagation speed of the discontinuity, and $\tilde{v}_0$, $\tilde{v}$ are the corresponding left state, right state, respectively. A discontinuity is a shock if it satisfies the entropy condition:

$$\sigma[\tilde{\eta}] - [\tilde{q}] \geq 0,$$

with

$$[\tilde{\eta}] = \tilde{\eta}(v) - \tilde{\eta}(v_0), \quad [\tilde{q}] = \tilde{q}(v) - \tilde{q}(v_0),$$

for any weak entropy pair $(\tilde{\eta}, \tilde{q})$.

Consider the Riemann problem of (3.4) with initial data

$$\tilde{v}|_{t=0} = \begin{cases} \tilde{v}_-, & x < x_0, \\ \tilde{v}_+, & x > x_0, \end{cases} \tag{3.5}$$

where $x_0 \in (0,1)$, $\tilde{v}_\pm = (\rho_\pm, m_\pm, \mathbf{q}_\pm)^\top$, and $\rho_\pm \geq 0$, $m_\pm$, and $\mathbf{q}_\pm$ are constants satisfying $|m_\pm/\rho_\pm| + |\mathbf{q}_\pm/\rho_\pm| < \infty$; and the Riemann initial-boundary problem of (3.4) with data:

$$\tilde{v}|_{t=0} = \tilde{v}_+, \quad m|_{x=0} = 0. \tag{3.6}$$

We have the following lemmas on the Riemann solutions.

Lemma 3.1. *There exists a piecewise smooth entropy solution $\tilde{v}(x,t)$ for each problem of (3.5) and (3.6) satisfying*

$$\begin{cases} w(\tilde{v}(x,t)) \leq \max(w(\tilde{v}_-), w(\tilde{v}_+)), \\ w(\tilde{v}(x,t)) - z(\tilde{v}(x,t)) \geq 0, \\ \min(q_{i-}, q_{i+}) \leq q_i(x,t) \leq \max((q_{i-}, q_{i+}), \quad i = 1, 2, \end{cases}$$

where q_i $(i = 1, 2)$ are the two components of $\mathbf{q} = (q_1, q_2)$. And

$$z(\tilde{v}(x,t)) \geq \min(z(\tilde{v}_-), z(\tilde{v}_+)), \quad \textit{for (3.5)},$$
$$z(\tilde{v}(x,t)) \geq \min(z(\tilde{v}_+), 0), \quad \textit{for (3.6)}.$$

Lemma 3.2. *For the Riemann problem (3.5), the region*

$$\sum = \{(\rho, m, \mathbf{q}) : w \le w_0, z \ge z_0, w - z \ge 0, q_{i-} \le q_i \le q_{i+}, i = 1, 2\}$$

is invariant. For the Riemann initial-boundary problem (3.6), the region

$$\begin{aligned}\sum = \{(\rho, m, \mathbf{q}) :& w \le w_0, z \ge z_0, w - z \ge 0, q_{i-} \le q_i \le q_{i+}, i = 1, 2\},\\ & z_0 \le 0 \le \frac{w_0 + z_0}{2}\end{aligned}$$

is invariant. That is, if the Riemann data lie in $\sum$, then the Riemann solutions $\tilde{v}(x,t) \in \sum$ and $\frac{1}{b-a}\int_a^b \tilde{v}(x,t)dx \in \sum$.

For the Riemann initial-boundary problem of (3.4) with data: $\tilde{v}|_{t=0} = v_-$, $m|_{x=1} = 0$, we have similar results to those for (3.6) in the above two lemmas. The Riemann problems associated with the linear conservation laws

$$\mathbf{a}_t + \mathbf{a}_x = 0, \qquad \mathbf{b}_t - \mathbf{b}_x = 0, \tag{3.7}$$

with boundary conditions in (2.8) are very basic, and we omit the discussion.

Lemma 3.3. *Assume $0 \le \rho \le M$, $|m/\rho| + |\mathbf{q}/\rho| \le M$, for some positive constant M. Then, for any weak entropy pair $(\tilde{\eta}, \tilde{q})$ of (3.4), there exists a positive constant C, depending only on $\tilde{\eta}$ and M, such that*

$$|\nabla\tilde{\eta}(\rho, m, \mathbf{q})| \le C, \quad |\nabla\tilde{q}(\rho, m, \mathbf{q})| \le C;$$

for any four-dimensional vector $\tilde{v}$ and any weak entropy $\tilde{\eta}$ of (3.4),

$$|\tilde{v}^\top \nabla^2 \tilde{\eta} \tilde{v}| \le C \tilde{v}^\top \nabla^2 \tilde{\eta}_* \tilde{v};$$

and the rate of entropy production of a shock for any weak entropy pair $(\tilde{\eta}, \tilde{q})$ of (3.4) is dominated by the associated rate of entropy production for $(\tilde{\eta}_*, \tilde{q}_*)$ in the sense:

$$|\sigma[\tilde{\eta}] - [\tilde{q}]| \le C\left(\sigma[\tilde{\eta}_*] - [\tilde{q}_*]\right),$$

where

$$\tilde{\eta}_* = \frac{m^2 + |\mathbf{q}|^2}{2\rho} + \frac{\rho^\gamma}{\gamma(\gamma-1)}, \quad \tilde{q}_* = m\left(\frac{m^2 + |\mathbf{q}|^2}{2\rho^2} + \frac{\rho^{\gamma-1}}{\gamma-1}\right),$$

is the mechanical energy-energy flux pair of (3.4).

Lemma 3.3 is also true for the system (3.1)-(3.2) with the following mechanical energy-energy flux pair:

$$\eta_* = \frac{m^2}{2\rho} + \frac{\rho^\gamma}{\gamma(\gamma-1)}, \quad q_* = m\left(\frac{m^2}{2\rho^2} + \frac{\rho^{\gamma-1}}{\gamma-1}\right).$$

The mechanical energy is strictly convex if $1 < \gamma \leq 2$, but is not strictly convex if $\gamma > 2$.

4 Approximate Solutions

In this section, we construct the approximate solutions $V^h = (\rho^h, m^h, \mathbf{q}^h, \mathbf{a}^h, \mathbf{b}^h)(x,t)$ to (2.7)-(2.8) in the strip $\Pi_T = [0,1] \times [0,T]$ for any fixed $T > 0$. Here $h = 1/M > 0$, with M a large positive integer, is the space mesh length. The time mesh length is $\Delta t > 0$. These mesh lengths satisfy the Courant-Friedrichs-Levy condition

$$\max_{\kappa=1,2,3}\left(\sup_{0\leq t\leq T} |\lambda_\kappa(v^h)|, 1\right) \leq \frac{h}{4\Delta t}.$$

Assume that $V^h(x,t)$ is defined for $t < n\Delta t$. Then we define V_j^n as:

$$\begin{aligned} V_1^n &= \frac{2}{3h}\int_0^{\frac{3}{2}h} V^h(x, n\Delta t - 0)dx, \\ V_j^n &= \frac{1}{h}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} V^h(x, n\Delta t - 0)dx, \qquad 2 \leq j \leq M-2, \\ V_{M-1}^n &= \frac{2}{3h}\int_{1-\frac{3}{2}h}^{1} V^h(x, n\Delta t - 0)dx. \end{aligned} \tag{4.1}$$

In the strip $n\Delta t \leq t < (n+1)\Delta t$, we define $V_0^h(x,t)$ as the solution to the Riemann problem for the following homogeneous equations:

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \left(\frac{m^2}{\rho} + p\right)_x = 0, \\ \mathbf{q}_t + \left(\frac{m\mathbf{q}}{\rho}\right)_x = 0, \\ \mathbf{a}_t + \mathbf{a}_x = 0, \qquad \mathbf{b}_t - \mathbf{b}_x = 0, \end{cases} \tag{4.2}$$

with initial data

$$
\begin{aligned}
V|_{t=n\Delta t} = V_1^n,\ m|_{x=0} = 0,\ \mathbf{a}|_{x=0} = \bar{\mathbf{a}}(t), &\quad 0 \le x \le h, \\
V|_{t=n\Delta t} = \begin{cases} V_j^n, & x < (j+\frac{1}{2})h, \\ V_{j+1}^n, & x > (j+\frac{1}{2})h, \end{cases} &\quad jh \le x \le (j+1)h,\ 1 \le j \le M-2, \\
V|_{t=n\Delta t} = V_{M-1}^n,\ m|_{x=1} = 0,\ \mathbf{b}|_{x=1} = \bar{\mathbf{b}}(t), &\quad 1-h \le x \le 1.
\end{aligned}
$$

Then we define the approximate solution $V^h(x,t) = (\rho^h, m^h, \mathbf{q}^h, \mathbf{a}^h, \mathbf{b}^h)(x,t)$ of (2.7)-(2.8) in the strip $n\Delta t \le t < (n+1)\Delta t$ by the fractional step procedure:

$$
\begin{cases}
\rho^h(x,t) = \rho_0^h(x,t), \\
m^h(x,t) = m_0^h(x,t) + G_2(V_0^h(x,t))(t - n\Delta t), \\
\mathbf{q}^h(x,t) = \mathbf{q}_0^h(x,t) + G_3(V_0^h(x,t))(t - n\Delta t), \\
\mathbf{a}^h(x,t) = \mathbf{a}_0^h(x,t) + f_3(V_0^h(x,t))(t - n\Delta t), \\
\mathbf{b}^h(x,t) = \mathbf{b}_0^h(x,t) + f_3(V_0^h(x,t))(t - n\Delta t),
\end{cases}
\tag{4.3}
$$

where

$$
G_2(V) = \rho(E + f_1) - \frac{m}{\tau}, \quad G_3(V) = \rho(\mathbf{e} + A\mathbf{w} + f_2) - \frac{\mathbf{q}}{\tau}.
$$

We will show that the approximate solutions V^h are uniformly bounded for any small $h > 0$, and $v^h = (r^h, m^h)$ satisfy certain H^{-1} compactness criteria. Therefore the theory of weak convergence and compensated compactness can be applied to show that the approximate solutions converge to the weak solution.

5 L^∞ Stability

In this section, we will make the L^∞ estimates to get the uniform bounds of the approximate solutions $V^h(x,t)$. For simplicity of notation, we denote by $C > 0$ a universal constant depending only on T.

From the construction of the approximate solutions,

$$
\int_0^1 \rho^h(x,t)dx = \int_0^1 \rho_0^h(x,t)dx \le C. \tag{5.1}
$$

We now make estimates on the Riemann invariants

$$
w(\rho^h, m^h) = \frac{m^h}{\rho^h} + \frac{(\rho^h)^\theta}{\theta}, \quad z(\rho^h, m^h) = \frac{m^h}{\rho^h} - \frac{(\rho^h)^\theta}{\theta}, \quad \mathbf{w}^h = \frac{\mathbf{q}^h}{\rho^h},
$$

$\mathbf{a}^h$, and $\mathbf{b}^h$. Assume

$$f_1(\mathbf{w},\mathbf{h}) = \mathbf{w}\cdot A\bar{\mathbf{h}} + \bar{\mathbf{w}}\cdot A\mathbf{h} - \bar{\mathbf{w}}\cdot A\bar{\mathbf{h}}, \tag{5.2}$$

$$|f_2(u,\mathbf{h})| \le C_1|u| + C_2|\mathbf{h}|, \quad |f_3(\rho,\mathbf{w})| \le C_1\rho^\theta + C_2|\mathbf{w}|, \tag{5.3}$$

where $\bar{\mathbf{w}} = \int_0^1 \mathbf{w}_0(x)dx$, $\bar{\mathbf{h}} = \int_0^1 \mathbf{h}_0(x)dx$. For $n\Delta t \le t < (n+1)\Delta t$, we have the following estimates from (4.3), (2.5) and (5.1),

$$\begin{aligned} w(\rho^h,m^h)(x,t) &= \left(1-\frac{t-n\Delta t}{2\tau}\right)w(\rho_0^h,m_0^h)(x,t) - \frac{t-n\Delta t}{2\tau}z(\rho_0^h,m_0^h)(x,t)\\ &\quad + \left(E_0(t) + \int_0^x (D(\xi)-\rho_0^h(\xi,t))d\xi\right)(t-n\Delta t)\\ &\quad + \left(A\bar{\mathbf{h}}\cdot\mathbf{w}_0^h(x,t) + \bar{\mathbf{w}}\cdot A^2(\mathbf{b}_0^h(x,t)-a_0^h(x,t))/2 - \bar{\mathbf{w}}\cdot A\bar{\mathbf{h}}\right)(t-n\Delta t)\\ &\le \left(1-\frac{t-n\Delta t}{2\tau}\right)w(\rho_0^h,m_0^h)(x,t) - \frac{t-n\Delta t}{2\tau}z(\rho_0^h,m_0^h)(x,t)\\ &\quad + C\left(|\mathbf{w}_0^h(x,t)| + |a_0^h(x,t)| + |\mathbf{b}_0^h(x,t)| + 1\right)(t-n\Delta t), \end{aligned}$$

and similarly,

$$\begin{aligned} z(\rho^h,m^h)(x,t) &\ge \left(1-\frac{t-n\Delta t}{2\tau}\right)z(\rho_0^h,m_0^h)(x,t) - \frac{t-n\Delta t}{2\tau}w(\rho_0^h,m_0^h)(x,t)\\ &\quad - \left(C|\mathbf{w}_0^h(x,t)| + C|a_0^h(x,t)| + C|\mathbf{b}_0^h(x,t)| + C\right)(t-n\Delta t). \end{aligned}$$

By (4.3) and (2.5),

$$\begin{aligned} \mathbf{w}^h(x,t) &= \mathbf{w}_0^h(x,t) + \left(\frac{\mathbf{a}_0^h(x,t)+\mathbf{b}_0^h(x,t)}{2} + A\mathbf{w}_0^h(x,t)\right)(t-n\Delta t)\\ &\quad + \left(f_2\left(u_0^h(x,t), A(\mathbf{b}_0^h(x,t)-\mathbf{a}_0^h(x,t))/2\right) - \frac{\mathbf{w}_0^h(x,t)}{\tau}\right)(t-n\Delta t), \end{aligned}$$

then, from (5.3) and the observation,

$$|u_0^h| = |m_0^h/\rho_0^h| \le \max\left(w(\rho_0^h,m_0^h), -z(\rho_0^h,m_0^h)\right),$$

we have

$$\begin{aligned} |\mathbf{w}^h(x,t)| &\le \left(1+\frac{\tau+1}{\tau}\Delta t\right)|\mathbf{w}_0^h(x,t)| + C\left(|\mathbf{a}_0^h(x,t)| + |\mathbf{b}_0^h(x,t)|\right)(t-n\Delta t)\\ &\quad + \max\left(w(\rho_0^h,m_0^h)(x,t), -z(\rho_0^h,m_0^h)(x,t)\right)(t-n\Delta t). \end{aligned}$$

We also have from (4.3),

$$|\mathbf{a}^h(x,t)| \le |\mathbf{a}_0^h(x,t)| + C\left(w(\rho_0^h, m_0^h)(x,t) - z(\rho_0^h, m_0^h)(x,t) + |\mathbf{w}_0^h(x,t)|\right)(t - n\Delta t).$$
$$|\mathbf{b}^h(x,t)| \le |\mathbf{b}_0^h(x,t)| + C\left(w(\rho_0^h, m_0^h)(x,t) - z(\rho_0^h, m_0^h)(x,t) + |\mathbf{w}_0^h(x,t)|\right)(t - n\Delta t).$$

Set

$$\begin{aligned} M_n = \max\Big(& \sup_x w(\rho_0^h, m_0^h)(x, n\Delta t + 0), -\inf_x z(\rho_0^h, m_0^h)(x, n\Delta t + 0), \\ & \sup_x |\mathbf{w}_0^h(x, n\Delta t + 0)|, \sup_x |\mathbf{a}_0^h(x, n\Delta t + 0)|, \sup_x |\mathbf{b}_0^h(x, n\Delta t + 0)|\Big). \end{aligned}$$

Then, from Lemma 3.1, we have, for $n\Delta t \le t < (n+1)\Delta t$,

$$\begin{gathered} w(\rho^h, m^h)(x,t) \le M_n(1 + C\Delta t) + C\Delta t, \\ -z(\rho^h, m^h)(x,t) \le M_n(1 + C\Delta t) + C\Delta t, \\ |\mathbf{w}^h(x,t)| \le M_n(1 + C\Delta t), \\ |\mathbf{a}^h(x,t)| \le M_n(1 + C\Delta t), \quad |\mathbf{b}^h(x,t)| \le M_n(1 + C\Delta t), \end{gathered}$$

and from Lemma 3.2, we further obtain the following estimate:

$$M_{n+1} \le M_n(1 + C\Delta t) + C\Delta t,$$

that is,

$$\frac{M_{n+1} - M_n}{\Delta t} \le C(1 + M_n). \tag{5.4}$$

Indeed, consider the corresponding ordinary differential equation:

$$\frac{dr}{dt} = C(1 + r), \qquad r(0) = M_0. \tag{5.5}$$

There exists a positive constant $C(T)$ such that

$$M_0 \le r(t) \le C(T), \qquad t \in [0, T]. \tag{5.6}$$

The integral curve $r = r(t)$ is convex since

$$\frac{d^2 r(t)}{dt^2} = C^2(1 + r) \ge 0.$$

It follows from (5.4)-(5.6) and the convexity that

$$M_n \le r(n\Delta t) \le C(T).$$

Indeed, one uses arguments based upon the convex curve lying above its tangent line, which in turn dominates the chord, with slope given by the right hand side of (5.6). This implies

$$\begin{aligned}
&w(\rho^h, m^h)(x,t) \le C(T), \quad -z(\rho^h, m^h)(x,t) \le C(T), \\
&w(\rho^h, m^h)(x,t) - z(\rho^h, m^h)(x,t) \ge 0, \\
&|\mathbf{w}^h(x,t)| \le C(T), \quad |\mathbf{a}^h(x,t)| \le C(T), \quad |\mathbf{b}^h(x,t)| \le C(T),
\end{aligned}$$

that is, for $x \in [0,1]$, $t \in [0,T]$,

$$0 \le \rho^h(x,t) \le C(T), \quad \left|\frac{m^h(x,t)}{\rho^h(x,t)}\right| \le C(T), \tag{5.7}$$

$$|\mathbf{w}^h(x,t)| \le C(T), \quad |\mathbf{a}^h(x,t)| \le C(T), \quad |\mathbf{b}^h(x,t)| \le C(T). \tag{5.8}$$

In summary, the following theorem holds.

Theorem 5.1. *Under the assumptions (5.2)-(5.3), there exists a positive constant $C(T)$ such that the approximate solutions $V^h(x,t)$ constructed in Section 4 satisfy the uniform estimates (5.7)-(5.8) for any $(x,t) \in \Pi_T = [0,1] \times [0,T]$.*

6 H^{-1} Estimates of the Entropy Dissipation Measures

In this section, we prove the H^{-1} compactness of the approximate solutions (ρ^h, m^h). The following lemma ([4, 18]) is required for the proof.

Lemma 6.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain. Then*

$$\begin{aligned}
&(\textit{compact set of } W^{-1,l}(\Omega)) \cap (\textit{ bounded set of } W^{-1,r}(\Omega)) \\
&\subset (\textit{ compact set of } W^{-1,2}_{loc}(\Omega)),
\end{aligned}$$

where l and r are constants satisfying $1 < l \le 2 < r < \infty$.

To prove the H^{-1} estimates, we will consider two cases: $1 < \gamma \le 2$ and $\gamma > 2$, since η_* is no longer strictly convex if $\gamma > 2$.

Theorem 6.1. *For the approximate solutions $v^h = (\rho^h, m^h)$, the measure sequence*

$$\eta(v^h)_t + q(v^h)_x$$

is a compact subset of $H^{-1}_{loc}(\Omega)$ for all weak entropy pairs (η, q) of (3.1)-(3.2), where Ω is any bounded and open set in Π_T.

Proof. For any test function $\phi \in C_0^1(\Omega)$, we have

$$\iint_{\Pi_T} (\eta(v^h)\phi_t + q(v^h)\phi_x)dxdt = S_1(\phi) + S_2(\phi) + S_3(\phi) + S_4(\phi), \tag{6.1}$$

with

$$S_1(\phi) = \iint_{\Pi_T} \left((\eta(v^h) - \eta(v_0^h))\phi_t + (q(v^h) - q(v_0^h))\phi_x\right) dxdt,$$

$$S_2(\phi) = \int_0^1 \phi(x,T)\eta(v_0^h(x,T))dx - \int_0^1 \phi(x,0)\eta(v_0^h(x,0))dx,$$

$$S_3(\phi) = \int_0^T \sum (\sigma[\eta] - [q])\,\phi(x(t),t)dt,$$

and

$$S_4(\phi) = \sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \left(\eta(v_{0-}^n) - \eta(v_j^n)\right)\phi(x, n\Delta t)dx,$$

where $v_{0-}^n = v_0^h(x, n\Delta t - 0)$, $\phi_j^n = \phi(1 + jh, n\Delta t)$, the summation in $S_3(\phi)$ is taken over all discontinuities in v_0^h at a fixed time t, σ is the propagating speed of the discontinuity, and

$$[\eta] = \eta(v_0^h(x(t)+0,t)) - \eta(v_0^h(x(t)-0,t)),$$

$$[q] = q(v_0^h(x(t)+0,t)) - q(v_0^h(x(t)-0,t)),$$

are the jumps of $\eta(v_0^h(x,t))$ and $q(v_0^h(x,t))$ across a discontinuity $(x(t),t)$ in $v_0^h(x,t)$.

First we have, from the fractional step procedure in (4.3),

$$|S_1(\phi)| \le \iint_{\Pi_T} (\|\nabla\eta\|_\infty + \|\nabla q\|_\infty)(|\phi_t| + |\phi_x|)|v^h - v_0^h|dxdt \le Ch\|\phi\|_{H_0^1(\Omega)}.$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, then

$$\|S_1\|_{H^{-1}_{loc}(\Omega)} \le Ch \to 0, \quad \text{as} \quad h \to 0,$$

and thus S_1 is compact in $H^{-1}_{loc}(\Omega)$.

We now use Lemma 6.1 to prove the H^{-1} compactness of the other terms. The uniform boundedness of v^h implies the boundedness of $S_2 + S_3 + S_4$ in $W^{-1,r}(\Omega)$ for any $r > 1$. Therefore it remains to show the compactness of these terms in $W^{-1,l}(\Omega)$ for some $l \in (1, 2]$.

Substituting

$$\eta = \eta_* = \frac{m^2}{2\rho} + \frac{\rho^\gamma}{\gamma(\gamma-1)}, \quad q = q_* = m\left(\frac{m^2}{2\rho^2} + \frac{\rho^{\gamma-1}}{\gamma-1}\right), \quad \phi \equiv 1,$$

in the equality (6.1), we obtain

$$\sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \left(\eta_*(v_{0-}^n) - \eta_*(v_j^n)\right) dx + \int_0^T \sum (\sigma[\eta_*] - [q_*])\, dt \le C_0. \tag{6.2}$$

Set

$$\Theta(\eta, s) = (1-s)(v_{0-}^n - v_j^n)^\top \nabla^2 \eta(v_j^n + s(v_{0-}^n - v_j^n))(v_{0-}^n - v_j^n).$$

From the construction of v^h, one has, by use of the Taylor expansion,

$$\begin{aligned}
&\sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} (\eta_*(v_{0-}^n) - \eta_*(v_j^n))dx \\
&= \sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \nabla\eta_*(v_j^n)(v_{0-}^n - v_j^n)dx + \sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \int_0^1 \Theta(\eta_*, s)dsdx \\
&= \nabla\eta_*(v_j^n) \sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} (v_{0-}^n - v^h(x, n\Delta t - 0))dx \\
&\quad + \sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \int_0^1 \Theta(\eta_*, s)dsdx.
\end{aligned} \tag{6.3}$$

From the fractional step procedure and Lemma 3.3 we deduce that

$$\left| \nabla\eta_*(v_j^n) \sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} (v_{0-}^n - v^h(x, n\Delta t - 0))dx \right| \le C.$$

Since $\sigma[\eta_*] - [q_*] \ge 0$ holds across the shock waves, we have the following inequalities from the above estimates and (6.2):

$$\sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \int_0^1 \Theta(\eta_*, s)dsdx \le C, \tag{6.4}$$

$$\int_0^T \sum (\sigma[\eta_*] - [q_*]) dt \le C, \tag{6.5}$$

and then

$$\sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} (\eta_*(v_{0-}^n) - \eta_*(v_j^n)) dx \le C, \tag{6.6}$$

for some constant $C > 0$.

From the fact $|\sigma[\eta] - [q]| \le C(\sigma[\eta_*] - [q_*])$ and (6.5), the following estimates hold immediately:

$$|S_2(\phi)| \le C\|\phi\|_{C_0(\Omega)}, \quad |S_3(\phi)| \le C\|\phi\|_{C_0(\Omega)}.$$

Rewrite $S_4(\phi) \equiv S_5(\phi) + S_6(\phi)$ with

$$S_5(\phi) = \sum_{j,n} \phi_j^n \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} (\eta(v_{0-}^n) - \eta(v_j^n)) dx,$$

$$S_6(\phi) = \sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} (\eta(v_{0-}^n) - \eta(v_j^n))(\phi - \phi_j^n) dx.$$

By Lemma 3.3 and (6.4), we obtain, in a manner similar to (6.3),

$$\begin{aligned}
|S_5(\phi)| &\le \|\phi\|_{C_0} \sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \int_0^1 \Theta(\eta, s) ds dx + O(1)\|\phi\|_{C_0} \\
&\le \|\phi\|_{C_0} \sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \int_0^1 \Theta(\eta_*, s) ds dx + O(1)\|\phi\|_{C_0} \\
&\le C\|\phi\|_{C_0}.
\end{aligned}$$

Then

$$|(S_2 + S_3 + S_5)(\phi)| \le C\|\phi\|_{C_0},$$

i. e.

$$||S_2 + S_3 + S_5||_{C_0^*} \le C.$$

By the embedding theorem, $(C_0(\Omega))^* \hookrightarrow W^{-1,l_1}$, for $1 < l_1 < 2$, the set $S_2 + S_3 + S_5$ is compact in $W^{-1,l_1}(\Omega)$.

To make estimates on $S_6(\phi)$, we have to consider two cases: $1 < \gamma \le 2$ and $\gamma > 2$.

(1). For $1 < \gamma \le 2$, the strict convexity of η_* yields from (6.4)

$$\sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} |v_{0-}^n - v_j^n|^2 dx \le C. \tag{6.7}$$

For any $\phi \in C_0^\alpha(\Omega)$, $\frac{1}{2} < \alpha < 1$, using Hölder's inequality and (6.7), we have

$$\begin{aligned} |S_6(\phi)| &\le C\|\phi\|_{C_0^\alpha} h^{\alpha-1/2} \|\nabla\eta\|_\infty \left(\sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} |v_{0-}^n - v_j^n|^2 dx \right)^{\frac{1}{2}} \\ &\le C h^{\alpha-1/2} \|\phi\|_{C_0^\alpha}. \end{aligned}$$

(2). For $\gamma > 2$, from (6.6) and $\eta_* = \frac{1}{2}\rho u^2 + \frac{1}{\gamma(\gamma-1)}\rho^\gamma$, we have

$$\begin{aligned} \sum_{j,n} \int_{(j-1/2)h}^{(j+1/2)h} \Big(&\rho_{0-}^n (u_{0-}^n - u_j^n)^2/2 \\ &+ \int_0^1 (1-r)(\rho_j^n + r(\rho_{0-}^n - \rho_j^n))^{\gamma-2} dr (\rho_{0-}^n - \rho_j^n)^2 \Big) dx \le C. \end{aligned} \tag{6.8}$$

If $\rho_j^n > \rho_{0-}^n$, then

$$\int_0^1 (1-r)(\rho_j^n + r(\rho_{0-}^n - \rho_j^n))^{\gamma-2} dr \ge \frac{1}{\gamma - 1} |\rho_j^n - \rho_{0-}^n|^{\gamma-2};$$

and, if $\rho_j^n \le \rho_{0-}^n$, then

$$\int_0^1 (1-r)(\rho_j^n + r(\rho_{0-}^n - \rho_j^n))^{\gamma-2} dr \ge \frac{1}{\gamma(\gamma - 1)} |\rho_j^n - \rho_{0-}^n|^{\gamma-2}.$$

Combining these estimates with (6.8), one has

$$\sum_{j,n} \int_{(j-1/2)h}^{(j+1/2)h} \left(\rho_{0-}^n (u_{0-}^n - u_j^n)^2 + |\rho_{0-}^n - \rho_j^n|^\gamma \right) dx \le C.$$

Therefore, by the Hölder inequality, we have

$$\begin{aligned} &\sum_{j,n} \int_{(j-1/2)h}^{(j+1/2)h} \rho_{0-}^n |u_{0-}^n - u_j^n| dx \\ &\le \left(\sum_{j,n} \int_{(j-1/2)h}^{(j+1/2)h} \rho_{0-}^n dx \right)^{1/2} \left(\sum_{j,n} \int_{(j-1/2)h}^{(j+1/2)h} \rho_{0-}^n |u_{0-}^n - u_j^n|^2 dx \right)^{1/2} \\ &\le C h^{-1/2}, \end{aligned}$$

and

$$\sum_{j,n}\int_{(j-1/2)h}^{(j+1/2)h}|\rho_{0-}^n-\rho_j^n|dx \le \left(\sum_{j,n}\int_{(j-1/2)h}^{(j+1/2)h}|\rho_{0-}^n-\rho_j^n|^\gamma dx\right)^{1/\gamma}\left(\sum_{j,n}h\right)^{(\gamma-1)/\gamma}$$
$$\le Ch^{(1-\gamma)/\gamma}.$$

That is, for $\gamma > 2$,

$$\begin{aligned}
&\sum_{j,n}\int_{(j-1/2)h}^{(j+1/2)h}\left|\rho_{0-}^n-\rho_j^n\right|dx \le Ch^{(1-\gamma)/\gamma},\\
&\sum_{j,n}\int_{(j-1/2)h}^{(j+1/2)h}\rho_{0-}^n\left|u_{0-}^n-u_j^n\right|dx \le Ch^{-1/2}.
\end{aligned} \tag{6.9}$$

For any $\phi \in C_0^\alpha(\Omega)$, $1-1/\gamma<\alpha<1$, we have from (6.9) that

$$\begin{aligned}
|S_6(\phi)| &\le \sum_{j,n}\int_{(j-1/2)h}^{(j+1/2)h}|\phi(x,n\Delta t)-\phi_j^n||\eta(v_{0-}^n)-\eta(v_j^n)|dx\\
&\le Ch^\alpha||\phi||_{C_0^\alpha}\sum_{j,n}\int_{(j-1/2)h}^{(j+1/2)h}\left(|\rho_{0-}^n-\rho_j^n|+\rho_{0-}^n|u_{0-}^n-u_j^n|\right)dx\\
&\le C||\phi||_{C_0^\alpha}h^{\alpha-1+1/\gamma}.
\end{aligned}$$

With the above estimates and Sobolev's theorem: $W_0^{1,p}(\Omega)\subset C_0^\alpha(\Omega)$, $0<\alpha<1-2/p$, one has

$$|S_6(\phi)| \le Ch^{\alpha_0-1/2}||\phi||_{W_0^{1,p}(\Omega)},\quad p>2/(1-\alpha_0),\quad \alpha_0=\max(\alpha/2,\alpha-1+1/\gamma).$$

That is,

$$||S_6||_{W^{-1,l_2}(\Omega)}\to 0,\quad h\to 0,$$

for $1<l_2<2/(1+\alpha_0)$. Therefore,

$$S_2+S_3+S_4 \quad \text{is compact in } W^{-1,l},$$

for $1<l\equiv\min(l_1,l_2)<2/(1+\alpha_0)<2$. The proof is completed. □

Remark: from the above proof, we see, by manipulation of the following integral,

$$\iint_{\Pi_T}\left((\tilde\eta_*(\tilde v^h)+|\mathbf{a}^h|^2+|\mathbf{b}^h|^2)\phi_t+(\tilde q_*(\tilde v^h)+|\mathbf{a}^h|^2-|\mathbf{b}^h|^2)\phi_x\right)dxdt,$$

in (6.1), we can obtain the following estimates similar to (6.7) and (6.9):

$$\sum_{j,n} \int_{(j-1/2)h}^{(j+1/2)h} |V_{0-}^n - V_j^n|^2 dx \le C, \quad \text{if} \quad 1 < \gamma \le 2, \tag{6.10}$$

$$\sum_{j,n} \int_{(j-1/2)h}^{(j+1/2)h} \left|V_{0-}^n - V_j^n\right| dx \le C h^{(1-\gamma)/\gamma}, \quad \text{if} \quad \gamma > 2, \tag{6.11}$$

where $V_{0-}^n = V_0^h(x, n\Delta t - 0)$.

7 Convergence and Consistency

In this section, we show that the approximate solutions $V^h(x,t)$ converge to the weak solution to (2.7)-(2.8). We need the following compensated compactness theorem to show the strong convergence of $v^h = (\rho^h, m^h)$.

Lemma 7.1. *Assume that a sequence of measurable functions $v^h = (\rho^h, m^h)$, defined on $\mathbf{R}_+^2$, satisfies:*

(1). There is a constant $C > 0$ such that $\quad 0 \le \rho^h \le C$, and $\quad |m^h/\rho^h| \le C$;

(2). The measure sequence $\eta(v^h)_t + q(v^h)_x$ is compact in $H_{loc}^{-1}(\mathbf{R}_+^2)$ for all weak entropy pairs (η, q) of (3.1)-(3.2).

Then, for $\gamma > 1$, there exists a convergent subsequence (still labeled) v^h, such that

$$v^h(x,t) \to v(x,t) = (\rho(x,t), m(x,t))$$

almost everywhere.

This compensated compactness theorem was proved in [10] for $\gamma = 1 + 2/K, K \ge 5$ odd, in [4] for the case of gases $1 < \gamma \le 5/3$, and in [13, 14] for the case $\gamma > 5/3$. (See [7] for Chen-LeFloch's recent work on the isentropic Euler equations with general pressure law.)

Theorem 7.1. *With the initial condition (2.9) and the hypotheses (5.2)-(5.3), there is a subsequence (still labeled) V^h in the approximate solutions $V^h(x,t) = (\rho^h, m^h, \mathbf{q}^h, \mathbf{a}^h, \mathbf{b}^h)(x,t)$ such that, as $h \to 0$,*

$$(\rho^h, m^h)(x,t) \to (\rho(x,t), m(x,t)), \quad a.e.$$

$$(\mathbf{q}^h, \mathbf{a}^h, \mathbf{b}^h)(x,t) \overset{*}{\rightharpoonup} (\mathbf{q}, \mathbf{a}, \mathbf{b})(x,t), \quad \textit{weak star in } L^\infty.$$

The vector function $(\rho, m, \mathbf{q}, \mathbf{a}, \mathbf{b})(x,t)$ is the weak solution to (2.7)-(2.8) in $\Pi_T = [0,1] \times [0,T]$ for any $T > 0$ in the sense of distributions and satisfies

$$0 \le (x,t) \le C(T), \quad |m(x,t)/\rho(x,t)| \le C(T),$$

$$|\mathbf{w}(x,t)| \le C(T), \quad |\mathbf{a}(x,t)| \le C(T), \quad |\mathbf{b}(x,t)| \le C(T),$$

for $(x,t) \in \Pi_T$ with $C(T)$ some positive constant.

Proof. The strong convergence of (ρ^h, m^h) follows from (5.7), Theorem 6.1 and Lemma 7.1. The weak-star convergence of $(\mathbf{q}^h, \mathbf{a}^h, \mathbf{b}^h)$ follows from the uniform boundedness (5.8). Now let $\phi \in C_0^1(\Pi_T)$ be any test function with $\phi(0,t) = \phi(1,t) = \phi(x,T) = 0$.

(1). By the construction of the approximate solutions and the Rankine-Hugoniot condition, we have

$$\begin{aligned}&\iint_{\Pi_T} \left(\rho^h \phi_t + m^h \phi_x\right) dxdt + \int_0^T \rho_0^h(x)\phi(x,0)dx \\ &\quad = \sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \phi(x, n\Delta t)(\rho_0^h(x, n\Delta t - 0) - \rho_j^n)dx + \iint_{\Pi_T} (m^h - m_0^h)\phi_x dxdt.\end{aligned} \tag{7.1}$$

From the fractional step procedure, one has

$$\left|\iint_{\Pi_T} (m^h - m_0^h)\phi_x dxdt\right| \le Ch\|\phi\|_{C_0^1} \to 0, \qquad \text{as} \quad h \to 0.$$

From the definition of the Godunov value ρ_j^n, we have

$$\begin{aligned}&\sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \phi(x, n\Delta t)(\rho_0^h(x, n\Delta t - 0) - \rho_j^n)dx \\ &= \sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} (\phi - \phi_j^n)(\rho_0^h(x, n\Delta t - 0) - \rho_j^n)dx.\end{aligned}$$

For $1 < \gamma \le 2$, by Hölder's inequality and (6.7),

$$\begin{aligned}
&\left|\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}(\phi-\phi_j^n)(\rho_0^h(x,n\Delta t-0)-\rho_j^n)dx\right| \\
&\le C\|\phi\|_{C_0^1}\sqrt{h}\left(\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}|\rho_0^h(x,n\Delta t-0)-\rho_j^n|^2dx\right)^{\frac{1}{2}} \\
&\le C\sqrt{h}\|\phi\|_{C_0^1}, \qquad \text{as} \quad h\to 0.
\end{aligned}$$

For $\gamma > 2$, by (6.9),

$$\begin{aligned}
&\left|\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}(\phi-\phi_j^n)(\rho_0^h(x,n\Delta t-0)-\rho_j^n)dx\right| \\
&\le C\|\phi\|_{C_0^1}h\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}|\rho_0^h(x,n\Delta t-0)-\rho_j^n|dx \\
&\le Ch^{1/\gamma}\|\phi\|_{C_0^1}, \qquad \text{as} \quad h\to 0.
\end{aligned}$$

Taking the limit $h\to 0$ in (7.1) and using the dominated convergence theorem, we have

$$\iint_{\Pi_T}(\rho\phi_t+m\phi_x)\,dxdt+\int_0^T\rho_0(x)\phi(x,0)dx=0.$$

(2). From the construction of the approximate solutions, and the Rankine-Hugoniot condition, we have

$$\begin{aligned}
&\iint_{\Pi_T}\left(m^h\phi_t+((m^h)^2/\rho^h+p(\rho^h))\phi_x+G_2(V^h)\phi\right)dxdt+\int_0^T m_0^h(x)\phi(x,0)dx \\
&=\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}(m_0^h(x,n\Delta t-0)-m_j^n)\phi(x,n\Delta t)dx+\iint_{\Pi_T}G_2(V_0^h)\phi dxdt \\
&\quad+\iint_{\Pi_T}\left((m^h-m_0^h)\phi_t+((m^h)^2/\rho^h+p(\rho^h)-(m_0^h)^2/\rho_0^h-p(\rho_0^h))\phi_x\right)dxdt \\
&\quad+\iint_{\Pi_T}\left(G_2(V^h)-G_2(V_0^h)\right)\phi dxdt \\
&=I_1+I_2+O(h)\|\phi\|_{C_0^1},
\end{aligned}\tag{7.2}$$

with

$$\begin{aligned}
I_1&=\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}(\phi(x,n\Delta t)-\phi_j^n)(m_0^h(x,n\Delta t-0)-m_j^n)dx, \\
I_2&=\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}(m_0^h(x,n\Delta t-0)-m_j^n)\phi_j^n dx+\iint_{\Pi_T}G_2(V_0^h)\phi dxdt.
\end{aligned}$$

For $1 < \gamma \le 2$, by Hölder's inequality and (6.7),

$$|I_1| \le C\|\phi\|_{C_0^1}\sqrt{h}\left(\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}|m_0^h(x,n\Delta t-0)-m_j^n|^2dx\right)^{1/2}$$
$$\le C\sqrt{h}\|\phi\|_{C_0^1},$$

and for $\gamma > 2$, by (6.9),

$$|I_1| \le C\|\phi\|_{C_0^1}h\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\left(\rho_{0-}|u_{0-}-u_j^n|+|\rho_{0-}-\rho_j^n|\right)dx$$
$$\le Ch^{1/\gamma}\|\phi\|_{C_0^1}.$$

We observe that estimates similar to [15] yield the following inequalities from (6.10) and (6.11):

$$\sum_n\int_{(n-1)\Delta t}^{n\Delta t}\int_0^1\left|V_0^h(x,t)-V_0^h(x,n\Delta t-0)\right|^2dxdt \le Ch, \quad \text{if} \quad 1<\gamma\le 2, \tag{7.3}$$

$$\sum_n\int_{(n-1)\Delta t}^{n\Delta t}\int_0^1\left|V_0^h(x,t)-V_0^h(x,n\Delta t-0)\right|dxdt \le Ch^{1/\gamma}, \quad \text{if} \quad \gamma>2. \tag{7.4}$$

From the fractional step procedure,

$$I_2 = \sum_{j,n}\int_{(n-1)\Delta t}^{n\Delta t}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\left(G_2(V_0^h(x,t))\phi(x,t)-G_2(V_0^h(x,n\Delta t-0))\phi_j^n\right)dxdt$$
$$= \sum_{j,n}\int_{(n-1)\Delta t}^{n\Delta t}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\left(G_2(V_0^h(x,t))-G_2(V_0^h(x,n\Delta t-0))\right)\phi(x,t)$$
$$+\sum_{j,n}\int_{(n-1)\Delta t}^{n\Delta t}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}G_2(V_0^h(x,n\Delta t-0))(\phi(x,t)-\phi_j^n)dxdt,$$

so that, for $1 < \gamma \le 2$, by Hölder's inequality and (7.3), one has

$$|I_2| \le C\|\phi\|_{C_0^1}\left(\sum_n\int_{(n-1)\Delta t}^{n\Delta t}\int_0^1\left|V_0^h(x,t)-V_0^h(x,n\Delta t-0)\right|^2dxdt\right)^{1/2}+O(h)\|\phi\|_{C_0^1}$$
$$\le Ch^{1/2}|\phi\|_{C_0^1},$$

and for $\gamma > 2$, by estimates similar to those in Section 6, one has,

$$|I_2| \le C\|\phi\|_{C_0^1}\sum_{j,n}\int_{(n-1)\Delta t}^{n\Delta t}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\left|V_0^h(x,t)-V_0^h(x,n\Delta t-0)\right|dxdt+O(h)\|\phi\|_{C_0^1}$$
$$\le Ch^{1/\gamma}\|\phi\|_{C_0^1}.$$

Taking the limit $h \to 0$ in (7.2) and using the dominated convergence theorem, we see that the second equation of (2.7) is satisfied in the sense of distributions. Similar estimates also show that the other equations in (2.7) are satisfied in this weak sense. The uniform boundedness (5.7)-(5.8) of the approximate solutions implies the boundedness of the weak solution. This completes the proof of Theorem 7.1. □

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