The Multidimensional Damped Wave Equation: Maximal Weak Solutions for Nonlinear Forcing via Semigroups and Approximation

Joseph W. Jerome*

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Abstract

The damped nonlinear wave equation, also known as the nonlinear telegraph equation, is studied within the framework of semigroups and eigenfunction approximation. The linear semigroup assumes a central role: it is bounded on the domain of its generator for all time $t \geq 0$. This permits eigenfunction approximation within the semigroup framework, as a tool for the study of weak solutions. The semigroup convolution formula, known to be rigorous on the generator domain, is extended to the interpretation of weak solution on an arbitrary time interval. A separate approximation theory can be developed by using the invariance of the semigroup on eigenspaces of the Laplacian as the system evolves.

For (locally) bounded continuous L^2 forcing, there is a natural derivation of a maximal solution, which can logically include a constraint on the solution as well. Operator forcing allows for the incorporation of concurrent physical processes. A significant feature of the proof in the nonlinear case is verification of successive approximation without standard fixed point analysis.

[1]Department of Mathematics, Northwestern University, Evanston, IL 60208.
In residence: George Washington University, Washington, D.C. 20052

1 Introduction.

The damped nonlinear wave equation, also known as the nonlinear telegraph equation, has assumed renewed importance in recent years (cf. [5, 8, 12, 18, 19, 20]). Reference [18] is novel because of its connection to MEMS modeling. Operator forcing, induced by a concurrent physical process, and constraints come into play in [18] (see Table 1, p. 472, $\gamma > 0, \beta = 0$, for an unresolved case within the framework we are studying).

In this article, we consider the following initial/boundary-value problem. The spatial domain $\Omega \subset \mathbb{R}^N$ represents a bounded convex domain or a bounded C^2 domain. For these domains, ∇^2 represents an algebraic and topological isomorphism of $H^2(\Omega) \cap H_0^1(\Omega)$ onto $L^2(\Omega)$ ([1, 10]). The system considered is the following, where subscripts represent (partial) derivatives with respect to t.

$$\begin{aligned} u_{tt} &= -\nu u_t + \kappa \nabla^2 u + \mathcal{F}(u), \\ u(\mathbf{x}, t) &= 0, \ \mathbf{x} \in \partial\Omega, t > 0, \\ (u(\mathbf{x}, 0), u_t(\mathbf{x}, 0)) &= (u_0(\mathbf{x}), v_0(\mathbf{x})), \ \mathbf{x} \in \Omega. \end{aligned} \tag{1}$$

Here, ν and κ are positive physical constants and u depends on the physical context. $\mathcal{F}$ represents a nonlinear forcing term. Typically, the nonlinear problem is posed locally on a closed bounded time interval, $J = [0, T_1]$, and is extended to a maximal interval $[0, \tau)$ in $\mathbb{R}$. We will also consider the impact on solutions of this system if a continuous operator constraint is adjoined. One requires, on a subinterval of the maximal time interval $[0, \tau)$:

$$\mathcal{G}(u(t), u_t(t)) > 0. \tag{2}$$

We now summarize the principal results in the article. In Proposition 2.1 we construct the semigroup $T(t)$ with generator U for the linear part of (1). Although this contractive semigroup has been extensively investigated, its global stability, indeed decay, on the *generator* domain appears to be recent. Generator spectral estimates are required. In section three, we establish the convolution formula on the frame space $\mathcal{H}$ to define a weak solution of the inhomogeneous problem $\mathcal{F}(u) = \tilde{F}$ in Theorem 3.1. The time interval is arbitrary, due to the global character of the semigroup, and facilitated by eigenfunction approximation. Global stability on the generator domain enters here. The eigenfunctions of the Laplacian enter in another fundamental way; their invariance under the semigroup gives rise to a rigorous construction of separation of variables and a corresponding useful approximation result, independent of the existence results. The relevant theorem is Theorem 3.2. In section four, we consider nonlinear forcing in L^2. We define a local solution in Theorem 4.1 via successive approximation, and use the local solution to define a maximal solution in Theorem 4.2. This leads to the result derived in Proposition 4.2 that every weak solution can be extended to a maximal solution. In addition, we prove uniqueness in the case of local Lipschitz forcing in Proposition 4.3. In this section, we also consider an application of the results to the constrained case and obtain a maximal subinterval on which the constraint holds in Theorem 4.3. The constraint may be applied to the pair (u, u_t). Summary remarks and an appendix close the article.

2 The Semigroup

We discuss an explicit construction of the semigroup associated with the linear part of the equation in (1). We will develop special features necessary for the analysis given here. The semigroup $T(t)$ will initially be defined on a Hilbert space $\mathcal{H}$. Much is known about the norm of $T(t)$ on this space. For example, the time decay of the semigroup for manifolds without boundary was established in [22]; the results of these authors also hold in one dimension for intervals. The stability of the semigroup is under study by many mathematicians. We are particularly interested here in solutions contained in the domain $D(U)$ of the semigroup generator. Instrumental to this is the study of the norm of $T(t)$ when restricted to this smoother space. This distinction assumes especial importance in the work of Kato [14, Ch. 6] in his construction of the evolution operator.

After a standard definition, we will concisely state and derive the required results. We will make extensive use of parts (2,3) of Proposition 2.1, stated below.

Define $v = u_t$, and

$$U = \begin{bmatrix} 0 & \mathcal{I} \\ \kappa\nabla^2 & -\nu\mathcal{I} \end{bmatrix}. \tag{3}$$

Here $\mathcal{I}$ represents the identity. Then the equation,

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = U \begin{bmatrix} u \\ v \end{bmatrix},$$

is a standard equivalent representation of $u_{tt} = -\nu u_t + \kappa\nabla^2 u$. In order to position this in an operator framework, we formulate the following definitions.

Definition 2.1. *Define the function space, $\mathcal{H} = H_0^1(\Omega)\times L^2(\Omega)$, and the domain $D(U)$ of U, $D(U) = (H^2(\Omega)\cap H_0^1(\Omega)) \times H_0^1(\Omega)$. We shall employ an equivalent norm on $H_0^1(\Omega)$ by utilizing the inner product:*

$$(u, w)_{H_0^1} = \kappa(\nabla u, \nabla w)_{L^2}. \tag{4}$$

For elements $f_j = \begin{bmatrix} u_j \\ v_j \end{bmatrix}$ in $D(U), j = 1, 2,$ we employ the $D(U)$ inner product given by the sum,

$$(f_1, f_2)_{D(U)} = (\nabla^2 u_1, \nabla^2 u_2)_{L^2} + (u_1, u_2)_{H_0^1} + (v_1, v_2)_{H_0^1}. \tag{5}$$

We employ the notation $\mathcal{D}(U)$ for the first components of the elements of $D(U)$. The corresponding inner product is defined by the truncation of (5):

$$(u_1, u_2)_{\mathcal{D}(U)} = (\nabla^2 u_1, \nabla^2 u_2)_{L^2} + (u_1, u_2)_{H_0^1}.$$

For clarity, throughout section two we will use vector notation to represent the components in the Cartesian products which represent $\mathcal{H}$ and $D(U)$. Throughout the remainder of the article, for X a Banach space, and J a compact time interval, $C(J;X)$ is the usual Banach space with norm $\|u\|_{C(J;X)} := \max_{t\in J} \|u(t)\|_X$.

The following proposition contains the results needed for the subsequent sections.

Proposition 2.1. *U is a closed linear operator in $\mathcal{H}$. Denote by $\{\lambda_n\}$ the (positive) eigenvalues of $-\kappa\nabla^2$ on the domain $H^2(\Omega) \cap H_0^1(\Omega)$. The following properties hold.*

1. *The resolvent set of U satisfies $\rho(U) \supset \{\lambda : \mathrm{Re}\ \lambda > \theta\}$, where, for $\theta_n = 4\lambda_n - \nu^2$,*

$$\theta = -\nu/2, \ \textit{if}\ \theta_n \geq 0, \forall n \geq 1, \tag{6}$$

$$\theta = \max_{n:\theta_n<0} \{-\nu/2 + \sqrt{-\theta_n}/2\}, \ \textit{otherwise}. \tag{7}$$

   In particular, the imaginary axis is contained in $\rho(U)$.

2. *U is the generator of a strongly continuous semigroup $T(t), t \geq 0$, on $\mathcal{H}$, which is contractive:*

$$\|T(t)\|_{\mathcal{H}} \leq 1, \ t \geq 0. \tag{8}$$

   Suppose $T_1 > 0$ is arbitrary. If $G \in D(U)$, and $F \in C(J; D(U))$, where $J = [0, T_1]$, then

$$V(t) = T(t)G + \int_0^t T(t-s)F(s)\, ds \tag{9}$$

   is in $C(J; D(U))$ and satisfies the initial value problem,

$$V_t = UV(t) + F(t), \ V(0) = G, \ 0 < t \leq T_1. \tag{10}$$

3. *$\|T(t)\|_{D(U)}$, defined with respect to $D(U)$, decays to zero as $t \to \infty$. In particular, there is a number ω such that this norm satisfies $\|T(t)\|_{D(U)} \leq \omega, t \geq 0$.*

Proof. The property that U is closed follows routinely from the definitions. If

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} \to \begin{bmatrix} u \\ v \end{bmatrix}, \ U \begin{bmatrix} u_n \\ v_n \end{bmatrix} \to \begin{bmatrix} w \\ z \end{bmatrix}, \ \text{in } \mathcal{H}, \ \begin{bmatrix} u_n \\ v_n \end{bmatrix} \in D(U),$$

then we conclude directly that $v = w$ and the L^2 limit of $\kappa\nabla^2 u_n$ is $z + \nu w$. From this relation, we conclude that ∇u has components with L^2 derivative, so that

$$\begin{bmatrix} u \\ v \end{bmatrix} \in D(U), \ \text{with image} \begin{bmatrix} w \\ z \end{bmatrix}.$$

It follows that U is closed.

To prove statement (1), suppose that $\mathrm{Re}\ \lambda > \theta$, and consider the formal system,

$$(\lambda\mathcal{I} - U) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} w \\ z \end{bmatrix}.$$

Since λ is permitted to be complex, for this part of the proof we interpret $\mathcal{H}$ and $D(U)$ as complex Hilbert spaces. We show that $\lambda\mathcal{I} - U$ is an algebraic isomorphism from $D(U)$ to $\mathcal{H}$, i.e. , the formal system is uniquely solvable. By using the definition of U, we see that $v = \lambda u - w$, provided u is determined. This means that it is necessary to show that the following differential equation, with homogeneous boundary values, determines a unique solution u in $H^2(\Omega) \cap H_0^1(\Omega)$: $-\kappa\nabla^2 u + \lambda(\lambda+\nu)u = z + (\lambda+\nu)w$, for this pair w, z. Now the spectrum of the self-adjoint operator $-\kappa\nabla^2$ consists of discrete positive eigenvalues λ_n, with limit ∞. Thus, it must be shown that the product $-\lambda(\lambda+\nu)$ excludes these numbers. A calculation, based upon proof by contradiction, shows that this is achieved for Re $\lambda > \theta$. We conclude that $\lambda\mathcal{I} - U$ is surjective and injective from $D(U)$ to $\mathcal{H}$. It is, by the definition of norms, a bounded linear operator between these spaces. Its algebraic inverse, $R(\lambda, U)$, is a bounded linear operator, as follows from the open mapping theorem. Indeed, a bound on the $D(U)$ norm, which dominates the $\mathcal{H}$ norm, is obtained.

We now prove part (2), and operate within real Hilbert spaces. We verify that the resolvent of U, $R(\lambda, U)$, satisfies the norm inequality in $\mathcal{H}$,

$$\|R(\lambda, U)\| \leq \frac{1}{\lambda},\ \lambda > 0. \tag{11}$$

The Hille-Yosida theorem [6, 9] then implies that U generates a contraction semigroup. To establish (11), it is sufficient to show that the inner products $(Uf, f) = (f, Uf)$ are nonpositive, as is seen by expanding the square of the $\mathcal{H}$-norms of both sides of $(\lambda - \theta)f - Uf = g$. Now we compute, for $f = \begin{bmatrix} u \\ v \end{bmatrix}$,

$$(Uf, f)_{\mathcal{H}} = (v, u)_{H_0^1} + (\kappa\nabla^2 u - \nu v, v)_{L^2}.$$

After integration by parts, the rhs reduces to $-\nu(v, v)_{L^2}$, which is nonpositive. Note that the cancelation involved here depends on the equivalent norm introduced in (4).

The statement that (9) is well defined and provides a solution of (10) depends fundamentally on the characterization of $D(U)$ in terms of the limit of difference quotients of $T(t)$. The result follows by direct computation. The details of the computation may be found in [14, Section 6.4], in the general case of the evolution operator $\mathcal{U}(t, s)$. For the current result, $\mathcal{U}(t, s) = T(t - s)$.

To establish part (3), we use the lemma cited at the conclusion of the proof, in conjunction with part (1). This concludes the proof. □

The following lemma is quoted from [25, Theorem 3].

Lemma 2.1. *A bounded C_0-semigroup e^{At} on a Banach space X with (unbounded) generator A satisfies*

$$\|e^{At}(A - \lambda\mathcal{I})^{-1}\| \to 0,\ \text{as}\ t \to \infty, \lambda \in \rho(A),$$

if and only if the imaginary axis is in the resolvent set of A: $i\mathbb{R} \subset \rho(A)$.

An equivalent form of this result was obtained earlier in [4]. The special case of this result for $\lambda = 0$ was obtained in [2, 3] and [26].

3 Weak Solutions for the Inhomogeneous Equation

In this section, we will construct weak solutions of the special case of (1), usually referred to as the linear inhomogeneous equation. This requires the introduction of the appropriate approximation spaces. We begin with these.

3.1 Approximation subspaces and projections

The eigenfunctions of the 'Laplacian' $-\kappa\nabla^2$, denoted $\{\phi_n\}_{n\geq 1}$, will occupy a significant role. The system is a complete orthogonal system in $L^2(\Omega)$ and $H_0^1(\Omega)$, and (normalized to be) orthonormal in $L^2(\Omega)$. A comprehensive survey of Laplacian eigenfunctions for important geometric domains may be found in [11]. In this section, $J = [0, T_1]$, where T_1 is an arbitrary terminal time.

Definition 3.1. *Denote by Q_n the orthogonal projection in $H_0^1(\Omega)$ onto the linear span $\mathcal{M}_n$ of $\{\phi_k\}_{1\leq k\leq n}$, and by $\tilde{Q}_n$ the corresponding orthogonal projection in $L^2(\Omega)$. Further, denote by $\mathcal{P}_n$ the (closed) linear subspace of $C(J; H_0^1(\Omega))$ defined by*

$$\mathcal{P}_n = \left\{ \sum_{k=1}^{n} \alpha_k(t)\phi_k(\mathbf{x}) : \alpha_k \in C(J), k = 1, \ldots, n \right\},$$

and write P_n for the projection in $C(J; H_0^1(\Omega))$ onto $\mathcal{P}_n$, and $\tilde{\mathcal{P}}_n, \tilde{P}_n$, for the spaces and projections in $C(J; L^2(\Omega))$.

A direct implementation of the definition shows that $Q_n u$ has the same formal representation as $\tilde{Q}_n u$. Furthermore, as verified in [15], $\mathcal{P}_n$ is closed, $P_n = P_n^2$ is a standard projection, and P_n can be interpreted, for each fixed $t \in J$, as the orthogonal projection Q_n onto $\mathcal{M}_n$. Similar remarks apply to $\tilde{P}_n$. Moreover, $P_n f$ converges to f in $C(J; H_0^1(\Omega))$ and $\tilde{P}_n g$ converges to g in $C(J; L^2(\Omega))$. These statements follow from an estimation of the maxima on J of the respective error functions.

3.2 Projections and classical solutions

Designate by F_n the pair $(0, \tilde{P}_n \tilde{F})$ which is a member of $C(J; D(U))$. Set $G = (u_0, v_0) \in \mathcal{H}$ and designate by G_n the pair $(Q_n u_0, \tilde{Q}_n v_0)$. It follows from Proposition 2.1, part (2), that

$$V_n(t) = T(t)G_n + \int_0^t T(t-s)F_n(s)\, ds \tag{12}$$

is a strong solution of the initial/boundary-value problem,

$$\begin{aligned} u_{tt} &= -\nu u_t + \kappa\nabla^2 u + \tilde{P}_n \tilde{F}, \\ u(\mathbf{x}, t) &= 0,\ \mathbf{x} \in \partial\Omega, 0 < t \leq T_1, \\ u(\mathbf{x}, 0) = Q_n u_0(\mathbf{x}), &\qquad u_t(\mathbf{x}, 0) = \tilde{Q}_n v_0(\mathbf{x}),\ \mathbf{x} \in \Omega. \end{aligned} \tag{13}$$

The V_n will serve as approximations to establish existence of weak solutions on J in section 3.3, and will be shown to have eigenfunction components in section 3.4.

3.3 Weak solutions for the linear inhomogeneous equation

Definition 3.2. *For $T_1 > 0$ arbitrary, define $J = [0, T_1]$ and suppose $\tilde{F} \in C(J; L^2(\Omega))$ and $(u_0, v_0) \in \mathcal{H}$ are given. Suppose there exists u such that:*

1. *(u, u_t) is continuous from J to $\mathcal{H}$, with u_{tt} continuous from J to $H^{-1}(\Omega)$.*
2. *The initial conditions are satisfied: $(u(0, \mathbf{x}), u_t(0, \mathbf{x})) = (u_0(\mathbf{x}), v_0(\mathbf{x}))$, $\mathbf{x} \in \Omega$.*
3. *$\forall\, \phi \in C(J; H_0^1(\Omega)), \forall\, 0 < t \leq T_1$,*

$$(u_{tt}, \phi)_{L^2} = -\nu(u_t, \phi)_{L^2} - \kappa(\nabla u, \nabla\phi)_{L^2} + (\tilde{F}, \phi)_{L^2}. \tag{14}$$

Then we say that u is a weak solution for the linear inhomogeneous equation, i. e., the version of the system (1) with $\mathcal{F}(u) = \tilde{F}$.

Theorem 3.1. *Given $\tilde{F} \in C(J; L^2)$, and $G = (u_0, v_0) \in \mathcal{H}$, the pair given by*

$$V(t) = T(t)G + \int_0^t T(t-s)(0, \tilde{F}(s))\, ds \tag{15}$$

is a unique weak solution, as defined in Definition 3.2.

Proof. The classical solutions V_n as defined by (12) are seen to be weak solutions of the system defined by projection, upon application of the Gauss-Green theorem, and converge uniformly in $\mathcal{H}$ to V. This convergence makes direct use of the contractive semigroup properties of $T(t)$ and the projection properties. To verify this, suppose that $(u_n, (u_n)_t) = V_n \in D(U)$ is given by (12). For $\phi \in C(J; C_0^\infty(\mathbb{R}^N))$, we obtain upon direct multiplication and integration,

$$((u_n)_{tt}, \phi)_{L^2} = -\nu((u_n)_t, \phi)_{L^2} + \kappa(\Delta u_n, \phi)_{L^2} + (F_n, \phi)_{L^2}. \tag{16}$$

An application of the Gauss-Green theorem, followed by the limit $n \to \infty$, gives

$$(W, \phi)_{L^2} = -\nu(u_t, \phi)_{L^2} - \kappa(\nabla u, \nabla\phi)_{L^2} + (\tilde{F}, \phi)_{L^2}. \tag{17}$$

Here we have used the convergence of V_n to V in $C(J; \mathcal{H})$. Also, we have identified W with a member of $C(J; H^{-1})$. Further,

$$(u_n)_{tt} \to W, \text{ in } C(J; H^{-1}).$$

Upon integration, it follows that

$$(u_n)_t - \tilde{Q}_n v_0 \to \int_0^t W\, ds, \text{ in } C(J; H^{-1}).$$

Independently, we have established that $(u_n)_t \to u_t$ in $C(J; L^2)$ and $\tilde{Q}_n v_0 \to v_0$ in L^2 It follows that $W = u_{tt}$. This concludes the proof of existence of weak solutions upon use of the denseness of the test functions $C_0^\infty(\Omega)$.

In order to verify uniqueness, we make some preliminary statements. Notice that the difference of weak solutions of the inhomogeneous initial/boundary-value problem is a weak solution of the homogeneous problem with homogeneous initial conditions. This has only the zero solution. Indeed, suppose $f \in C(J; \mathcal{H})$ is a weak solution of $df/dt = Uf$, with homogeneous initial conditions. For the time integrated version, we may take L^2 inner products:

$$(f(t), f(t))_{L^2} = \int_0^t (Uf(s), f(s))_{L^2} \, ds.$$

The rhs is nonpositive, as shown in section two in the proof of part (2) of Proposition 2.1. Since the lhs is nonnegative, we conclude that $f = 0$. This establishes uniqueness of weak solutions. □

3.4 Approximation by eigenfunctions

In this section, we examine more closely the explicit role of eigenfunction approximation. The section is independent of the rest of the article, and is included to show that when approximation of the inhomogeneous term is carried out by members of $\tilde{\mathcal{P}}_n$, the solution formula generates approximations from this space. This is accomplished by a rigorous implementation of classical separation of variables. In particular, by direct calculation, we verify the semigroup invariance on the approximation spaces.

Definition 3.3. *We say that $\psi_n(t)\phi_n(\mathbf{x})$ is a separated solution of the damped wave equation,*

$$u_{tt} = -\nu u_t + \kappa \nabla^2 u,$$

with homogeneous boundary conditions and zero forcing if ψ_n satisfies the ordinary differential equation,

$$\psi_n'' + \nu \psi_n' + \lambda_n \psi_n = 0, \tag{18}$$

on the time interval $t > 0$. Here, (λ_n, ϕ_n) are an eigenvalue/eigenfunction pair for the operator $-\kappa\nabla^2$ with Dirichlet boundary conditions on Ω.

Lemma 3.1. *Separated solutions satisfy the damped wave equation as strong solutions and are members of the regularity class $C([0,\infty); D(U))$. They can be explicitly written as follows.*

1. $4\lambda_k - \nu^2 > 0$. *In this case, the separated solutions are of the form, $(aC_k(t) + bS_k(t))\phi_k(x)$, where*

$$C_k(t) := \exp(-(\nu/2)t)\cos(\omega_k t), \qquad S_k(t) := \exp(-(\nu/2)t)\sin(\omega_k t),$$
$$dC_k/dt = (-\nu/2)C_k(t) - \omega_k S_k(t), \qquad dS_k/dt = (-\nu/2)S_k(t) + \omega_k C_k(t),$$
$$\omega_k = \frac{\sqrt{4\lambda_k - \nu^2}}{2}.$$

2. $4\lambda_k - \nu^2 = 0$. *In this case, the separated solutions are constructed by replacing* C_k *and* S_k *by* c_k *and* s_k*, where*

$$\begin{aligned} c_k(t) &:= \exp(-(\nu/2)t), & s_k(t) &:= t\exp(-(\nu/2)t), \\ dc_k/dt &= (-\nu/2)c_k(t), & ds_k/dt &= c_k(t) - (\nu/2)s_k(t). \end{aligned}$$

3. $4\lambda_k - \nu^2 < 0$. *Here, the replacements are* D_k *and* E_k*, where*

$$\begin{aligned} D_k(t) &:= \exp((-\nu/2 + \rho_k)t), & E_k(t) &:= \exp((-\nu/2 - \rho_k)t), \\ dD_k/dt &= (-\nu/2 + \rho_k)D_k(t), & dE_k/dt &= (-\nu/2 - \rho_k)E_k(t). \end{aligned}$$

   Here, $\rho_k = \sqrt{\nu^2 - 4\lambda_k}/2$.

Proof. The regularity class for the separated solutions follows directly from their definition, given in the statement of the lemma. A direct substitution into the damped wave equation of $\psi_n\phi_n$ verifies the first statement when (18) is employed for ψ_n and the eigenfunction property is employed for ϕ_n. The stated solution trichotomy depends upon the solutions of the associated characteristic equation,

$$m^2 + \nu m + \lambda_n = 0,$$

which leads to the above cases, depending on the discriminant $D = \nu^2 - 4\lambda_n$. These are determined by $D < 0, D = 0, D > 0$, resp. □

Lemma 3.2. *Suppose that the pair* $(0, \phi_n)$ *is given as an initial condition for the damped wave equation. There is a unique separated solution pair* (u_*, v_*)*, with* $v_* = du_*/dt$*, such that the pair* (u_*, v_*) *assumes the evaluation* $(0, \phi_n)$ *at* $t = 0$*. In particular, this pair is precisely* $T(t)(0, \phi_n)$*. We have the following formulas.*

1. $4\lambda_k - \nu^2 > 0$. *We have* $T(t)(0, \phi_k) = (u_*, v_*)$*, where*

$$u_*(\mathbf{x}, t) = (1/\omega_k)S_k(t)\phi_k(\mathbf{x}),\ v_*(\mathbf{x}, t) = (C_k(t) - \nu/(2\omega_k)S_k(t))\phi_k(\mathbf{x}).$$

2. $4\lambda_k - \nu^2 = 0$. *We have* $T(t)(0, \phi_k) = (u_*, v_*)$*, where*

$$u_*(\mathbf{x}, t) = s_k(t)\phi_k(\mathbf{x}),\ v_*(\mathbf{x}, t) = (c_k(t) - (\nu/2)s_k(t))\phi_k(\mathbf{x}).$$

3. $4\lambda_k - \nu^2 < 0$. *We have* $T(t)(0, \phi_k) = (u_*, v_*)$*, where*

$$u_*(\mathbf{x}, t) = 1/(2\rho_k)(D_k(t) - E_k(t))\phi_k(\mathbf{x}),$$

$$v_*(\mathbf{x}, t) = (1/2)[(1 - \nu/(2\rho_k))D_k(t) + (1 + \nu/(2\rho_k))E_k(t)]\phi_k(\mathbf{x}).$$

Proof. The semigroup is uniquely determined by the conditions that $T(0) = \mathcal{I}$, $(d/dt)T(t) = UT(t)$. The formulas cited, derived from the previous lemma, satisfy these conditions when evaluated at $(0, \phi_n)$. □

Theorem 3.2. *The components of V_n are in $C(J; \mathcal{P}_n \times \tilde{\mathcal{P}}_n)$. In particular, any weak solution V of the linear inhomogeneous equation can be approximated in $C(J; \mathcal{H})$ by eigenfunctions via a standard Galerkin procedure.*

Proof. The previous lemmas have allowed us to examine the action of $T(t)$ on $(0, \phi_k)$. Linearity of the semigroup T permits extension to members of the form $\sum_{k=1}^{n} \alpha_k(s)(0, \phi_k(x))$. Indeed, motivated by the representation for the semigroup action, we compute

$$T(t-s)\sum_{k=1}^{n} \alpha_k(s)(0, \phi_k(\mathbf{x})) = \sum_{k=1}^{n} \alpha_k(s) T(t-s)(0, \phi_k(\mathbf{x}))$$

$$= \sum_{k=1}^{n} \alpha_k(s)(\beta_k(t-s), \gamma_k(t-s))\phi_k(\mathbf{x}),$$

where the functions β_k, γ_k are specified in the preceding lemma. In all three cases, since formula (12) involves integration in the variable s, we conclude that both components are functions from J to $\mathcal{M}_n$. This argument also holds for the approximation of the initial conditions. It follows that the convergence of V_n to V can be interpreted as convergence of the Faedo-Galerkin method. □

Remark 3.1. *As mentioned earlier, the other results of this article are independent of the preceding theorem.*

3.5 Solution stability as $t \to \infty$

Suppose $\tilde{F} \in C([0, \infty); L^2)$. We give a sufficient condition on the norm growth of $\tilde{F}$ so that the $\mathcal{H}$ norm of V, defined on each interval $[0, T_1]$ by Theorem 3.1, remains bounded in norm as $t \to \infty$.

Corollary 3.1. *Suppose $\tilde{F} \in C([0, \infty); L^2) \cap L^1([0, \infty); L^2)$. Then $\|V\|_{C([0,T_1];\mathcal{H})}$ remains bounded as $T_1 \to \infty$.*

Proof. This is implied by the contractive property of the semigroup on $\mathcal{H}$. □

Remark 3.2. *A similar result can be formulated for strong solutions, with the appropriate assumptions on u_0, v_0, F. Interestingly, because of the decay of $\|T(t)\|_{D(U)}$, one has the additional asymptotic property that the term in $V(t)$ due to the initial data decays to zero as $t \to \infty$.*

4 The Nonlinear System with Operator Forcing

In this section we construct weak solutions for the general nonlinear system (1). We begin by defining the properties of the forcing term.

Definition 4.1. *We consider an operator $\mathcal{F}$ with the following properties.*

1. *$\mathcal{F}$ is defined on $L^2(\Omega)$ with range in $L^2(\Omega)$.*

2. *$\mathcal{F}$ is locally bounded: For every bounded set B in $L^2(\Omega)$, $\mathcal{F}(B)$ is bounded in $L^2(\Omega)$.*

3. *$\mathcal{F}$ is continuous from $L^2(\Omega)$ to $L^2(\Omega)$.*

Remark 4.1. *We observe that Lipschitz continuity, even local Lipschitz continuity, is not assumed for $\mathcal{F}$, since many significant applications do not satisfy this property. The preceding hypothesis is discussed by Krasnosel'skii [17, Th. 2.1, p. 22]. The class of Nemytskii operators with growth conditions is known to imply these conditions [23].*

Remark 4.2. *The hypotheses for the nonlinear forcing term are not as restrictive as first appears. Although forcing by function composition is typical in the literature, other types of forcing are permitted, including convolution such as occurs in mollification. For example, $\mathcal{F}(u) = \phi_\epsilon * |u|^2$ satisfies the required hypotheses, where ϕ_ϵ is a classical smooth mollifier. In fact, by Young's convolution inequality, one sees that $\mathcal{F}$ is locally Lipschitz continuous on L^2. When appropriate compactness is available, one can use this theory to extend results to nonlinearities stronger than those which can be treated by Nemytskii operators.*

Definition 4.2. *Suppose there exist T_1 and u such that*

1. *(u, u_t) is continuous from J to $\mathcal{H}$, with u_{tt} continuous from J to $H^{-1}(\Omega)$.*

2. *The initial conditions are satisfied: $(u(\mathbf{x}, 0), u_t(\mathbf{x}, 0)) = (u_0(\mathbf{x}), v_0(\mathbf{x}))$, for $\mathbf{x} \in \Omega$.*

3. *$\forall\, \phi \in C(J; H_0^1(\Omega)), \forall\, 0 < t \leq T_1$,*

$$(u_{tt}, \phi)_{L^2} \quad = \quad -\nu(u_t, \phi)_{L^2} - \kappa(\nabla u, \nabla \phi)_{L^2} + (\mathcal{F}(u), \phi)_{L^2}. \tag{19}$$

Then u is said to be a weak solution of the system (1). If u is defined on an interval $[0, \tau)$, with the property that its restriction to any closed interval $[0, T_1]$ is a weak solution, then u is called maximal if it has no proper extension which is a weak solution.

4.1 Operator properties

We begin with a definition, which identifies two fundamental constants.

Definition 4.3. *Define ω_0:*

$$\omega_0 = \sup_{u : \|u - u_0\|_{L^2} \leq 1} \|\mathcal{F}(u)\|_{L^2}. \tag{20}$$

Define s_0 (by continuity):

$$\|T(t)(u_0, v_0) - (u_0, v_0)\|_{\mathcal{H}} \leq 1/2, \ 0 \leq t \leq s_0. \tag{21}$$

Proposition 4.1. *There is a time $t_1 > 0$ such that the operator,*

$$\mathcal{V}(u,v)(t) = V(t), 0 \leq t \leq t_1,$$

where $V(t)$ is defined by Theorem 3.1 with $T_1 = t_1$, and $\tilde{F}(t) = \mathcal{F}(u(t))$, maps

$$\mathcal{B}_0 = \{(u,v) \in C(J_1; \mathcal{H}) : \|(u,v) - (u_0, v_0)\|_{C(J_1;\mathcal{H})} \leq 1\}$$

into itself. Here, $J_1 = [0, t_1]$. Furthermore, the operator $\mathcal{V} : \mathcal{B}_0 \mapsto \mathcal{B}_0$ is uniformly equicontinuous on J_1 in the sense that its range is a uniformly equicontinuous family on J_1. The time t_1 can be explicitly represented as,

$$t_1 = \min(s_0, 1/(2\omega_0)). \tag{22}$$

Proof. Since T_1 as used in the previous section is arbitrary, it follows that the mapping $\mathcal{V}$ is well-defined, for any choice of t_1. Theorem 3.1 is also used to verify the invariance statement. Indeed, we estimate $\|V(t) - G\|_{\mathcal{H}}$ as follows. For each $t \in J_1$,

$$\|V(t) - G\|_{\mathcal{H}} \leq \|T(t)G - G\|_{\mathcal{H}} + \int_0^t \|\mathcal{F}(u(s))\|_{L^2}\, ds.$$

Here, we have used the contractive property of T on $\mathcal{H}$. The first term is bounded by 1/2 by (21), while the second term is also bounded by 1/2 by the combination of (20) and (22). In order to establish uniform equicontinuity, let $0 \leq \tau_1 < \tau_2 \leq t_1$ be given in J_1 and $(u,v) \in \mathcal{B}_0$. We first write the representation,

$$V(\tau_2) - V(\tau_1) = [T(\tau_2) - T(\tau_1)]G \; + \int_0^{\tau_1} [T(\tau_2 - s) - T(\tau_1 - s)](0, \tilde{F}(s))\, ds$$

$$+ \int_{\tau_1}^{\tau_2} T(\tau_2 - s)(0, \tilde{F}(s))\, ds.$$

As previously, we use (20) and the contractive property of T to conclude that

$$\|V(\tau_2) - V(\tau_1)\|_{\mathcal{H}} \leq \|T(\tau_2) - T(\tau_1)\|_{\mathcal{H}} \|G\|_{\mathcal{H}}$$

$$+(\omega_0/2)[t_1 \sup_{0 \leq t \leq t_1 - \delta} \|T(t+\delta) - T(t)\|_{\mathcal{H}} + \delta].$$

Here, $\delta = \tau_2 - \tau_1$. When the uniform operator continuity property of $\|T(t)\|_{\mathcal{H}}$ is utilized, one obtains the uniform equicontinuity. □

4.2 The local solution and the maximal solution

We will use the method of successive approximation to establish the existence of a local solution. Although it would be possible to use the Schauder fixed point theorem on the eigenspaces, followed by a convergence analysis, the use of successive approximation appears to be a more constructive approach. Moreover, it provides a proof independent of Theorem 3.2.

Theorem 4.1. *There is a weak solution satisfying (19) on* $[0, t_1]$*. Here,* t_1 *is defined in (22). The solution may be obtained by subsequential convergence, derived from successive solution of linear problems (see (23) below).*

Proof. Define the sequence of successive approximations,

$$(u_k, v_k) = \mathcal{V}(u_{k-1}, v_{k-1}), k \geq 1. \tag{23}$$

The mapping $\mathcal{V}$ is defined in Proposition 4.1, and its fixed points are weak solutions. This sequence is contained in $\mathcal{B}_0$, hence is bounded in $\mathcal{H}$, uniformly in $t \in J_1$. By the first proposition of the appendix, we obtain a subsequence, weakly convergent in $\mathcal{H}$ pointwise in $t \in J_1$, to an element, (u, v). This follows from the identifications $X = Y = \mathcal{H}$ and the equicontinuity established in Proposition 4.1. We claim that (u, v) is a fixed point, $\mathcal{V}(u, v) = (u, v)$, hence a weak solution. The principal question is the uniform L^2 convergence in t of u_k to u. This sequence is already known to be L^2 convergent, pointwise in t, by the Rellich theorem. We will show that a subsequence of u_k is convergent to u in $C(J_1; L^2)$, which allows the continuity of $\mathcal{F}$ to be applied within the rhs of the representation sequence for $\mathcal{V}$. In fact, the solutions of the inhomogeneous equations are given by (cf. Theorem 3.1)

$$\mathcal{V}(u_k, v_k)(t) = T(t)G + \int_0^t T(t-s)(0, \tilde{F}(u_{k-1})(s))\, ds \tag{24}$$

The rhs converges in $C(J_1; L^2)$ to $\mathcal{V}(u, v)$, under the assumption of the uniform convergence of $u_{k-1}(s)$. We establish this now. Consider the decomposition,

$$u - u_k = (u - \tilde{P}_n u) + (\tilde{P}_n u - \tilde{P}_n u_k) + (\tilde{P}_n u_k - u_k).$$

Suppose a sequence $\epsilon_\ell \to 0$ is specified. The first term can be estimated in $C(J_1; L^2)$ for sufficiently large n, not to exceed $\epsilon_\ell/3$. It can be shown (see the following Lemma 4.2) that a similar statement holds, uniformly in k, for the third term. The estimation of the second term is more complicated, and is carried out by a variant of Cantor's diagonalization argument. We first set up the tableau $\{\mathcal{R}_j\}_{j\geq 1}$ of nested sequences $\mathcal{R}_j$. These will be chosen to have the property that

$$\lim_{m\to\infty} \tilde{P}_j u_{jm} = \tilde{P}_j u, \; j \geq 1.$$

Here, $\{\mathcal{R}_j\} = \{u_{jm}\}_{m\geq 1}, \; j = 1, 2, \dots$. Suppose that the first $n-1$ nested rows $\mathcal{R}_j$ of the tableau have been defined, with the stated property. Observe that $\tilde{P}_n \mathcal{R}_{n-1}$ is a precompact set in $C(J_1; L^2)$, which is a consequence of Lemma 4.1 to follow. Select a convergent subsequence $\tilde{P}_n \mathcal{R}_n$. Pointwise convergence established earlier shows that the necessary limit is $\tilde{P}_n u$. Now define the nth row of the tableau to be $\{\mathcal{R}_n\}$. Proceed inductively to obtain the tableau.

We are now ready for the variant of Cantor selection. Define (the subsequence) $\{u_{n_\ell}\}$ as follows. Given ϵ_ℓ, choose $n = n_\ell$ such that the first and third terms of the decomposition are estimated, as discussed earlier. The appropriate ℓth element of the sequence is selected from row n_ℓ so that its projection (and those of its successors) lies within $\epsilon_\ell/3$ in norm, of the projection of u. This inductively defines a subsequence of (the first components of) the original weakly convergent sequence, which is uniformly convergent when viewed in L^2. Uniform convergence is preserved when $\mathcal{F}$ is applied. Earlier arguments establish the fixed point. $\square$

We now state and prove the lemmas used in the preceding proof.

Lemma 4.1. *Let (u_k, v_k) be a sequence in $\mathcal{V}(\mathcal{B}_0)$. Then, for each fixed positive integer n,*

$$\{\tilde{P}_n u_k\}$$

is a precompact set in $C(J_1; L^2)$.

Proof. We shall make use of the second proposition of the appendix. Define $B = \{u \in H_0^1 : \|u - u_0\|_{H_0^1} \leq 1\}$. Now make the identifications,

$$X = J_1,\ Y = L^2,\ C = \tilde{Q}_n B,\ \mathcal{F}_0 = \{\tilde{P}_n u_k\}.$$

The uniform equicontinuity has been established in Proposition 4.1 in the (stronger) case where the $\mathcal{H}$ norm is employed. It therefore holds for L^2. Since C is a closed bounded subset of a finite dimensional space, it is compact as required.The precompactness now follows from Proposition A.2. $\square$

Lemma 4.2. *Suppose that $\{h_k\}$ is a sequence which is bounded in $C(J; H_0^1(\Omega))$. Then, uniformly in k, $\tilde{P}_n h_k \to h_k,\ n \to \infty,$ in $C(J_1; L^2)$.*

Proof. Suppose that b is a bound for h_k in $C(J, H_0^1(\Omega))$. For each fixed t, consider the self-adjoint operator $R = -(\kappa/b^2)\nabla^2$ on $\mathcal{D}(U)$, and the ellipsoid in $L^2(\Omega)$ defined by

$$\mathcal{R}(t) = \{u \in \mathcal{D}(U) : (Ru, u)_{L^2} \leq 1\}.$$

Then the $L^2(\Omega)$ n-width of $\mathcal{R}(t)$ is attained by the subspace $\mathcal{M}_n$ for each fixed t [13], so that, by closure, this holds true for the widths d_n of the closed $H_0^1(\Omega)$-ball of radius b. The latter are directly computable [13] in terms of the eigenvalues of R: $d_n = b\lambda_{n+1}^{-1/2}$, where we retain the earlier meaning of λ_n. This implies that, in $L^2(\Omega)$, uniformly in t, and in k, $\tilde{Q}_n h_k - h_k \to 0,\ n \to \infty$. Equivalently, uniformly in k,

$$\tilde{P}_n h_k - h_k \to 0,\ n \to \infty.$$

$\square$

The following corollary follows from the preceding arguments.

Corollary 4.1. *Suppose that $(u_1, v_1) = (u(t_1), v(t_1))$ and define $\omega_1, t_2, \mathcal{V}$ in analogy with the procedure of Theorem 4.1. This extends the solution to $[0, t_1 + t_2]$, and the procedure can be repeated by induction to any interval $[0, \tau_k]$, where*

$$\tau_k = \sum_{j=1}^{k} t_j,\ k \geq 1.$$

Proof. Given the local solution as defined in Theorem 4.1, one again defines a local solution on $[t_1, t_1 + t_2]$ by the same method. It remains to verify that the two local solutions are restrictions of a global solution on $[0, t_1 + t_2]$. This is immediate from Definition 4.2, however, together with the equivalence of a weak solution with the representation of the operator $\mathcal{V}$. The induction proceeds similarly. □

Theorem 4.2. *There is a maximal solution, i. e., a solution with no proper extension. The time interval for the maximal solution is* $[0, \tau)$*, where*

$$\tau = \lim_{k \to \infty} \tau_k.$$

The solution, designated V*, satisfies Definition 4.2 on every compact subinterval of* $[0, \tau)$*.*

Proof. By Corollary 4.1, applied inductively for $k = 1, 2, \dots,$, we conclude that there is a solution V defined on $[0, \tau)$. Since $[0, \tau) = \cup_{k=1}^{\infty} [0, \tau_k]$, the solution satisfies Definition 4.2 on every compact subinterval of $[0, \tau)$. The only statement which is not immediate from the construction is that there is no proper extension of V, which is a weak solution. Suppose, for a contradiction, that V_{ext} is a proper extension of V. In particular, the following are uniformly continuous functions on $[0, \tau]$ into L^2:

$$u_{\text{ext}}, \mathcal{F}(u_{\text{ext}}).$$

Further,

$$V_{\text{ext}}(\tau) = \lim_{k \to \infty} V(\tau_k). \tag{25}$$

Since $\tau < \infty$,

$$t_{k+1} - t_k \to 0, \ k \to \infty.$$

There are two possibilities in this case. Either the numbers ω_k, defined in (20) for $k = 0$, and selected inductively for each interval $[t_k, t_{k+1}]$, must be an unbounded sequence; or, the numbers s_k, defined in (21) for s_0, and defined more generally by the formula,

$$\|T(t)(u_k, v_k) - (u_k, v_k)\|_{\mathcal{H}} \leq 1/2, \ 0 \leq t \leq s_k, \tag{26}$$

must possess a subsequence which converges to zero. Neither of these possibilities can occur. The first is excluded since $\mathcal{F}$ has bounded range when evaluated on the compact set $K = \{V_{\text{ext}}(t) : 0 \leq t \leq \tau\}$. The second is excluded since $\|T(t) - I\|$ is uniformly continuous on the compact set K cited above, so that s_k may be chosen independently of k. □

A question of interest, since uniqueness has not been resolved in the general case of L^2 forcing, is whether every solution in the sense of Definition 4.2, has a maximal extension. The answer is affirmative.

Proposition 4.2. *Every solution satisfying Definition 4.2 has a maximal extension. It may be constructed by the inductive procedure defined in Corollary 4.1 and Theorem 4.2.*

Proof. The proof begins with the assumed solution on $[0, T_1]$, and proceeds identically as in the corollary and the theorem. □

4.3 Further properties I: Uniqueness

The semigroup representation of the weak solution is sufficient to prove uniqueness of the maximal solution if $\mathcal{F}$ is locally Lipschitz. We have the following.

Proposition 4.3. *If $\mathcal{F}$ is locally Lipschitz, then the maximal solution is unique.*

Proof. Suppose that there exist two distinct maximal solutions V_1 and V_2, defined on $[0, \tau_1), [0, \tau_2)$, resp. Consider any compact interval $J = [0, T_1]$ common to these intervals. It is immediate that the $\mathcal{H}$ norm of $V_1 - V_2$ on J satisfies Gronwall's inequality under the locally Lipschitz assumption. It follows that one of these functions is a proper extension of the other, which contradicts maximality. □

4.4 Further properties II: The constrained equation

We now add a constraint to the system (19). The framework permits the constraint to be applied to (u, u_t), not simply to u. We have the following result.

Theorem 4.3. *Suppose that $\mathcal{G}$ is a continuous real-valued functional defined on $\mathcal{H}$, which is positive for the initial values. For any maximal solution, there is a positive number $\tau' \leq \tau$ such that $[0, \tau')$ is maximal for the constraint. In particular, if $\tau' < \tau$, the constraint fails at $t = \tau'$.*

Proof. Define τ' to be the (possibly infinite) supremum of those t for which the constraint holds. If $\tau' < \tau$, the $\mathcal{H}$ compactness of $\{V(t) : 0 \leq t \leq \tau'\}$ implies that $\mathcal{G}(u, u_t)(\tau') = 0$, otherwise, the constraint interval may be extended within the maximal interval for the solution. □

5 Concluding Remarks

We have analyzed an operator forcing for the nonlinear damped wave equation (equivalently, telegraph equation), which reflects recent studies of this equation, allowing for an intermediate or concurrent physical process. The forcing is continuous and (locally) bounded on L^2. It includes, but is not restricted to, function composition. As mentioned earlier, it includes smoothing of quadratic nonlinearities, and may be capable of producing existence when combined with compactness methods. This approach has been utilized in [16] in a different context. For nonlinearities defined by negative powers, new ideas are required in the case of homogeneous Dirichlet boundary conditions, as considered here. However, in certain cases involving inhomogeneous boundary conditions, it is expected that the approaches of this article apply to the case of negative powers in one dimension. We have defined maximal solutions for the nonlinear forcing, obtained locally by successive approximation. The nonlinear theory is based upon a global linear weak solution theory, and approximation of the forcing. An auxiliary result, not required for the existence, details a rigorous separation of variables, associated with eigenfunction approximation; this reduces to familiar modal approximation in one dimension. Furthermore, the semigroup is invariant on the individual eigenfunctions, which is maintained by the convolution formula. The addition of a continuous constraint is compatible with the existence of a maximal solution.

Uniqueness holds in the locally Lipschitz case. It is not clear whether uniqueness holds more generally. If it is ultimately demonstrated that uniqueness fails, then the model would require further physical principles, likely from thermodynamics. In the cited MEMS application [18], the forcing is the result of a complex physical process involving elastic/electrostatic interactions, and is best described via operator composition. Also, the wave motion tracked by the model is constrained to avoid 'touchdown'. This constraint is readily handled by the choice $\mathcal{G}(u) = 1 + u$ in one spatial dimension. Our framework addresses this case.

For a pedagogical introduction to the telegraph equation, cf. [21]. Its derivation dates to the 1880s, when it was derived by Heaviside to describe attenuated electrical transmission.

We conclude with some comparison of our hypotheses with those appearing in the literature. In [19, 20], bounded solutions are derived. These results are more general than appears, since they are derived in combination with maximum principles. It appears that the maximum principles are dimension dependent. Absent maximum principles, the restriction on the nonlinear forcing appears to be somewhat strong. In [5], the forcing is assumed bounded and periodic solutions are derived via the Leray-Schauder theorem. As previously mentioned, the forcing of this article is not restricted to function composition, and may have greater applicability. Furthermore, by avoiding the use of classical fixed point theorems, and relying on subsequential convergence, we are able to minimize the required hypotheses.

A Subsequential Convergence for Bounded Families

In section 4, we applied two basic compactness results, taken from [7] and [24]. Here, we quote the underlying results for the reader's convenience. The first is cited from [7, Proposition 1.1.2(i)].

Proposition A.1 (Cazenave)**.** *Let I be a bounded open interval of $\mathbb{R}$, let $X \hookrightarrow Y$ be Banach spaces. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $C(\bar{I};Y)$. Assume that $f_n(t) \in X$ $\forall (n,t) \in \mathbb{N}\times I$ and that* $\sup\{\|f_n(t)\|_X, (n,t) \in \mathbb{N}\times I\} = K < \infty$. *Assume further that f_n is uniformly equicontinuous in Y. If X is reflexive, then the following holds. There exists a function $f \in C(\bar{I};Y)$ which is weakly continuous $\bar{I} \mapsto X$ and a subsequence n_k such that*

$$\forall t \in \bar{I},\ f_{n_k}(t) \rightharpoonup f(t), k \to \infty,\ \textit{in } X.$$

It is not asserted that convergence is uniform in t.

The next result is cited from [24, Theorem 2.3.14]. It is a generalized Arzela-Ascoli theorem.

Proposition A.2 (Simon)**.** *Let X be a separable metric space and Y a complete metric space, with $C \subset Y$ compact. Let $\mathcal{F}_0$ be a family of uniformly equicontinuous functions from X to Y with Range$(f) \subset C$ for every $f \in \mathcal{F}_0$. Then any sequence in $\mathcal{F}_0$ has a subsequence converging at each $x \in X$. If X is compact, then $\mathcal{F}_0$ is precompact in the uniform topology.*

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