Functional Analytic Methods for Evolution Systems: Local Smooth Theory Stable Under Singular Limits

Joseph W. Jerome*

Abstract

By using two prototypical applications, the hydrodynamic-Maxwell system and the Navier-Stokes/charge transport system, we discuss the current relevance of local smooth theories for the Cauchy problem based on semigroup methods, and inspired by the Friedrichs and Kato inequalities. There appear to be three major advantages to the use of this theory: stability under the vanishing of diffusion or viscosity terms; flexibility in handling block systems, which are only partially symmetrizable; the use of implicit semidiscretization to determine estimates relating the size of the initial datum and the admissible terminal time.

Keywords Hydrodynamic-Maxwell Systems, symmetrized formulation, vanishing heat flux, Navier-Stokes systems, Poisson-Nernst Planck systems, vanishing viscosity, Cauchy problem, smooth solutions, semigroups of operators, resolvent stability, semidiscretization

AMS Subject Classification Numbers Primary: 35K22, 35L90, 35M10, 35Q05, 35Q10, 47D60; Secondary: 82D15, 82D37

Acknowledgements: The author is supported by the National Science Foundation under grant DMS-0311263.

*Department of Mathematics, Northwestern University, Evanston, IL 60208

1 Introduction

The Lax-Majda theory for symmetrizable hyperbolic systems has been of significant importance for establishing local well-posedness in the class of smooth solutions of the Cauchy problem (see [21], [2]). The theory is not directly designed to deal with incompletely parabolic systems, which are singular perturbations of symmetrizable hyperbolic systems. An example is a charged compressible fluid conducting heat, and the limit of vanishing heat conduction. The theory also does not lend itself to coupled systems written in block matrix form, of which only one block component along the principal diagonal is a symmetrizable hyperbolic system, or a singular perturbation of such a system. An example is a coupled Navier-Stokes/charge transport system, remaining stable under the singular limit of vanishing viscosity. In this concise survey, we describe an adaptation of Kato's semigroup theory applicable to these cases. The theory we describe has the added advantage that it can balance the restriction on the initial datum with that on the terminal time in terms of an analytic inequality.

2 First Example: Hydrodynamic-Maxwell System

This model treats the propagation of electrons in a semiconductor device as the flow of a compressible, charged, heat-conducting fluid. Coupling to electrostatic fields has been well studied, involving equations for the conservation of density, momentum and energy, coupled to Poisson's equation for the electrostatic potential. When semiconductor devices are operated under high frequency conditions (including technologies such as microwave devices, electro-optics, spintronics, and semiconductor lasers), magnetic fields are generated by moving charges inside the device, and the charge transport interacts with the propagating electromagnetic waves. In this case, the electromagnetic field satisfies Maxwell's equations, which are coupled to the transport system. Therefore, the hydrodynamic model for high-frequency charge transport in semiconductors consists of the conservation laws, coupled to Maxwell's equations for the electric and magnetic fields.

The evolution system assumes the following (nonconservative) form:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \mathbf{v}\cdot \nabla\rho + \rho\nabla\cdot \mathbf{v} = 0, \\ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{k}{m}\nabla\mathcal{T} + \frac{k\mathcal{T}}{m\rho}\nabla\rho = -\frac{q}{m}\mathbf{F} - \frac{\mathbf{v}}{\tau_{\mathbf{p}}}, \\ \frac{\partial \mathcal{T}}{\partial t} - \frac{\kappa_0}{\rho}\nabla\cdot(\rho\nabla\mathcal{T}) + \mathbf{v}\cdot\nabla\mathcal{T} + \frac{2}{3}\mathcal{T}\nabla\cdot\mathbf{v} = -\frac{2m|\mathbf{v}|^2}{3k}\left(\frac{1}{2\tau_w} - \frac{1}{\tau_{\mathbf{p}}}\right) - \frac{\mathcal{T}-\mathcal{T}_*}{\tau_w}, \\ \varepsilon\mathbf{E}_t - \nabla\times\mathbf{H} + \mathbf{J} = 0, \\ \mu\mathbf{H}_t + \nabla\times\mathbf{E} = 0, \\ -\varepsilon\nabla\cdot\mathbf{E} = \frac{q}{m}\rho - D(\mathbf{x}), \quad \nabla\cdot\mathbf{B} = 0, \\ \mathbf{B} = \mu\mathbf{H}, \quad \mathbf{J} = -\frac{q}{m}\rho\mathbf{v}, \quad \mathbf{x}\in\mathbb{R}^3, \quad t>0, \end{cases} \tag{2.1}$$

where ρ is the electron mass density, $\mathbf{v} \in \mathbb{R}^3$ is the electron velocity, $\mathcal{T}$ is the electron temperature, $\mathbf{E} \in \mathbb{R}^3$ is the electric field, $\mathbf{H} \in \mathbb{R}^3$ is the magnetic field, $\mathbf{J} \in \mathbb{R}^3$ is the current density, $\mathbf{B} \in \mathbb{R}^3$ is the magnetic induction, $\mathbf{F} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$ and $-q\mathbf{F}$ is the Lorentz force, D is the permanent charge profile, q is the electronic charge modulus, m is the effective electron mass, k is Boltzmann's constant, τ_p is the momentum relaxation time, τ_w is the energy relaxation time, μ is the permeability of the medium, ε is the permittivity of the medium, κ_0 is a constant multiplier (with the variable density) of heat conduction, and $\mathcal{T}_*$ is the ambient temperature.

Define the vector $\mathbf{u}$ by

$$\mathbf{u} = \left[\begin{array}{c} \rho \\ \mathbf{v} \\ \mathcal{T} \\ \hline \mathbf{E} \\ \mathbf{B} \end{array}\right] = \left[\begin{array}{c} \mathbf{y} \\ \hline \mathbf{z} \end{array}\right]. \tag{2.2}$$

Choose units in which the following *numerical* relationships hold: $q/m = 1, k/m = 1, \varepsilon\mu = 1$. The system (2.1) as defined above has matrix multipliers of $\frac{\partial \mathbf{y}}{\partial x_j},\ j = 1, 2, 3$, given by

$$\tilde{C}_j = \left[\begin{array}{c|ccc|c} v_j & \rho\delta_{1j} & \rho\delta_{2j} & \rho\delta_{3j} & 0 \\ \hline \frac{\mathcal{T}}{\rho}\delta_{1j} & v_j & 0 & 0 & 1\delta_{1j} \\ \frac{\mathcal{T}}{\rho}\delta_{2j} & 0 & v_j & 0 & 1\delta_{2j} \\ \frac{\mathcal{T}}{\rho}\delta_{3j} & 0 & 0 & v_j & 1\delta_{3j} \\ \hline 0 & \frac{2\mathcal{T}}{3}\delta_{1j} & \frac{2\mathcal{T}}{3}\delta_{2j} & \frac{2\mathcal{T}}{3}\delta_{3j} & v_j \end{array}\right], \ (\delta_{ij} = 1,\ i = j; 0, i \neq j). \tag{2.3}$$

The matrix multipliers of $\frac{\partial \mathbf{z}}{\partial x_j},\ j = 1, 2, 3$, are given by the matrices D_j:

$$D_j = \left[\begin{array}{c|c} \mathbf{0} & G_j \\ \hline G_j^t & \mathbf{0} \end{array}\right], \tag{2.4}$$

where

$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, G_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.5}$$

Here, and throughout, $\mathbf{0}$ denotes an appropriate (possibly non-square) zero matrix, possibly a row or column vector. Note the presence of heat conduction in the system. The hydrodynamic-Maxwell equations are more intricate than the Euler-Poisson equations. The only rigorous studies appear to be that made by Chen, Wang and the author in [3], where a global weak solution is proved in one spatial dimension, and by the author in [14], which is surveyed here as pertains to local theories. References to the applied literature are furnished in these two references.

2.1 The Symmetrizer

The symmetrizer of $\tilde{C}_j$ is then given by:

$$C_0 = \left[\begin{array}{c|c|c} \frac{\mathcal{T}}{\rho} & \mathbf{0} & 0 \\ \hline \mathbf{0} & \rho I_3 & \mathbf{0} \\ \hline 0 & \mathbf{0} & \frac{3\rho}{2\mathcal{T}} \end{array}\right], \tag{2.6}$$

where I_3 is the identity matrix of order 3. C_0 is symmetrizing in the following sense:

$$C_j = C_0 \tilde{C}_j = \left[\begin{array}{c|ccc|c} \frac{\mathcal{T} v_j}{\rho} & \mathcal{T}\delta_{1j} & \mathcal{T}\delta_{2j} & \mathcal{T}\delta_{3j} & 0 \\ \hline \mathcal{T}\delta_{1j} & \rho v_j & 0 & 0 & \rho\delta_{1j} \\ \mathcal{T}\delta_{2j} & 0 & \rho v_j & 0 & \rho\delta_{2j} \\ \mathcal{T}\delta_{3j} & 0 & 0 & \rho v_j & \rho\delta_{3j} \\ \hline 0 & \rho\delta_{1j} & \rho\delta_{2j} & \rho\delta_{3j} & \frac{3\rho v_j}{2\mathcal{T}} \end{array}\right] \tag{2.7}$$

is symmetric for each $j = 1, 2, 3$. We may then define the system symmetrizer and the symmetric multipliers via

$$a_0 = \left[\begin{array}{c|c} C_0 & \mathbf{0} \\ \hline \mathbf{0} & I_6 \end{array}\right], \; a_j = \left[\begin{array}{c|c} C_j & \mathbf{0} \\ \hline \mathbf{0} & D_j \end{array}\right]. \tag{2.8}$$

We then obtain:

$$a_0(\mathbf{u})\mathbf{u}_t + L(\mathbf{u})\mathbf{u} + \left[\sum_{j=1}^{3} a_j(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_j} + b(\mathbf{u})\mathbf{u}\right] = 0, \tag{2.9}$$

where

$$L(\mathbf{u}) = -\mathrm{diag}(0, \mathbf{0}, \gamma_0/\mathcal{T}, \mathbf{0}) \nabla\cdot(\rho\nabla), \; \gamma_0 = \frac{3}{2}\kappa_0, \; c = \left(\frac{1}{2\tau_w} - \frac{1}{\tau_p}\right),$$

$$b = \left[\begin{array}{c|c|c|c} 0 & \mathbf{0} & 0 & \mathbf{0} \\ \hline \mathbf{F} & \frac{\rho}{\tau_p} I_3 & \mathbf{0} & \mathbf{0} \\ \hline \frac{3(1-\frac{\mathcal{T}_*}{\mathcal{T}})}{2\tau_w} & c\frac{\rho \mathbf{v}}{\mathcal{T}} & 0 & \mathbf{0} \\ \hline \mathbf{0} & -\mu\rho I_3 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]. \tag{2.10}$$

It remains to discuss the divergence conditions in the Maxwell system, expressed in terms of $\mathbf{E}$ and $\mathbf{B}$. The latter condition is imposed by requiring $\mathbf{B}$ to belong to a divergence free space. In regard to $\mathbf{E}$, it is enough to impose the appropriate condition on the initial electric field; by taking the divergence of the equation involving $\mathbf{E}_t$, we infer that

$$\varepsilon(\nabla\cdot\mathbf{E})_t = -\nabla\cdot\mathbf{J} = \nabla\cdot(\rho\mathbf{v}),$$

and the latter is given, by the conservation of particle density equation, by

$$-\rho_t = -(\rho - D(\mathbf{x}))_t,$$

so that the equality of $\varepsilon\nabla\cdot\mathbf{E}$ and $-(\rho - D(\mathbf{x}))$ at $t = 0$ implies equality for $0 \le t \le T$. The initial condition for the Cauchy problem is then given by,

$$\mathbf{u}(\cdot\,, 0) = \mathbf{u}_0, \tag{2.11}$$

for a given function, $\mathbf{u}_0 \in H^s(\mathbb{R}^3; \mathbb{R}^{11})$, satisfying certain positivity conditions reflecting restriction of vacuum states. The density-dependent heat conduction requires $s > 7/2$. The components corresponding to $\mathbf{B}$ are required to have zero L_2 divergence in the sense of distributions; those corresponding to $\mathbf{E}$ must satisfy the divergence condition at $t = 0$.

3 Second Example: Fluid Transport System

Modeling of electrodiffusion in electrolytes is a problem of major scientific interest [22]. At the present time, it finds application in biology (ion channels), chemistry (electro-osmosis), and pharmacology (transdermal iontophoresis). Self-consistent charge transport is represented by the Poisson-Nernst-Planck system, and the fluid motions by a Navier-Stokes system with forcing terms. The current densities are given on $\mathbb{R}^m$ in terms of the electron density n, the hole density p, the electrostatic potential ϕ, and the fluid velocity $\mathbf{v}$, by

$$\mathbf{J}_n = qD_n \nabla n - q\mu_n n \nabla\phi - q\mathbf{v}n, \tag{3.1}$$
$$\mathbf{J}_p = -qD_p \nabla p - q\mu_p p \nabla\phi + q\mathbf{v}p. \tag{3.2}$$

The respective diffusion and mobility constants are denoted by D_n, D_p, μ_n, μ_p. The charge-transport component of the system is given by

$$\frac{q\partial n}{\partial t} - \nabla\cdot \mathbf{J}_n = 0, \qquad \frac{q\partial p}{\partial t} + \nabla\cdot \mathbf{J}_p = 0, \tag{3.3}$$
$$\mathbf{E} = -\nabla\phi, \tag{3.4}$$
$$\nabla\cdot(\varepsilon\nabla\phi) = q(n-p) - D \quad \text{(Poisson equation)}. \tag{3.5}$$

The velocity of the electrolyte is determined by the Navier-Stokes equations:

$$\rho(\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v}) - \eta\Delta\mathbf{v} = -\nabla P_f - q(p-n)\nabla\phi, \tag{3.6}$$
$$\nabla\cdot\mathbf{v} = 0, \tag{3.7}$$

where ρ is the (mass) density of the electrolyte, P_f denotes fluid pressure, and η is the dynamic viscosity. The constants q, ε have retained their meaning from the first example. These equations have been introduced by Rubinstein [22]. We shall make use of the kinematic viscosity, $\nu = \eta/\rho$, in the statement of the mathematical model.

3.1 The Mathematical Model

It has been traditional since the observations of Leray in 1933–34, to consider a reduced Navier-Stokes system, in tandem with the projection P onto divergence free distributions. The idea, discussed by Temam in [24, Chapter 1, §1,2], is to solve the equation of the pressure free part of the system, projected onto divergence free functions; it follows by the DeRham property that the reduced system is the gradient of a function (pressure). It is also required for well-posedness of the problem that the concentrations n and p be nonnegative. This is easily handled within the present framework as follows. One requires that

$$n_0 \geq \alpha_0 > 0, \; p_0 \geq \beta_0 > 0, \tag{3.8}$$

where $n(\cdot\,,0) = n_0, \; p(\cdot\,,0) = p_0$. Since the solution regularity implies that the vector solution is uniformly continuous on $[0,T] \times \mathbb{R}^m$, we select $T' \leq T$ so that $n \geq 0, \; p \geq 0$, i., e., the physical solution can be taken as an appropriate restriction of the solution of the mathematical model developed here.

Define the $m+2$-vector $\mathbf{u}$ by

$$\mathbf{u} = \begin{bmatrix} \mathbf{v} \\ qn \\ qp \end{bmatrix}. \tag{3.9}$$

The initial condition for the Cauchy problem on $\mathbb{R}^m$ is given by,

$$\mathbf{u}(\cdot\,,0) = \mathbf{u}_0,$$

for a given function,

$$\mathbf{u}_0 \in H^s(\mathbb{R}^m;\mathbb{R}^{m+2}), s > m/2+1.$$

We require a block system format. Thus, if $\mathbf{u}_1$ denotes the first m components of $\mathbf{u}$, and $\mathbf{u}_2$ denotes the remaining 2 components, we rewrite the system as

$$\frac{d\mathbf{u}}{dt} + A\mathbf{u} = \frac{d}{dt}\left[\begin{array}{c}\mathbf{u}_1 \\ \hline \mathbf{u}_2\end{array}\right] + \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22}\end{array}\right]\left[\begin{array}{c}\mathbf{u}_1 \\ \hline \mathbf{u}_2\end{array}\right] = \left[\begin{array}{c}\mathbf{F}_1 \\ \hline \mathbf{F}_2\end{array}\right] = \mathbf{F}. \tag{3.10}$$

We have permitted an external forcing term $\mathbf{F}$. The nonlinear dependence of A is given by the operator representations:

$$\begin{aligned}
A_{11}(\mathbf{u}_1) &= -\nu I\Delta + \mathbf{P}\mathbf{u}_1\cdot\nabla, \\
A_{12}(\mathbf{u}_2) &= \mathbf{P}[\ \rho^{-1}(-\phi_{x_1},\ldots,-\phi_{x_m})^T \mid \rho^{-1}(\phi_{x_1},\ldots,\phi_{x_m})^T\ ], \\
A_{22}(\mathbf{u}_1,\mathbf{u}_2) &= -\mathrm{diag}(D_n,D_p)\,\Delta + \mathrm{diag}(\mu_n\Delta\phi, -\mu_p\Delta\phi) \\
+\sum_{i=1}^{m}\mathrm{diag}(u_i &+ \mu_n\phi_{x_i}, u_i - \mu_p\phi_{x_i})(\partial/\partial x_i).
\end{aligned} \tag{3.11}$$

In the above system, the function ϕ has been used implicitly in its dependence upon n,p. We make this explicit:

$$\phi = \Phi(\mathbf{u}_2), \quad \text{where} \quad -\varepsilon\nabla^2\phi = \mathbf{u}_2\cdot(-1,1)^T + D. \tag{3.12}$$

The assumptions on the smoothing map Φ are specified later. It is most convenient to rewrite the entire system in operator/vector format.

If we define the diagonal matrix D_{m+2} by

$$D_{m+2} = \mathrm{diag}\ (\nu,\ldots,\nu,D_n,D_p), \tag{3.13}$$

and the matrices a_i and b by

$$a_i(\mathbf{u}) = \mathrm{diag}\ (\mathbf{u}_1, u_i + \mu_n\phi_{x_i}, u_i - \mu_p\phi_{x_i}), i = 1,\ldots,m,$$

$$b(\mathbf{u}) = b(\mathbf{u}_2) = \begin{bmatrix} 0 & \rho^{-1}(-\phi_{x_1},\ldots,-\phi_{x_m})^T & \rho^{-1}(\phi_{x_1},\ldots,\phi_{x_m})^T \\ 0 & (\mu_n\Delta\phi, 0)^T & (0,-\mu_p\Delta\phi)^T \end{bmatrix},$$

then the system may be written,

$$\mathbf{u}_t - D_{m+2}\Delta\mathbf{u} + \mathcal{P}E(\mathbf{u})\mathbf{u} = \mathbf{F}(t,\mathbf{u}), \tag{3.14}$$

where $A(\mathbf{u}) = -D_{m+2}\Delta + \mathcal{P}E(\mathbf{u})$, and

$$E(\mathbf{u}) = \left[\sum_{i=1}^{m} a_i(\mathbf{u})\frac{\partial}{\partial x_i} + b(\mathbf{u})\right], \mathcal{P} = \begin{bmatrix} \mathrm{P} & 0 \\ 0 & \mathrm{I}_2 \end{bmatrix}, \tag{3.15}$$

with I_2 the identity matrix of order two. Finally, the following assumption on $\mathbf{F}$ is made for consistency: $\mathcal{P}\mathbf{F} = \mathbf{F}$. This model was first analyzed in [13].

4 A Semigroup Framework

4.1 Basic Facts

We begin with some familiar terminology.

Definition 4.1. *Let U be a closed linear operator with domain and range dense in a Banach space X. Denote by $R(\lambda, U)$ the resolvent $(\lambda I - U)^{-1}$ for λ in the resolvent set of U. For $M > 0$ and $\omega \in R$ denote by $G(X, M, \omega)$ the set of all operators $A = -U$ such that*

$$\|[R(\lambda, U)]^r\| \le M(\lambda - \omega)^{-r},\ r \ge 1,\ \lambda > \omega.$$

Finally,

$$G(X) = \cup_{\omega,M} G(X, M, \omega).$$

The operators U are generators of strongly continuous semigroups on X. Systematic theories date to the volumes of Hille [8] and Hille-Phillips [9]. Canonical examples include self-adjoint extensions (weak = strong) of symmetric, semi-bounded operators [6]. Friedrichs' work, continued in [7], represented a major advance in the applications of semigroup theory to partial differential equations. Later, Kato considered the question of when the generation property extends to smooth spaces Y embedded in X. The following lemma is the cornerstone.

Proposition 4.1. *Suppose Y is a Banach space densely and continuously embedded in X and $S : Y \mapsto X$ is an isomorphism. We write $\|v\|_Y = \|Sv\|_X$. Suppose $A \in G(X, M, \omega)$ such that*

$$A_1 = SAS^{-1} = A + B, \tag{4.1}$$

where B is a bounded linear operator on X and

$$\mathcal{D}(A_1) = \{v : AS^{-1}v \in Y\}.$$

Then the semigroup generated by $-A$, restricted to Y, is the semigroup generated by the restriction of $-A$ to $\{v \in Y \cap \mathcal{D}(A) : Av \in Y\}$. In fact, $Se^{-tA}S^{-1} = e^{-tA_1}$ holds. It follows that $A_1 \in G(X, M, \omega + M\|B\|)$ so that $A \in G(Y, M, \omega_1)$, with $\omega_1 = \omega + M\|B\|$.

Kato used this idea to construct the evolution operators [15, 16]. This theory, as improved by Dorroh [5], is presented in [12, Ch. 6].

4.2 The General Initial-Value Problem

The evolution operators constructed for the linear theory were used in conjunction with appropriate fixed point theory to analyze nonlinear Cauchy problems in a reflexive Banach space X, of the form,

$$\frac{du}{dt} + A(t,u)u = F(t,u), \quad u(0) = u_0, \tag{4.2}$$

where $A(t,u) \in G(X, M, \omega)$ for u restricted to a subset of a 'smooth' reflexive Banach space Y, densely and continuously embedded in X. The function $F(t,u)$ is required to be in Y, and a solution $u(t) \in Y,\ 0 \leq t \leq T$ is sought with derivative, du/dt required to belong to an intermediate space, V. The mapping $(t,u) \mapsto A(t,u)$ is required to be continuous into $B[Y,X]$ and $(t,u) \mapsto F(t,u)$ is required to be strongly continuous into X. It is required that $A(t,u)$ and $F(t,u)$ satisfy a (uniform in t) Lipschitz condition:

$$\|A(t,u) - A(t,v)\|_{Y,X} \leq C_A \|u - v\|_X, \quad \|F(t,u) - F(t,v)\|_X \leq C_F \|u - v\|_X.$$

A similarity relation connecting A to A_1 as in (4.1) is required for this smooth theory. Kato developed these ideas in [17, 18], and an illustration of the versatility of the theory was demonstrated in [10] where the vacuum field equations of general relativity and the general problem of elastodynamics were studied. A presentation of the nonlinear Kato theory was given in [12, Ch. 7]. Later extensions of the theory, with some discussion of the initial/boundary problem, appeared in [19, 20]. When symmetrizers are required to expedite the semigroup theory, as is the case in the first example, a more general initial value problem of the form,

$$A_0(t,u)\frac{du}{dt} + A(t,u)u = F(t,u), \quad u(0) = u_0,$$

must be considered. The Lipschitz properties are appropriately generalized, and equivalent norms involving the symmetrizer are necessary. This technique appears most notably in [10].

4.3 The Rothe Method

Use of the Kato framework, but with the substitution of Rothe's method of lines for the evolution operators, first appeared in [11] (see also [4]). The models presented earlier in this paper were analyzed in this way in [13, 14]. There is a distinct advantage to this approach in terms of estimating explicit sufficient conditions for local well-posedness. This is described in the following subsections.

If Δt is given as the ratio T/N, then the method of horizontal lines applied to (4.2) yields a semidiscrete set of implicit equations,

$$A(t_k, u_k^N)u_k^N + (1/\Delta t)u_k^N = (1/\Delta t)u_{k-1}^N + F(t_k, u_k^N),\ k = 1, \ldots, N. \tag{4.3}$$

If we set $\mu^2 = 1/\Delta t = N/T$, then the u_k^N can be characterized formally as fixed points of

$$Qv = Q_k^N v = -R(\mu^2 - 1, -A(t_k, v))v + \mu^2 R(\mu^2 - 1, -A(t_k, v))u_{k-1}^N +$$

$$+R(\mu^2-1,-A(t_k,v))F(t_k,v). \tag{4.4}$$

By repeated back substitution, one obtains the following useful formula for u^N_{k-1}:

$$u^N_{k-1}=\prod_{j=1}^{k-1}\mu^2R(\mu^2,-A(t_j,u^N_j))u_0+\sum_{j=1}^{k-1}(\mu^2)^{k-1-j}\prod_{i=j}^{k-1}R(\mu^2,-A(t_i,u^N_i))F(t_j,u^N_j). \tag{4.5}$$

Pivotal to the entire study is the demonstration of the existence of fixed points for this map within an appropriately smooth ball. The concept of stability proves useful in estimating the contraction constant for the mapping Q on both X and Y when (4.4, 4.5) are used.

4.4 The Invariance and Lipschitz Constant for Q

The invariant set B on which Q is defined is described:

$$B=\{u\in Y:\|u\|_Y\le\sigma\|u_0\|_Y,\ \|u\|_X\le\sigma\|u_0\|_X\}.$$

For convenience, we take $\|\cdot\|_X\le\|\cdot\|_Y$. Here, $\sigma>1$ is to be determined, and represents the a parameter related to the local nature of the analysis. Since Y is assumed reflexive, B is a complete metric subspace of X. One must show that $QB\subset B$, independent of k,N, and that Q is a strict $X-$ contraction. As shown, for example, in [13], one can set

$$\sigma=\sigma(\delta,\rho,M,\omega),$$

where the stability constants M,ω are natural extensions to products of resolvents on X and Y of the constants introduced in Definition 4.1, and where δ,ρ are selectable parameters. For the mapping Q, there are four terms to estimate in verifying invariance. We illustrate by studying one of these terms: For given $\delta>0$ and $\rho>0$, if the integer N satisfies:

$$\mu^2=\frac{1}{\Delta t}=\frac{N}{T}>[2(1+\delta^{-1})M+(\rho+1)(\omega+1)], \tag{4.6}$$

then $\mu^2\ge(1+\rho)(\omega+1)$ so that

$$\frac{\mu^2}{\mu^2-\omega-1}\left(\frac{\mu^2}{\mu^2-\omega}\right)^{k-1}\le\left(\frac{\mu^2}{\mu^2-\omega-1}\right)^N\le e^{(1+1/\rho)(1+\omega)T}=\frac{\sigma}{2M(1+\delta)}$$

if σ is defined by

$$\sigma=2(1+\delta)Me^{(1+1/\rho)(1+\omega)T}.$$

The term to be estimated is actually the product of this term with $M\|u_0\|_X$, so that the final upper bound for this term is of the form

$$\frac{\sigma}{2(1+\delta)}\|u_0\|_X.$$

There is a second upper estimate of this form, and two of the form

$$\frac{\sigma\delta}{2(1+\delta)}\|u_0\|_X,$$

so that one finally arrives at an upper bound of $r = \sigma\|u_0\|_X$. The estimates have thus been designed so that $X-$norm invariance holds. By employing the similarity transformation, one also obtains $Y-$norm invariance. The estimation of the contraction constant in the $X-$norm (which is all that is required) reveals proportionality to $1/(\mu^2 - \omega - 1)$, which can be made arbitrarily small by the choice of Δt.

4.5 Parameter Selection

M is generally determined as a function of the admissible radius r of B in Y and the terminal time T, while ω is typically a function of r. In terms of these, one can define numbers δ and ρ. Set $\gamma = 1 + \omega$, and select ρ satisfying

$$2M(r,T)\|\mathbf{u}_0\|_Y e^{(1+1/\rho)\gamma(r)T} < r.$$

This is possible if the *assumption*,

$$2M(r,T)\|\mathbf{u}_0\|_Y e^{1\gamma(r)T} < r, \tag{4.7}$$

holds for a particular pair r, T. For such r, T, define δ by the relation:

$$1 + \delta = re^{-(1+1/\rho)\gamma T}/(2M\|\mathbf{u}_0\|_Y).$$

It is immediate that

$$2(1+\delta)Me^{(1+1/\rho)\gamma T}\|\mathbf{u}_0\|_Y = r,$$

which permits the definition of σ consistent with the previous subsection:

$$\sigma = 2(1+\delta)M(r,T)e^{(1+1/\rho)\gamma(r)T}.$$

5 Discussion of the Examples

We concisely describe how the theory outlined in the previous section is interpreted for the examples discussed earlier in the paper.

5.1 The Function Spaces and the Isomorphism

We introduce the classical Bessel potential space $H^s(R^m; R^k)$. It can be characterized, via the isometric Fourier transform $\mathcal{F}$, as the linear space of functions v with norm,

$$\|v\|_{H^s}^2 = \int_{R^m} (1+|x|^2)^s |\mathcal{F}v(x)|^2 \, dx, \; s > 0.$$

It follows from the definition that the diagonal operator $S = I_k(I - \Delta)^{s/2}$ induces an isometry of $H^s(R^m; R^k)$ onto $L_2(R^m; R^k)$.

We may now define, for the second example,

$$X = X_1 \otimes X_2, \quad X_1 = \mathrm{P}L_2(R^m; R^m), \; X_2 = L_2(R^m; R^2),$$

$$Y = Y_1 \otimes Y_2, \quad Y_1 = \mathrm{P}H^s(R^m; R^m), \; Y_2 = H^s(R^m; R^2).$$

P denotes the projection onto divergence free distributions. The projection for the first example is $\mathrm{P} = \mathrm{I}_8 \otimes \mathrm{P}_3$, where P_3 projects the final three components onto divergence free distributions. Thus, for example one,

$$X = \mathrm{P}L_2(R^3; R^{11}), \; Y = \mathrm{P}H^s(R^3; R^{11}).$$

5.2 The Friedrichs and Kato Inequalities

The Friedrichs inequality [6] derives a semibounded relation for a first order differential operator $A(t)$ with symmetric (matrix) coefficients $a_j = a_j(\mathbf{x}, t), j = 1, \ldots, m$, of $\partial/\partial x_j$ and zeroth order coefficient $b(\mathbf{x}, t)$:

$$(A(t)\mathbf{u}, \mathbf{u})_{L_2} \geq -\omega_t(\mathbf{u}, \mathbf{u})_{L_2},$$

where

$$\omega_t = \frac{1}{2} \sum_{j=1}^{m} \|a_j(\cdot\,, t)\|_{C_b^1} + \|b\|_{C_b}.$$

The subscript b indicates 'boundedness'. The operator $A(t)$ can be thought of as arising in a nonlinear theory via the freezing of coefficients. This allows one to use the Hille-Yosida theorem in the applications to infer the generator property $-A(t) \in G(X, 1, \omega_t)$. In the general nonlinear theory, ω_t depends on the radius r of the admissible balls in Y. This can be combined with the theory of relatively bounded perturbations to cover the case of second order generators. What is *significant* is that the constants ω_t do not depend on the diffusion or viscosity coefficients. This ultimately accounts for stability when these parameters tend to zero.

The commutator estimate is used to deduce the similarity relation,

$$SA(t)S^{-1} = A(t) + B(t), \; B \in B[X],$$

for a first order operator, and this can easily be extended to the case where the first order operator is a relatively bounded perturbation of a second order operator. The result for first order operators uses the relation,

$$SA(t)S^{-1} = A(t) + \sum_{j=1}^{m} [S, a_j] \left(\frac{\partial}{\partial x_j} \right) S^{-1} + [S, b]S^{-1} = A(t) + B(t),$$

where $[\cdot\,]$ denotes the commutator. Kato used a result of Calderón [1] to deduce the L_2 boundedness of $[S, a_j]$ with bound not exceeding $c\|a_j\|_{C_b^1}$, hence the fact that $B \in B[X]$ when $s > dim/2 + 1$. Here, dim denotes the dimension of the relevant Euclidean space. The condition on s carries over to simple perturbations with constant diffusion or viscosity. However, the variable heat conductivity of the first example requires a more stringent assumption on s.

5.3 The Block Resolvent

The second example requires the use of a block resolvent. Since

$$\lambda I + A = \left[\begin{array}{c|c} \lambda I + A_{11} & A_{12} \\ \hline 0 & \lambda I + A_{22} \end{array}\right], \tag{5.1}$$

we have by a standard invertibility result for the block resolvent:

$$R(\lambda, -A) = \left[\begin{array}{c|c} R(\lambda, -A_{11}) & -R(\lambda, -A_{11})A_{12}R(\lambda, -A_{22}) \\ \hline 0 & R(\lambda, -A_{22}) \end{array}\right]. \tag{5.2}$$

The Friedrichs and Kato inequalities are applied to the block diagonal terms in the analysis. The off-diagonal block must be separately estimated.

5.4 Regularization Near the Vacuum

The symmetrization employed for the first example introduces terms which can become singular at vacuum and absolute zero. Regularization is employed and it is the regularized problem which is analyzed. An 'a posteriori' study related to the positivity assumptions on the initial conditions then allows for (local) well-posedness of the original Cauchy problem.

5.5 Solutions on the Space-Time Domain

The semidiscrete solutions are interpolated by piecewise linear (in time) functions. Limit points of this set as $\Delta t \to 0$ can be shown to be solutions in the models of interest. The arguments are technical, but now standard, making use of the Aubin lemma [23]. The regularity property,

$$\mathbf{u} \in C([0,T]; H^s),$$

requires a postconvergence proof, analogous to that contained in [21, pp. 44–46], to strengthen the regularity statement derived from sequential convergence:

$$\mathbf{u} \in L_\infty([0,T]; H^s).$$

5.6 Explicit Criteria for Local Existence

We will illustrate the local hypothesis for the second example. One estimates

$$\omega(r) = a + br,$$

for positive constants a, b. M is a more complicated expression which we may write as

$$M(r,T) = \frac{(c + dr)e^{(1+1/\rho_0)\omega(r)T}}{\omega(r)},$$

for appropriate constants, c, d, ρ_0. This is inclusive as can be seen from the following cases.

1. T **is given**

   If we write,

$$H(r,T) = \frac{1}{2}e^{-(1+\alpha\omega(r))T}h(r), \quad h(r) = \frac{r(a+br)}{c+dr},$$

   where $\alpha = 2 + 1/\rho_0$, then we may maximize $H(\cdot\,,T)$ as a function of r. We find that r is determined by

$$\frac{h'(r)}{h(r)} = \alpha bT. \tag{5.3}$$

   By direct computation,

$$\frac{h'(r)}{h(r)} = \frac{c(a+br) + br(c+dr)}{r(a+br)(c+dr)}.$$

   The latter function is strictly decreasing on $(0,\infty)$, and satisfies

$$\frac{h'(r)}{h(r)} \to \infty,\ r \to 0+;\ \frac{h'(r)}{h(r)} \to 0,\ r \to \infty. \tag{5.4}$$

   It follows that (5.3) has a unique solution, $r(T)$. In this case where T is arbitrary, the condition (4.7) reduces to: $\|\mathbf{u}_0\|_Y < H(r(T),T)$.

2. $\|\mathbf{u}_0\|$ **is given**

   Since $h(r)$ is a strictly increasing mapping of $(0,\infty)$ onto itself, there is a unique r_0 such that

$$\|\mathbf{u}_0\|_Y = \frac{1}{2}h(r_0) = H(r_0,0).$$

   It follows that, for each $r > r_0$, there is a $T_0 = T_0(r)$ such that

$$\|\mathbf{u}_0\|_Y < H(r,T),\ \text{for } T < T_0,\quad r \text{ fixed}.$$

   This gives an admissible range of r and T which satisfy (4.7) in this case where $\|\mathbf{u}_0\|_Y$ is arbitrary.

5.7 Summation

The condition (4.7) provides a range of r, T for which the evolution system possesses a smooth solution which remains stable under possible singular limits; the convergence can be measured in L_2 norms, if desired. More precisely, the rate in the norm $C([0,T]; L_2(\mathbb{R}^k)$ is proportional to the parameters ν or κ_0 as these tend to zero. Although (4.7) is sufficient, and not necessary, it does suggest a possible parametric relation to investigate for potential blowup; see [21] for the formulation of such a principle.

References

[1] A.P. Calderón Commutators of singular integral operators, *Proc. Nat. Acad. Sci.*, **53**: 1092–1099, 1965.

[2] Gui-Qiang Chen and Dehua Wang. The Cauchy problem for the Euler equations for compressible fluids. *Handbook of Mathematical Fluid Dynamics*, **1**:421–453, North-Holland, 2002.

[3] G.-Q. Chen, J.W. Jerome, and D. Wang. Compressible Euler-Maxwell equations, *Transport Theory and Statistical Physics* **29**: 311–331, 2000.

[4] M. Crandell and P. Souganidis. Convergence of Difference Approximations of Quasilinear Evolution Equations. *Nonlinear Anal.*, **10**:425–445, 1986.

[5] J.R. Dorroh. A simplified proof of a theorem of Kato on linear evolution equations, *J. Math. Soc. Japan*, **27**:474–478, 1975.

[6] K.O. Friedrichs. The identity of weak and strong extensions of differential operators. *Trans. Amer. Math. Soc.*, **55**:132–151, 1944.

[7] K.O. Friedrichs. Symmetric positive linear differential equations. *Comm. Pure Appl. Math.*, **11**:333–418, 1958.

[8] E. Hille. *Functional Analysis and Semigroups*, Amer. Math. Soc. Col. Publ. **31**, 1948.

[9] E. Hille and R.S.Phillips. *Functional Analysis and Semigroups*, Amer. Math. Soc. Col. Publ. **31** (revised ed.), 1957.

[10] T. Hughes, T. Kato, and J. Marsden. Well-posed quasilinear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Arch. Rat. Mech. Anal.*, **63**:273–294, 1977.

[11] J.W. Jerome. Quasilinear hyperbolic and parabolic systems: contractive semidiscretizations and convergence of the discrete viscosity method. *J. Math. Anal. Appl.*, **90**:185–206, 1982.

[12] J.W. Jerome. *Approximation of Nonlinear Evolution Systems*. Academic Press, 1983.

[13] J.W. Jerome. Analytical approaches to charge transport in a moving medium. *Transport Theory and Statistical Physics*, **31**:333–366, 2002.

[14] J.W. Jerome. The Cauchy problem for compressible hydrodynamic-Maxwell systems: A local theory for smooth solutions, *Differential and Integral Equations* **16**:1345–1368, 2003.

[15] T. Kato. Linear equations of hyperbolic type. *J. Fac. Sc. Univ. Tokyo*, **17**:241–258, 1970.

[16] T. Kato. Linear evolution equations of hyperbolic type II. *J. Math. Soc. Japan*, **25**:648–666, 1973.

[17] T. Kato. Quasi-linear equations of evolution, with applications to partial differential equations, in, *Spectral Theory and Differential Equations* (W. Everitt, ed.), Springer Lecture Notes in Mathematics 448, 1975, pp. 25–70.

[18] T. Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. *Arch. Rational Mech. Anal.*, **58**:181–205, 1975.

[19] T. Kato. Nonlinear equations of evolution in Banach spaces, in, *Nonlinear Functional Analysis and its Applications*, Proc. Sympos. Pure Math., **45**, Part 2, Amer. Math. Soc., 1986, pp. 9–23.

[20] T. Kato. Abstract evolution equations, linear and quasilinear, revisited, in, *Functional Analysis and Related Topics* (H. Komatsu, ed.), Springer Lecture Notes in Mathematics 1540, Springer-Verlag, 1991, pp. 103–125.

[21] A. Majda. *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*. Appl. Math. Sc. **53**, Springer, 1984.

[22] I. Rubinstein. *Electro-Diffusion of Ions*. SIAM Studies in Applied Mathematics, SIAM, Philadelphia, 1990.

[23] R.E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Vol. 49, AMS, Providence, 1997.

[24] R. Temam. *Navier-Stokes Equations* (rev. ed.). North-Holland Publ., 1979.