Mixed-hybrid discretization methods for the linear Boltzmann transport equation

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Abstract

The linear Boltzmann transport equation is discretized using a finite element technique for the spatial variable and a spherical harmonic technique for the angular variable. Based on an even- and odd- angular parity flux decomposition, mixed-hybrid methods combine the advantages of mixed (simultaneous approximation of even- and odd-parity fluxes) and hybrid (use of Lagrange multipliers to enforce interface regularity conditions) methods. Existence and uniqueness are proved for the resulting problems. Beside the well-known primal/dual distinction induced by the spatial variable, the angular variable yields an even/odd distinction for the spherical harmonic expansion order.

Key words: Mixed-hybrid discretization methods, Linear Boltzmann transport equation, Spherical harmonics, P_N approximation

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1 Introduction

The linear Boltzmann transport equation is an integro-differential equation modelizing neutral and charged particle transport. It is extensively used in computational neutron transport for the analysis of nuclear reactors and radiation shields. Such nuclear engineering applications constitute the authors' background and determine the notations used throughout this paper. Assuming isotropic scattering and sources, the mono-energetic time-independent version of the Boltzmann equation for the angular flux $\Psi(\mathbf{r}, \mathbf{\Omega})$ reads [9]

$$\mathbf{\Omega} \cdot \nabla\Psi(\mathbf{r}, \mathbf{\Omega}) + \sigma(\mathbf{r})\Psi(\mathbf{r}, \mathbf{\Omega}) = \sigma_s(\mathbf{r}) \int_S d\Omega \Psi(\mathbf{r}, \mathbf{\Omega}) + s(\mathbf{r}). \tag{1}$$

In (1), the unit vector $\mathbf{\Omega}$ represents the traveling direction of a neutron, S represents the unit sphere in $\mathbb{R}^3$, the nabla operator ∇ acts on the spatial variable $\mathbf{r}$ only, $\sigma(\mathbf{r})$ and $\sigma_s(\mathbf{r})$ are the macroscopic total and scattering cross sections (i.e.reaction probability per unit length) with $\sigma_s \leq \sigma$, and $s(\mathbf{r}, \mathbf{\Omega})$ is the source term. The direction $\mathbf{\Omega}$ is expressed in terms of (θ, ϕ) in spherical coordinates, with θ the colatitude ($\theta \in [0, \pi]$) and ϕ the azimuthal (or polar) angle ($\phi \in [0, 2\pi]$). We adopt the convention that $\int_S d\Omega = 1$. As for boundary conditions, we consider vacuum and reflected boundaries, corresponding respectively to a zero incoming flux in the domain and an incoming flux equals to the spectrally reflected outgoing flux. To avoid re-entrant fluxes, we consider convex domains.

A widely used angular discretization technique for (1) is based on the spherical harmonics, that form an orthogonal basis for square integrable functions on the unit sphere in three dimensions. Angular dependences are then expanded in spherical harmonics series, truncated at order N, leading to a P_N approximation [4,9]. A previous paper [19] dealt with the lowest-order ($N = 1$) of these angular discretizations, that is the P_1 approximation, where no explicit angular dependence remains in the equations. We here intend to generalize this approach to the general P_N approximation.

The transport equation in its integro-differential first-order form (1) was investigated mathematically in [6, Ch.XXI]. A second-order (with respect to the spatial variable) form can be obtained using the even- and odd- (angular) parity decomposition for the angular flux $\Psi(\mathbf{r}, \mathbf{\Omega})$ introduced by Vladimirov [20] (who also gives credit for it to Kuznetsov [8]). This decomposition reads

$$\Psi^{\pm}(\mathbf{r}, \mathbf{\Omega}) = \frac{1}{2} \left(\Psi(\mathbf{r}, \mathbf{\Omega}) \pm \Psi(\mathbf{r}, -\mathbf{\Omega})\right)$$

and yields the following coupled pair of first order equations

$$\mathbf{\Omega} \cdot \nabla \Psi^-(\mathbf{r}, \mathbf{\Omega}) + \sigma \Psi^+(\mathbf{r}, \mathbf{\Omega}) = \sigma_s \phi(\mathbf{r}) + s(\mathbf{r}) \tag{2}$$

$$\mathbf{\Omega} \cdot \nabla \Psi^+(\mathbf{r}, \mathbf{\Omega}) + \sigma \Psi^-(\mathbf{r}, \mathbf{\Omega}) = 0 \tag{3}$$

where $\phi(\mathbf{r}) = \int_S d\Omega \Psi(\mathbf{r}, \mathbf{\Omega}) = \int_S d\Omega \Psi^+(\mathbf{r}, \mathbf{\Omega})$ is the scalar flux. The current vector is given by $\mathbf{J}(\mathbf{r}) = \int_S d\Omega \mathbf{\Omega} \Psi(\mathbf{r}, \mathbf{\Omega}) = \int_S d\Omega \mathbf{\Omega} \Psi^-(\mathbf{r}, \mathbf{\Omega})$. Eliminating the even- or odd-parity flux from (2) or (3) yields a second-order equation in the remaining unknown.

Mixed methods can be obtained based on equations (2-3). These methods yield simultaneous approximations of Ψ^+ and Ψ^-, thus of flux and current (the physically interesting values in computational neutron transport) avoiding errors to propagate from one to the other. Note that, opposite to the traditional mixed methods for purely spatial problems, both unknowns Ψ^+ and Ψ^- are here scalar quantities. Besides, hybrid methods using Lagrange multipliers to enforce interface regularity constraints have been proved useful in nuclear engineering codes [11]. Non-standard finite element discretization methods such as mixed and hybrid methods have been widely studied in the finite element literature for second-order elliptic problems [15,5]. This work aims at generalizing such approaches to the Boltzmann equation in the form (2-3). The complexity of the finite element discretization for the spatial variable is then supplemented with the presence of the P_N discretization for the angular variable.

Specifically, we establish a mathematical setting for mixed-hybrid (i.e. simultaneously mixed and hybrid) discretization methods for the Boltzmann equation, and use this setting to investigate the well-posedness of the resulting problems. This way, we provide insight for both purely hybrid and purely mixed methods, while the mixed-hybrid methods presented here can also be used as such. Mixed-hybrid (primal) methods were first introduced for purely spatial second-order PDE's by Babŭska et al. [3], whose paper strongly influenced the present work.

The paper is organized as follows. Some useful notations and results are first mentioned in section 2. Then a mathematical setting is introduced in section 3, that provides the necessary framework for the investigation in section 4 of the continuous mixed-hybrid problems, that is the abstract problems posed in infinite dimensional spaces. The discrete problems are investigated in section 5. Both continuous and discrete well-posedness results are proved in appendix. Furthermore, illustrative numerical results are given in section 6.

2 Preliminaries

With $n \leq 3$ the number of space dimensions, let V be a convex open bounded Lipschitzian domain of the Euclidean space $\mathbb{R}^n$ with piecewise smooth boundary ∂V. We thus use neutronic standard notations by denoting the considered spatial domain V and keeping the letter $\mathbf{\Omega}$ for the neutron traveling direction. We denote by $L^2(V) = H^0(V)$, $H^1(V)$ and $H(div, V)$ the usual Lebesgue and Sobolev spaces on V with their elements understood in the sense of distributions. In this section, $\|\cdot\|$, $\|\cdot\|_1$ and $\|\cdot\|_{div}$ denote their respective associated norms.

The following theorems [15,17] define the spaces $H^{1/2}(\partial V)$ and $H^{-1/2}(\partial V)$ that respectively contain the trace of functions in $H^1(V)$ and $H(div, V)$. Their correspondence with Sobolev spaces of fractional order is proved in [10]. Also, $\bar{V}$ denotes the closure of V, and $\mathbf{n}$ denotes the unit outward normal vector to the considered (sub)domain.

Theorem 1 (Trace mapping theorem) *The map $v \to v|_{\partial V}$ defined a priori for functions v continuous on $\bar{V}$, can be extended to a continuous linear mapping called the trace map of $H^1(V)$ into $L^2(\partial V)$. The kernel of the trace mapping is denoted $H_0^1(V)$, and its range $H^{1/2}(\partial V)$ is a Hilbert space, subset of $L^2(\partial V)$, equipped with the norm*

$$\|\psi\|_{1/2,\partial V} = \inf_{\{\nu \in H^1(V): \nu|_{\partial V} = \psi\}} \|\nu\|_{1,V}.$$

Theorem 2 (Normal trace mapping theorem) *The map $\mathbf{q} \to \mathbf{n} \cdot \mathbf{q}$ defined a priori for vector functions $\mathbf{q}$ from $(H^1(V))^n$ into $L^2(\partial V)$ can be extended to a continuous linear mapping from $H(div, V)$ onto $H^{-1/2}(\partial V)$, the dual space of $H^{1/2}(\partial V)$, called the normal trace mapping. $H^{-1/2}(\partial V)$ is a Hilbert space with norm*

$$\|\chi\|_{-1/2,\partial V} = \sup_{\{\psi \in H^{1/2}(\partial V): \|\psi\|_{1/2,\partial V} = 1\}} \langle \psi, \chi \rangle$$

where $\langle \psi, \chi \rangle = \int_{\partial V} \psi\, \chi\, d\Gamma$. We also have the characterization

$$\|\chi\|_{-1/2,\partial V} = \inf_{\{\mathbf{q} \in H(div, V):\ \mathbf{n} \cdot \mathbf{q}|_{\partial V} = \chi\}} \|\mathbf{q}\|_{div,V}.$$

The kernel of the normal trace mapping is denoted $H_0(div, V)$.

We have $H^{1/2}(\partial V) \subset L^2(\partial V) \subset H^{-1/2}(\partial V)$.

3 Mathematical setting for transport

3.1 Definitions

We introduce the Lebesgue space $L^2(S \times V)$ defined on the spatio-angular domain. From now on, $(\cdot,\cdot)$ and $\|\cdot\|$ denote the scalar product and norm in $L^2(S \times V)$. Also, we often emphasize the domain where this and subsequent norms are taken, by writing $\|\cdot\|_V$, shorthand notation for $\|\cdot\|_{S\times V}$. Furthermore, we define

$$\begin{aligned}
H(grad, S \times V) &= \{\Psi \in L^2(S \times V) \text{ such that } \forall \mathbf{\Omega} \in S, \nabla\Psi \in L^2(S \times V)\},\\
H(div, S \times V) &= \{\mathbf{\Psi} \in L^2(S \times V) \text{ such that } \forall \mathbf{\Omega} \in S, \nabla \cdot \mathbf{\Psi} \in L^2(S \times V)\},\\
L^2(\mathbf{\Omega}, S \times V) &= \{\Psi \in L^2(S \times V) \text{ such that } \forall \mathbf{\Omega} \in S, \mathbf{\Omega}\Psi \in L^2(S \times V)\}, \text{ and}\\
H(\mathbf{\Omega}\cdot\nabla, S \times V) &= \{\Psi \in L^2(\mathbf{\Omega}, S \times V) \text{ such that } \forall \mathbf{\Omega} \in S, \mathbf{\Omega}\cdot\nabla\Psi \in L^2(S \times V)\},
\end{aligned}$$

The scalar product in $L^2(\mathbf{\Omega}, S \times V)$ is defined as

$$(\Psi_1, \Psi_2)_\Omega = \int_S \int_V (\mathbf{\Omega}\Psi_1)\cdot(\mathbf{\Omega}\Psi_2) dV d\Omega = (\Psi_1, \Psi_2)$$

since $\mathbf{\Omega}$ is a unit vector, and thus $L^2(\mathbf{\Omega}, S \times V) = L^2(S \times V)$. We nevertheless go on using both notations. The other scalar products are defined as follows: in $H(grad, S \times V)$

$$(\Psi_1, \Psi_2)_{grad} = \int_S \int_V \nabla\,\Psi_1 \cdot \nabla\,\Psi_2 \,dV d\Omega + \int_S \int_V \Psi_1\,\Psi_2\,dV d\Omega,$$

in $H(div, S \times V)$

$$(\mathbf{\Psi}_1, \mathbf{\Psi}_2)_{div} = \int_S \int_V \nabla\cdot\mathbf{\Psi}_1\,\nabla\cdot\mathbf{\Psi}_2\,dV d\Omega + \int_S \int_V \mathbf{\Psi}_1\cdot\mathbf{\Psi}_2\,dV d\Omega,$$

and in $H(\mathbf{\Omega}\cdot\nabla, S \times V)$

$$(\Psi_1, \Psi_2)_{\mathbf{\Omega}\cdot\nabla} = \int_S \int_V (\mathbf{\Omega}\cdot\nabla\Psi_1)(\mathbf{\Omega}\cdot\nabla\Psi_2) dV d\Omega + \int_S \int_V \Psi_1\Psi_2 dV d\Omega.$$

The corresponding norms are thus $\|\Psi\|^2_{grad} = \|\Psi\|^2 + \sum_i \|\partial_i\Psi\|^2$ in $H(grad, S \times V)$, $\|\mathbf{\Psi}\|^2_{div} = \sum_i \|\mathbf{\Psi}_i\|^2 + \|\nabla\cdot\mathbf{\Psi}\|^2$ in $H(div, S\times V)$, and $\|\Psi\|^2_{\mathbf{\Omega}\cdot\nabla} = \|\Psi\|^2 + \|\mathbf{\Omega}\cdot\nabla\Psi\|^2$ in $H(\mathbf{\Omega}\cdot\nabla, S\times V)$. Note that $H(grad, S \times V)$ could have been denoted $H^1(S \times V)$ as in Sobolev space theory, but here the first-order derivative has to be taken with respect to only part of the variables, and besides the choice of $H(grad, S \times V)$ makes a more homogeneous set of notation together with $H(div, S \times V)$. These two spaces are Hilbert spaces, and we also have

Theorem 3 *$H(\mathbf{\Omega}\cdot\nabla)$ is a Hilbert space, and $H(grad) \subset H(\mathbf{\Omega}\cdot\nabla) \subset L^2(S \times V)$.*

Proof of theorem 3 One has to show the completeness of $H(\mathbf{\Omega}\cdot\nabla)$. In this view, one can proceed similarly to what is usually done to prove the completeness of the Sobolev space H^1, that is take a Cauchy sequence $\{\Psi_n\}$ in $H(\mathbf{\Omega}\cdot\nabla)$, note that $\{\Psi_n\}$ and $\{\mathbf{\Omega}\cdot\nabla\Psi_n\}$ are Cauchy sequences in $L^2(S\times V)$, and, denoting by Ψ^0 and Ψ^1 their respective limits (in $L^2(S\times V)$), prove using integration by parts that $\mathbf{\Omega}\cdot\nabla\Psi^0 = \Psi^1$, that is $\Psi^0 \in H(\mathbf{\Omega}\cdot\nabla)$. The second part of the theorem follows from $\|\Psi\|_{\mathbf{\Omega}\cdot\nabla} \leq \|\Psi\|_{grad}$. $\square$

Note that an advection operator similar to $\mathbf{\Omega}\cdot\nabla$ was already introduced and studied in [6, XXI §2.2].

3.2 Trace theorems

The following two trace theorems generalize the theorems 1 and 2:

Theorem 4 *There exists a continuous linear map $\Psi \to \Psi|_{\partial V}$ from $H(grad, S\times V)$ into $L^2(S\times\partial V)$. The kernel of this mapping is denoted $H_0(grad, S\times V)$, and its range, denoted $H^{1/2}(S\times\partial V)$, is endowed with the norm*

$$\|\Psi^\alpha\|_{1/2,\partial V} = \inf_{\{\Psi\in H(grad,S\times V):\ \Psi|_{S\times\partial V}=\Psi^\alpha\}} \|\Psi\|_{grad,V}. \tag{4}$$

Proof of theorem 4 Apply theorem 1, the angular variables playing a passive role. $\square$

Theorem 5 *There exists a continuous linear map $\Psi \to \Psi|_{\partial V}$ from $H(\mathbf{\Omega}\cdot\nabla, S\times V)$ onto $H_\Omega^{-1/2}(S\times\partial V)$ where*

$$H_\Omega^{-1/2}(S\times\partial V) = \{\Psi \text{ defined on } S\times\partial V \quad \text{such that } \mathbf{n}\cdot\mathbf{\Omega}\Psi \in H^{-1/2}(S\times\partial V)\},$$

with $H^{-1/2}(S\times\partial V)$ the dual of $H^{1/2}(S\times\partial V)$. The kernel of this mapping is denoted $H_0(\mathbf{\Omega}\cdot\nabla, S\times V)$, and its range $H_\Omega^{-1/2}(S\times\partial V)$ is endowed with the norm

$$\|\Psi\|_{\Omega,-1/2,\partial V} = \|\mathbf{n}\cdot\mathbf{\Omega}\Psi\|_{-1/2,\partial V}, \tag{5}$$

with

$$\|\Psi^\alpha\|_{-1/2,\partial V} = \inf_{\{\mathbf{\Psi}^q\in H(div,S\times V):\ \mathbf{n}\cdot\mathbf{\Psi}^q|_{S\times\partial V}=\Psi^\alpha\}} \|\mathbf{\Psi}^q\|_{div,V}. \tag{6}$$

Proof of theorem 5 $\Psi \in H(\mathbf{\Omega}\cdot\nabla, S\times V)$ if and only if $\mathbf{\Omega}\Psi \in H(div, S\times V)$, and one can apply the normal trace mapping theorem 2. □

3.3 Domain decomposition and tools for hybrid methods

To obtain a hybrid method, we need to subdivide the spatial domain V into a finite family of elements V_l such that $V_l \bigcap V_k = \emptyset$ if $l \neq k$, and $\bar{V} = \bigcup_{l=1}^{L} \bar{V}_l$, with L positive integer. In the sequel, we assume that all the considered elements V_l are open subsets of $\mathbb{R}^n$ with piecewise smooth boundaries. We also define

$$\Gamma = \bigcup_l \partial V_l.$$

The spaces $H_0^{1/2}(S\times\Gamma)$ and $H_{\Omega,0}^{-1/2}(S\times\Gamma)$ respectively designate the subsets of $H^{1/2}(S\times\Gamma)$ and $H_\Omega^{-1/2}(S\times\Gamma)$ whose elements vanish on the outer boundary ∂V. Besides, with the subdivision just described, norms in $H_{(0)}^{1/2}(S\times\Gamma)$ and $H_{\Omega,(0)}^{-1/2}(S\times\Gamma)$ are respectively $\|\Psi\|_{1/2,\Gamma}^2 = \sum_l \|\Psi\|_{1/2,\partial V_l}^2$ and $\|\Psi\|_{\Omega,-1/2,\Gamma}^2 = \sum_l \|\Psi\|_{\Omega,-1/2,\partial V_l}^2$. Furthermore, we introduce

$$\begin{aligned} X &= \{\Psi \in L^2(S\times V) \text{ such that } \forall l, \Psi|_{S\times V_l} \in H(grad, S\times V_l)\} \text{ and} \\ X_\Omega &= \{\Psi \in L^2(\mathbf{\Omega}, S\times V) \text{ such that } \forall l, \Psi|_{S\times V_l} \in H(\mathbf{\Omega}\cdot\nabla, S\times V_l)\} \end{aligned}$$

with corresponding norms $\|\Psi\|_X = \sum_l \|\Psi\|_{grad,V_l}$ and $\|\Psi\|_{X_\Omega} = \sum_l \|\Psi\|_{\mathbf{\Omega}\cdot\nabla,V_l}$. The following two theorems (generalizing propositions 1.1 and 1.2 of [5, p.95]) tell us how to enforce the interface regularity (with respect to the spatial variable) of members of $X_{(\Omega)}$.

Theorem 6

$$\begin{aligned} H(grad, S\times V) = \Big\{\Psi^\pm \in X : \sum_l \int_S d\Omega \int_{\partial V_l} d\Gamma\, \mathbf{\Omega}\cdot\mathbf{n}\, \tilde{\Psi}^\alpha\, \Psi^\pm = 0 \\ \forall \tilde{\Psi}^\alpha \in H_{\Omega,0}^{-1/2}(S\times\Gamma)\ \Big\}. \end{aligned}$$

Proof of theorem 6 Following theorem 5, there exists a Ψ in $H(\mathbf{\Omega}\cdot\nabla, S\times V)$ whose restrictions to each $S\times V_l$ have traces equal to the corresponding restrictions of $\tilde{\Psi}^{\alpha} \in H_{\Omega,0}^{-1/2}(S \times \Gamma)$. Thus $\Psi \in H_0(\mathbf{\Omega} \cdot \nabla, S \times V)$. Clearly, $H(grad, S \times V) \subset X$, and if $\Psi^{\pm} \in H(grad, S \times V)$, we obtain successively

$$\begin{aligned}\sum_l \int_S d\Omega \int_{\partial V_l} d\Gamma\, \mathbf{\Omega} \cdot \mathbf{n}\, \tilde{\Psi}^{\alpha}\, \Psi^{\pm} &= \sum_l \int_S d\Omega \int_{V_l} dV\, \mathbf{\Omega} \cdot \nabla \left(\Psi\, \Psi^{\pm}\right) \\ &= \int_S d\Omega \int_V dV\, \mathbf{\Omega} \cdot \nabla \left(\Psi\, \Psi^{\pm}\right) \\ &= \int_S d\Omega \int_{\partial V} d\Gamma\, \mathbf{\Omega} \cdot \mathbf{n}\, \tilde{\Psi}^{\alpha}\, \Psi^{\pm},\end{aligned}$$

which vanishes for any $\tilde{\Psi}^{\alpha} \in H_{\Omega,0}^{-1/2}(S \times \Gamma)$. Conversely, the proposed characterization of $H(grad, S \times V)$ implies using integration by parts

$$\int_S d\Omega \int_V dV\, \Psi^{\pm}\, (\mathbf{\Omega} \cdot \nabla\, \Psi) = -\int_S d\Omega \int_V dV\, \left(\nabla\, \Psi^{\pm}\right) \cdot \mathbf{\Omega}\, \Psi \qquad \forall \Psi \in H_0(\mathbf{\Omega} \cdot \nabla, S \times V),$$

and thus for all Ψ in $C_0^{\infty}(V)$ (i.e. compactly supported infinitely continuously derivable),

$$\left| \int_S d\Omega \int_V dV\, \Psi^{\pm}\, (\mathbf{\Omega} \cdot \nabla\, \Psi) \right| \leq \left(\sum_l \|\, \Psi^{\pm}\, \|^2_{grad, V_l} \right)^{1/2} \|\mathbf{\Omega}\, \Psi\|.$$

Since $(\nabla\, \Psi^{\pm}, \mathbf{\Omega}\, \Psi) = -(\Psi^{\pm}, \nabla \cdot \mathbf{\Omega}\, \Psi)$ in the distribution sense, $(\nabla\, \Psi^{\pm}, \cdot)$ is thus a bounded linear functional that can be extended to $L^2(S \times V)$. Hence $\Psi^{\pm} \in H(grad, S \times V)$. $\square$

Similar arguments lead to

Theorem 7

$$H(\mathbf{\Omega} \cdot \nabla, S \times V) = \Big\{ \Psi^{\pm} \in X_{\Omega} : \sum_l \int_S d\Omega \int_{\partial V_l} d\Gamma\, \mathbf{\Omega} \cdot \mathbf{n}\, \tilde{\Psi}^{\alpha}\, \Psi^{\pm} = 0 \qquad \forall \tilde{\Psi}^{\alpha} \in H_0^{1/2}(S \times \Gamma)\ \Big\}.$$

4 Continuous mixed-hybrid problems

4.1 Weak form derivation

We apply the distributions in equations (2) and (3) respectively to test functions $\tilde{\Psi}^{+}(\mathbf{r}, \mathbf{\Omega})$ and $\tilde{\Psi}^{-}(\mathbf{r}, \mathbf{\Omega})$. These test functions are respectively of even-and odd- angular parity. Integrating over space and angle, we obtain:

$$\int_S d\Omega \int_V dV \tilde{\Psi}^+ \Big(\mathbf{\Omega} \cdot \nabla \Psi^- + \sigma \Psi^+ - \sigma_s \int_S d\Omega' \Psi^+ \Big) = \int_S d\Omega \int_V dV \; s \; \tilde{\Psi}^+ \tag{7}$$

and

$$\int_S d\Omega \int_V dV \tilde{\Psi}^- \Big(\mathbf{\Omega} \cdot \nabla \Psi^+ + \sigma \Psi^- \Big) = 0. \tag{8}$$

Integrating by parts (with respect to $\mathbf{r}$), and introducing the notations $\Psi^\psi(\mathbf{\Omega}, \mathbf{r})$ and $\Psi^\chi(\mathbf{\Omega}, \mathbf{r})$ to respectively represent the traces on ∂V of Ψ^+ and Ψ^-, we get successively

$$\begin{aligned} &- \int_S d\Omega \int_V dV \mathbf{\Omega} \cdot \nabla \tilde{\Psi}^+ \Psi^- + \int_S d\Omega \int_{\partial V} d\Gamma \, \mathbf{\Omega} \cdot \mathbf{n} \tilde{\Psi}^+ \Psi^\chi \\ &+ \int_S d\Omega \int_V dV \tilde{\Psi}^+ \left(\sigma \Psi^+ - \sigma_s \int_S d\Omega' \Psi^+ \right) = \int_S d\Omega \int_V dV \; s \; \tilde{\Psi}^+ \end{aligned} \tag{9}$$

and

$$\begin{aligned} &- \int_S d\Omega \int_V dV \; \Psi^+ \mathbf{\Omega} \cdot \nabla \tilde{\Psi}^- + \int_S d\Omega \int_{\partial V} d\Gamma \, \mathbf{\Omega} \cdot \mathbf{n} \tilde{\Psi}^- \Psi^\psi \\ &+ \int_S d\Omega \int_V dV \sigma \tilde{\Psi}^- \Psi^- = 0. \end{aligned} \tag{10}$$

Equations (8) and (9) lead to the mixed primal weak form, while (7) and (10) lead to the mixed dual weak form. We now introduce hybridization, keeping only weak (but natural) regularity conditions at the interfaces of the spatially decomposed domain. In this view, we restrict the above equations to $S \times V_l$ (that is spatial integrations are now taken on V_l and ∂V_l), and sum them up. In the choice of approximation spaces, regularity requirements can then be restricted to each $S \times V_l$ separately, and interface regularity conditions are enforced by a third equation arising from theorem 6 or 7. In the sequel, we require $\sigma_l(\mathbf{r})$ and $\sigma_{s,l}(\mathbf{r})$ to be in $C^\infty(V_l)$. Using a subscript l to denote restrictions to $S \times V_l$, the mixed-hybrid primal weak form equations are from (8) and (9)

$$\begin{cases} - \sum_l \int_S d\Omega \int_{V_l} dV \; \Psi_l^- \, \mathbf{\Omega} \cdot \nabla \tilde{\Psi}_l^+ + \int_S d\Omega \int_{\partial V_l} d\Gamma \, \mathbf{\Omega} \cdot \mathbf{n} \, \tilde{\Psi}_l^+ \Psi_l^\chi \\ \quad + \int_S d\Omega \int_{V_l} dV \tilde{\Psi}_l^+ \left(\sigma_l \Psi_l^+ - \sigma_{s,l} \int_S d\Omega' \Psi_l^+ \right) = \sum_l \int_S d\Omega \int_{V_l} dV \; s \; \tilde{\Psi}_l^+ \\ \sum_l \int_S d\Omega \int_{V_l} dV \tilde{\Psi}_l^- \mathbf{\Omega} \cdot \nabla \Psi_l^+ + \int_S d\Omega \int_{V_l} dV \sigma_l \tilde{\Psi}_l^- \Psi_l^- = 0 \\ \sum_l \int_S d\Omega \int_{\partial V_l} d\Gamma \, \mathbf{\Omega} \cdot \mathbf{n} \, \tilde{\Psi}_l^\chi \Psi_l^+ = 0, \end{cases} \tag{11}$$

and we consider two possibilities for the choice of spaces where to find the unknowns (the denominations introduced here will be motivated by the angular discretization; they refer to the parity of the angular expansion order N in section 5):

- even-order primal choice: we look for $(\Psi^+, \Psi^-, \Psi^\chi)$ in $X \times L^2(\mathbf{\Omega}, S \times V) \times H_\Omega^{-1/2}(S \times \Gamma)$ for all $(\tilde{\Psi}^+, \tilde{\Psi}^-, \tilde{\Psi}^\chi)$ in $X \times L^2(\mathbf{\Omega}, S \times V) \times H_{\Omega,0}^{-1/2}(S \times \Gamma)$ (in this case we require s to be in X'),
- odd-order primal choice: we look for $(\Psi^+, \Psi^-, \Psi^\chi)$ in $X_\Omega \times L^2(S \times V) \times H^{1/2}(S \times \Gamma)$ for all $(\tilde{\Psi}^+, \tilde{\Psi}^-, \tilde{\Psi}^\chi)$ in $X_\Omega \times L^2(S \times V) \times H_0^{1/2}(S \times \Gamma)$ (in this case we require s to be in X'_Ω).

Besides, we require Ψ^- and Ψ^χ to be odd in $\mathbf{\Omega}$, and Ψ^+ to be even in $\mathbf{\Omega}$. These angular parity properties are assumed throughout the remaining of this study even if they are not explicitely included in the notations (to avoid making them too cumbersome). For the mixed-hybrid dual formulation, we obtain from (7) and (10)

$$\begin{cases} \sum_l \int_S d\Omega \int_{V_l} dV \, \tilde{\Psi}_l^+ \left(\mathbf{\Omega} \cdot \nabla \Psi_l^- + \sigma_l \, \Psi_l^+ - \sigma_{s,l} \, \tilde{\Psi}_l^+ \int_S d\Omega' \Psi_l^+ \right) \\ \quad = \sum_l \int_S d\Omega \int_{V_l} dV \, s \, \tilde{\Psi}_l^+ \\ - \sum_l \int_S d\Omega \int_{V_l} dV \, \Psi_l^+ \, \mathbf{\Omega} \cdot \nabla \, \tilde{\Psi}_l^- + \int_S d\Omega \int_{\partial V_l} d\Gamma \, \mathbf{\Omega} \cdot \mathbf{n} \, \tilde{\Psi}_l^- \Psi_l^\psi \\ \quad + \int_S d\Omega \int_{V_l} dV \sigma_l \, \tilde{\Psi}_l^- \Psi_l^- = 0 \\ \sum_l \int_S d\Omega \int_{\partial V_l} d\Gamma \, \mathbf{\Omega} \cdot \mathbf{n} \, \tilde{\Psi}_l^\psi \Psi_l^- = 0, \end{cases} \tag{12}$$

and the two choices of spaces are here:

- even-order dual choice: we look for $(\Psi^-, \Psi^+, \Psi^\psi)$ in $X_\Omega \times L^2(S \times V) \times H^{1/2}(S \times \Gamma)$ for all $(\tilde{\Psi}^-, \tilde{\Psi}^+, \tilde{\Psi}^\psi)$ in $X_\Omega \times L^2(S \times V) \times H_0^{1/2}(S \times \Gamma)$ (then $s \in L^2(S \times V)$),
- odd-order dual choice: we look for $(\Psi^-, \Psi^+, \Psi^\psi)$ in $X \times L^2(\mathbf{\Omega}, S \times V) \times H_\Omega^{-1/2}(S \times \Gamma)$ for all $(\tilde{\Psi}^-, \tilde{\Psi}^+, \tilde{\Psi}^\psi)$ in $X \times L^2(\mathbf{\Omega}, S \times V) \times H_{\Omega,0}^{-1/2}(S \times \Gamma)$ (then $s \in L^2(\mathbf{\Omega}, S \times V)$).

Besides, we require Ψ^+ and Ψ^ψ to be even in $\mathbf{\Omega}$, and Ψ^- to be odd in $\mathbf{\Omega}$. Note that $\tilde{\Psi}^\chi$ and $\tilde{\Psi}^\psi$ can be interpreted as Lagrange multipliers in the third equation of the primal and dual weak forms (respectively), enforcing the interface continuity of Ψ^+ and Ψ^- (again respectively).

4.2 Well-posedness

Without loss of generality, any flux $\Psi(\mathbf{r}, \mathbf{\Omega})$ can be extended in spherical harmonics Y_{nm} according to $\Psi = \sum_{n=0}^{\infty} \sum_{|m|<n} (\Psi, Y_{nm})_\Omega \, Y_{nm}$, where the Y_{nm} form an orthonormal basis and $(\Psi, Y_{nm})_\Omega = \int_S d\Omega \, \Psi Y_{nm}$. Clearly, the coefficients $(\Psi, Y_{nm})_\Omega$ must decrease for $n \to \infty$ in order to meet the square integrable requirement with respect to the angular variable. We make this behavior more precise in the next definition.

Definition 8 *We say that a flux* $\Psi(\mathbf{r},\mathbf{\Omega})$ *is* mildly anisotropic *when*

$$(1+\alpha_n)\sum_{|m|<n}\|(\Psi,Y_{nm})_\Omega\|^2 \geq \sum_{|m|<n}\|(\Psi,Y_{n+1,m})_\Omega\|^2 \text{ for any } n\geq 1, n \text{ odd}$$

where

$$\alpha_n \to 0 \text{ when } n\to\infty, \text{ and } \alpha_n < \alpha^* \quad \forall n$$

with $\alpha^* = 2\frac{\sigma_l}{\sigma_{s,l}} - 1$.

Besides, we consider vacuum and reflected boundary conditions, which cover all practical applications. The neutron flux vanishes on vacuum boundaries, and thus, for any $\mathbf{r}$ on such a boundary, one has $\Psi(\mathbf{r},\mathbf{\Omega})=0$, that is, in terms of even- and odd-parity fluxes,

$$\Psi^+(\mathbf{r},\mathbf{\Omega}) = \begin{cases}\Psi^-(\mathbf{r},\mathbf{\Omega}) & \mathbf{\Omega}\cdot\mathbf{n}>0\\ -\Psi^-(\mathbf{r},\mathbf{\Omega}) & \mathbf{\Omega}\cdot\mathbf{n}<0\end{cases} \tag{13}$$

where $\mathbf{n}$ is the outward unit normal on the vacuum boundary. On a reflected boundary perpendicular to the prescribed normal, $\Psi^\pm(\mathbf{r},\theta,\phi) = \Psi^\pm(\mathbf{r},\Pi-\theta,\phi)$.

The following theorem is proved in appendices A.1 (primal case) and A.2 (dual case).

Theorem 9 *Assume boundary or reflected boundary conditions on the external boundary* ∂V. *Assume* σ_l *and* $\sigma_{s,l}$ *constant on each element* V_l, *and* $\sigma_l > \sigma_{s,l} > 0$ *for all* l. *The mixed-hybrid continuous primal and dual weak forms (11) and (12), together with the odd- or even-order choice of spaces described in section 4.1, have a unique solution provided the flux* $\Psi(\mathbf{r},\mathbf{\Omega}) = \Psi^+(\mathbf{r},\mathbf{\Omega}) + \Psi^-(\mathbf{r},\mathbf{\Omega})$ *is mildly anisotropic in each element.*

The hypothesis $\sigma_l > \sigma_{s,l} > 0$ excludes void (i.e. $\sigma_l = 0$), pure scattering ($\sigma_{s,l} = \sigma_l$) and pure absorbing ($\sigma_{s,l} = 0$) media.

Finally, an alternative proof is given in appendices A.1 and A.2, showing that the mild anisotropy assumption in the previous theorem can be replaced by the assumption $\sigma_{s,l} < \frac{4}{5}\sigma_l$, for all l.

5 Discrete mixed-hybrid problems

5.1 Choice of finite-dimensional spaces

The subscript h is used to distinguish the unknowns that have been discretized in both space and angle. We denote by P_h, M_h and B_h (respectively for Plus, Minus and Boundary) the finite-dimensional approximation spaces. In the primal case, we look for a triple $(\Psi_h^+, \Psi_h^-, \Psi_h^\chi) \in P_h \times M_h \times B_h$ that verifies the primal weak form equations (11) for any $(\tilde{\Psi}_h^+, \tilde{\Psi}_h^-, \tilde{\Psi}_h^\chi) \in P_h \times M_h \times B_{h,0}$ [1] with in the even-order formulation

$$P_h \subset X, \quad M_h \subset L^2(\mathbf{\Omega}, S \times V), \quad \text{and} \quad B_{h,(0)} \subset H^{-1/2}_{\Omega,(0)}(S \times \Gamma),$$

and in the odd-order formulation

$$P_h \subset X_\Omega, \quad M_h \subset L^2(S \times V), \quad \text{and} \quad B_{h,(0)} \subset H^{1/2}_{(0)}(S \times \Gamma).$$

Similarly in the dual case, we look for a triple $(\Psi_h^-, \Psi_h^+, \Psi_h^\psi) \in M_h \times P_h \times B_h$ verifying the dual weak form equations (12) for any $(\tilde{\Psi}_h^-, \tilde{\Psi}_h^+, \tilde{\Psi}_h^\psi) \in M_h \times P_h \times B_{h,0}$, with in the even-order formulation

$$M_h \subset X_\Omega, \quad P_h \subset L^2(S \times V), \quad \text{and} \quad B_{h,(0)} \subset H^{1/2}_{(0)}(S \times \Gamma),$$

and in the odd-order formulation

$$M_h \subset X, \quad P_h \subset L^2(\mathbf{\Omega}, S \times V), \quad \text{and} \quad B_{h,(0)} \subset H^{-1/2}_{\Omega,(0)}(S \times \Gamma).$$

We need to introduce both spatial and angular discrete expansions to build P_h, M_h, and B_h.

For the spatial variable, with regularity requirements made natural in our mixed-hybrid formulations, we use limited polynomial expansions whose moments become the coefficients to be determined. Note that this differs from the finite element technique commonly used for standard methods, that leads to determine coefficients that are grid point values (and possibly derivatives at these points) of the unknown(s), consequently enforcing essential regularity requirements. For historical reasons, the technique we use here is called "nodal finite elements" in the nuclear engineering community. We denote by $\mathcal{P}_z(V_l)$

[1] $B_{h,0}$ is the subset of B_h where all the functions vanish on the external boundary ∂V.

the space of polynomials of (total) order lower or equal to z on V_l. Also, for the interface unknowns, we write $\Gamma = \bigcup_g \Gamma_g$ where Γ_g is any smooth closed arc on the interface between two elements or on an element side that is part of the external boundary. Then $\mathcal{P}_b(\Gamma_l)$ is the space of polynomials of (total) order lower or equal to b on Γ_l.

For the angular expansions, we use the spherical harmonics, well-known to form an orthogonal basis for square integrable functions on the unit sphere. As in [11], Y_{nm} ($|m| < n$) designates the real and imaginary parts of the spherical harmonics, with the convention that values $m \geq 0$ refer to the real (cosine) part, and values $m < 0$ refer to the imaginary (sine) part. We do a so-called P_N approximation, that is truncate the series at the value $n = N$ in Y_{nm}. We denote by Y_N^+ and Y_N^- the sets spanned by the spherical harmonics Y_{nm} up to order $n = N$ that are respectively even or odd in $\mathbf{\Omega}$, that is

$$Y_N^+ = span\{Y_{nm}, n \leq N, n \text{ even}\}, \text{ and } Y_N^- = span\{Y_{nm}, n \leq N, n \text{ odd}\}$$

where if N is even $N^+ = N = N^- + 1$ and if N is odd $N^- = N = N^+ + 1$. These sets are used to expand the angular dependence of the internal unknowns $\Psi^{\pm}$. For the interface unknowns Ψ^χ and Ψ^ψ, we define the sets Y_N^χ and Y_N^ψ, and for parity reasons $Y_N^\chi \subset Y_N^-$ and $Y_N^\psi \subset Y_N^+$.

Total expansions are obtained by tensor (or Kronecker) product of spatial and angular expansions, and we are now prepared to define our approximation spaces:

$$\begin{aligned}
P_h &= P_h^{N,p}(V) = \{\Psi^+ : \Psi^+ \in Y_N^+ \otimes \mathcal{P}_p(V_l), \forall l\}, \\
M_h &= M_h^{N,m}(V) = \{\Psi^- : \Psi^- \in Y_N^- \otimes \mathcal{P}_m(V_l), \forall l\}, \\
B_h &= B_h^{N,b}(\Gamma) = \{\Psi^{\chi \text{ or } \psi} : \Psi^{\chi \text{ or } \psi} \in Y_N^{\chi \text{ or } \psi} \otimes \mathcal{P}_b(\Gamma_g), \forall g\}, \text{ and} \\
B_{h,0} &= B_{h,0}^{N,b}(\Gamma) = \{\Psi^{\chi \text{ or } \psi} : \Psi^{\chi \text{ or } \psi} \in B_{h,0}^{N,b}(\Gamma),\ \Psi^{\chi \text{ or } \psi} = 0 \text{ on } \partial V\}.
\end{aligned}$$

Finally, plugging the above described expansions into the weak forms (11) and (12) leads to a linear system in the expansion coefficients.

5.2 Well-posedness

The following theorem is proved in appendices B.1 and B.2.

Theorem 10 *Assume boundary or reflected boundary conditions on the external boundary ∂V. Assume σ_l and $\sigma_{s,l}$ constant on each element V_l, and $\sigma_l > \sigma_{s,l} > 0$ for all l. With the notation introduced above, assume furthermore that the primal mixed-hybrid discrete problems verify either*

- $\nabla P_h^{N,p} \subset \mathbf{\Omega} M_h^{N,m}$ *(true if* $p-1 \leq m$ *and* $N^+ \leq N^- + 1$*), or*
- $\mathbf{\Omega} \cdot \nabla P_h^{N,p} \subset M_h^{N,m}$ *(true if* $p-1 \leq m$ *and* $N^+ + 1 \leq N^-$*),*

and assume that the dual mixed-hybrid discrete problems verify either

- $\mathbf{\Omega} \cdot \nabla M_h^{N,m} \subset P_h^{N,p}$ *(true if* $m-1 \leq p$ *and* $N^- + 1 \leq N^+$ *), or*
- $\nabla M_h^{N,m} \subset \mathbf{\Omega} P_h^{N,p}$ *(true if* $m-1 \leq p$ *and* $N^- \leq N^+ + 1$*).*

Then, these discrete problems have a unique solution provided the flux $\Psi(\mathbf{r},\mathbf{\Omega}) = \Psi^+(\mathbf{r},\mathbf{\Omega}) + \Psi^-(\mathbf{r},\mathbf{\Omega})$ *is mildly anisotropic in each element, and in the primal case, for any* $\Psi^\chi_{h,l} \in B_h^{N,b}(V_l)$ *in any* V_l*,*

$$\int_{\mathbf{\Omega}} d\Omega \int_{\partial V_l} d\Gamma \, \mathbf{\Omega} \cdot \mathbf{n} \, \Psi^\chi_{h,l} \, \Psi^+_{h,l} = 0 \qquad \forall \, \Psi^+ \in P_h^{N,p}(V_l) \;\; \textit{implies that } \Psi^\chi_{h,l} = 0, \tag{14}$$

while in the dual case, for any $\Psi^\psi_{h,l} \in B_h^{N,b}(V_l)$ *in any* V_l*,*

$$\int_{\mathbf{\Omega}} d\Omega \int_{\partial V_l} d\Gamma \, \mathbf{\Omega} \cdot \mathbf{n} \, \Psi^\psi_{h,l} \, \Psi^-_{h,l} = 0 \qquad \forall \, \Psi^- \in M_h^{N,m}(V_l) \;\; \textit{implies that } \Psi^\psi_{h,l} = 0. \tag{15}$$

Conditions (14) and (15) are in fact equivalent to rank conditions. If the mixed-hybrid weak forms are given a matrix system shape, such conditions typically requires the matrix coupling internal and interface expansions in each V_l to have maximum rank. Since the total expansions are obtained by tensor product of spatial and angular expansions, both conditions (14) and (15) lead here to a spatial and an angular rank condition.

The spatial rank condition requires the matrix coupling internal and interface spatial expansions in each V_l to have maximum rank. This condition is well-known in the hybrid finite element literature [14], and was detailed in [19]. It establishes a dependence between the main internal expansion order (p or m) and the interface expansion order (b). Thus it characterizes the interface spatial expansion once the internal spatial expansions are defined.

The angular rank condition requires the matrix coupling internal and interface angular expansions to have maximum rank. It characterizes the interface angular expansions (thus $Y_N^{\chi \text{ or } \psi}$) once the internal angular expansions are defined. Although the choice is not unique, this condition can be satisfied using the following interface angular expansions:

$$Y_N^{\chi} = span\Big\{ \sum_{l_j,m_j} (\int_S d\Omega\, \mathbf{\Omega}\cdot\mathbf{n}\, Y_{pq}\, Y_{l_j\, m_j})\, Y_{l_j\, m_j} \quad \text{with } |m_j| \le l_j \le N,$$

$$\text{such that } \int_S d\Omega\, \mathbf{\Omega}\cdot\mathbf{n}\, Y_{pq}\, Y_{l_j m_j} \ne 0, \text{ where } Y_{pq} \in Y_N^{+} \Big\}, \qquad (16)$$

$$Y_N^{\psi} = span\Big\{ \sum_{l_j,m_j} (\int_S d\Omega\, \mathbf{\Omega}\cdot\mathbf{n}\, Y_{pq}\, Y_{l_j\, m_j})\, Y_{l_j\, m_j} \quad \text{with } |m_j| \le l_j \le N,$$

$$\text{such that } \int_S d\Omega\, \mathbf{\Omega}\cdot\mathbf{n}\, Y_{pq}\, Y_{l_j m_j} \ne 0, \text{ where } Y_{pq} \in Y_N^{-} \Big\}. \qquad (17)$$

These expansions correspond to what is known in the nuclear engineering community as the Rumyantsev boundary conditions [16]. The l_j in the characterization (16) of Y_N^{χ} are necessarily odd, while those in (17) are necessarily even.

While the rank conditions are typical of hybrid methods, inclusion conditions of the type $\nabla P_h \subset \mathbf{\Omega} M_h$, $\mathbf{\Omega}\cdot\nabla P_h \subset M_h, \ldots$ are typical of mixed methods. The main difference with the usual condition for mixed formulations of (purely spatial) second order elliptic equations is that our conditions involve not only the nabla operator ∇, but also $\mathbf{\Omega}$.

Taking the maximal spatial and angular order allowed by theorem 10, we can define four methods as

- even-order primal ($p = m + 1$ and N even),
- odd-order primal ($p = m + 1$ and N odd),
- even-order dual ($m = p + 1$ and N even), and
- odd-order dual ($m = p + 1$ and N odd).

The primal/dual distinction related to the spatial variable and therefore well-known in the second-order PDE's literature, is thus completed with an even/odd order P_N approximation distinction related to the angular variable. Even-order P_N approximations have long been neglected in computational neutron transport. This work shows that equal interest should in fact be given to even- and odd-order P_N approximations.

Note that the mild anisotrpy assumption is not very restrictive since such an assumption is implicetely made whenever a truncated spherical harmonic series is used to extend an angularly dependent function.

6 Numerical results

We consider here a shielding problem known as the Azmy [1] benchmark. This two-dimensional problem is based on a "square in a square" geometry depicted on figure 1: a square whose bottom left quarter (zone 1) has a constant

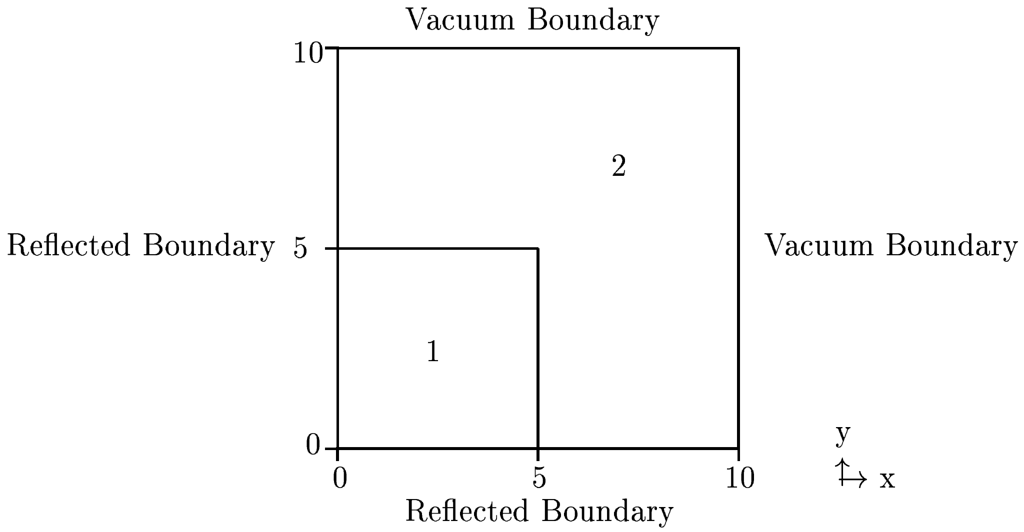

Fig. 1. Azmy benchmark geometry

non-zero source and whose three other quarters (zone 2) are source-free. The cross-section data are detailed in [1]. Boundary conditions are reflected on the bottom and left edges, and vacuum on the right and top edges. Square elements forming a 32×32 grid were used to decompose the domain.

To concentrate on the effect of the different P_N approximations, we used fixed spatial polynomial expansion order, namely 6th order inside ($p = 6 = m + 1$ in the primal case), and 2nd order on interfaces ($b = 2$). Such choice verify the spatial rank condition described in section 5.2. Also, we used the interface angular expansions (16) and (17) corresponding to the Rumyantsev conditions, such that the angular rank condition is verified.

Figure 2 displays the mixed-hybrid primal flux (i.e. Y_{00} component) along the line $y = 9.84375$, together with corresponding results from the well-known neutronic code VARIANT [11]. This code uses a primal (non-mixed) hybrid formulation and provides the interface flux as output. As the other existing neutronic codes using spherical harmonic expansions, it is restricted to odd-order P_N approximations. Figure 2 shows that for the odd-order P_N approximations, the match is fine between mixed-hybrid and VARIANT results. As for the even-order P_N approximations, they yield here intermediate values.

7 Conclusions and perspectives

We developed mixed-hybrid discretization methods for the linear Boltzmann transport equation. With constant $\sigma_l > \sigma_{s,l} > 0$ in each element V_l, and introducing a mostly non-restrictive mild anisotropy assumption, we first proved

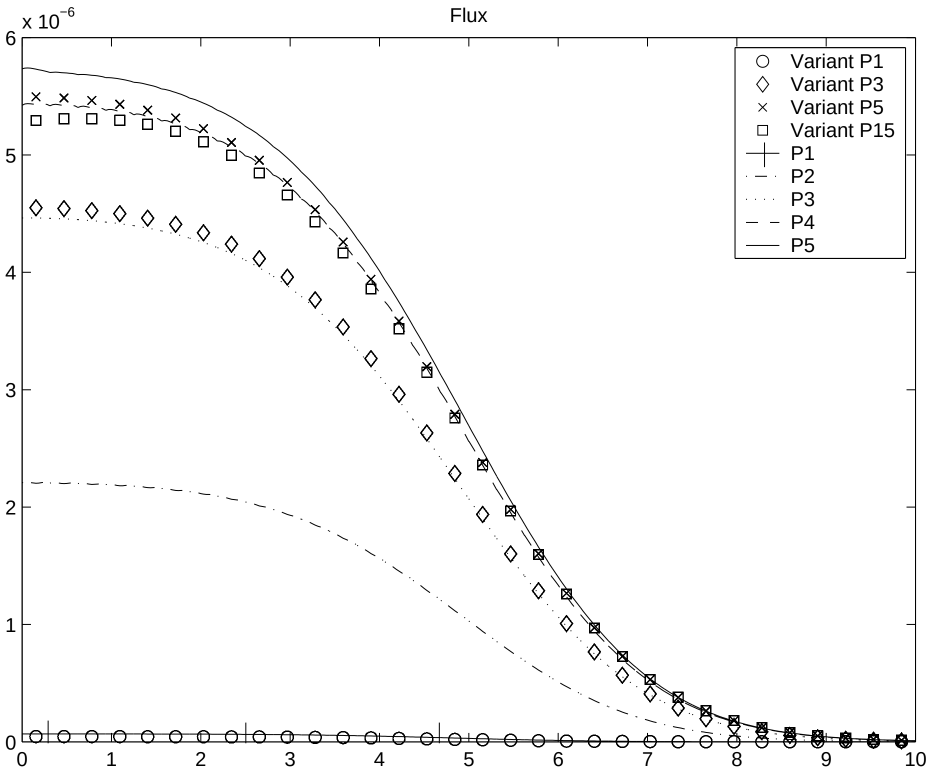

Fig. 2. Mixed-hybrid primal and VARIANT flux along $y = 9.84375$ for the Azmy benchmark.

the existence and uniqueness of a solution in the continuous (infinite-dimensional) problems. Next, discrete (finite-dimensional) problems were obtained using a finite element technique in space, and a P_N spherical harmonic technique in angle. Approximation space inclusion conditions involving both ∇ and $\mathbf{\Omega}$, and spatio-angular rank conditions were proved to guarantee the existence and uniqueness of a discrete solution. Our well-posedness investigation showed that, beside the well-known primal/dual distinction induced by the spatial variable, the angular variable leads to an even/odd distinction for the spherical harmonic expansion order N. While odd-order methods have been widely employed in computational neutron transport, our work shows the usability of even-order methods, also in numerical experiments.

As mentioned in appendix B, uniform stability and error estimates could be further investigated based on what was done in [3] and [19] for purely spatial PDE's. Furthermore, the mathematical setting introduced here could be used to particularize our results to purely mixed or purely hybrid formulations. Future research also includes the development of efficient solution techniques for the linear system arising after discretization. Early results show that a condensation approach such as the one developed in [5, V.1] could be generalized to the transport case. Finally, time dependence could be incorporated in the Boltzmann equation, and our results used to establish a well-posedness theory in a semi-group framework. Such a study was performed in [6, XXI] for the time-dependent integro-differential first-order form of the transport equation. Our results could be used to generalize this theory to mixed and/or hybrid formulations arising from the even- and odd-parity flux decomposition.

A Proofs for the continuous problems (Theorem 9)

A.1 Proofs for the primal case

A.1.1 Even-order primal method

We define

$$\begin{aligned}\Lambda &= X \times L^2(\mathbf{\Omega}, S \times V) \times H_{\Omega}^{-1/2}(S \times \Gamma),\\ \Lambda_0 &= X \times L^2(\mathbf{\Omega}, S \times V) \times H_{\Omega,0}^{-1/2}(S \times \Gamma),\end{aligned}$$

$\lambda = (\Psi^+, \Psi^-, \Psi^\chi) \in \Lambda$ and $\tilde{\lambda} = (\tilde{\Psi}^+, \tilde{\Psi}^-, \tilde{\Psi}^\chi) \in \Lambda_0$ [2]. As for norms, we have $\|\lambda\|_\Lambda^2 = \|\Psi^+\|_X^2 + \|\Psi^-\|^2 + \|\Psi^\chi\|_{\Omega,-1/2,\Gamma}^2$. Then we introduce the symmetric bilinear form

$$K(\lambda, \tilde{\lambda}) = \sum_l K_l(\lambda, \tilde{\lambda}),$$

where

$$\begin{aligned}K_l(\lambda, \tilde{\lambda}) = &- \int_S d\Omega \int_{V_l} dV\, \Psi_l^- \,\mathbf{\Omega} \cdot \nabla \tilde{\Psi}_l^+ + \int_S d\Omega \int_{\partial V_l} d\Gamma\, \mathbf{\Omega} \cdot \mathbf{n}\, \tilde{\Psi}_l^+ \Psi_l^\chi\\ &+ \int_S d\Omega \int_{V_l} dV \sigma_l \tilde{\Psi}_l^+ \Psi_l^+ - \int_S d\Omega \int_{V_l} dV \left(\sigma_{s,l} \tilde{\Psi}_l^+ \int_S d\Omega' \Psi_l^+\right)\\ &- \int_S d\Omega \int_{V_l} dV \tilde{\Psi}_l^- \mathbf{\Omega} \cdot \nabla \Psi_l^+ - \int_S d\Omega \int_{V_l} dV \sigma_l \tilde{\Psi}_l^- \Psi_l^-\\ &+ \int_S d\Omega \int_{\partial V_l} d\Gamma\, \mathbf{\Omega} \cdot \mathbf{n}\, \tilde{\Psi}_l^\chi \Psi_l^+.\end{aligned}$$

With

$$\langle s, \tilde{\Psi}^+ \rangle = \sum_l \int_S d\Omega \int_{V_l} dV\, s\, \tilde{\Psi}_l^+,$$

we have to examine the problem: find $\lambda \in \Lambda$ such that

[2] We as well assume the usual parity properties.

$$K(\lambda,\tilde{\lambda}) = \langle s, \tilde{\Psi}^+\rangle, \qquad \forall\tilde{\lambda}\in\Lambda_0.$$

According to the generalization of the Lax-Milgram theorem for the non-coercive operators, existence and uniqueness of a solution is ensured provided we can demonstrate the continuity of K, as well as the "Ladyshenskaya-Babuška-Brezzi" (LBB, or just BB) or "inf-sup" condition for K. See for instance [15, Theorem 9.1], [3, Theorem 2.1], and [2, Theorem 2.1]. We therefore prove here the existence of strictly positive constants C_1 and C_2 such that

$$K_l(\lambda_l,\tilde{\lambda}_l) \leq C_1 \, \|\lambda_l\|_{\Lambda_l}\|\tilde{\lambda}_l\|_{\Lambda_{0,l}} \tag{A.1}$$

and

$$\inf_{\lambda\in\Lambda,\lambda\neq 0}\ \sup_{\tilde{\lambda}\in\Lambda_0,\tilde{\lambda}\neq 0} \frac{K(\lambda,\tilde{\lambda})}{\|\lambda\|_\Lambda\ \|\tilde{\lambda}\|_{\Lambda_0}} \geq C_2 \tag{A.2}$$

as well as

$$\sup_{\lambda\in\Lambda,\lambda\neq 0} K(\lambda,\tilde{\lambda}) > 0 \qquad \forall\tilde{\lambda}\in\Lambda_0 \quad \textit{with} \quad \tilde{\lambda}\neq 0. \tag{A.3}$$

To check (A.1), we introduce for each V_l a $\Psi_l^y \in H(grad, S\times V_l)$ as the weak solution of the auxiliary problem

$$\begin{cases} -\Delta\Psi_l^y + \Psi_l^y = 0 \text{ in } V_l \\ \\ \mathbf{n}\cdot\nabla\Psi_l^y = \mathbf{n}\cdot\mathbf{\Omega}\,\Psi_l^\chi \text{ on } \partial V_l \end{cases}$$

such that

$$\int_S d\Omega \int_{\partial V_l} d\Gamma\,\mathbf{\Omega}\cdot\mathbf{n}\,\Psi_l^+\,\Psi_l^\chi = (\Psi_l^+,\Psi_l^y)_{grad,V_l}. \tag{A.4}$$

Letting $\mathbf{\Psi}_l^u = \nabla\Psi_l^y$, we get

$$\begin{cases} -\nabla\cdot\mathbf{\Psi}_l^u + \Psi_l^y = 0 \text{ in } V_l \\ \\ \mathbf{n}\cdot\mathbf{\Psi}_l^u = \mathbf{n}\cdot\mathbf{\Omega}\Psi_l^\chi \text{ on } \partial V_l \end{cases}$$

and therefore $\Psi_l^y = \nabla\cdot\mathbf{\Psi}_l^u$ in V_l, and

$$\begin{cases} -\Delta\mathbf{\Psi}_l^u + \mathbf{\Psi}_l^u = 0 \text{ in } V_l \\ \\ \mathbf{n}\cdot\mathbf{\Psi}_l^u = \mathbf{n}\cdot\mathbf{\Omega}\Psi_l^\chi \text{ on } \partial V_l \end{cases}$$

where $\Delta\mathbf{\Psi}_l^u = \nabla(\nabla\cdot\mathbf{\Psi}_l^u)$. This implies

$$
\begin{aligned}
\|\mathbf{\Psi}_l^u\|_{div,V_l} &= \inf_{\{\mathbf{\Psi}^q \in H(div,S\times V):\ \mathbf{n}\cdot\mathbf{\Psi}^q|_{\partial V}=\mathbf{n}\cdot\mathbf{\Omega}\Psi_l^\chi\}} \|\mathbf{\Psi}^q\|_{div,V} = \|\mathbf{n}\cdot\mathbf{\Omega}\Psi_l^\chi\|_{-1/2,\partial V_l} \\
&= \|\Psi_l^\chi\|_{\Omega,-1/2,\partial V_l}
\end{aligned}
$$

where we used (5) and (6). Since

$$
\|\mathbf{\Psi}_l^u\|_{div,V_l}^2 = \int_S d\Omega \int_{\partial V_l} d\Gamma\, \mathbf{\Omega}\cdot\mathbf{n}\, \Psi_l^y\, \Psi_l^\chi = \|\Psi_l^y\|_{grad,V_l}^2,
$$

we have

$$
\|\Psi_l^\chi\|_{\Omega,-1/2,\partial V_l} = \|\Psi_l^y\|_{grad,V_l}. \tag{A.5}
$$

Using Schwartz inequality, we can now write from (A.4)

$$
\int_S d\Omega \int_{\partial V_l} d\Gamma\, \mathbf{\Omega}\cdot\mathbf{n}\, \Psi_l^+\, \Psi_l^\chi \le \|\Psi_l^+\|\ \|\Psi_l^\chi\|_{\Omega,-1/2,\partial V_l}
$$

and, assuming σ_l and $\sigma_{s,l}$ constant in each V_l,

$$
\begin{aligned}
K_l(\lambda,\tilde\lambda) &\le \|\nabla\tilde\Psi_l^+\|\ \|\mathbf{\Omega}\Psi_l^-\| + \|\Psi_l^\chi\|_{\Omega,-1/2,\partial V_l}\|\tilde\Psi_l^+\|_{grad,V_l} + (\sigma_l+\sigma_{s,l})\|\tilde\Psi_l^+\|\ \|\Psi_l^+\| \\
&\quad + \|\mathbf{\Omega}\tilde\Psi_l^-\|\ \|\nabla\Psi_l^+\| + \sigma_l\|\tilde\Psi_l^-\|\ \|\Psi_l^-\| + \|\tilde\Psi_l^\chi\|_{\Omega,-1/2,\partial V_l}\|\Psi_l^+\|_{grad,V_l} \\
&\le \Big(\|\mathbf{\Omega}\Psi_l^-\|^2 + \|\Psi_l^\chi\|^2_{\Omega,-1/2,\partial V_l} + (\sigma_l+\sigma_{s,l})\|\Psi_l^+\|^2 + \|\nabla\Psi_l^+\|^2 + \sigma_l\|\Psi_l^-\|^2 \\
&\qquad\qquad + \|\Psi_l^+\|^2_{grad,V_l}\Big)^{1/2} \\
&\quad \Big(\|\nabla\tilde\Psi_l^+\|^2 + \|\tilde\Psi_l^+\|^2_{grad,V_l} + (\sigma_l+\sigma_{s,l})\|\tilde\Psi_l^+\|^2 + \|\mathbf{\Omega}\tilde\Psi_l^-\|^2 + \sigma_l\|\tilde\Psi_l^-\|^2 \\
&\qquad\qquad + \|\tilde\Psi_l^\chi\|^2_{\Omega,-1/2,\partial V_l}\Big)^{1/2} \\
&\le (2+\sigma_l+\sigma_{s,l})\ \|\lambda_l\|_{\Lambda_l}\ \|\tilde\lambda_l\|_{\Lambda_{0,l}}
\end{aligned}
$$

so that the continuity of K, i.e. (A.1), is proved.

For the LBB condition (A.2), assume $\sigma_l > \sigma_{s,l} > 0$ and consider $\hat\lambda = (\hat\Psi_l^+, \hat\Psi_l^-, \hat\Psi_l^\chi) \in \Lambda_0$ with [3]

$$
\begin{aligned}
\hat\Psi_l^+ &= 2\Psi_l^+ + \Psi_l^z \\
\mathbf{\Omega}\hat\Psi_l^- &= -\frac{1}{\sigma_l}\nabla\Psi_l^+ - \mathbf{\Omega}\Psi_l^- - \frac{1}{\sigma_l}\nabla\Psi_l^z \\
\hat\Psi_l^\chi &= (-3 + \frac{\sigma_s}{\sigma_l})\Psi_l^\chi|_{\partial V_l,0},
\end{aligned}
$$

where $\Psi_l^\chi|_{\partial V_l,0}$ is the restriction of Ψ_l^χ to $\partial V_l \setminus \partial V$ (extended by zero on $\partial V \bigcap \partial V_l$), and $\Psi_l^z \in H(grad, S\times V_l)$ is the weak solution of the auxiliary problem

[3] The presence of dimensional factors is understood.

$$
\begin{cases}
-\Delta\Psi_l^z + \sigma_l(\sigma_l - \sigma_{s,l})\,\Psi_l^z = 0 \text{ in } V_l \\
\\
\mathbf{n}\cdot\nabla\Psi_l^z = (\sigma_l - \sigma_{s,l})\,\mathbf{n}\cdot\mathbf{\Omega}\Psi_l^\chi \text{ on } \partial V_l.
\end{cases}
$$

Then, after some manipulations (including the addition and subtraction of the term $\int_S d\Omega \int_{V_l} dV\,\sigma_{s,l}\,\Psi_l^+\,\Psi_l^z$), we obtain

$$
\begin{aligned}
K_l(\lambda,\hat{\lambda}) = 2\sigma_l\|\Psi_l^+\|^2 + \frac{1}{\sigma_l}\|\nabla\Psi_l^+\|^2 + \sigma_l\|\Psi_l^-\|^2 + \int_S d\Omega \int_{\partial V_l} d\Gamma\,\mathbf{\Omega}\cdot\mathbf{n}\,\Psi_l^\chi\Psi_l^z \\
- \int_S d\Omega \int_{V_l} dV\,2\,\sigma_{s,l}\,\Psi_l^+ \int_S d\Omega'\Psi_l^+ \\
- \int_S d\Omega \int_{V_l} dV\,\sigma_{s,l}\,\Psi_l^z\,(\int_S d\Omega'\Psi_l^+ - \Psi_l^+) \\
+ \int_S d\Omega \int_{\partial V_l \bigcap \partial V} d\Gamma\,\mathbf{\Omega}\cdot\mathbf{n}\,(3 - \frac{\sigma_{s,l}}{\sigma_l})\,\Psi_l^\chi\Psi_l^+.
\end{aligned}
$$

The last term is non-negative as soon as vacuum or reflected boundary conditions are enforced. Indeed, (13) shows that $\mathbf{\Omega}\cdot\mathbf{n}\,\Psi_l^\chi\Psi_l^+ = |\mathbf{\Omega}\cdot\mathbf{n}|\,(\Psi_l^\chi)^2$ on a vacuum boundary, while on a reflected boundary, the last term vanishes since

$$
\begin{aligned}
\int_{\partial V_l \bigcap \partial V} d\Gamma \int_S d\Omega\,\mathbf{\Omega}\cdot\mathbf{n}\,\Psi_l^\chi\Psi_l^+ \\
= \frac{1}{4\pi}\int_{\partial V_l \bigcap \partial V} d\Gamma \int_0^{2\pi} d\phi\Big[\int_0^{\pi/2} d\theta \sin\theta\cos\theta\,\Psi_l^\chi(\theta,\phi)\Psi_l^+(\theta,\phi) \\
+ \int_{\pi/2}^{\pi} d\theta \sin\theta\cos\theta\,\Psi_l^\chi(\theta,\phi)\Psi_l^+(\theta,\phi)\Big]
\end{aligned}
$$

and the reflection condition $\Psi_l^{\chi(resp.+)}(\theta,\phi) = \Psi_l^{\chi(resp.+)}(\pi-\theta,\phi)$ implies

$$
\int_{\pi/2}^{\pi} d\theta \sin\theta\cos\theta\Psi_l^\chi(\theta,\phi)\Psi_l^+(\theta,\phi) = \int_{\pi/2}^{0} d\theta \sin\theta\cos\theta\Psi_l^\chi(\theta,\phi)\Psi_l^+(\theta,\phi).
$$

Defining $\|\Psi_l\|^2_{grad,V_l,*} = \|\nabla\Psi_l\|^2 + \sigma_l(\sigma_l - \sigma_{s,l})\,\|\Psi_l\|^2$, $\|\mathbf{\Psi}_l\|^2_{div,V_l,*} = \|\nabla\cdot\mathbf{\Psi}_l\|^2 + \sigma_l(\sigma_l - \sigma_{s,l})\,\|\mathbf{\Psi}_l\|^2$, and consequently $\|\Psi_l^\chi\|_{\Omega,-1/2,\partial V_l,*}$ as in (5) and (6), with starred norms replacing unstarred ones, we have that

$$
\int_S d\Omega \int_{\partial V_l} d\Gamma\,\mathbf{\Omega}\cdot\mathbf{n}\,(\sigma_l - \sigma_{s,l})\,\Psi_l^\chi\Psi_l^z = \|\Psi_l^z\|^2_{grad,V_l,*}.
$$

Then,

$$
\begin{aligned}
K_l(\lambda,\hat{\lambda}) \geq\, & 2(\sigma_l-\sigma_{s,l})\,\|\Psi_l^+\|^2+\frac{1}{\sigma_l}\|\nabla\Psi_l^+\|^2+\sigma_l\|\Psi_l^-\|^2\\
&+\frac{1}{\sigma_l-\sigma_{s,l}}\|\Psi_l^z\|^2_{grad,V_l,*}-\sigma_{s,l}\,\|\Psi_l^z\|\,\|\phi_l-\Psi_l^+\|,
\end{aligned}
$$

where $\phi_l = \int_S d\Omega\Psi_l^+$. Now, since for any Ψ, we can write without loss of generality $\Psi = \sum_{n=0}^{\infty}\sum_{|m|<n}(\Psi, Y_{nm})_\Omega Y_{nm}$, we have

$$
\begin{aligned}
\|\Psi_l^-\|^2 &= \sum_{n=1,\ n \text{ odd}}^{\infty}\ \sum_{|m|<n}\|(\Psi_l^-, Y_{nm})_\Omega\|^2,\\
\|\phi_l-\Psi_l^+\|^2 &= \sum_{n=2,\ n \text{ even}}^{\infty}\ \sum_{|m|<n}\|(\Psi_l^+, Y_{nm})_\Omega\|^2.
\end{aligned}
$$

Thus if

$$
(1+\alpha_n)^2\sum_{|m|<n}\|(\Psi_l, Y_{nm})_\Omega\|^2 \geq \sum_{|m|<n}\|(\Psi_l, Y_{n+1,m})_\Omega\|^2
$$

for any $n \geq 1$, n odd, then $(1+\alpha_n)\,\|\Psi_l^-\| \geq \|\phi_l-\Psi_l^+\|$, and we can write

$$
\begin{aligned}
K_l(\lambda,\hat{\lambda}) \geq\, & 2(\sigma_l-\sigma_{s,l})\,\|\Psi_l^+\|^2+\frac{1}{\sigma_l}\|\nabla\Psi_l^+\|^2+\sigma_l\|\Psi_l^-\|^2\\
&+\frac{1}{\sigma_l-\sigma_{s,l}}\|\Psi_l^z\|^2_{grad,V_l,*}-\frac{1}{2}\sigma_{s,l}\left(\|\Psi_l^z\|^2+(1+\alpha_n)^2\,\|\Psi_l^-\|^2\right),
\end{aligned}
$$

The coefficient of $\|\Psi^-\|^2$ is thus $\sigma_l - \frac{1}{2}\sigma_{s,l}(1+\alpha_n)^2$ which is strictly positive if the angular flux Ψ_l is supposed mildly anisotropic. With the starred norms introduced above, we have $\|\Psi_l^z\|^2 \leq \frac{1}{\sigma_l(\sigma_l-\sigma_{s,l})}\|\Psi_l^z\|^2_{grad,V_l,*}$ and the coefficient of $\|\Psi_l^z\|^2_{grad,V_l,*}$ is $\frac{1}{\sigma_l-\sigma_{s,l}}(1-\frac{\sigma_{s,l}}{2\sigma_l})$ and thus as well strictly positive. Moreover, proceeding as previously to obtain (A.5), we here obtain

$$
\sigma_l(\sigma_l-\sigma_{s,l})\|\Psi_l^z\|^2_{grad,V_l,*} = \|(\sigma_l-\sigma_{s,l})\Psi_l^\chi\|^2_{\mathbf{\Omega},-1/2,\partial V_l,*}. \tag{A.6}
$$

Furthermore, $\|\cdot\|_{\mathbf{\Omega},-1/2,\partial V_l,*}$ and $\|\cdot\|_{\mathbf{\Omega},-1/2,\partial V_l}$ are equivalent norms. Thus, there is a strictly positive constant C_l such that

$$
K_l(\lambda,\hat{\lambda}) \geq C_l\|\lambda_l\|^2_{\Lambda_0,l}
$$

provided we can assume mild anisotropy in each V_l. Since we also have $\|\hat{\lambda}_l\|^2_{\Lambda_0,l} \leq C_l'\|\lambda_l\|^2_{\Lambda_l}$ for another strictly positive constant C_l', we obtain

$$
K_l(\lambda,\hat{\lambda}) \geq \frac{C_l}{\sqrt{C_l'}}\|\lambda_l\|_{\Lambda_l}\|\hat{\lambda}_l\|_{\Lambda_{0,l}}.
$$

Thus there is a constant $C_2 > 0$ such that

$$\sup_{\tilde\lambda\in\Lambda_0,\tilde\lambda\neq 0}\frac{K(\lambda,\tilde\lambda)}{\|\tilde\lambda\|_{\Lambda_0}}\geq C_2\|\lambda\|_\Lambda \qquad \forall\lambda\in\Lambda,$$

and the LBB condition (A.2) is verified. Besides, the symmetry of $K(\lambda,\tilde\lambda)$ leads to (A.3).

A.1.1.1 ***Alternative proof in the even-order primal case*** In the absence of the mild anisotropy assumption, well-posedness can be proved with the assumption $0<\sigma_{s,l}<\frac{4}{5}\sigma_l$ for all l. Indeed, consider $\hat\lambda=(\hat\Psi_l^+,\hat\Psi_l^-,\hat\Psi_l^\chi)\in\Lambda_0$ with

$$\begin{aligned}\hat\Psi_l^+ &= 2\Psi_l^+ + \sigma_l\Psi_l^z\\ \boldsymbol{\Omega}\hat\Psi_l^- &= -\frac{1}{\sigma_l}\nabla\Psi_l^+ - \boldsymbol{\Omega}\Psi_l^- - \frac{1}{\sigma_l}\nabla\Psi_l^z\\ \hat\Psi_l^\chi &= -3\,\Psi_l^\chi|_{\partial V_l,0},\end{aligned}$$

where $\Psi_l^z\in H(grad,S\times V_l)$ is the weak solution of the auxiliary problem

$$\begin{cases}-\Delta\Psi_l^z+\sigma_l^2\,\Psi_l^z=0 \text{ in } V_l\\ \\ \mathbf{n}\cdot\nabla\Psi_l^z=\sigma_l\,\mathbf{n}\cdot\boldsymbol{\Omega}\Psi_l^\chi \text{ on } \partial V_l.\end{cases}$$

Then, some manipulations yield

$$\begin{aligned}K_l(\lambda,\hat\lambda) =& 2\sigma_l\|\Psi_l^+\|^2+\frac{1}{\sigma_l}\|\nabla\Psi_l^+\|^2+\sigma_l\|\Psi_l^-\|^2+\int_S d\Omega\int_{\partial V_l}d\Gamma\,\boldsymbol{\Omega}\cdot\mathbf{n}\,\Psi_l^\chi\Psi_l^z\\ &-\int_S d\Omega\int_{V_l}dV\left(\sigma_{s,l}[2\Psi_l^+ + \Psi_l^z]\int_S d\Omega\Psi_l^+\right)\\ &+\int_S d\Omega\int_{\partial V_l\bigcap\partial V}d\Gamma\,\boldsymbol{\Omega}\cdot\mathbf{n}\,(3\,\sigma_l)\,\Psi_l^\chi\Psi_l^+.\end{aligned}$$

The last term is again non-negative as soon as vacuum or reflected boundary conditions are enforced. Defining this time $\|\Psi_l\|^2_{grad,V_l,*}=\|\nabla\Psi_l\|^2+\sigma_l^2\|\Psi_l\|^2$, $\|\Psi_l\|^2_{div,V_l,*}=\|\nabla\Psi_l\|^2+\sigma_l^2\|\Psi_l\|^2$ (and consequently $\|\Psi^\chi\|_{\Omega,-1/2,\partial V_l,*}$ as in (5) and (6) with starred norms replacing unstarred ones), we have

$$\int_S d\Omega\int_{\partial V_l}d\Gamma\,\boldsymbol{\Omega}\cdot\mathbf{n}\,\sigma_l\,\Psi_l^\chi\Psi_l^z=\|\Psi_l^z\|^2_{grad,V_l,*}.$$

Then,

$$
\begin{aligned}
K_l(\lambda,\hat{\lambda}) \geq\, & 2\,(\sigma_l - \sigma_{s,l})\|\Psi_l^+\|^2 + \frac{1}{\sigma_l}\|\nabla\Psi_l^+\|^2 + \sigma_l\|\Psi_l^-\|^2 \\
& + \frac{1}{\sigma_l}\|\Psi_l^z\|^2_{grad,V_l,*} - \sigma_{s,l}\int_{V_l} dV\Big(\int_S d\Omega\Psi_l^+\int_S d\Omega\Psi_l^z\Big).
\end{aligned}
$$

Now, with

$$
\int_{V_l} dV\Big(\int_S d\Omega\Psi_l^+\int_S d\Omega\Psi_l^z\Big) \leq \|\Psi_l^+\|\,\|\Psi_l^z\| \leq \frac{1}{2}(\|\Psi_l^+\|^2 + \|\Psi_l^z\|^2),
$$

and $\|\Psi_l^z\|^2 \leq \frac{1}{\sigma_l^2}\,\|\Psi_l^z\|^2_{grad,V_l,*}$, we have

$$
\begin{aligned}
K_l(\lambda,\hat{\lambda}) \geq\, & (2\sigma_l - \frac{5}{2}\sigma_{s,l})\,\|\Psi_l^+\|^2 + \frac{1}{\sigma_l}\|\nabla\Psi_l^+\|^2 + \sigma_l\|\Psi_l^-\|^2 \\
& + \frac{1}{\sigma_l}(1 - \frac{1}{2}\frac{\sigma_{s,l}}{\sigma_l})\,\|\Psi_l^z\|^2_{grad,V_l,*}.
\end{aligned}
$$

We can also show that $\|\Psi_l^z\|^2_{grad,V_l,*} = \frac{1}{\sigma}\|\Psi_l^\chi\|_{\Omega,-1/2,\partial V_l,*}$, and since $\|\cdot\|_{\Omega,-1/2,\partial V_l,*}$ and $\|\cdot\|_{\Omega,-1/2,\partial V_l}$ are equivalent norms, there is thus a strictly positive constant C_l such that

$$
K_l(\lambda,\hat{\lambda}) \geq C_l\|\lambda_l\|^2_{\Lambda_0,l}
$$

provided $0 < \sigma_{s,l} < \frac{4}{5}\sigma_l$. We conclude as before that the LBB condition is satisfied under this condition.

A.1.2 Odd-order primal method

Define

$$
\begin{aligned}
\Lambda &= X_\Omega \times L^2(S\times V) \times H^{1/2}(S\times\Gamma), \\
\Lambda_0 &= X_\Omega \times L^2(S\times V) \times H_0^{1/2}(S\times\Gamma),
\end{aligned}
$$

$\lambda = (\Psi^+,\Psi^-,\Psi^\chi) \in \Lambda$ and $\tilde{\lambda} = (\tilde{\Psi}^+,\tilde{\Psi}^-,\tilde{\Psi}^\chi) \in \Lambda_0$. As for norms, we have $\|\lambda\|^2_\Lambda = \|\Psi^+\|^2_{X_\Omega} + \|\Psi^-\|^2 + \|\Psi^\chi\|^2_{1/2,\Gamma}$. The bilinear form $K(\lambda,\tilde{\lambda})$, as well as the problem to solve, are formally the same as in the even-order case, but the functional spaces are different. We thus check the continuity and the LBB condition for K within this new framework.

First we introduce for each V_l a $\Psi_l^y \in H(\mathbf{\Omega}\cdot\nabla, S\times V_l)$ as the weak solution of the auxiliary problem

$$\begin{cases} -\nabla(\mathbf{\Omega}\cdot\nabla\Psi_l^y) + \mathbf{\Omega}\Psi_l^y = 0 & \text{in } V_l \\ \\ \mathbf{\Omega}\cdot\nabla\Psi_l^y = \Psi_l^\chi & \text{on } \partial V_l, \end{cases}$$

such that

$$\int_S d\Omega \int_{\partial V_l} d\Gamma\, \mathbf{\Omega}\cdot\mathbf{n}\, \Psi_l^+\, \Psi_l^\chi = (\Psi_l^+, \Psi_l^y)_{\mathbf{\Omega}\cdot\nabla, V_l}.$$

Then, letting $\Psi_l^u = \mathbf{\Omega}\cdot\nabla\Psi_l^y$, we can show similarly to the even-order case

$$\|\Psi_l^y\|_{\mathbf{\Omega}\cdot\nabla,V_l} = \|\Psi_l^u\|_{grad,V_l} = \|\Psi_l^\chi\|_{1/2,\partial V_l}.$$

We then can write using Schwartz inequality

$$\begin{aligned} K_l(\lambda,\tilde\lambda) \le\, & \|\mathbf{\Omega}\cdot\nabla\tilde\Psi_l^+\|\,\|\Psi_l^-\| + \|\Psi_l^\chi\|_{1/2,\partial V_l}\|\tilde\Psi_l^+\|_{\Omega,1,l} + (\sigma_l+\sigma_{s,l})\|\tilde\Psi_l^+\|\,\|\Psi_l^+\| \\ & + \|\tilde\Psi_l^-\|\,\|\mathbf{\Omega}\cdot\nabla\Psi_l^+\| + \sigma_l\|\tilde\Psi_l^-\|\,\|\Psi_l^-\| + \|\Psi_l^\chi\|_{1/2,\partial V_l}\|\Psi_l^+\|_{\Omega,1,l} \\ \le\, & constant\|\lambda\|_{\Lambda_l}\,\|\tilde\lambda\|_{\Lambda_{0,l}} \end{aligned}$$

so that the continuity of K is verified.

Next, the LBB condition can be proved adapting what is done in the even-order case. The $\hat\lambda$ to consider here is

$$\begin{aligned} \hat\Psi_l^+ &= 2\,\Psi_l^+ + \Psi_l^z \\ \hat\Psi_l^- &= -\frac{1}{\sigma_l}\mathbf{\Omega}\cdot\nabla\Psi_l^+ - \Psi_l^- - \frac{1}{\sigma_l}\mathbf{\Omega}\cdot\nabla\Psi_l^z \\ \hat\Psi_l^\chi &= (-3 + \frac{\sigma_l}{\sigma_{s,l}})\Psi_l^\chi|_{\partial V_l,0}, \end{aligned}$$

where $\Psi_l^z \in H(\mathbf{\Omega}\cdot\nabla, S\times V_l)$ is the weak solution of the auxiliary problem

$$\begin{cases} -\nabla(\mathbf{\Omega}\cdot\nabla\Psi_l^z) + \sigma_l(\sigma_{s,l}-\sigma_l)\mathbf{\Omega}\Psi_l^z = 0 & \text{in } V_l \\ \\ \mathbf{\Omega}\cdot\nabla\Psi_l^z = (\sigma_l - \sigma_{s,l})\,\Psi_l^\chi & \text{on } \partial V_l. \end{cases}$$

A.2 Proofs for the dual case

A.2.1 Even-order dual method

Here we define

$$\Lambda = X_\Omega \times L^2(S \times V_l) \times H^{1/2}(S \times \Gamma),$$
$$\Lambda_0 = X_\Omega \times L^2(S \times V_l) \times H_0^{1/2}(S \times \Gamma),$$

$\lambda = (\Psi^-, \Psi^+, \Psi^\psi) \in \Lambda$ and $\tilde\lambda = (\tilde\Psi^-, \tilde\Psi^+, \tilde\Psi^\psi) \in \Lambda_0$. As for norms, we have $\|\lambda\|_\Lambda^2 = \|\Psi^-\|_{X_\Omega}^2 + \|\Psi^+\|^2 + \|\Psi^\psi\|_{1/2,\Gamma}^2$. Define then

$$K(\lambda, \tilde\lambda) = \sum_l K_l(\lambda, \tilde\lambda),$$

where

$$\begin{aligned} K_l(\lambda,\tilde\lambda) = &- \int_S d\Omega \int_{V_l} dV\, \tilde\Psi_l^+\, \mathbf{\Omega}\cdot\nabla\Psi_l^- - \int_S d\Omega \int_{V_l} dV \sigma_l\, \tilde\Psi_l^+ \Psi_l^+ \\ &+ \int_S d\Omega \int_{V_l} dV \sigma_{s,l} \left(\tilde\Psi_l^+ \int_S d\Omega' \Psi_l^+\right) - \int_S d\Omega \int_{V_l} dV\, \Psi_l^+\, \mathbf{\Omega}\cdot\nabla\,\tilde\Psi_l^- \\ &+ \int_S d\Omega \int_{\partial V_l} d\Gamma\, \mathbf{\Omega}\cdot\mathbf{n}\, \tilde\Psi_l^- \Psi_l^\psi + \int_S d\Omega \int_{V_l} dV \sigma_l\, \tilde\Psi_l^- \Psi_l^- \\ &+ \int_S d\Omega \int_{\partial V_l} d\Gamma\, \mathbf{\Omega}\cdot\mathbf{n}\, \tilde\Psi_l^\psi \Psi_l^-. \end{aligned}$$

With

$$\langle s, \tilde\Psi^+\rangle = \sum_l \int_S d\Omega \int_{V_l} dV\; s\, \tilde\Psi_l^+,$$

we have to examine the problem: find $\lambda \in \Lambda$ such that

$$K(\lambda, \tilde\lambda) = \langle s, \tilde\Psi^+\rangle, \qquad \forall\tilde\lambda \in \Lambda_0.$$

From there, we can check the continuity and the LBB condition for K in a way similar to what we did in the odd-order primal case.

A.2.2 Odd-order dual method

Define

$$\Lambda = X \times L^2(\mathbf{\Omega}, S \times V) \times H_\Omega^{-1/2}(S \times \Gamma),$$
$$\Lambda_0 = X \times L^2(\mathbf{\Omega}, S \times V) \times H_{\Omega,0}^{-1/2}(S \times \Gamma),$$

$\lambda = (\Psi^-, \Psi^+, \Psi^\chi) \in \Lambda$ and $\tilde\lambda = (\tilde\Psi^-, \tilde\Psi^+, \tilde\Psi^\chi) \in \Lambda_0$. As for norms, we have $\|\lambda\|_\Lambda^2 = \|\Psi^+\|_X^2 + \|\Psi^-\|^2 + \|\Psi^\chi\|_{\Omega,-1/2,\Gamma}^2$. The bilinear form $K(\lambda, \tilde\lambda)$ is formally the same as in the even-order dual case. The continuity and LBB condition for

K can be proved in a way similar to what was done in the even-order primal case.

B Proofs for the discrete problems (Theorem 10)

B.1 Proofs for the primal case

B.1.1 Even-order primal method

We introduce

$$\Lambda_h = P_h^{N,p} \times M_h^{N,m} \times B_h^{N,b} \text{ and } \Lambda_{0,h} = P_h^{N,p} \times M_h^{N,m} \times B_{0,h}^{N,b},$$

where the different finite-dimensional spaces were defined in section 5.1. We define Π_l^+ and Π_l^- as the orthogonal projections of $H(grad, S \times V_l)$ and $[L^2(S \times V_l)]^n$ (with respect to their scalar product) respectively onto $P_h^{N,p}$ and $\mathbf{\Omega}\, M_h^{N,m}$.

We need to verify the LBB condition in the discrete case ([15, Theorem 9.2 p.564], [3, Theorem 2.2], and [2, Theorem 2.2]). In this view, we define $\hat{\lambda}_h = (\hat{\Psi}_h^+, \hat{\Psi}_h^-, \hat{\Psi}_h^\chi) \in \Lambda_{0,h}$ with

$$\begin{aligned}
\hat{\Psi}_{h,l}^+ &= 2\Psi_{h,l}^+ + \Pi_l^+(\Psi_{h,l}^z) \\
\mathbf{\Omega}\hat{\Psi}_{h,l}^- &= -\frac{1}{\sigma_l}\Pi_l^-(\nabla\Psi_{h,l}^+) - \mathbf{\Omega}\Psi_{h,l}^- - \frac{1}{\sigma_l}\Pi_l^-(\nabla\Pi_l^+(\Psi_{h,l}^z)) \\
\hat{\Psi}_{h,l}^\chi &= (-3 + \frac{\sigma_{s,l}}{\sigma_l})\Psi_{h,l}^\chi|_{\partial V_l,0}
\end{aligned}$$

where $\Psi_{h,l}^z \in H(grad, S \times V_l)$ is the weak solution of

$$\begin{cases}
-\Delta\Psi_{h,l}^z + \sigma_l\,(\sigma_l - \sigma_{s,l})\,\Psi_{h,l}^z = 0 & \text{in } S \times V_l \\
\mathbf{n}\cdot\nabla\Psi_{h,l}^z = (\sigma_l - \sigma_{s,l})\,\mathbf{n}\cdot\mathbf{\Omega}\Psi_{h,l}^\chi & \text{on } S \times \partial V_l.
\end{cases}$$

We define as in the primal even continuous case $\|\Psi_l\|_{grad,V_l,*}^2 = \|\nabla\Psi_l\|^2 + \sigma_l(\sigma_l - \sigma_{s,l})\,\|\Psi_l\|^2$ such that

$$\|\Pi_l^+\Psi_{h,l}^z\|_{grad,V_l,*}^2 = (\sigma_l - \sigma_{s,l})\int_S d\Omega\int_{V_l} dV\,\mathbf{n}\cdot\mathbf{\Omega}\,\Psi_{h,l}^\chi\,\Pi_l^+(\Psi_{h,l}^z).$$

Orthogonal projections properties provide

$$
\begin{aligned}
&(\nabla\Psi^+_{h,l},\Pi^-_l(\nabla\Psi^+_{h,l})) = \|\Pi^-_l(\nabla\Psi^+_{h,l})\|^2 \qquad \forall\Psi^+_{h,l}\in P^{N,p}_h(V_l)\\
&(\mathbf{\Omega}\,\Psi^-_{h,l},\nabla\Psi^+_{h,l}-\Pi^-_l(\nabla\Psi^+_{h,l})) = 0 \qquad \forall\Psi^-_{h,l}\in M^{N,m}_h(V_l)\\
&(\mathbf{\Omega}\,\Psi^-_{h,l},\nabla\Pi^+_l(\Psi^z_{h,l})-\Pi^-_l(\nabla\Pi^+_l(\Psi^z_{h,l}))) = 0 \qquad \forall\Psi^-_{h,l}\in M^{N,m}_h(V_l).
\end{aligned}
$$

Then some manipulations using these properties lead to

$$
\begin{aligned}
K_l(\lambda_{h,l},\hat\lambda_{h,l}) = {} & 2\sigma_l\|\Psi^+_{h,l}\|^2 + \frac{1}{\sigma_l}\|\Pi^-_l(\nabla\Psi^+_{h,l})\|^2 + \sigma_l\|\Psi^-_{h,l}\|^2\\
&+\frac{1}{\sigma_l-\sigma_{s,l}}\|\Pi^+_l(\Psi^z_{h,l})\|^2_{grad,V_l,*}\\
&+\frac{1}{\sigma_l}\int_S d\Omega\int_{V_l} dV\,(\nabla\Psi^+_{h,l})\,(\Pi^-_l(\nabla\Pi^+_l(\Psi^z_{h,l}))-\nabla\Pi^+_l(\Psi^z_{h,l}))\\
&-\int_S d\Omega\int_{V_l} dV\,2\,\sigma_{s,l}\,\Psi^+_{h,l}\int_S d\Omega'\Psi^+_{h,l}\\
&-\int_S d\Omega\int_{V_l} dV\,\sigma_{s,l}\,\Pi^+_l(\Psi^z_{h,l})\,(\int_S d\Omega\Psi^+_{h,l}-\Psi^+_{h,l})\\
&+\int_S d\Omega\int_{\partial V_l\cap\partial V} d\Gamma\,(3-\frac{\sigma_{s,l}}{\sigma_l})\,\Psi^\chi_{h,l}\,\Psi^+_{h,l}.
\end{aligned}
$$

The last term can again be proved non-negative when reflected or vacuum boundary conditions are imposed. Besides, mild anisotropy implies as before $(1+\alpha_n)\,\|\Psi^-_{h,l}\|\geq\|\phi_{h,l}-\Psi^+_{h,l}\|$, and since $\|\Psi^z_{h,l}\|^2\leq\frac{1}{\sigma_l(\sigma_l-\sigma_{s,l})}\|\Psi^z_{h,l}\|^2_{grad,V_l,*}$ for any $\Psi_l\in H(grad,S\times V_l)$, we can write

$$
\begin{aligned}
K_l(\lambda_{h,l},\hat\lambda_{h,l}) \geq {} & 2\,(\sigma_l-\sigma_{s,l})\,\|\Psi^+_{h,l}\|^2 + \frac{1}{\sigma_l}\|\Pi^-_l(\nabla\Psi^+_{h,l})\|^2\\
&+\left(\sigma_l-\frac{1}{2}(1+\alpha_n)^2\,\sigma_{s,l}\right)\,\|\Psi^-_{h,l}\|^2\\
&+\frac{1}{\sigma_l-\sigma_{s,l}}\left(1-\frac{1}{2}\frac{\sigma_{s,l}}{\sigma_l}\right)\,\|\Pi^+_l(\Psi^z_{h,l})\|^2_{grad,V_l,*}\\
&+\frac{1}{\sigma_l}\int_S d\Omega\int_{V_l} dV\,(\nabla\Psi^+_{h,l})\,\left(\Pi^-_l(\nabla\Pi^+_l(\Psi^z_{h,l}))-\nabla\Pi^+_l(\Psi^z_{h,l})\right).
\end{aligned}
$$

Thus there exist a constant $C_l>0$ such that

$$
\begin{aligned}
K_l(\lambda_{h,l},\hat\lambda_{h,l}) \geq {} & C_l\,\left(\|\Psi^+_{h,l}\|^2+\|\Pi^-_l(\nabla\Psi^+_{h,l})\|^2+\|\Psi^-_{h,l}\|^2+\|\Pi^+_l(\Psi^z_{h,l})\|^2_{grad,V_l,*}\right)\\
&+\frac{1}{\sigma_l}\int_S d\Omega\int_{V_l} dV\,(\nabla\Psi^+_{h,l})\,(\Pi^-_l(\nabla\Pi^+_l(\Psi^z_{h,l}))-\nabla\Pi^+_l(\Psi^z_{h,l})).
\end{aligned}
$$

We now introduce the parameters

$$\mu_l = \mu_l(P_h^{N,p}, B_h^{N,b}) = \inf_{\Psi^\chi_{h,l} \in B_h^{N,b}} \frac{\|\Pi_l^+(\Psi^z_{h,l})\|^2_{grad,V_l,*}}{\|\Psi^\chi_{h,l}\|^2_{\Omega,-1/2,\partial V_l}},$$

$$\nu_l = \nu_l(P_h^{N,p}, M_h^{N,m}) = \inf_{\Psi^+_{h,l} \in P_h^{N,p}} \frac{\|\Pi_l^-(\nabla\Psi^+_{h,l})\|^2}{\|\nabla\Psi^+_{h,l}\|^2},$$

$$\gamma_l = \gamma_l(P_h^{N,p}, M_h^{N,m}) = \sup_{\Psi^+_{h,l} \in P_h^{N,p}} \frac{\|\nabla\Psi^+_{h,l} - \Pi_l^-(\nabla\Psi^+_{h,l})\|}{\|\nabla\Psi^+_{h,l}\|},$$

and note that they are all contained in the interval $[0,1]$. Then

$$K_l(\lambda_{h,l}, \hat\lambda_{h,l}) \geq C_l \left(\|\Psi^+_{h,l}\|^2 + \nu_l \,\|\nabla\Psi^+_{h,l}\|^2 + \|\Psi^-_{h,l}\|^2 + \mu_l \,\|\Psi^\chi_{h,l}\|^2_{\Omega,-1/2,l} \right) \\ -\gamma_l \,\|\nabla\Psi^+_{h,l}\| \,\|\nabla\Pi_l^+(\Psi^z_{h,l}))\|.$$

Using $\|\nabla\Pi_l^+(\Psi^z_{h,l}))\| \leq \|\Psi^z_{h,l}\|_{grad,V_l,*}$, equation (A.6), and the equivalence of the norms $\|\cdot\|^2_{\Omega,-1/2,\partial V_l,*}$ and $\|\cdot\|^2_{\Omega,-1/2,\partial V_l}$, we obtain that there exists a constant $k > 0$ such that

$$K_l(\lambda_{h,l}, \hat\lambda_{h,l}) \geq C_l \Big(\|\Psi^+_{h,l}\|^2 + (\nu_l - \frac{1}{2}\gamma_l) \,\|\nabla\Psi^+_{h,l}\|^2 + \|\Psi^-_{h,l}\|^2 + \\ (\mu_l - k\gamma_l) \,\|\Psi^\chi_{h,l}\|^2_{\Omega,-1/2,\partial V_l,*} \Big)$$

that is

$$K_l(\lambda_{h,l}, \hat\lambda_{h,l}) \geq C_l \min(1, \nu_l - \frac{1}{2}\gamma_l, \mu_l - k\gamma_l) \,\|\lambda\|^2_\Lambda.$$

Also there is a constant $C_l' > 0$ such that

$$\|\hat\lambda_l\|^2_{\Lambda_{0,l}} \leq C_l' \,\|\lambda_l\|^2_{\Lambda_l}.$$

The LBB condition is thus verified provided

$$\min(1, \nu_l - \frac{1}{2}\gamma_l, \mu_l - k\gamma_l) > 0. \tag{B.1}$$

In case $\nabla P_h^{N,p} \subset \mathbf{\Omega} M_h^{N,m}$ in any V_l, which occurs as soon as $p-1 \leq m$ and $N^+ \leq N^- + 1$, we get $\nu_l = 1$ and $\gamma_l = 0$. Then, the condition becomes $\mu_l > 0$. Similarly to what is proved in [3], we have:

Lemma 11 *The parameter $\mu_l > 0$ if and only if, for any $\Psi^\chi_{h,l} \in B_h^{N,b}(V_l)$,*

$$\int_S d\Omega \int_{\partial V_l} d\Gamma\, \mathbf{\Omega}\cdot\mathbf{n}\, \Psi^\chi_{h,l}\, \Psi^+_{h,l} = 0 \qquad \forall\, \Psi^+ \in P_h^{N,p}(V_l) \;\; \textit{implies that} \;\; \Psi^\chi_{h,l} = 0. \tag{B.2}$$

Proof of lemma 11 If we suppose $\mu_l = 0$, then by the definition of μ_l there exist a non-zero $\Psi^\chi_{h,l} \in B_h^{N,b}(V_l)$ such that

$$\|\Pi_l^+(\Psi^z_{h,l})\|^2_{grad,V_l,*} = 0. \tag{B.3}$$

Since $\int_S d\Omega \int_{\partial V_l} d\Gamma\, \mathbf{\Omega} \cdot \mathbf{n}\, \Psi^\chi_{h,l}\, \Psi^+_{h,l} = \frac{1}{\sigma_l - \sigma_{s,l}}(\Psi^+_{h,l}, \Pi^+(\Psi^z_{h,l}))_{grad,V_l,*}$, (B.2) and (B.3) imply $\Psi^\chi_{h,l} = 0$, a contradiction. Now take $\mu_l > 0$. Then, if $\Psi^\chi_{h,l} \neq 0$,

$$\int_S d\Omega \int_{\partial V_l} d\Gamma\, \mathbf{\Omega} \cdot \mathbf{n}\, \Psi^\chi_{h,l}\, \Pi_l^+(\Psi^z_{h,l}) \neq 0,$$

which is the contrapositive of (B.2). $\square$

As discussed in section 5.2, condition (B.2) is in fact equivalent to a spatio-angular rank condition.

In (B.1), ν_l and γ_l do not depend on h, but μ_l does. To ensure uniform stability of the method, we should therefore show the existence of a lower bound for μ_l as $h \to 0$. This can probably be obtained adapting what is done in [3] and [19] for purely spatial PDE's, that is using a master element of unit diameter. Also, error estimates could then be derived.

B.1.2 Odd-order primal method

The spaces Λ_h and $\Lambda_{0,h}$ are formally the same as in the even-order case, but the definition of the approximation spaces $P_h^{N,p}$, $M_h^{N,m}$ and $B_h^{N,b}$ differ as explained in section 5.1. We here define Π_l^+ and Π_l^- as the orthogonal projections of $H(\mathbf{\Omega} \cdot \nabla, S \times V_l)$ and $L^2(S \times V_l)$ respectively onto $P_h^{N,p}$ and $M_h^{N,m}$.

The LBB condition can be proved here using $\hat{\lambda}_h = (\hat{\Psi}^+_h, \hat{\Psi}^-_h, \hat{\Psi}^\chi_h) \in \Lambda_{0,h}$ with

$$\begin{aligned}
\hat{\Psi}^+_{h,l} &= 2\,\Psi^+_l + \Pi_l^+(\Psi^z_{h,l}) \\
\hat{\Psi}^-_{h,l} &= -\frac{1}{\sigma_l}\Pi_l^-(\mathbf{\Omega} \cdot \nabla \Psi^+_l) - \Psi^-_l - \frac{1}{\sigma_l}\Pi_l^-(\mathbf{\Omega} \cdot \nabla \Pi_l^+(\Psi^z_{h,l})) \\
\hat{\Psi}^\chi_{h,l} &= -(3 + \frac{\sigma_{s,l}}{\sigma_l})\Psi^\chi_l|_{\partial V_l,0},
\end{aligned}$$

where $\Psi^z \in H(\mathbf{\Omega} \cdot \nabla, S \times V_l)$ is the weak solution of the auxiliary problem

$$\begin{cases}
-\nabla(\mathbf{\Omega} \cdot \nabla \Psi^z) + \sigma_l(\sigma_{s,l} - \sigma_l)\mathbf{\Omega}\Psi^z = 0 & \text{in } V_l \\
\\
\mathbf{\Omega} \cdot \nabla \Psi^z = (\sigma_l - \sigma_{s,l})\,\Psi^\chi & \text{on } \partial V_l.
\end{cases}$$

The proof follows the same lines as in B.1.1. In case $\mathbf{\Omega}\cdot\nabla P_h^{N,p} \subset M_h^{N,m}$, which occurs as soon as $p-1 \leq m$ and $N^+ + 1 \leq N^-$, we obtain that the LBB condition becomes $\mu_l > 0$ where

$$\mu_l = \mu_l(P_h^{N,p}, B_h^{N,b}) = \inf_{\Psi_{h,l}^\chi \in B_h^{N,b}} \frac{\|\Pi_l^+(\Psi_{h,l}^z)\|^2_{\mathbf{\Omega}\cdot\nabla,V_l,*}}{\|\Psi_{h,l}^\chi\|^2_{1/2,\partial V_l}}.$$

Lemma 11 remains valid in this case.

B.2 Proofs for the dual case

Roughly speaking, the roles of $P_h^{N,p}$ and $M_h^{N,m}$ are swapped when going from primal to dual. Working with the finite-dimensional subspaces

$$\Lambda_h = M_h^{N,m} \times P_h^{N,p} \times B_h^{N,b} \text{ and } \Lambda_{0,h} = M_h^{N,m} \times P_h^{N,p} \times B_{0,h}^{N,b},$$

we can again verify the LBB condition and obtain theorem 10.

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References

[1] Y. Azmy, Multidimensional linear-linear nodal transport methods in weighted diamond difference form, in: Proceeding of the International Topical Meeting on advances in reactor physics, mathematics and computation, Paris (1987) 1005–1019.

[2] I. Babŭska, Error-bounds for finite element method, Numer. Math. 16 (1971) 322–333.

[3] I. Babŭska, J. Oden, J. Lee, Mixed-hybrid finite element approximations of second-order elliptic boundary-value problems, Comp. Meth. Appl. Mech. Eng. 11 (1977) 175–206.

[4] G. Bell, S. Glasstone, Nuclear rector theory (Van Nostrand Reinhold Company, 1970).

[5] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Elements Methods, Vol. 15 of Springer Series in Computational Mathematics, (Springer Verlag, New York, 1991).

[6] R. Dautray, J.-L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 6, (Springer-Verlag, 1993).

[7] S. Kaplan, J. Davis, Canonical and involutory transformations of the variational problems of transport theory, Nuclear Science and Engineering 28 (2) (1967) 166–176.

[8] E. Kuznetsov, Radiative equilibrium of a gaseous sphere surrounding an absolutely black sphere, Izv.An.SSSR Ser. Geophys. (1951) 69–93.

[9] E. Lewis, W. Miller, Jr., Computational methods of neutron transport, (John Wiley & Sons, 1984).

[10] J. Lions, E. Magenes, Problèmes aux limites non homogènes et applications (Dunod, Paris, 1968).

[11] G. Palmiotti, E. Lewis, C. Carrico, VARIANT: VARIational Anisotropic Nodal Transport for multidimensional cartesian and hexagonal geometry calculations, Tech. Rep. ANL-95/40, Argonne National Laboratory, 1995.

[12] G. Pomraning, M. Clark,Jr, The variational method applied to the monoenergetic boltzmann equation (part 2), Nuclear Science and Engineering 16 (2) (1963) 155–164.

[13] P. Raviart, J. Thomas, A mixed finite element method for second-order elliptic problems, in: Mathematical aspects of the finite element method, Vol. 606 of Lecture Notes in Mathematics (Springer-Verlag, Berlin, 1977) 292–315.

[14] P. Raviart, J. Thomas, Primal hybrid finite element methods for 2nd order elliptic equations, Mathematics of Computations 31 (138) (1977) 391–413.

[15] J. Roberts, J.-M. Thomas, Mixed and hybrid methods, in: P.G.Ciarlet, J.L.Lions (Eds.), Handbook of Numerical Analysis, Vol. 2, (North Holland, Amsterdam, 1991) 523–633.

[16] G. Rumyantsev, Boundary conditions in the spherical harmonic method, Reactor Science and Technology (Journal of Nuclear Energy Parts A/B) 16 (1962) 111–118.

[17] R. Temam, Navier-Stokes Equations, Vol. 2 of Studies in Mathematics and Its Application (North-Holland, Amsterdam, 1979).

[18] T. Toivanen, A self-adjoint variational principle for deriving vacuum and interface boundary conditions in the spherical harmonics method, Nuclear Science and Engineering 25 (3) (1966) 275–284.

[19] S. Van Criekingen, R. Beauwens, Mixed-hybrid discretization methods for the P_1 equations, submitted to Applied Numerical Analysis (2004).

[20] V. Vladimirov, Mathematical problems in the one-velocity theory of particle transport, Tech. rep., Atomic Energy of Canada Ltd., Ontario, 1963; translated from Transactions of the V.A. Steklov Mathematical Institute, **61**, 1961.