The Cauchy Problem for Compressible Hydrodynamic-Maxwell Systems: A Local Theory for Smooth Solutions

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Abstract

The Hydrodynamic-Maxwell equations are studied, as a compressible model of charge transport induced by an electromagnetic field in semiconductors. A local smooth solution theory for the Cauchy problem is established by the author's modification of the classical semigroup-resolvent approach of Kato. The author's theory has three noteworthy features: (1) stability under vanishing heat flux, which is not derivable from other theories; (2) accommodation to arbitrarily specified terminal time for the regularized problem; and, (3) constructive in nature, in that it is based upon time semidiscretization, and the solution of these semidiscrete problems determines the localization theory criteria. The regularization is employed to avoid vacuum states, and eliminated for the final results which may contract the admissible time interval. We also provide a symmetrized formulation in matrix form which is useful for applications and simulation. The theory uses the generalized energy estimates of Friedrichs on the ground function space, and leverages them to the smooth space via Kato's commutator estimate.

Keywords Hydrodynamic-Maxwell Systems, Symmetrized Formulation, Vanishing Heat Flux, Cauchy Problem, Smooth Solutions, Semigroups, Resolvent Stability, Semidiscretization, Vacuum States

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1 Introduction

In this paper, we consider the hydrodynamic model of charge transport of semiconductors in an electromagnetic field. The hydrodynamic model treats the propagation of electrons in a semiconductor device as the flow of a compressible charged fluid. Coupling to electrostatic fields has been well studied, involving equations for the conservation of density, momentum and energy, coupled to Poisson's equation for the electrostatic potential. See [5, 20, 21, 9, 19, 12, 8, 14] and the references therein for issues of underlying physical derivation, simulation, and mathematical analysis of this system. The third reference discusses the important topic of the thermodynamic limit and the Onsager relations. These references cover both the steady-state and dynamic cases. The final two cited references deal with weak and smooth solutions, respectively, for the evolution system. The case of electrostatic coupling has also been investigated in the context of current flow in cellular ion channels, and possible implications for temperature variation (see [6] for a comprehensive such simulation study). The dynamic version is of potential usefulness in this case for the study of bio-effects, when intrinsically generated fields are augmented by external pulsing.

When semiconductor devices are operated under high frequency conditions (including technologies such as microwave devices, electro-optics, spintronics, and semiconductor lasers), magnetic fields are generated by moving charges inside the device, and the charge transport interacts with the propagating electromagnetic waves. In this case, the electromagnetic field satisfies Maxwell's equations, which are coupled to the transport system. Therefore, the hydrodynamic model for high-frequency charge transport in semiconductors consists of the conservation laws, coupled to Maxwell's equations for the electric and magnetic fields.

This system assumes the following (nonconservative) form ([2, 3, 19]):

$$
\begin{cases}
\frac{\partial n}{\partial t} + \mathbf{v}\cdot \nabla n + n\nabla\cdot \mathbf{v} = 0, \\
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{k}{m}\nabla\mathcal{T} + \frac{k\mathcal{T}}{mn}\nabla n = -\frac{q}{m}\mathbf{F} - \frac{\mathbf{v}}{\tau_{\mathbf{p}}}, \\
\frac{\partial \mathcal{T}}{\partial t} - \frac{\kappa_0}{n}\nabla\cdot(n\nabla\mathcal{T}) + \mathbf{v}\cdot\nabla\mathcal{T} + \frac{2}{3}\mathcal{T}\nabla\cdot\mathbf{v} = -\frac{2m|\mathbf{v}|^2}{3k}\left(\frac{1}{2\tau_w} - \frac{1}{\tau_{\mathbf{p}}}\right) - \frac{\mathcal{T}-\mathcal{T}_*}{\tau_w}, \\
\varepsilon\mathbf{E}_t - \nabla\times\mathbf{H} + \mathbf{J} = 0, \\
\mu\mathbf{H}_t + \nabla\times\mathbf{E} = 0, \\
-\varepsilon\nabla\cdot\mathbf{E} = \frac{q}{m}n - D(\vec{x}), \quad \nabla\cdot\mathbf{B} = 0, \\
\mathbf{B} = \mu\mathbf{H}, \quad \mathbf{J} = -\frac{q}{m}n\mathbf{v}, \quad \vec{x}\in R^3, \quad t>0,
\end{cases}
\tag{1.1}
$$

where n is the electron mass density, $\mathbf{v} \in R^3$ is the electron velocity, $\mathcal{T}$ is the electron temperature, $\mathbf{E} \in R^3$ is the electric field, $\mathbf{H} \in R^3$ is the magnetic field, $\mathbf{J} \in R^3$ is the current density, $\mathbf{B} \in R^3$ is the magnetic induction, $\mathbf{F} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$ and $-q\mathbf{F}$ is the Lorentz force, D is the doping profile, q is the electronic charge, m is the effective electron mass, k is Boltzmann's constant, τ_p is the momentum relaxation time, τ_w is the energy relaxation time, μ is the permeability of the medium and ε is the permittivity of the medium.

Note the presence of heat conduction in the system. The Hydrodynamic-Maxwell equations are more intricate than the Euler-Poisson equations, because of the complicated coupling of the Lorentz force. There have been numerical simulations ([2, 3]), but the only rigorous study appears to be that made by Chen, Wang and the author in [7], where a global weak solution is proved in one spatial dimension. In this paper, we demonstrate local classical solutions of the Cauchy problem on R^3 by use of an adaptation of Kato's theory of evolution operators. The method is stable under the singular limit of vanishing heat flux. Such results are not typically deduced from smooth theories. In the process, we provide a symmetrized formulation which is important for applications. An issue relevant for computation is that the so-called Gauss equation in the Maxwell system is satisfied for all time if it is incorporated into the initial condition. The zero divergence of the magnetic induction must be maintained for all time, but this condition is simpler (see [22]). We have relegated the more routine elements of the proofs to the appendices so as to minimize any redundancy with respect to other published work by the author. The key result is Theorem 5.1. The traditional Friedrichs inequalities, or generalized energy estimates, are employed on the ground space for the semidiscrete solutions; they are extended to the smooth space by Kato's commutator estimate. We are thus able to extend classical methods quite generally. The stability of (quasi-numerical) time semidiscretization also follows from this approach.

2 System Reformulation and Functional Framework

We shall consider the Cauchy problem on Euclidean space of dimension 3. Define the vector $\mathbf{u}$ by

$$\mathbf{u} = \begin{bmatrix} n \\ \mathbf{v} \\ \mathcal{T} \\ \mathbf{E} \\ \mathbf{B} \end{bmatrix}. \tag{2.1}$$

It is conceptually simpler, in formulating the matrix version of the Hydrodynamic-Maxwell system, to use a block matrix approach. Thus, we write

$$\mathbf{u} = \left[\begin{array}{c} n \\ \mathbf{v} \\ \mathcal{T} \\ \hline \mathbf{E} \\ \mathbf{B} \end{array}\right] = \left[\begin{array}{c} \mathbf{y} \\ \hline \mathbf{z} \end{array}\right]. \tag{2.2}$$

For mathematical simplification, we shall choose units in which the following *numerical* relationships hold:

$$q/m = 1, k/m = 1, \varepsilon\mu = 1.$$

The system (1.1) as defined above has matrix multipliers of $\frac{\partial \mathbf{y}}{\partial x_j}$, $j = 1, 2, 3$, given by the matrices (where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise):

$$\tilde{C}_j = \left[\begin{array}{c|ccc|c} v_j & n\delta_{1j} & n\delta_{2j} & n\delta_{3j} & 0 \\ \hline \frac{\mathcal{T}}{n}\delta_{1j} & v_j & 0 & 0 & 1\delta_{1j} \\ \frac{\mathcal{T}}{n}\delta_{2j} & 0 & v_j & 0 & 1\delta_{2j} \\ \frac{\mathcal{T}}{n}\delta_{3j} & 0 & 0 & v_j & 1\delta_{3j} \\ \hline 0 & \frac{2\mathcal{T}}{3}\delta_{1j} & \frac{2\mathcal{T}}{3}\delta_{2j} & \frac{2\mathcal{T}}{3}\delta_{3j} & v_j \end{array}\right]. \tag{2.3}$$

The matrix multipliers of $\frac{\partial \mathbf{z}}{\partial x_j}$, $j = 1, 2, 3$, are given by the matrices D_j:

$$D_j = \left[\begin{array}{c|c} \mathbf{0} & G_j \\ \hline G_j^t & \mathbf{0} \end{array}\right], \tag{2.4}$$

where

$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, G_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.5}$$

Here, and throughout, $\mathbf{0}$ denotes an appropriate (possibly non-square) zero matrix, possibly a row or column vector.

2.1 The Symmetrizer

The symmetrizer of $\tilde{C}_j$ is then given by:

$$C_0 = \left[\begin{array}{c|c|c} \frac{\mathcal{T}}{n} & \mathbf{0} & 0 \\ \hline \mathbf{0} & nI_3 & \mathbf{0} \\ \hline 0 & \mathbf{0} & \frac{3n}{2\mathcal{T}} \end{array}\right], \tag{2.6}$$

where I_3 is the identity matrix of order 3. C_0 is symmetrizing in the following sense:

$$C_j = C_0\tilde{C}_j = \left[\begin{array}{c|ccc|c} \frac{\mathcal{T}v_j}{n} & \mathcal{T}\delta_{1j} & \mathcal{T}\delta_{2j} & \mathcal{T}\delta_{3j} & 0 \\ \hline \mathcal{T}\delta_{1j} & nv_j & 0 & 0 & n\delta_{1j} \\ \mathcal{T}\delta_{2j} & 0 & nv_j & 0 & n\delta_{2j} \\ \mathcal{T}\delta_{3j} & 0 & 0 & nv_j & n\delta_{3j} \\ \hline 0 & n\delta_{1j} & n\delta_{2j} & n\delta_{3j} & \frac{3nv_j}{2\mathcal{T}} \end{array}\right] \tag{2.7}$$

is symmetric for each $j = 1, 2, 3$. We may then define the system symmetrizer via

$$a_0 = \left[\begin{array}{c|c} C_0 & \mathbf{0} \\ \hline \mathbf{0} & I_6 \end{array}\right] \tag{2.8}$$

and the symmetric multipliers via

$$a_j = \left[\begin{array}{c|c} C_j & \mathbf{0} \\ \hline \mathbf{0} & D_j \end{array}\right]. \tag{2.9}$$

We then obtain:

$$a_0(\mathbf{u})\mathbf{u}_t + L(\mathbf{u})\mathbf{u} + \left[\sum_{j=1}^{3} a_j(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_j} + b(\mathbf{u})\mathbf{u}\right] = 0, \tag{2.10}$$

where

$$L(\mathbf{u}) = -\mathrm{diag}(0, \mathbf{0}, \gamma_0/\mathcal{T}, \mathbf{0})\nabla\cdot(n\nabla),\ \gamma_0 = \frac{3}{2}\kappa_0,\ c = \left(\frac{1}{2\tau_w} - \frac{1}{\tau_{\mathbf{p}}}\right),$$

and

$$b = \left[\begin{array}{c|c|c|c} 0 & \mathbf{0} & 0 & \mathbf{0} \\ \hline \mathbf{F} & \frac{n}{\tau_{\mathbf{p}}}I_3 & \mathbf{0} & \mathbf{0} \\ \hline \frac{3(1-\frac{\mathcal{T}_*}{\mathcal{T}})}{2\tau_w} & c\frac{n\mathbf{v}}{\mathcal{T}} & 0 & \mathbf{0} \\ \hline \mathbf{0} & -\mu nI_3 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]. \tag{2.11}$$

It remains to discuss the divergence conditions expressed in terms of **E** and **H**. The latter condition is imposed by requiring **B** to belong to a divergence free space. We shall make this specific shortly. In regard to **E**, it is enough to impose the appropriate condition on the initial electric field; by taking the divergence of the equation involving $\mathbf{E}_t$, we infer that

$$\varepsilon(\nabla\cdot \mathbf{E})_t = -\nabla\cdot \mathbf{J} = \nabla\cdot (n\mathbf{v}),$$

and the latter is given, by the conservation of particle density equation, by

$$-n_t = -(n - D(\vec{x}))_t,$$

so that the equality of $\varepsilon\nabla\cdot \mathbf{E}$ and $-(n - D(\vec{x}))$ at $t = 0$ implies equality for $0 \leq t \leq T$. The initial condition for the Cauchy problem is then given by,

$$\mathbf{u}(\cdot\,, 0) = \mathbf{u}_0, \tag{2.12}$$

for a given function, $\mathbf{u}_0 \in H^s(R^3; R^{11}),\ s > 7/2$ (see (2.14) for positivity conditions). The components corresponding to **B** are required to have zero L_2 divergence in the sense of distributions; the components corresponding to **E** must satisfy the divergence condition described above. The spaces H^s are defined in §2.3 to follow. We could relax the condition on s to $s > 5/2$ if the heat conductivity had a simpler structure (constant). The complete Cauchy problem is defined by (2.10, 2.12). We shall pose it as a Cauchy problem in Hilbert space. The major result is Theorem 5.1.

2.2 The Regularization of a_0, a_j, and L

Let ζ be a non-decreasing C^∞ function satisfying:

$$\zeta(s) = \begin{cases} 0, & s \leq 0, \\ 1, & s \geq 1. \end{cases} \tag{2.13}$$

An example of such a function is given in [17, p. 36]. Given the initial conditions, $n_0, \mathcal{T}_0$, and constant threshold values,

$$0 < n_{00} < \inf n_0, 0 < \mathcal{T}_{00} < \inf \mathcal{T}_0, \tag{2.14}$$

define the regularizations by setting

$$\tilde{n}(n) = \frac{n}{2}\left(1 + \zeta(n/n_{00})\right), \ \tilde{\mathcal{T}}(\mathcal{T}) = \frac{\mathcal{T}}{2}\left(1 + \zeta(\mathcal{T}/\mathcal{T}_{00})\right).$$

Then, $\tilde{n}(n) = n,\ n \geq n_{00}$, $\tilde{\mathcal{T}}(\mathcal{T}) = \mathcal{T},\ \mathcal{T} \geq \mathcal{T}_{00}$, and $n_{00}/2$ and $\mathcal{T}_{00}/2$ are lower bounds.

regularization of a_0

Define the regularization a_{00} of a_0 to be the diagonal matrix obtained by the replacements:

$$n \mapsto \tilde{n},\ \mathbf{v} \mapsto \mathbf{v},\ \mathcal{T} \mapsto \tilde{\mathcal{T}}.$$

regularization of L

Similarly, for the regularization L_0 of L, we have: $n \mapsto \tilde{n},\ \mathbf{v} \mapsto \mathbf{v},\ \mathcal{T} \mapsto \tilde{\mathcal{T}}$.

regularization of a_j and b

For the regularization a_{j0} of a_j, the quotients in the first and final diagonal positions of C_j are regularized, and for the regularization b_0 of b, the elements involving division by $\mathcal{T}$ are regularized. Specifically,

$$\frac{\mathcal{T} v_j}{n} \mapsto \frac{\mathcal{T} v_j}{\tilde{n}},\ \frac{3nv_j}{2\mathcal{T}} \mapsto \frac{3nv_j}{2\tilde{\mathcal{T}}},\ \frac{3(1-\frac{\mathcal{T}_0}{\mathcal{T}})}{2\tau_w} \mapsto \frac{3(1-\frac{\mathcal{T}_0}{\tilde{\mathcal{T}}})}{2\tau_w},\ c\frac{n\mathbf{v}}{\mathcal{T}} \mapsto c\frac{n\mathbf{v}}{\tilde{\mathcal{T}}}.$$

We are interested in the Cauchy problem for the regularized system,

$$a_{00}(\mathbf{u})\mathbf{u}_t - L_0(\mathbf{u})\mathbf{u} + \left[\sum_{j=1}^{3} a_{j0}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_j} + b_0(\mathbf{u})\mathbf{u}\right] = 0, \tag{2.15}$$

2.3 A Framework for Analysis

We introduce the classical Bessel potential space $H^s(R^3; R^k)$ [1]. It can be characterized, via the isometric Fourier transform $\mathcal{F}$, as the linear space of functions v with norm,

$$\|v\|_{H^s}^2 = \int_{R^3} (1+|x|^2)^s |\mathcal{F}v(x)|^2\, dx.$$

It follows from the definition that the diagonal operator $S = I(I-\Delta)^{s/2}$ induces an isometry of $H^s(R^3; R^k)$ onto $L_2(R^3; R^k)$. Here, Δ denotes the Laplacian.

We may now define:

$$X = \mathrm{P}L_2(R^3; R^{11}),\ Y = \mathrm{P}H^s(R^3; R^{11}).$$

We understand by P the orthogonal projection $\mathrm{I}_8 \bigotimes \mathrm{P}_3$, which leaves invariant the first eight of the components of $\mathbf{u}$, and projects the final three components, via P_3, onto the subspace of $L_2(R^3; R^3)$ consisting of functions with divergence free distributional derivatives. Another type of space required for our analysis is the class of uniformly local spaces [16], [11, p. 252]. We recall their definition here.

Definition 2.1. *For $\mathcal{H}$ a fixed Hilbert space (including the special case of Euclidean space), let $L_{2,u\ell}(R^d;\mathcal{H})$ denote the set of all (equivalence classes of) $\mathcal{H}$-valued strongly measurable functions u such that*

$$\|u\| = \sup_{x\in R^d}\left(\int_{|y-x|<1}|u(y)|^2dy\right)^{1/2} < \infty.$$

This space is called a uniformly local L_2 space. For each integer $s \geq 0$, denote by $H^s_{u\ell}$ the set of $u \in L_{2,u\ell}$ such that the distribution derivatives $D^\alpha u$ of order $|\alpha| \leq s$ are in $L_{2,u\ell}$. The norm in $H^s_{u\ell}$ is given by

$$\|u\|_{H^s_{u\ell}} = \sup_{|\alpha|\leq s}\|D^\alpha u\|_{L_{2,u\ell}}.$$

Interpolation space theory may be used to extend the definition to non-integral s.

We state two essential properties of the uniformly local H^s spaces which will be used in the course of our analysis.

inclusion relations

If $s > 3/2 + k$, k a nonnegative integer, and $m \geq 1$ is an integer, then

$$H^s(R^3;R^m) \subset H^s_{u\ell}(R^3;R^m) \subset C^k_b(R^3;R^m),$$

where the inclusions are continuous. Here, the subscript b indicates that derivatives through order k are bounded.

multiplier relations

If $s > 3/2$, then multiplication induces continuous bilinear maps:

$$H^{\tau+\sigma}(R^3;R^1) \times H^{s-\sigma}_{u\ell}(R^3;R^m) \mapsto H^\tau(R^3;R^m),$$

for $0 \leq \sigma \leq s, 0 \leq \tau \leq s - \sigma$.

The regularized matrix a_{00} contains entries obtained by taking reciprocals, which are not, in general, H^s functions. We have introduced the uniformly local spaces precisely to cover this situation. The regularized reciprocals of H^s functions are in $H^s_{u\ell}$, and hence are (invariant) multipliers. This pertains as well to the regularizations of L and a_j.

3 The General Cauchy Problem

3.1 Preliminaries

We begin with the standard description of the class of operators related to the infinitesimal generators of semigroups to be used in the sequel [4].

Definition 3.1. *Let U be a closed linear operator with domain and range dense in a Hilbert space X. Denote by $R(\lambda, U)$ the resolvent $(\lambda I - U)^{-1}$ for λ in the resolvent set $\rho(U)$. For $M > 0$ and $\omega \in R$ denote by $G(X, M, \omega)$ the set of all operators $A = -U$ such that*

$$\|[R(\lambda, U)]^r\| \leq M(\lambda - \omega)^{-r},\ r \geq 1,\ \lambda > \omega.$$

Finally,

$$G(X) = \cup_{\omega, M} G(X, M, \omega).$$

We now pass to the core result for the theory. There is a criterion due to Kato (see [15], [11]), which permits one to deduce semigroup generation on a smooth space Y via stability on X. More precisely, the criterion is directed more fundamentally at transferring the property $A \in G(X, M, \omega)$ to $A \in G(Y, M, \omega_1)$. It is particularly useful when $M = 1$. We quote the relevant result. It follows from [11, Propositions 6.2.3 and 6.2.4], which are based on [15].

Proposition 3.1. *Suppose Y is a Hilbert space densely and continuously embedded in X and $S : Y \mapsto X$ is an isomorphism. We write $\|v\|_Y = \|Sv\|_X$. Suppose $A \in G(X, M, \omega)$ such that*

$$A_1 = SAS^{-1} = A + B, \tag{3.1}$$

where B is a bounded linear operator on X and

$$\mathcal{D}(A_1) = \{v : AS^{-1}v \in Y\}.$$

Then the semigroup generated by $-A$, restricted to Y, is the semigroup generated by the restriction of $-A$ to $\{v \in Y \cap \mathcal{D}(A) : Av \in Y\}$. It follows that $A_1 \in G(X, M, \omega + M\|B\|)$, or, equivalently, $A \in G(Y, M, \omega_1)$, with $\omega_1 = \omega + M\|B\|$. In this case, $Se^{-tA}S^{-1} = e^{-tA_1}$.

Note that use is made of the well-known fact that $A \in G(X, M, \omega)$ and B bounded on X imply

$$A + B \in G(X, M, \omega + M\|B\|).$$

3.2 The Abstract Cauchy Problem

We are interested in solving an initial value problem, in a Hilbert space X,

$$A_0(u)\frac{du}{dt} + A(u)u = 0, \quad u(0) = u_0, \tag{3.2}$$

where $A(u) \in G(X, M, \omega)$ for u restricted to a subset of a 'smooth' Hilbert space Y, densely and continuously embedded in X, and where A_0 and A_0^{-1} are bounded on X for u suitably restricted. We seek a solution $u(t) \in Y$, $0 \leq t \leq T$. The derivative, du/dt, is required to belong to an intermediate space, V. Certain Lipschitz continuity conditions, to be described later, are also required. Among these, we require:

$$\|A_0^{-1}(v)A(u) - A_0^{-1}(v)A(w)\|_{Y,X} \leq C\|u - w\|_X,$$

uniformly in v, for u, w suitably restricted in norm.

3.3 The Implicit Semidiscretization in Time

If Δt is given as the ratio T/N, then the method of horizontal lines, applied to (3.2), yields a semidiscrete set of implicit equations, explicit in A_0:

$$A(u_k^N)u_k^N + (1/\Delta t)A_0(u_{k-1}^N)u_k^N = (1/\Delta t)A_0(u_{k-1}^N)u_{k-1}^N, \; k = 1, \ldots, N. \tag{3.3}$$

If we set $\mu^2 = 1/\Delta t = N/T$, then the u_k^N can be characterized formally as fixed points of the mapping

$$Qv = Q_k^N v = -R(\mu^2 - 1, -A_0^{-1}(u_{k-1}^N)A(v))v + \mu^2 R(\mu^2 - 1, -A_0^{-1}(u_{k-1}^N)A(v))u_{k-1}^N. \tag{3.4}$$

By repeated back substitution, one obtains the following useful formula for u_{k-1}^N:

$$u_{k-1}^N = \prod_{j=1}^{k-1} \mu^2 R(\mu^2, -\tilde{A}(u_j^N))u_0, \tag{3.5}$$

where we have written,

$$\tilde{A}(u_j^N) = A_0^{-1}(u_{j-1}^N)A(u_j^N).$$

Pivotal to the entire study is the demonstration of the existence of fixed points for this map within an appropriately smooth set. We are now able to introduce the type of stability which is appropriate for this purpose.

Definition 3.2. *Let X be a Hilbert space, suppose that $\Delta t = T/N$ is given, and a partition $t_j = j\Delta t, j = 0, \ldots, N$, is specified. Suppose that a family $\{\tilde{A}(u)\}$ is given as above, and that $u_1^N, \ldots, u_k^N$ are obtained recursively via (3.3), where $k \leq N$. The family is said to be stable if there are constants M, ω, and c, independent of k and N, but depending on the radius r of a ball in Y containing $u_1^N, \ldots, u_{k-1}^N$, such that*

$$\| \prod_{j=1}^{k} [\tilde{A}(u_j^N) + \lambda_j]^{-1} \| \leq M \prod_{j=1}^{k} (\lambda_j - \omega) \exp{(cT)}\,, \quad \lambda_j > \omega. \tag{3.6}$$

In order to obtain smooth solutions, it is required that stability hold on X and Y, with constants M, ω c, and M_1, ω_1, c_1, respectively.

3.4 The Invariance and Lipschitz Continuity of Q

For ω and ω_1 introduced through Definition 3.2 above, we define: $\bar{\omega} = \max(\omega, \omega_1)$. $\bar{M}$ and $\bar{c}$ are defined similarly. Suppose that δ and ρ are fixed positive constants, and that

$$\sigma = (1 + \delta)\bar{M} e^{(1+1/\rho)(1+\bar{\omega}+\bar{c})T}, \tag{3.7}$$

where T is a fixed terminal time to be specified. We define

$$\bar{W}_0 = \{u \in Y : \|u\|_Y \leq \sigma \|u_0\|_Y, \; \|u\|_X \leq \sigma \|u_0\|_X\}.$$

This is the invariant set on which Q acts. A precise statement is now given.

Proposition 3.2. *Suppose that a family $\{\tilde{A}(u)\}$ and $\Delta t = T/N$ are given, and a partition $t_j = j\Delta t, j = 0, \ldots, N$, is thereby specified. If $u_1^N, \ldots, u_{k-1}^N \in \bar{W}_0$ are inductively defined solutions of* (3.3)*, suppose that the family $\{\tilde{A}(u_j^N)\}$ is defined as in Definition* 3.2 *and is stable on X and Y. If the integer N satisfies:*

$$\frac{N}{T} > [(1 + \delta^{-1})\bar{M} + (\rho + 1)(1 + \bar{\omega} + \bar{c})], \tag{3.8}$$

then the mappings $Q = Q_k^N$ of (3.4) *are mappings of $\bar{W}_0$ into itself.*

The proof is deferred to Appendix A. We shall next describe the Lipschitz continuity of Q. This will close the induction, and give the existence of u_k^N, for Δt sufficiently small.

Proposition 3.3. *Under the assumptions of Proposition* 3.2*, the mappings* $Q = Q_k^N$ *of* (3.4) *are Lipschitz continuous mappings in the topology of* X *with Lipschitz constant*

$$C_Q = \frac{\bar{M}}{\mu^2 - 1 - \bar{\omega}} \left[1 + C(1 + \bar{M}(1 + 1/\rho))\sigma\|u_0\|_X\right]. \tag{3.9}$$

Here, C *is the Lipschitz constant cited earlier. If* N *is sufficiently large, then* $C_Q < 1$ *and* Q *has a unique fixed point in* $\bar{W}_0$*.*

The proof is given in Appendix A. The reader will notice that $\bar{W}_0$ is viewed as a complete metric subspace of X in this realization of the Banach contraction mapping principle.

3.5 Technical Results Related to Stability of Norms

The use of the symmetrizer introduces a technical complication related to stability. This is dealt with by use of a family of equivalent norms. The following result is recorded in a form useful for later use. Its routine proof is omitted.

Lemma 3.1. *Given* $u_i^N, i = 0, \ldots, k-1$*, with* $A(u_i^N) \in G(X, 1, \omega)$*, we define a family of norms, indexed by* i*:*

$$\|f\|_i = \|A_0(u_{i-1}^N)f\|_X, \; i = 1, \ldots, k, \tag{3.10}$$

where A, A_0 *have been given previously. Suppose that there is a constant* K*,*

$$\|A_0(u_i^N)\| \leq K, \; \|A_0^{-1}(u_i^N)\| \leq K, \; all\, i. \tag{3.11}$$

Suppose also that

$$\|A_0(u_i^N)A_0^{-1}(u_{i-1}^N)\| \leq (1 + C_0\Delta t)^2. \; all\, i. \tag{3.12}$$

Then the norms as defined in (3.10) *satisfy the stability condition with* $M = K^2$ *and* $C = 2C_0$*.*

As will be seen, the constants K and C_0 depend upon the radius of the set on which the mapping Q acts. We present an additional result which will be useful for deriving stability on Y. Suppose we have the setup of the previous lemma, and suppose $Y \subset X$ are related by an isomorphism, $S : Y \mapsto X$, and that S remains an isomorphism from Y_i to X_i, where

$$\|u\|_{Y_i} = \|Su\|_{X_i}, \; i = 1, \ldots, k.$$

Here, the subscript i denotes the equivalent norm of the lemma. We have the following.

Proposition 3.4. *Suppose the conditions of the previous lemma hold, and suppose* (3.1) *holds for each member of the family* $\{\tilde{A}(\cdot)\}$ *(the argument* $\cdot$ *represents any of the* u_i^N*):*

$$\tilde{A}_1(\cdot) = S\tilde{A}(\cdot)S^{-1} = \tilde{A}(\cdot) + \tilde{B}(\cdot), \tag{3.13}$$

where $\|\tilde{B}(\cdot)\|_X$ *is uniformly bounded. If* $\tilde{A}_1(\cdot) \in G(X_i, 1, \omega), \forall i \in \{1, \dots, k\}$*, then* $\{\tilde{A}(\cdot)\}$ *is stable, with* $M_1 = 1$*,* $c_1 = 2c_0$*,* $\omega_1 = \omega + \|B(\cdot)\|_i$*, with respect to* Y_i*, for each* $i \in \{1, \dots, k\}$*.*

4 Generators for the Model and their Properties

The preceding theory has been described as an appropriate template for the Hydrodynamic-Maxwell model. What makes this theory applicable to our model is that the differential operators which we have defined in §2 are semigroup generators on X and Y. For the reader's convenience, we state these properties in Appendix B. The operator A is given in (B.4). The techniques make essential use of the Friedrichs framework for generation by A on X and the Kato framework, via commutator estimates, for generation on Y. By employing the appendix in this way, we have minimized redundancy with respect to reference [14]. Here, we proceed directly to the stability results. These and the Lipschitz continuity are central to the application of the preceding theory.

4.1 Stability on X and Y

We suppose that we are in the inductive situation of Definition 3.2. We shall now deduce stability for the regularized problem by formulating a natural criterion in terms of the semidiscrete solutions.

Lemma 4.1. *In the context of Definition 3.2, suppose that there is a constant* c'*, not depending on* $k = 1, \dots, N$*, such that*

$$\|\mathbf{u}_i^N - \mathbf{u}_{i-1}^N\|_C \leq c'\Delta t, \; i = 1, \dots, k-1. \tag{4.1}$$

Then the norms are stable. Here, the subscript C *refers to the norm in the space of uniformly continuous, bounded functions on* R^3*.*

Proof. According to Lemma 3.1, we must examine

$$\|a_{00}(u_i^N)a_{00}^{-1}(u_{i-1}^N)\|. \tag{4.2}$$

The individual main-diagonal entries of the diagonal matrix, $a_{00}(u_i^N)a_{00}^{-1}(u_{i-1}^N)$, are either unit entries or quotients, or a simple product of quotients. These basic quotients are of the form,

$$\frac{\tilde{n}(n(t))}{\tilde{n}(n(s))}, \ \frac{\tilde{\mathcal{T}}(\mathcal{T}(t))}{\tilde{\mathcal{T}}(\mathcal{T}(s))},$$

or their reciprocals. We use a simple algebraic relation to estimate these quotients:

$$\left|\frac{\alpha}{\beta}\right| \leq (1 + C|\alpha - \beta|), \tag{4.3}$$

where $C \geq 1/\beta$. It will be enough to consider certain case distinctions. These are made so that the term $|\alpha - \beta|$ is easily computable. For concreteness, we consider the quotient,

$$\frac{\tilde{n}(n_i)}{\tilde{n}(n_{i-1})}.$$

Its simple estimation is bounded above via (4.3) and the definitions of $\tilde{n}, a_{00}$ by:

$$(1 + C_1\|n_i - n_{i-1}\|_C), \ 1 \leq i \leq N - 1,$$

for some constant C_1. This makes use of the Sobolev embedding theorem and the smoothness of the function ζ. A similar estimate is derivable for the reciprocal, and for the expressions in $\mathcal{T}$. We have derived an upper bound of at most

$$(1 + C_0\Delta t)^2,$$

for the individual entries of $a_{00}(u_i^N)a_{00}^{-1}(u_{i-1}^N)$, upon use of the hypothesis (4.1). This completes the proof. □

Proposition 4.1. *Suppose r is the radius of the admissible $\bar{W}_0$ in Y on which Q acts. Then* (4.1) *holds, with c' a sixth degree polynomial of r. In the context of Definition* 3.2 *we have stability on X and Y, if a fixed r is chosen independently of k. The constants have values $M_t = 1, \omega_t = Cr^3$ on any X_t, so that, on X, M is proportional to r^2 and $\omega = Cr^3$. On Y, we have M_1 proportional to r^2 and ω_1 a quintic function of r. The stability constant c is a constant multiple of c' for both X and Y.*

Proof. The characterization of c' as a sixth polynomial is a consequence of the detailed analysis of Lemma 4.2 to follow: a direct estimate of the semidiscrete equation gives a product of a quintic polynomial estimate with r, as an estimate for the differences $\|\mathbf{u}_j^N - \mathbf{u}_{j-1}^N\|_C/\Delta t$. The remaining statements follow from Lemma 3.1 and Lemma B.2. Note that $\|\tilde{B}(t)\|_X$ is a constant multiple of r^5. □

4.2 The Lipschitz Properties of $\tilde{A}(\mathbf{u})$

We make use of the definitions of E and A given in (B.1, B.4). We have reserved the notation E for the first order operator part of A. Again, we define $\tilde{A} = a_{00}^{-1}A$. A precise statement of the latter is given in the following lemma.

Lemma 4.2. *The mapping* $\mathbf{w} \mapsto E(\mathbf{w}) \in B(H^s, H^\tau)$ *is Lipschitz continuous in the norm topology for* $0 \le \tau \le s-1$ *for* $s > 5/2$*:*

$$\|E(\mathbf{w}) - E(\mathbf{w}')\|_{H^s,H^\tau} \le C\|\mathbf{w} - \mathbf{w}'\|_{H^\tau}, \ \ \mathbf{w}, \mathbf{w}' \in \bar{W}.$$

The constant C *is proportional to a cubic function of the radius* r *of* $\bar{W}_0 \subset Y$*. Similarly, if* $s > 7/2$*, the mapping* $\mathbf{w} \mapsto A(\mathbf{w}) \in B(H^s, H^\tau)$ *is Lipschitz continuous in the norm topology for* $0 \le \tau \le s-2$*:*

$$\|A(\mathbf{w}) - A(\mathbf{w}')\|_{H^s,H^\tau} \le C\|\mathbf{w} - \mathbf{w}'\|_{H^\tau}, \ \ \mathbf{w}, \mathbf{w}' \in \bar{W}.$$

The dependence on C *is cubic in* r*. Finally, for fixed* $\mathbf{v} \in H^s$*, the mapping* $\mathbf{w} \mapsto \tilde{A}_{\mathbf{v}}(\mathbf{w}) \in B(H^s, H^\tau)$ *is Lipschitz continuous in the norm topology for* $0 \le \tau \le s-2$*. Here,* $\tilde{A}_{\mathbf{v}}(\mathbf{w}) = a_{00}^{-1}(\mathbf{v})A(\mathbf{w})$*. In this case, the dependence is quintic in* r*.*

Proof. We first note the inequalities,

$$\|a_{j0}(\mathbf{w}) - a_{j0}(\mathbf{w}')\|_{H^\tau} \le c_1\|\mathbf{w} - \mathbf{w}'\|_{H^\tau},$$

$$\|b_0(\mathbf{w}) - b_0(\mathbf{w}')\|_{H^\tau} \le c_2\|\mathbf{w} - \mathbf{w}'\|_{H^\tau}.$$

These inequalities use the definitions of the matrices $a_{j0}, j = 1, 2, 3, and\, the matrix b_0$. The constants c_1, c_2 depend quadratically upon r. Now, since H^{s-1} functions are multipliers on H^τ, we have:

$$\|E(\mathbf{w})\mathbf{v} - E(\mathbf{w}')\mathbf{v}\|_{H^s,H^\tau} \le c(\sum_{j=1}^{d} \|a_{j0}(\mathbf{w}) - a_{j0}(\mathbf{w}')\|_{H^\tau}\|\frac{\partial \mathbf{v}}{\partial x_j}\|_{H^{s-1}}+$$

$$\|b_0(\mathbf{w}) - b_0(\mathbf{w}')\|_{H^\tau}\|\mathbf{v}\|_{H^{s-1}}) \leq C'\|\mathbf{w} - \mathbf{w}'\|_{H^\tau}\|\mathbf{v}\|_{H^s}.$$

This gives the statement of the lemma regarding E. To obtain the statement regarding A, we examine L_0:

$$\|L_0(\mathbf{w})\mathbf{v} - L_0(\mathbf{w}')\mathbf{v}\|_{H^s,H^\tau} \leq c\{\|D_0(\mathbf{w}) - D_0(\mathbf{w}')\|_{H^\tau}\|\nabla\cdot(n\nabla\mathcal{T})\|_{H^{s-2}}$$
$$+\|D_0(\mathbf{w}')(\nabla(n-n')\cdot\nabla\mathcal{T} + (n-n')\Delta\mathcal{T})\|_{H^\tau}\}$$
$$\leq C'\|\mathbf{w} - \mathbf{w}'\|_{H^\tau},$$

provided $s > 7/2$. Here, we have made use of the multiplier relations. The same observations concerning the cubic growth in r apply. Finally, in order to analyze $\tilde{A}$, we note that the explicit operator a_{00}^{-1} is a multiplier on each H^τ space; the estimate contributes an additional factor proportional to r^2. □

4.3 The Fixed Point Theorem for the Semidiscrete Euler-Maxwell System

As in the discussion of §3, we define $\bar{\omega}$ to be the maximum of the constants ω and ω_1, derived from the classes $G(X, M, \omega)$ and $G(Y, M, \omega_1)$. These functions have polynomial dependence on r. We retain the notation M and c, which also have polynomial dependence.

Local Assumption on $\|\mathbf{u}_0\|$ and T

If $M(r)$ is the stability constant of Definition 3.2, and $c = c(r)$ is given in this definition, both analyzed in Proposition 4.1, we require:

$$\|\mathbf{u}_0\|_Y < \frac{r}{M(r)}e^{-(1+\bar{\omega}(r)+c(r))T} := H(r,T). \tag{4.4}$$

(4.4) is the general inequality which must be satisfied by $\|\mathbf{u}_0\|_Y, T$, and the radius r of the admissible ball in Y. This is quite general: T may be given arbitrarily.

We now define numbers δ and ρ which allow us to connect (4.4) with the theoretical analysis of Propositions 3.2 and 3.3. Set $\gamma = 1 + \bar{\omega} + c$, and select ρ satisfying

$$M(r)\|\mathbf{u}_0\|_Y e^{(1+1/\rho)\gamma(r)T} < r,$$

which is possible by (4.4). Define:

$$\delta = re^{-(1+1/\rho)\gamma(r)T}/(M(r)\|\mathbf{u}_0\|_Y) - 1.$$

It is immediate that

$$(1+\delta)M(r)e^{(1+1/\rho)\gamma(r)T}\|\mathbf{u}_0\|_Y = r.$$

We further define:

$$\sigma = (1+\delta)M(r)e^{(1+1/\rho)\gamma(r)T}.$$

These definitions then describe the framework investigated in Propositions 3.2 and 3.3. In particular,

$$\sigma\|\mathbf{u}_0\|_Y = r.$$

We then have the following theorem, which follows in a direct manner from the framework we have developed..

Theorem 4.1. *If* (4.4) *holds, and N is sufficiently large, then the mapping Q, with Lipschitz constant C_Q given by* (3.9)*, is a strict contraction on $\bar{W}_0$. In this case, Q has a unique fixed point, denoted $\mathbf{u}_k^N$.*

5 Well-Posedness Theorem and Summation

The details of the transition from the semidiscrete analysis to the evolution system are presented in Appendix C. Combinatorial and compactness arguments lead first to a weak, then a strong solution of the evolution system. The strong solution is unique, and stable under the limit $\kappa_0 \to 0$. All of this is initially carried out for the regularized problem. Here, we complete the analysis of the well-posedness of the model by removing the presence of the regularization. We do require, however, the specification of threshold parameters. More precisely, we assume that (2.14) is satisfied, which implicitly requires the uniform boundedness of $n_0, \mathcal{T}_0$ away from zero, and also sets lower bounds of half the threshold values.

Theorem 5.1. *If* (2.14) *is satisfied, and* $\mathbf{u}$ *is the unique solution on* $[0,T]$ *satisfying Corollary* C.1 *and Proposition* C.1*, then there is a maximum subinterval* $[0,T']$ *of* $[0,T]$ *such that $n \geq n_{00}, \mathcal{T} \geq \mathcal{T}_{00}$ on this subinterval and such that the regularized problem is identical to the given system.*

Proof. The proof is immediate because of the regularity class (C.13) and the Sobolev embedding theorem, which together guarantee that $\mathbf{u}(x,t)$ is bounded and uniformly continuous as a function of x, t. Thus, given a unique strong solution $\mathbf{u}$ of (2.15) on $[0,T]$, one defines:

$$T' = \max\{t' : u_1(\cdot\,,t) \geq n_{00},\ 0 \leq t \leq t',\ u_{d+2}(\cdot\,,t) \geq \mathcal{T}_{00},\ 0 \leq t \leq t'\}.$$

□

Remark 5.1. *This is the first analysis of the Euler-Maxwell system in this complete generality. As noted in the introduction, the use of the fully implicit method of horizontal lines has allowed for the formulation of a precise condition (see* (4.4)*) for the local assumption. By using semigroup methods, we are able to pass to the limit of vanishing heat conduction. The first use of these ideas in the context of the Kato semigroup framework is described in* [11, Section 7.5]*. A weak solution study of the isentropic subsystem in one variable was given in [7] for the two-carrier model with geometric terms. This alternative approach permits shock formation. Finally, the reader who has followed the arguments of the paper will realize that the case of temperature dependent relaxation terms is elementary, so long as singularity formation is avoided. One can regularize such expressions as in (2.15), and proceed much the way we did above.*

Appendices

A The Properties of Q and the Fixed Point Theorem

For completeness, we give here the proofs of Propositions 3.2 and 3.3.

Proof of Proposition 3.2:

For fixed N, we assume inductively that (3.3) has a solution u_ℓ^N for $\ell < k$, where $1 \leq k \leq N$. We can estimate $\|Qv\|_X$. From (3.4), we have, by use of the stability property:

$$\|Qv\|_X \leq \frac{M}{\mu^2 - \omega - 1}\sigma\|u_0\|_X + \frac{M\mu^2}{\mu^2 - \omega - 1}\left(\frac{\mu^2}{\mu^2 - \omega}\right)^{k-1} \exp(cT)\|u_0\|_X. \tag{A.1}$$

Here we have used (3.5). For the first term, we estimate, by the choice of N,

$$\frac{M}{\mu^2 - \omega - 1} \leq \frac{\delta}{(1+\delta)}.$$

An observation required for the estimate of the second term is given by

$$\frac{\mu^2}{\mu^2-\omega-1} \le (1+1/\rho) \tag{A.2}$$

if $\mu^2 \ge (1+\rho)(\omega+1)$. When this is combined with the standard inequality,

$$(1+s)^N \le e^{sN}, \quad s = \frac{\omega+1}{\mu^2-\omega-1},$$

we arrive at a chain of inequalities for the second term. By the choice of N and σ,

$$\begin{aligned}\frac{\mu^2}{\mu^2-\omega-1}\left(\frac{\mu^2}{\mu^2-\omega}\right)^{k-1}\exp(cT) \le \left(\frac{\mu^2}{\mu^2-\omega-1}\right)^{N}\exp(cT) \le e^{(1+1/\rho)(1+\omega+c)T} \\ \le \frac{\sigma}{M(1+\delta)}.\end{aligned}$$

If we apply each of the estimates, we have the estimate that $\|Qv\|_X \le \sigma\|u_0\|_X$.

The estimate for $\|Qv\|_Y$ is similar and completes the proof.

Proof of Proposition 3.3:

The critical representation is the identity,

$$R(\lambda, -\tilde{A}(w)) - R(\lambda, -\tilde{A}(v)) = R(\lambda, -\tilde{A}(w))[\tilde{A}(v) - \tilde{A}(w)]R(\lambda, -\tilde{A}(v)).$$

We obtain:

$$\begin{aligned}\|R(\lambda, -\tilde{A}(w)) - R(\lambda, -\tilde{A}(v))\|_X &\le \|R(\lambda, -\tilde{A}(w))\|_X C\|v-w\|_X \|R(\lambda, -\tilde{A}(v))\|_Y \\ &\le C_1 \frac{\|v-w\|_X}{(\lambda-\omega)(\lambda-\omega_1)},\end{aligned}$$

where $C_1 = MM_1C$. This leads to the estimate, for $\lambda = \mu^2 - 1$,

$$\|Qv - Qw\|_X \le \frac{1}{\mu^2-1-\omega}\left[M + \frac{C_1\sigma\|u_0\|_X}{\mu^2-1-\omega_1} + \frac{C_1\sigma\|u_0\|_X\,\mu^2}{\mu^2-1-\omega_1}\right]\|v-w\|_X.$$

Here, we have used the inductive assumption that $\|u^N_{k-1}\|_X \le \sigma\|u_0\|_X$. By using the estimates of the proof of Proposition 3.2, we obtain

$$\|Qv - Qw\|_X \le \frac{1}{\mu^2-1-\omega}\left[M + \frac{MC\sigma\delta\|u_0\|_X}{(1+\delta)} + MM_1C(1+1/\rho)\sigma\|u_0\|_X\right]\|v-w\|_X.$$

This yields the estimate (3.9) of the proposition. Since Y is assumed to be an embedded Hilbert space, $\bar{W}_0$ is a complete metric subspace of X, and the final statement follows from the contraction mapping theorem.

B The Semigroup Generation Properties

In this appendix, we retain the meanings of X and Y as defined in §2.3 in order to analyze the regularized Cauchy problem as defined in §2.2. We begin with semigroup generation on the ground space X, and follow the classical approach of Friedrichs.

Lemma B.1. *Let* $\mathbf{w} \in Y, s > 5/2$. *Let*

$$E(\mathbf{w}) = \mathrm{P}\left[\sum_{j=1}^{3} a_{j0}(\mathbf{w})\frac{\partial}{\partial x_j} + b_0(\mathbf{w})\right], \tag{B.1}$$

where a_j, b *are defined in* (2.9), (2.11) *and* a_{j0}, b_0 *are the regularizations. Then the following hold.*

1. $E = E(\mathbf{w})$ *may be identified with the closed linear operator, defined by the formal adjoint relation,*

$$(\mathbf{u}, E^*\psi)_X = (\mathbf{v}, \psi)_X, \;\; \forall \psi \in \mathrm{P}C_0^\infty(R^3; R^{11}),$$

where $\mathbf{v} = E\mathbf{u}, \; \mathbf{u} \in \mathcal{D}(E)$.

2. *The relation,*

$$(E\psi, \psi)_X = \left(\frac{1}{2}\left(b_0 + b_0^* - \sum_{j=1}^{3}\frac{\partial a_{j0}}{\partial x_j}\right)\psi, \psi\right)_X, \tag{B.2}$$

holds. In particular, the energy inequality,

$$(E\mathbf{u}, \mathbf{u})_X \geq -\omega_0(\mathbf{u}, \mathbf{u})_X,$$

holds for all $\mathbf{u} \in \mathcal{D}(E)$, *where*

$$\omega_0 = \frac{1}{2}\sum_{j=1}^{3}\|a_{j0}\|_{C^1} + \|b_0\|_C. \tag{B.3}$$

3. $E \in G(X, 1, \omega_0)$.

Moreover, the sum,

$$A(\mathbf{w}) = L_0(\mathbf{w}) + \mathrm{P}\left[\sum_{j=1}^{3} a_{j0}(\mathbf{w})\frac{\partial}{\partial x_j} + b_0(\mathbf{w})\right], \tag{B.4}$$

is in $G(X, 1, \omega)$. *Here,* ω *is proportional to the cube of the radius* r *in* Y *of the ball from which the coefficients of* L_0, a_{j0}, b_0 *are taken.*

Proof. We give only a sketch, referring the reader to the proper references. Property (1) is due to Friedrichs [10]. Property (2) is a direct calculation, and uses the Hermitian symmetry of the matrices a_{j0}. The energy inequality is an immediate consequence, and, in turn, implies the generation property for $-E$, via the Hille-Yosida theorem. We now discuss the complete spatial operator. This is straightforward after an initial observation. One observes that the energy inequality holds for the augmented operator, on a pre-domain of compact support functions. We illustrate this by examining the action of the nontrivial part of $L_0(\mathbf{w})$. Let $\psi \in C_0^\infty(R^3; R)$, and consider the L_2 inner product,

$$(-\nabla\cdot(\tilde{n}\nabla\psi), \psi/\tilde{\mathcal{T}})_{L_2},$$

where $n, \mathcal{T}$ are components of $\mathbf{w}$. After integration by parts and an application of the product rule to $\psi/\tilde{\mathcal{T}}$, we obtain

$$(-\nabla\cdot(\tilde{n}\nabla\psi), \psi/\tilde{\mathcal{T}})_{L_2} = ((\tilde{n}/\tilde{\mathcal{T}})\nabla\psi, \nabla\psi)_{L_2} - ((\tilde{n}/\tilde{\mathcal{T}}^2)\nabla\psi, \psi\nabla\tilde{\mathcal{T}})_{L_2}.$$

The second term on the right hand side is estimated via:

$$((\tilde{n}/\tilde{\mathcal{T}}^2)\nabla\psi, \psi\nabla\tilde{\mathcal{T}})_{L_2} \geq -\delta((\tilde{n}/\tilde{\mathcal{T}})\nabla\psi, \nabla\psi)_{L_2} - C_\delta(\psi,\psi)_{L_2},$$

where δ can be made arbitrarily small. Here, C_δ depends upon r as stated in the lemma. One forms the sum, $A(\mathbf{w})$, deduces its action on $C_0^\infty(R^3; R^{11})$ by the earlier part of the lemma, and then takes the closure of this operator. □

The transfer of these generation properties to Y is now discussed. We quote here the fundamental commutator result of Kato [15, Appendix], as adapted to our present context.

Lemma B.2. *For functions $\mathbf{v}, \mathbf{w} \in \mathrm{P}H^s(R^3; R^k)$, where $k = 11$ in this paper, and an operator of the form,*

$$\tilde{A}(\mathbf{v}, \mathbf{w}) = a_{00}^{-1}(\mathbf{w}) \left\{ L_0(\mathbf{v}) + \mathrm{P}\left[\sum_{j=1}^{3} a_{j0}(\mathbf{v})\frac{\partial}{\partial x_i} + b_0(\mathbf{v})\right]\right\}, \tag{B.5}$$

we have

$$S\tilde{A}(\mathbf{v}, \mathbf{w})S^{-1} = \tilde{A}(\mathbf{v}, \mathbf{w})) +$$

$$\left[-[S, a_{00}^{-1}D_0]\Lambda^{1-(s-1)}(\Lambda^{-2}\nabla\cdot(\tilde{n}\nabla)) + \mathrm{P}\sum_{j=1}^{3}[S, a_{00}^{-1}a_{j0}]\Lambda^{1-s}\left(\frac{\partial}{\partial x_j}\right)\Lambda^{-1} + [S, a_{00}^{-1}b_0]\Lambda^{1-s}\Lambda^{-1}\right],$$

where $D_0 = \mathrm{diag}(0, \mathbf{0}, 1/\tilde{\mathcal{T}}, \mathbf{0})$, $\Lambda = (I - \Delta)^{1/2}$, $[\cdot\,,\cdot]$ *denotes the commutator, and* $S = I_k \Lambda^s$*; here* I_k *is the identity matrix of order* k*. In particular, in the notation of Proposition* 3.4*, we have*

$$\tilde{B} = \left[-[S, a_{00}^{-1} D_0] \Lambda^{1-(s-1)} \Lambda^{-2} (\nabla \cdot (\tilde{n} \nabla)) + \mathrm{P} \sum_{j=1}^{3} [S, a_{00}^{-1} a_{j0}] \Lambda^{1-s} \left(\frac{\partial}{\partial x_j} \right) \Lambda^{-1} + [S, a_{00}^{-1} b_0] \Lambda^{1-s} \Lambda^{-1} \right].$$

If $s > 7/2$*, then* B *is a bounded operator on* $L_2(R^3; R^k)$ *with bound:*

$$\|\tilde{B}\| \leq C \left(\|\mathrm{grad}\ a_{00}^{-1} D_0\|_{H^{s-2}} + \sum_{j=1}^{d} \|\mathrm{grad}\ a_{00}^{-1} a_{j0}\|_{H^{s-1}} + \|\mathrm{grad}\ a_{00}^{-1} b_0\|_{H^{s-1}} \right).$$

Note that we have used the fact that $\Lambda^{-2}(\nabla \cdot (\tilde{n} \nabla)$ can be extended to be a bounded linear operator on $L_2(R^3; R^k)$. This is where we need the increase (by one) of s.

C Analysis on the Space-Time Domain

This appendix is provided for readers unfamiliar with the details of the limiting procedure associated with the semidiscrete method. For the most part, we do not give proofs here. Amplification may be found in [14].

The idea is to define step function or piecewise linear sequences in time which make use of the semidiscrete spatial solutions. Weak and local strong compactness allow one to obtain a unique limit. This limit is first shown in the theory to be a weak solution, and then a strong solution in certain regularity classes. The strong solution is not only unique; it is invariant under $\kappa_0 \to 0$.

We begin by defining the relevant sequences which make use of the semidiscrete solutions.

Definition C.1. *For* $\Delta t = T/N$, $t_k = k\Delta t$*, and* $0 \leq t \leq T$*, set*

$$\theta_k^N(t) = \begin{cases} 1, \ t_{k-1} \leq t < t_k, \ k = 1, \ldots, N, \\ 0, \ \text{otherwise.} \end{cases}$$

Then for $x \in R^3$, define:

$$\mathbf{u}_{PL}^N(x,t) = \mathbf{u}_k^N(x) + \frac{t-t_k}{\Delta t}(\mathbf{u}_k^N(x) - \mathbf{u}_{k-1}^N(x)),\ t_{k-1} \le t < t_k,\ k = 1,\dots,N, \quad \text{(C.1)}$$

$$\mathbf{u}_S^N(x,t) = \sum_{k=1}^N \mathbf{u}_k^N(x)\theta_k^N(t), \quad \text{(C.2)}$$

$$a_{j0}^N(x,t) = a_{j0}(\mathbf{u}_S^N(x,t)), \quad \text{(C.3)}$$

$$b_0^N(x,t) = b_0(\mathbf{u}_S^N(x,t)). \quad \text{(C.4)}$$

We also require function space notation:

$$\mathcal{Y} = W_\infty^1((0,T); H^{s-2}(R^3; R^{11})) \cap L_\infty((0,T); Y).$$

The following lemma relates the sequences and the function spaces via norm boundedness.

Lemma C.1. *The sequence $\{\mathbf{u}_S^N\}$ is bounded in norm in $L_\infty((0,T);Y)$. The sequence $\{\mathbf{u}_{PL}^N\}$ is bounded in norm in $\mathcal{Y}$.*

C.1 The Weak Solution

Compactness arguments applied to the preceding lemma are used to obtain a weak solution. We have the following.

Theorem C.1. *There are subsequences, denoted $\mathbf{u}_{PL}^{N_j}$, $\mathbf{u}_S^{N_j}$, and a function $\mathbf{u} \in \mathcal{Y}$, such that:*

$$\mathbf{u}_{PL}^{N_j} \rightharpoonup \mathbf{u} \text{ weakly in } L_2((0,T);Y), \quad \text{(C.5)}$$

$$\mathbf{u}_{PL}^{N_j} \rightharpoonup^* \mathbf{u} \text{ weak-* in } W_\infty^1((0,T);H^{s-2}(R^d)) \cap L_\infty((0,T);Y), \quad \text{(C.6)}$$

$$\mathbf{u}_{PL}^{N_j} \to \mathbf{u} \text{ in } L_{2,\mathrm{loc}}(\mathcal{D}), \quad \text{(C.7)}$$

$$\mathbf{u}_S^{N_j} \rightharpoonup \mathbf{u} \text{ weakly in } L_2((0,T);Y), \quad \text{(C.8)}$$

$$\mathbf{u}_S^{N_j} \to \mathbf{u} \text{ in } L_{2,\mathrm{loc}}(\mathcal{D}), \quad \text{(C.9)}$$

$$a_{j0}^{N_j} \to a(\cdot\,,\mathbf{u}) \text{ in } L_{2,\mathrm{loc}}(\mathcal{D}), \quad \text{(C.10)}$$

$$b^{N_j} \to b(\cdot\,,\mathbf{u}) \text{ in } L_{2,\mathrm{loc}}(\mathcal{D}). \quad \text{(C.11)}$$

The function $\mathbf{u}$ *is a weak solution of the Cauchy problem: if* $\psi \in C^\infty([0,T]; C_0^\infty(R^d, R^{d+2}))$, *and* $T' \leq T$, *then, for* $\mathcal{D}_{T'} = R^d \times (0, T')$,

$$\int_{D_{T'}} \{\mathbf{u}\psi_t - A(\cdot\,, \mathbf{u})\mathbf{u}\psi\}\ dxdt + \int_{R^d \times \{0\}} \mathbf{u}_0 \psi\ dx - \int_{R^d \times \{T'\}} \mathbf{u}\psi\ dx = 0. \tag{C.12}$$

C.2 Existence and Uniqueness of Strong Solutions

Corollary C.1. *The solution* $\mathbf{u}$ *of* (C.12) *is a strong solution. Specifically,*

$$\mathbf{u}_t \in C([0,T]; H^{s-2}(R^d)), \tag{C.13}$$

and the equation (2.10) *holds in the strong sense described by* (C.13). *Moreover,*

$$\mathbf{u} \in C([0,T]; H^s(R^d)). \tag{C.14}$$

Proof. The regularity (C.13) follows from (C.6) of Theorem C.1 and the fundamental theorem of calculus in reflexive Banach spaces. This also validates an integration by parts, and hence the strong form of the evolution equation. Note that each of the terms in (2.10) (more precisely, in the regularized system (2.15)) is in the class (C.13) (see Lemma 4.2). The regularity (C.14) is more subtle and can be deduced from

$$\mathbf{u} \in L_\infty((0,T); H^s(R^d)),$$

established in Theorem C.1, in a manner similar to that employed in [18, pp. 44–46], where it is noted that right continuity at zero suffices to establish the continuity on $[0,T]$. The technique to establish right continuity at zero relies on establishing an estimate of the form,

$$\|\mathbf{u}(t)\|_{H^s}^2 \leq \|\mathbf{u}(0)\|_{H^s}^2 + \int_0^t f(\tau)\ d\tau,$$

where f is an L_1 function. One may now proceed as in [18]. □

The important property of uniqueness is noted.

Proposition C.1. *The strong solution of* (2.10) *described by Corollary* C.1 *is unique.*

C.3 Stability Under the Limit of Vanishing Heat Flux

An important feature of the semigroup-based theory which we have presented in this paper is its ability to permit the passage to the case of the strict gas dynamics limit for the charged fluid; i. e., the passage under the limit $\kappa_0 \to 0$. Results of this type were obtained in [13] for incompressible charged fluids, where the relevant parameter is the kinematic viscosity. It was noted in this reference that Kato developed his theory to cover a range of applications. The only distinction required when $\kappa_0 = 0$ is discussed in [15, p. 55] for the semigroup generation on L_2 (though the application is different, the procedure is the same). In fact, there is only one modification: the domain of $A \mapsto E$ is larger, with regularity index decreased by one. The rigorous argument appears in the proof of Lemma B.1. However, this has no impact upon the invariance results and fixed point arguments for Q, nor upon the arguments in the space-time domain.

The following result is a natural consequence of our arguments.

Proposition C.2. *There is a strong solution, in the regularity classes defined by* (C.13), (C.14)*, of the regularized system* (2.15)*, with* $\kappa_0 = 0$*. Moreover, the solution interval is stable under the inviscid limit* $\kappa_0 \to 0$*. More precisely, if* $\mathbf{u}_1, \mathbf{u}_2$*, are solutions of* (2.15) *for values* $\kappa_0' \geq \kappa_0'' \geq 0$*, then there is a constant* C *such that*

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{C([0,T];L_2(R^d))} \leq C(\kappa_0' - \kappa_0'').$$

The terminal time T *is independent of* κ_0*.*

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