Analytical Approaches to Charge Transport in a Moving Medium

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Abstract

We consider electrodiffusion in an incompressible electrolyte medium which is in motion. The Cauchy problem is governed by a coupled Navier-Stokes/Poisson-Nernst-Planck system. We prove the existence of a unique smooth local solution for smooth initial data, with nonnegativity preserved for the ion concentrations. We make use of semigroup ideas, originally introduced by T. Kato in the 1970s for quasi-linear hyperbolic systems. The time interval is invariant under the inviscid limit to the Euler/Poisson-Nernst-Planck system.

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1 Introduction

Modeling of electrodiffusion in electrolytes is a problem of major scientific interest [18]. At the present time, it finds application in biology (ion channels), chemistry (electro-osmosis), and pharmacology (transdermal iontophoresis). We shall study such a model in this paper, where self-consistent charge transport is represented by the Poisson-Nernst-Planck system, and the fluid motions by a Navier-Stokes system with forcing terms.

1.1 The Fluid/Transport System

The usual Poisson-Nernst-Planck model on a domain in R^m may be written for M carriers with concentration n_i, current density $\vec{J}_i$, (signed) charge e_i, $i = 1, \cdots, M$, permanent charge $d = d(x)$, and dielectric ϵ, as follows:

$$\begin{aligned} \frac{e_i \partial n_i}{\partial t} + \nabla\cdot \vec{J}_i &= 0, && (1.1) \\ \vec{E} &= -\nabla\phi, && (1.2) \\ \nabla\cdot(\epsilon\nabla\phi) &= -\sum e_i n_i - d \quad \text{(Poisson equation)}. && (1.3) \end{aligned}$$

In the case of two carriers (anion and cation carriers), that is, for $M = 2$, $n_1 = n, e_1 = -e$, and $n_2 = p, e_2 = e$, the classical drift-diffusion theory gives the following constitutive current relations:

$$\begin{aligned} \vec{J}_n &= eD_n\nabla n - e\mu_n n\nabla\phi, && (1.4) \\ \vec{J}_p &= -eD_p\nabla p - e\mu_p p\nabla\phi. && (1.5) \end{aligned}$$

Note that the *displacement current* is omitted from the current densities, and that the drift terms (called conduction current terms),

$$-e\mu_n n\nabla\phi, \ -e\mu_p p\nabla\phi,$$

are proportional to the respective drift velocities.

The use of the Einstein relations is common. These link the mobility coefficients μ_n, μ_p, the diffusion coefficients D_n, D_p, and the ambient temperature, T_0:

$$\begin{aligned} D_n &= (kT_0/e)\mu_n, && (1.6) \\ D_p &= (kT_0/e)\mu_p. && (1.7) \end{aligned}$$

In the velocity extended version on $R^m, m \geq 2$, the convection terms,

$$-e\vec{v}n, \quad e\vec{v}p,$$

are added to $\vec{J}_n, \vec{J}_p$, respectively, where $\vec{v}$ is the velocity of the electrolyte. Thus, the current densities are given by

$$\begin{aligned} \vec{J}_n &= eD_n\nabla n - e\mu_n n\nabla\phi - e\vec{v}n, && (1.8) \\ \vec{J}_p &= -eD_p\nabla p - e\mu_p p\nabla\phi + e\vec{v}p. && (1.9) \end{aligned}$$

The velocity of the electrolyte is determined by the Navier-Stokes equations:

$$\rho(\vec{v}_t + \vec{v}\cdot\nabla\vec{v}) - \eta\Delta\vec{v} = -\nabla P_f - e(p-n)\nabla\phi, \tag{1.10}$$
$$\nabla\cdot\vec{v} = 0, \tag{1.11}$$

where ρ is the (mass) density of the electrolyte, P_f denotes fluid pressure, and η is the dynamic viscosity. These equations have been introduced by Rubinstein [18]. Note that d has been neglected in the electric 'volume force' term. We shall make use of the kinematic viscosity, $\nu = \eta/\rho$, in the statement of the mathematical model.

1.2 The Mathematical Model

It has been traditional since the observations of Leray in 1933–34, to consider a reduced Navier-Stokes system, in tandem with the projection P onto divergence free distributions. The idea, discussed fully by Temam in [19, Chapter 1, §1,2], is to solve the equation of the pressure free part of the system, projected onto divergence free functions; it follows by the DeRham property that the reduced system is the gradient of a function (pressure). This reduction is also discussed in [7, pp. 34-35]. It is also required for well-posedness of the problem that the concentrations n and p be nonnegative. This is easily handled within the present framework as follows. One requires that

$$n_0 \geq \alpha_0 > 0, \; p_0 \geq \beta_0 > 0, \tag{1.12}$$

where $n(\cdot\,,0) = n_0, \; p(\cdot\,,0) = p_0$. Since the solution regularity so obtained implies that the vector solution is uniformly continuous on $[0,T] \times R^m$, we may select $T' \leq T$ so that $n \geq 0, \; p \geq 0$, i., e., the physical solution can be taken as an appropriate restriction of the solution of the mathematical model developed here. We make this precise in Theorem 4.2. The presence of the convection terms in the current densities makes an 'a posteriori' nonnegativity analysis, similar to that for self-consistent drift-diffusion (see [8]), problematical.

Define the $m+2$-vector $\mathbf{u}$ by

$$\mathbf{u} = \begin{bmatrix} \vec{v} \\ en \\ ep \end{bmatrix}. \tag{1.13}$$

The initial condition for the Cauchy problem on R^m is given by,

$$\mathbf{u}(\cdot\,,0) = \mathbf{u}_0,$$

for a given function,

$$\mathbf{u}_0 \in H^s(R^m; R^{m+2}),$$

The function spaces are defined in §3.1. We require a block system format. Thus, if $\mathbf{u}_1$ denotes the first m components of $\mathbf{u}$, and $\mathbf{u}_2$ denotes the remaining 2 components, we rewrite the system as

$$\frac{d\mathbf{u}}{dt} + A\mathbf{u} = \frac{d}{dt}\left[\begin{array}{c} \mathbf{u}_1 \\ \hline \mathbf{u}_2 \end{array}\right] + \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]\left[\begin{array}{c} \mathbf{u}_1 \\ \hline \mathbf{u}_2 \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_1 \\ \hline \mathbf{F}_2 \end{array}\right] = \mathbf{F}. \tag{1.14}$$

We have permitted an external forcing term $\mathbf{F}$. The nonlinear dependence of A is given by the operator representations:

$$
\begin{aligned}
A_{11}(\mathbf{u}_1) &= -\nu I\Delta + \mathrm{P}\mathbf{u}_1\cdot\nabla, \\
A_{12}(\mathbf{u}_2) &= \mathrm{P}[\ \rho^{-1}(-\phi_{x_1},\ldots,-\phi_{x_m})^T \mid \rho^{-1}(\phi_{x_1},\ldots,\phi_{x_m})^T\ ], \\
A_{22}(\mathbf{u}_1,\mathbf{u}_2) &= -\mathrm{diag}(D_n, D_p)\,\Delta + \mathrm{diag}(\mu_n\Delta\phi, -\mu_p\Delta\phi) \\
+\sum_{i=1}^{m}\mathrm{diag}(u_i &+ \mu_n\phi_{x_i}, u_i - \mu_p\phi_{x_i})(\partial/\partial x_i).
\end{aligned}
\tag{1.15}
$$

In the above system, the function ϕ has been used implicitly in its dependence upon n, p. We make this explicit:

$$
\phi = \Phi(\mathbf{u}_2), \quad \text{where} \quad -\epsilon\nabla^2\phi = \mathbf{u}_2\cdot(-1,1)^T + d. \tag{1.16}
$$

The assumptions on the smoothing map Φ are specified later. It is most convenient to rewrite the entire system in operator/vector format.

If we define the diagonal matrix D by

$$
D = \mathrm{diag}\ (\nu,\ldots,\nu, D_n, D_p), \tag{1.17}
$$

and the matrices a_i and b by

$$
a_i(\mathbf{u}) = \mathrm{diag}\ (\mathbf{u}_1, u_i + \mu_n\phi_{x_i}, u_i - \mu_p\phi_{x_i}), i = 1,\ldots,m,
$$

$$
b(\mathbf{u}) = b(\mathbf{u}_2) = \left[\begin{array}{lcc} 0 & \rho^{-1}(-\phi_{x_1},\ldots,-\phi_{x_m})^T & \rho^{-1}(\phi_{x_1},\ldots,\phi_{x_m})^T \\ 0 & (\mu_n\Delta\phi, 0)^T & (0, -\mu_p\Delta\phi)^T \end{array}\right],
$$

then the system may be written,

$$
\mathbf{u}_t - D\Delta\mathbf{u} + \mathcal{P}E(\mathbf{u})\mathbf{u} = \mathbf{F}(t,\mathbf{u}), \tag{1.18}
$$

where $A(\mathbf{u}) = -D\Delta + \mathcal{P}E(\mathbf{u})$, and

$$
E(\mathbf{u}) = \left[\sum_{i=1}^{m} a_i(\mathbf{u})\frac{\partial}{\partial x_i} + b(\mathbf{u})\right], \mathcal{P} = \left[\begin{array}{cc} \mathrm{P} & 0 \\ 0 & \mathrm{I}_2 \end{array}\right], \tag{1.19}
$$

with I_2 the identity matrix of order two. Finally, the following assumption on $\mathbf{F}$ is made for consistency: $\mathcal{P}\mathbf{F} = \mathbf{F}$. Our existence result for the Cauchy problem for (1.18) is presented in Corollary 4.1 and uniqueness in Proposition 4.2. In the following sections we shall develop the necessary theory, which is based upon Kato's semigroup ideas for evolution systems. The principal competing theory for local existence is the classical theory, due to Lax and Majda [17]. The latter theory, based upon symmetrization, mollification, and linearization, is very powerful when applicable, and is accompanied by a continuation principle, the breakdown of which is associated with shock formation or blowup. However, the classical theory is not as sharp as the semigroup theory in relation to the precise condition available for local existence. This condition, derived by the author in the use of the method of horizontal lines in conjunction with the semigroup theory (see (3.16) to follow) relates: (1) the norm of the initial datum; (2) the terminal time T; and, (3) the radius of the admissible ball gauging the size of the solution. This local existence inequality follows precisely from the semidiscrete method, and is also not evident from Kato's approach via the construction of evolution operators. Another important advantage of the semigroup approach is the incorporation of damping or frictional forces in an advantageous manner. This is especially significant when relaxation models are employed. Although the present paper does not make specific use of relaxation, future work will. Finally, the semigroup theory permits the natural passage to the inviscid limit (see Proposition 4.3).

2 A Semigroup Framework

2.1 Basic Facts

Definition 2.1. *Let U be a closed linear operator with domain and range dense in a Banach space X. Denote by $R(\lambda, U)$ the resolvent $(\lambda I - U)^{-1}$ for λ in the resolvent set $\rho(U)$. For $M > 0$ and $\omega \in R$ denote by $G(X, M, \omega)$ the set of all operators $A = -U$ such that*

$$\|[R(\lambda, U)]^r\| \leq M(\lambda - \omega)^{-r}, \ r \geq 1, \ \lambda > \omega.$$

Finally,

$$G(X) = \cup_{\omega, M} G(X, M, \omega).$$

We recall a result which gives characterizing conditions for $A \in G(X, 1, 0)$ [2, p. 143].

Proposition 2.1. *Suppose X is a real Hilbert space and A is a closed, densely-defined operator on X. Then $A \in G(X, 1, \omega)$ if and only if*

$$(Af, f) \geq -\omega(f, f),$$

and

$$\lambda \in \rho(A), \ \forall \lambda < -\omega.$$

Another useful result is a perturbation result, which requires the notion of relatively bounded perturbation. We first state the definition. Throughout the remainder of §2.1, X is a Banach space.

Definition 2.2. *Let T and A be closed linear operators on X with $\mathcal{D}(T) \subset \mathcal{D}(A)$. Then A is relatively bounded with respect to T, or simply T-bounded, if there exist nonnegative constants a, b such that*

$$\|Au\| \leq a\|u\| + b\|Tu\|, \ \forall u \in \mathcal{D}(T).$$

Moreover, the greatest lower bound of all admissible b is called the T-bound of A.

Sufficient conditions for a perturbed operator to remain in $G(X, 1, \omega)$ are given by the following proposition [14, Problems 2.8-2.9, p. 502].

Proposition 2.2. *If $T \in G(X, 1, \omega)$, $A \in G(X, 1, \omega')$ and A is relatively bounded with respect to T, with T-bound $< 1/2$ (< 1 if X is a Hilbert space), then*

$$T + A \in G(X, 1, \omega + \omega').$$

Next, we quote a result for perturbation by bounded linear operators [14, Theorem 2.1, p. 497], [2, Theorem 5.1, p. 179].

Proposition 2.3. *Let A be in $G(X, M, \omega)$, and let B be bounded on X. Then*

$$A + B \in G(X, M, \omega + M\|B\|).$$

Finally, we describe the dynamic induced by the embedding of a smooth subspace Y.

Proposition 2.4. *Suppose Y is a Banach space densely and continuously embedded in X and $S : Y \mapsto X$ is an isomorphism. We write $\|v\|_Y = \|Sv\|_X$. Suppose $A \in G(X, M, \omega)$ such that*

$$A_1 = SAS^{-1} = A + B, \tag{2.1}$$

where B is a bounded linear operator on X and

$$\mathcal{D}(A_1) = \{v : AS^{-1}v \in Y\}.$$

Then the semigroup generated by $-A$, restricted to Y, is the semigroup generated by the restriction of $-A$ to $\{v \in Y \cap \mathcal{D}(A) : Av \in Y\}$. In fact, $Se^{-tA}S^{-1} = e^{-tA_1}$ holds. It follows that $A_1 \in G(X, M, \omega + M\|B\|)$ so that $A \in G(Y, M, \omega_1)$, with $\omega_1 = \omega + M\|B\|$.

Proof. This result follows from [7, Propositions 6.2.3 and 6.2.4], which are based on Kato's work [12, 13, 15]. The key idea is the equivalence of $A_1 \in G(X, M, \omega + M\|B\|)$ with $A \in G(Y, M, \omega + M\|B\|)$. Note that Proposition 2.3 implies the former. □

2.2 The General Initial-Value Problem

We are interested in solving an initial value problem, in a reflexive Banach space X,

$$\frac{du}{dt} + A(t, u)u = F(t, u), \quad u(0) = u_0,$$

where $A(t, u) \in G(X, M, \omega)$ for u restricted to a subset of a 'smooth' reflexive Banach space Y, densely and continuously embedded in X. We require the function $F(t, u)$ to be in Y, and we seek a solution $u(t) \in Y$, $0 \le t \le T$. The derivative, du/dt, is required to belong to an intermediate space, V. The mapping $(t, u) \mapsto A(t, u)$ is required to be continuous into $B[Y, X]$ and $(t, u) \mapsto F(t, u)$ is required to be strongly continuous into X. We also require that $A(t, u)$ and $F(t, u)$ satisfy a (uniform in t) Lipschitz condition:

$$\|A(t, u) - A(t, v)\|_{Y,X} \le C_A\|u - v\|_X, \quad \|F(t, u) - F(t, v)\|_X \le C_F\|u - v\|_X.$$

In addition, we shall require a similarity relation connecting A to A_1 as in (2.1).

If Δt is given as the ratio T/N, then the method of horizontal lines yields a semidiscrete set of implicit equations,

$$A(t_k, u_k^N)u_k^N + (1/\Delta t)u_k^N = (1/\Delta t)u_{k-1}^N + F(t_k, u_k^N),\ k = 1, \ldots, N. \tag{2.2}$$

If we set $\mu^2 = 1/\Delta t = N/T$, then the u_k^N can be characterized formally as fixed points of the mapping

$$\begin{aligned} Qv = Q_k^N v = & -R(\mu^2 - 1, -A(t_k, v))v + \mu^2 R(\mu^2 - 1, -A(t_k, v))u_{k-1}^N \\ & + R(\mu^2 - 1, -A(t_k, v))F(t_k, v). \end{aligned} \tag{2.3}$$

By repeated back substitution, one obtains the following useful formula for u_{k-1}^N:

$$u_{k-1}^N = \prod_{j=1}^{k-1} \mu^2 R(\mu^2, -A(t_j, u_j^N))u_0 + \sum_{j=1}^{k-1} (\mu^2)^{k-1-j} \prod_{i=j}^{k-1} R(\mu^2, -A(t_i, u_i^N))F(t_j, u_j^N). \tag{2.4}$$

Pivotal to the entire study is the demonstration of the existence of fixed points for this map within an appropriately smooth ball. The concept of stability proves useful.

2.3 Stable Families of Generators and Invariant Action of Q

We recall a notion due originally to Kato [10, 11] (see also [7, Definition 6.2.1]).

Definition 2.3. *Let X be a Banach space, and suppose that a family $A(t) \in G(X)$ of linear operators is given on $0 \leq t \leq T$. The family $\{A(t)\}$ is said to be stable if there are constants M and ω such that*

$$\| \prod_{j=1}^{k} [A(t_j) + \lambda]^{-1} \| \leq M(\lambda - \omega)^{-k}, \quad \lambda > \omega, \tag{2.5}$$

for any non-decreasing family $\{t_j\}_{j=1}^{k}$. Moreover, $\prod$ is time-ordered from right to left.

Stability is automatically satisfied if $A(t) \in G(X, 1, \omega)$. Also, stability is equivalent to the apparently stronger property [7, Proposition 6.2.1]:

Proposition 2.5. *The family $\{A(t)\}$ is stable if and only if*

$$\| \prod_{j=1}^{k} [A(t_j) + \lambda_j]^{-1} \| \leq M \prod_{j=1}^{k} (\lambda_j - \omega)^{-1}, \quad \lambda_j > \omega. \tag{2.6}$$

The conclusion of Proposition 2.5 holds, even if the stability criterion (2.5) is weakened so that it is assumed to hold only for λ sufficiently large, say, $\lambda \geq \lambda_0$. In this case, λ_0 is even permitted to depend on the integer k in the product (2.5). This is due to the circle of equivalences involved in the proof of [7, Proposition 6.2.1]. We shall only require the kind of stability defined in Proposition 2.6 to follow.

It also follows that, if Y is a smooth space as in the previous proposition, and (2.1) holds, then $A(t)$ is stable on Y with stability constants M, ω_1, as is seen by an application of (2.11) to follow. We shall require stability on X and Y.

For ω and ω_1 introduced above, we define: $\bar{\omega} = \max(\omega, \omega_1)$. $M \geq 1$ will have the meaning of the previous definition. Suppose that δ and ρ are fixed positive constants, and that

$$\sigma = 2(1+\delta)Me^{(1+1/\rho)(1+\bar{\omega})T}, \tag{2.7}$$

where T is a fixed terminal time. We define

$$\bar{W} = \{u \in Y : \|u\|_Y \leq \sigma \|u_0\|_Y\},\ \bar{W}_0 = \{u \in Y : \|u\|_Y \leq \sigma \|u_0\|_Y,\ \|u\|_X \leq \sigma \|u_0\|_X\}.$$

Proposition 2.6. *Suppose that $\Delta t = T/N$ is given, and a partition $t_j = j\Delta t, j = 0, \ldots, N$, is specified. Suppose that a family $\{A(t,u)\}$ is given as in §2.2, and that $u_1^N, \ldots, u_{k-1}^N, \in \bar{W}_0$ are solutions of (2.2). Suppose that $A(t_j), j = 1, \ldots, k$, is stable on X and Y in the sense of satisfying (2.6), for any (consecutive) subset of $\{t_j\}$, where $A(t_j) = A(t_j, u_j^N), j = 1, \ldots, k-1$, and $A(t_k) = A(t_k, u)$, for arbitrary fixed $u \in \bar{W}$. Suppose $f(t) = e^{(1+1/\rho)\bar{\omega}t}$ and that $F(t,u)$ has X-norm and Y-norm not exceeding $\|u_0\|_X f(t)$ and $\|u_0\|_Y f(t)$, respectively, for each $0 \leq t \leq T$ and $u \in \bar{W}$. If the integer N satisfies:*

$$\frac{N}{T} > [2(1+\delta^{-1})M + (\rho+1)(\bar{\omega}+1)], \tag{2.8}$$

then the mappings $Q = Q_k^N$ of (2.3) are mappings of $\bar{W}_0$ into itself.

Proof. The proof significantly generalizes that of [6, Theorem 3.1]. Thus, for fixed N, we assume inductively that (2.2) has a solution u_ℓ^N for $\ell < k$, where $1 \leq k \leq N$. We can estimate $\|Qv\|_X$. From (2.3), we have, by use of the strong stability property (2.6),

$$\begin{aligned} \|Qv\|_X \leq{}& \frac{M}{\mu^2-\omega-1}\sigma\|u_0\|_X + \frac{M\mu^2}{\mu^2-\omega-1}\left(\frac{\mu^2}{\mu^2-\omega}\right)^{k-1}\|u_0\|_X \\ &+ \frac{M}{\mu^2-\omega-1}\sum_{j=1}^{k-1}\left(\frac{\mu^2}{\mu^2-\omega}\right)^{k-j}\|F(t_j,u_j^N)\|_X + \frac{M}{\mu^2-\omega-1}\|F(t_k,v)\|_X. \end{aligned} \tag{2.9}$$

For the first and fourth terms, we estimate, by the choice of N,

$$\frac{M}{\mu^2-\omega-1} \leq \frac{\delta}{2(1+\delta)}.$$

An observation required for the estimate of the second term is given by

$$\frac{\mu^2}{\mu^2-\omega-1} \leq (1+1/\rho) \tag{2.10}$$

if $\mu^2 \geq (1+\rho)(\omega+1)$. When this is combined with the standard inequality,

$$(1+s)^N \leq e^{sN}, \quad s = \frac{\omega+1}{\mu^2-\omega-1},$$

we arrive at a chain of inequalities for the second term. By the choice of N and σ,

$$\frac{\mu^2}{\mu^2-\omega-1}\left(\frac{\mu^2}{\mu^2-\omega}\right)^{k-1} \leq \left(\frac{\mu^2}{\mu^2-\omega-1}\right)^{N} \leq e^{(1+1/\rho)(1+\omega)T} \leq \frac{\sigma}{2M(1+\delta)}.$$

The third term is more complicated. For notational simplicity, we set

$$\alpha = (1+1/\rho)(1+\bar{\omega}), \quad \beta = (1+1/\rho)\bar{\omega}.$$

We have the estimate, since $\mu^2 = 1/\Delta t$,

$$\sum_{j=1}^{k-1}\left(\frac{\mu^2}{\mu^2-\omega}\right)^{k-j}\|F(t_j, u_j^N)\|_X \leq \mu^2\|u_0\|_X \sum_{j=1}^{k-1} e^{\alpha(k-j)(T/N)}e^{\beta j(T/N)}\Delta t$$

$$\leq \mu^2\|u_0\|_X e^{\alpha T}\sum_{j=1}^{k-1} e^{(\beta-\alpha)j(T/N)}\Delta t \leq \mu^2\|u_0\|_X e^{\alpha T}\int_0^T e^{-(\alpha-\beta)t}dt$$

$$\leq \mu^2\|u_0\|_X e^{\alpha T}(1/(\alpha-\beta)) \leq \mu^2\frac{\sigma\|u_0\|_X}{2M(1+\delta)(\alpha-\beta)}.$$

We must still account for the leading factor,

$$\frac{M}{\mu^2-\omega-1},$$

in the third term. We estimate:

$$\left(\frac{\mu^2 M}{\mu^2-\omega-1}\right)\left(\frac{\sigma\|u_0\|_X}{2M(1+\delta)(\alpha-\beta)}\right) \leq \frac{\sigma\|u_0\|_X}{2(1+\delta)}.$$

Here we have used the fact that

$$\frac{\mu^2}{\mu^2-\omega-1} \leq \alpha-\beta = 1+1/\rho,$$

which is implied by (2.10). If we apply each of the four estimates, together with the observation that $f(t_k) \leq \sigma$, we have the estimate that $\|Qv\|_X \leq \sigma\|u_0\|_X$.

The estimate for $\|Qv\|_Y$ is similar. We outline the approach. By (2.1), and the domain characterization of A_1, we conclude that

$$R(\lambda, -A_1(t,u)) = SR(\lambda, -A(t,u))S^{-1}. \tag{2.11}$$

Applying S to (2.3) and using (2.11) yield

$$SQv = -R(\mu^2-1, -A_1(t_k,v))Sv + \mu^2 R(\mu^2-1,-A_1(t_k,v))Su^N_{k-1}$$
$$+R(\mu^2-1,-A_1(t_k,v))SF(t_k,v), \tag{2.12}$$

where

$$Su^N_{k-1} = \prod_{j=1}^{k-1}\mu^2 R(\mu^2,-A_1(t_j,u^N_j))Su_0 + \sum_{j=1}^{k-1}(\mu^2)^{k-1-j}\prod_{i=j}^{k-1}R(\mu^2,-A_1(t_i,u^N_i))SF(t_j,u^N_j). \tag{2.13}$$

In particular, we obtain the estimate,

$$\|SQv\|_X \le \frac{M}{\mu^2-\omega_1-1}\sigma\|Su_0\|_X + \frac{M\mu^2}{\mu^2-\omega_1-1}\left(\frac{\mu^2}{\mu^2-\omega_1}\right)^{k-1}\|Su_0\|_X$$
$$+\frac{M}{\mu^2-\omega_1-1}\sum_{j=1}^{k-1}\left(\frac{\mu^2}{\mu^2-\omega_1}\right)^{k-j}\|SF(t_j,u^N_j)\|_X + \frac{M}{\mu^2-\omega_1-1}\|SF(t_k,v)\|_X, \tag{2.14}$$

so that, by the same arguments as above,

$$\|SQv\|_X \le \sigma\|SQu_0\|_X.$$

□

2.4 Lipschitz Continuity of Q

We shall next establish Lipschitz continuity of Q. This will close the induction, and give the existence of u^N_k, for Δt sufficiently small.

Proposition 2.7. *Under the assumptions of Proposition 2.6, the mappings $Q = Q^N_k$ of (2.3) are Lipschitz continuous mappings in the topology of X with Lipschitz constant,*

$$C_Q = \frac{M}{\mu^2-1-\bar{\omega}}\left[1 + C_A(1+M(1+1/\rho))\sigma\|u_0\|_X + C_F\right]. \tag{2.15}$$

Here, C_A and C_F are the Lipschitz constants cited earlier in §2.2. If N is sufficently large, then $C_Q < 1$ and Q has a unique fixed point in $\bar{W}_0$.

Proof. The critical representation is the identity,

$$R(\lambda,-A(t,w)) - R(\lambda,-A(t,v)) = R(\lambda,-A(t,w))[A(t,v)-A(t,w)]R(\lambda,-A(t,v)).$$

We obtain:

$$\begin{aligned}\|R(\lambda,-A(t,w)) - R(\lambda,-A(t,v))\|_X &\le \|R(\lambda,-A(t,w))\|_X C_A\|v-w\|_X\|R(\lambda,-A(t,v))\|_Y \\ &\le C_1\frac{\|v-w\|_X}{(\lambda-\omega)(\lambda-\omega_1)},\end{aligned}$$

where $C_1 = M^2 C_A$. This leads to the estimate, for $\lambda = \mu^2 - 1$,

$$\|Qv - Qw\| \leq \frac{1}{\mu^2 - 1 - \omega}\left[M + \frac{C_1\sigma\|u_0\|_X}{\mu^2 - 1 - \omega_1} + \frac{C_1\sigma\|u_0\|_X\ \mu^2}{\mu^2 - 1 - \omega_1} + MC_F + \frac{C_1\sigma\|u_0\|_X}{\mu^2 - 1 - \omega_1}\right]\|v - w\|_X.$$

Here, we have used the inductive assumption that $\|u^N_{k-1}\|_X \leq \sigma\|u_0\|_X$. By using the estimates of the proof of Proposition 2.6, we obtain

$$\|Qv - Qw\| \leq \frac{1}{\mu^2 - 1 - \omega}\left[M + \frac{MC_A\sigma\delta\|u_0\|_X}{(1+\delta)} + M^2 C_A(1 + 1/\rho)\sigma\|u_0\|_X + MC_F\right]\|v - w\|_X.$$

This yields the estimate (2.15) of the proposition. Since Y is assumed reflexive, $\bar{W}_0$ is a complete metric subspace of X, and the final statement follows from the contraction mapping theorem. □

3 System Properties for the Semidiscrete Problem

We are now prepared to apply the general theory of §2 to the model of §1.2. We begin by describing the function spaces and the isomorphism S.

3.1 The Function Spaces and the Isomorphism

We introduce the classical Bessel potential space $H^s(R^m; R^k)$ [1]. It can be characterized, via the isometric Fourier transform $\mathcal{F}$, as the linear space of functions v with norm,

$$\|v\|^2_{H^s} = \int_{R^m} (1 + |x|^2)^s |\mathcal{F}v(x)|^2\, dx.$$

It follows from the definition that the diagonal operator $S = I(I - \Delta)^{s/2}$ induces an isometry of $H^s(R^m; R^k)$ onto $L_2(R^m; R^k)$.

We may now define:

$$X = X_1 \otimes X_2,\ \ X_1 = \mathrm{P}L_2(R^m; R^m),\ X_2 = L_2(R^m; R^2),$$

$$Y = Y_1 \otimes Y_2,\ \ Y_1 = \mathrm{P}H^s(R^m; R^m),\ Y_2 = H^s(R^m; R^2).$$

3.2 The Block Resolvent

We shall proceed in stages, via a systematic study of the resolvent. We first write the representation of the block resolvent. Since

$$\lambda I + A = \left[\begin{array}{c|c} \lambda I + A_{11} & A_{12} \\ \hline 0 & \lambda I + A_{22} \end{array}\right], \tag{3.1}$$

we have by a standard invertibility result for the block resolvent of §1.2:

$$R(\lambda, -A) = \left[\begin{array}{c|c} R(\lambda, -A_{11}) & -R(\lambda, -A_{11})A_{12}R(\lambda, -A_{22}) \\ \hline 0 & R(\lambda, -A_{22}) \end{array}\right]. \tag{3.2}$$

3.3 Resolvent Estimates on X_1 and X_2

We have seen, via the interpretation of (3.2), that critical roles are played by $R(\lambda, -A_{11}(\mathbf{w}_1))$ and $R(\lambda, -A_{22}(\mathbf{w}_1, \mathbf{w}_2))$. Thus, we shall examine the Navier-Stokes operator separately from the charge transport operator.

Lemma 3.1. *Suppose* $\mathbf{w} \in H^s(R^m; R^m), s > m/2$, *with* $\nabla\cdot \mathbf{w} = 0$. *Then* $A_{11}(\mathbf{w}) \in G(\mathrm{P}L_2(R^m; R^m), 1, 0)$. *Furthermore,* $\mathcal{D}(A_{11}(\mathbf{w})) = \mathrm{P}\ H^2(R^m; R^m)$.

Proof. The proof follows [12, p. 55]. Thus, $-\nu\Delta I \in G(\mathrm{P}L_2(R^m; R^m), 1, 0)$ if we take the domain as given in the statement of the lemma. In this case, it coincides with the Friedrichs extension of the operator defined on $\mathrm{P}C_0^\infty(R^m; R^m)$. The perturbing term, $\mathrm{P}\mathbf{w}\cdot\nabla$, of A_{11} is dissipative on $\mathrm{P}C_0^\infty(R^m; R^m)$. We may take the closure, which is thus a member of $G(\mathrm{P}L_2(R^m; R^m), 1, 0)$. This operator is relatively bounded with respect to $\nu\Delta I$, with relative bound zero, since $\mathbf{w}$ is essentially bounded. It follows from Proposition 2.2 that $A_{11}(\mathbf{w}) \in G(\mathrm{P}L_2(R^m; R^m), 1, 0)$ with $\mathcal{D} = \mathrm{P}\ H^2(R^m; R^m)$. $\square$

Prior to the statement of the next lemma, we formally state the smoothing assumption regarding the mapping Φ of (1.16).

Remark 3.1. *It is assumed that the affine mapping* Φ *depends smoothly upon* $\mathbf{u}_2$ *in the following sense: Given* $\mathbf{u}_2 \in H^\tau(R^m; R^2)$, $0 \le \tau \le s$, *the solution* ϕ *of* (1.16) *is in* $H^{\tau+2}(R^m)$, *and the norm of* ϕ *in this space is affinely dominated by the* H^τ *norm of* $\mathbf{u}_2$, *with constants independent of* τ *and* $\mathbf{u}_2$.

Lemma 3.2. *Suppose* $\mathbf{w} \in H^s(R^m; R^{m+2}), s > m/2+1$. *Then* $A_{22}(\mathbf{w}) \in G(L_2(R^m; R^2), 1, \omega)$, *where* ω *depends affinely upon* $\|\mathbf{w}\|_{H^s}$. *Furthermore,* $\mathcal{D}(A_{22}) = H^2(R^m; R^2)$.

Proof. The proof follows the logical structure of the preceding proof.

$$-\mathrm{diag}(D_n, D_p)\Delta \in G(L_2(R^m; R^2), 1, 0), \tag{3.3}$$

and coincides with the Friedrichs extension of the restriction operator defined on $C_0^\infty(R^m; R^2)$. The perturbing operator,

$$\sum_{i=1}^{m} \mathrm{diag}(w_i + \mu_n \phi_{x_i}, w_i - \mu_p \phi_{x_i}) \frac{\partial}{\partial x_i} + \mathrm{diag}(\mu_n \Delta\phi, -\mu_p \Delta\phi), \tag{3.4}$$

is in $G(X, 1, \omega)$ by the Friedrichs theory (see [4], [12, p. 51], [7, Lemma 6.4.3]) and ω can be estimated by 1/2 the sum of the C-norms of

$$\nabla\mathrm{diag}(w_i + \mu_n \phi_{x_i}, w_i - \mu_p \phi_{x_i}), \mathrm{diag}(\mu_n \Delta\phi, -\mu_p \Delta\phi),$$

thus, by an affine function of the H^s norm of $\mathbf{w}$, where we have not required the full smoothing of Φ. The remainder of the proof follows as before. $\square$

Although it is possible in principle to derive an estimate on X from the two preceding lemmas, we shall not do this. This is due to the fact that it is *products* of terms of the form (3.2) which must be estimated (asymptotic stability), and this is effectively done through estimating subblock products.

3.4 Resolvent Estimates on Y_1 and Y_2

We now investigate the similarity relation (2.1), as applied to A_{11} and A_{22}. We first state the relevant result of Kato [12] which governs the estimates. It will be applied, not to A, but separately to A_{11} and A_{22}.

Lemma 3.3. *For a function* $\mathbf{v} \in H^s(R^m; R^k)$, *a projection* P, *and an operator of the form,*

$$A(t, \mathbf{v}) = -D\Delta + \mathrm{P}\left[\sum_{i=1}^{m} a_i(t, \mathbf{v})\frac{\partial}{\partial x_i} + b(t, \mathbf{v})\right], \tag{3.5}$$

we have

$$SA(t, \mathbf{v})S^{-1} = A(t, \mathbf{v}) + \mathrm{P}\left[\sum_{j=1}^{m}[S, a_j]\Lambda^{1-s}\left(\frac{\partial}{\partial x_j}\right)\Lambda^{-1} + [S, b]\Lambda^{1-s}\Lambda^{-1}\right],$$

where $\Lambda = (I - \Delta)^{1/2}$, $[\cdot\,,\cdot]$ *denotes the commutator, and* $S = I_k\Lambda^s$; *here* I_k *is the identity matrix of order* k. *We have assumed that* P *commutes with* S. *In particular, in the notation of Proposition* 2.4, *we have*

$$B = \mathrm{P}\left[\sum_{j=1}^{m}[S, a_j]\Lambda^{1-s}\left(\frac{\partial}{\partial x_j}\right)\Lambda^{-1} + [S, b]\Lambda^{1-s}\Lambda^{-1}\right].$$

B *is a bounded operator on* $L_2(R^m; R^k)$ *with bound:*

$$\|B\| \leq C\left(\sum_{j=1}^{m}\|\text{grad } a_j\|_{H^{s-1}} + \|\text{grad } b\|_{H^{s-1}}\right).$$

Proof. Note that Λ^s commutes with the Laplacian. The estimate depends upon the following fundamental result of Kato [12, Lemma A2]. For $s > m/2 + 1$:

$$\text{If } f \text{ is a multiplier, defining an operator } m_f, \text{ then } \|[\Lambda^s, m_f]\Lambda^{1-s}\|_{L_2,L_2} \leq c\|\text{grad} f\|_{H^{s-1}}, \tag{3.6}$$

where c is a positive constant. In particular, for some $c > 0$,

$$\|[S, a_j]\Lambda^{1-s}\|_{L_2,L_2} \leq c\|\text{grad } a_j\|_{H^{s-1}},$$

$$\|[S, b]\Lambda^{1-s}\|_{L_2,L_2} \leq c\|\text{grad } b\|_{H^{s-1}},$$

hold. Since $\left(\frac{\partial}{\partial x_j}\right)\Lambda^{-1}$ and Λ^{-1} are bounded on $L_2(R^m; R^k)$, it follows that B is bounded. This completes the generic estimate. □

Lemma 3.4. *The operator* A_{11} *of Lemma* 3.1 *satisfies* $A_{11}(\mathbf{w}_1) \in G(Y_1, 1, \beta_1)$, *where*

$$\beta_1 = C\|\mathbf{w}_1\|_{H^s}.$$

Here, C *is a constant which is defined by the Sobolev embedding theorem.*

Proof. The result is immediate upon an application of Lemma 3.3. ☐

Lemma 3.5. *The operator A_{22} of Lemma 3.2 satisfies $A_{22}(\mathbf{w}) \in G(Y_2, 1, \beta_2)$, where β_2 is an affine function of $\|\mathbf{w}\|_{H^s}$.*

Proof. The result is immediate upon an application of Lemma 3.3 and use of Remark 3.1. ☐

Corollary 3.1. *For the system (1.14), we have*

$$SA(\mathbf{w})S^{-1} = A(\mathbf{w}) + B(\mathbf{w}),$$

where each of the blocks $B_{ij}(\mathbf{w})$ is a bounded operator, and $B_{21} = 0$.

3.5 The Stability of $A(t_j)$ on X and Y

We denote by r the undetermined radius in Y of the ball $\bar{W}$ on which Q acts. Let $\Delta t = T/N$ be specified, and let $\mathbf{w}_j, j = 1, \ldots, N$, be fixed in $\bar{W}$. For $t_j = j\Delta t$, we shall use the notation $A(t_j)$ for $A(\mathbf{w}_j)$. In the light of Propositions 2.6 and 2.7, we will eventually interpret $A(t_k)$ as $A(\mathbf{w})$ for $\mathbf{w}$ fixed in Y, while $A(t_j)$, for $j < k$, will be identified with $A(\mathbf{u}_j^N)$. We shall actually verify stability in the sense of (2.5), where repetitions of the $\{t_j\}$ are permitted. By the arguments of [7, Proposition 6.2.1], this implies stability in the sense of (2.6).

The verification of stability is subtle. It does not suffice for our purposes to proceed via the norm of the block resolvent. Rather, we must first form the operator product as a block matrix, and then apply a norm estimate. This is critical. Thus, we use the representation (3.2). The following algebraic result ensues.

Lemma 3.6.

$$\prod_{j=1}^{k} [A(t_j) + \lambda]^{-1} = \left[\begin{array}{c|c} \prod_{j=1}^{k} R(\lambda, -A_{11}(t_j)) & -\sum_{j=1}^{k} A_j^{\text{left}} A_{12}(t_j) A_j^{\text{right}} \\ \hline 0 & \prod_{j=1}^{k} R(\lambda, -A_{22}(t_j)) \end{array}\right]. \tag{3.7}$$

where

$$A_j^{\text{left}} = \prod_{i=j,\ldots,k} R(\lambda, -A_{11}(t_i)), \quad A_j^{\text{right}} = \prod_{i=1,\ldots,j} R(\lambda, -A_{22}(t_i)). \tag{3.8}$$

Proof. One uses induction on k, together with the convention that products are (increasingly) time-ordered from right to left. ☐

Proposition 3.1. *The family $A(t_j)$ is stable on X in the sense of Proposition 2.6. The stability constants M_X, ω_X are given by (3.11) below.*

Proof. We estimate the operator X norm of a k-product according to (2.5). The adjustments for repetitions of, and consecutive subsets of, $\{t_j\}$, are elementary. Since X is the tensor product, $X = X_1 \otimes X_2$, the operator X norm is bounded from above, as in ordinary matrix theory, by the Euclidean block operator norm. Upon forming the Euclidean norm of the iterated block resolvent, we obtain, after an application of a form of the triangle inequality:

$$\| \prod_{j=1}^{k} [A(t_j) + \lambda]^{-1} \| \leq \frac{1}{\lambda^k} + C(c_1 + c_2 r) \sum_{j=1}^{k} \frac{1}{\lambda^{k-j+1}(\lambda - \omega)^j} + \frac{1}{(\lambda - \omega)^k}. \tag{3.9}$$

In this inequality, C is an ordinary Sobolev embedding constant, and $c_1 + c_2 r$ is an affine function of the radius r of the ball in Y_2 from which $\mathbf{u}_2$ is selected; this results from the H^{s-1} estimation of the gradient of ϕ, which serves as a multiplier. ω is defined in Lemma 3.2. It remains to estimate the sum,

$$\sum_{j=1}^{k} \frac{1}{\lambda^{k-j+1}(\lambda - \omega)^j}.$$

We first write,

$$\sum_{j=1}^{k} \frac{1}{\lambda^{k-j+1}(\lambda - \omega)^j} = \frac{1}{\lambda^k} \sum_{j=1}^{k} \frac{\lambda^j}{(\lambda - \omega)^j} \left(\frac{1}{\lambda} \right).$$

We use an idea similar to that used in estimating the third term in the proof of Proposition 2.6. This will also entail use of (2.10). First, for any prescribed $\rho > 0$, we define N_0 to be the smallest integer satisfying

$$N_0 \geq T(1 + \rho)\omega. \tag{3.10}$$

We shall select $\rho = \rho_0$ later in the proof. For $N \geq N_0$, $k \leq N$, and $\lambda \geq N/T$, we have

$$\sum_{j=1}^{k} \left(\frac{\lambda}{\lambda - \omega} \right)^j \left(\frac{1}{\lambda} \right) \leq \sum_{j=1}^{N} \left(\frac{\lambda}{\lambda - \omega} \right)^j \left(\frac{1}{\lambda} \right) \leq \sum_{j=1}^{N} e^{(1+1/\rho)\omega j \Delta t} \Delta t,$$

where $\Delta t = T/N$. This serves as a lower Riemann sum for

$$\int_{1}^{T+\Delta t} e^{(1+1/\rho)\omega t} \, dt,$$

and has an upper bound of $\frac{e^{(1+1/\rho)\omega T} e^{1/\rho}}{(1+1/\rho)\omega}$. We select $\rho = \rho_0$ such that

$$\frac{e^{1/\rho}}{(1 + 1/\rho)} = 2.$$

We conclude that $A(t_j)$ is stable on X with stability constants,

$$M_X = 3 \max \left(1, C(c_1 + c_2 r) \frac{2e^{(1+1/\rho_0)\omega T}}{\omega} \right), \ \omega_X = \omega. \tag{3.11}$$

$\square$

Proposition 3.2. *The family $A(t_j)$ is stable on Y in the sense of Proposition* 2.6. *The stability constants M_Y, ω_Y are given by* (3.14) *below.*

Proof. We shall consider k-products as before. The relation (2.11) permits us to estimate,

$$\prod_{j=1}^{k}[A_1(t_j)+\lambda]^{-1} = S\left[\begin{array}{c|c} \prod_{j=1}^{k} R(\lambda,-A_{11}(t_j)) & -\sum_{j=1}^{k} A_j^{\text{left}} A_{12}(t_j) A_j^{\text{right}} \\ \hline 0 & \prod_{j=1}^{k} R(\lambda,-A_{22}(t_j)) \end{array}\right] S^{-1}. \quad (3.12)$$

Here, S is the block matrix,

$$S = \left[\begin{array}{c|c} I_m\Lambda^s & 0 \\ \hline 0 & I_2\Lambda^s \end{array}\right] = \left[\begin{array}{c|c} S_m & 0 \\ \hline 0 & S_2 \end{array}\right]$$

When (3.12) is consolidated via multiplication by S and S^{-1}, the block representation is written as:

$$\left[\begin{array}{c|c} S_m\prod_{j=1}^{k} R(\lambda,-A_{11}(t_j))S_m^{-1} & -\sum_{j=1}^{k} S_m A_j^{\text{left}} S_m^{-1} S_m A_{12}(t_j) S_2^{-1} S_2 A_j^{\text{right}} S_2^{-1} \\ \hline 0 & S_2\prod_{j=1}^{k} R(\lambda,-A_{22}(t_j))S_2^{-1} \end{array}\right],$$

where we have used subscripts to designate the order of the matrix operator. Now Lemmas 3.4 and 3.5 provide the estimates for the first and fourth blocks, and also for the estimation of

$$S_m A_j^{\text{left}} S_m^{-1},\ S_2 A_j^{\text{right}} S_2^{-1}.$$

The estimation of

$$S_m A_{12}(t_j) S_2^{-1}$$

is a direct consequence of Kato's commutator estimates, described in Lemma 3.3 (see especially (3.6)), and leads to

$$\|S_m A_{12}(t_j) S_2^{-1}\|_{L_2,L_2} \le c\|\phi\|_{H^{s+1}}.$$

By hypothesis, the latter is bounded by an affine function of $\|\mathbf{u}_2\|_{H^{s-1}}$. Altogether, we may proceed as in the proof of the previous proposition with the following replacement:

$$\|\prod_{j=1}^{k}[A_1(t_j)+\lambda]^{-1}\| \le \frac{1}{(\lambda-\beta_1)^k} + C(c_1+c_2r)\sum_{j=1}^{k}\frac{1}{(\lambda-\beta_1)^{k-j+1}(\lambda-\beta_2)^j} + \frac{1}{(\lambda-\beta_2)^k}. \quad (3.13)$$

As in Proposition 3.1 we obtain the stability constants,

$$M_Y = 3\max\left(1, C(c_1+c_2r)\frac{2e^{(1+1/\rho_0)\omega_Y T}}{\omega_Y}\right),\ \omega_Y = \max(\beta_1,\beta_2). \quad (3.14)$$

The integer N_0 may be defined by,

$$N_0 \ge T(1+\rho_0)\omega_Y. \quad (3.15)$$

□

3.6 The Lipschitz Properties of $E(\mathbf{u})$

Recall the definition of E given in (1.18, 1.19). The Lipschitz continuity of E is standard, but we shall quote and prove the result for completeness (see [6, Lemma 2.3] and [5, p. 283]).

Lemma 3.7. *The mapping $\mathbf{w} \mapsto E(\mathbf{w}) \in B(H^s, H^\tau)$ is Lipschitz continuous in the norm topology for $0 \le \tau \le s-1$:*

$$\|E(\mathbf{w}) - E(\mathbf{w}')\|_{H^s,H^\tau} \le C\|\mathbf{w} - \mathbf{w}'\|_{H^\tau}, \quad \mathbf{w}, \mathbf{w}' \in \bar{W}.$$

The constant C is proportional to the radius r of $\bar{W}$.

Proof. We first note the inequalities,

$$\|a_j(\mathbf{w}) - a_j(\mathbf{w}')\|_{H^\tau} \le c_1\|\mathbf{w} - \mathbf{w}'\|_{H^\tau},$$

$$\|b(\mathbf{w}) - b(\mathbf{w}')\|_{H^\tau} \le c_2\|\mathbf{w} - \mathbf{w}'\|_{H^\tau}.$$

These inequalities use the definitions of the matrices $a_j, j = 1, \dots, m,\ b$, as well as the assumed properties of the mapping Φ. Because of the linear dependence of the elements in $a_j, j = 1, \dots, N$, and b, the constants c_1, c_2 do not depend upon $\mathbf{w}, \mathbf{w}'$. Now, since H^{s-1} functions are multipliers on H^τ, we have:

$$\|E(\mathbf{w})\mathbf{v} - E(\mathbf{w}')\mathbf{v}\|_{H^s,H^\tau} \le c(\sum_{j=1}^{m} \|a_j(\mathbf{w}) - a_j(\mathbf{w}')\|_{H^\tau}\|\frac{\partial \mathbf{v}}{\partial x_j}\|_{H^{s-1}} +$$

$$\|b(\mathbf{w}) - b(\mathbf{w}')\|_{H^\tau}\|\mathbf{v}\|_{H^{s-1}}) \le C'\|\mathbf{w} - \mathbf{w}'\|_{H^\tau}\|\mathbf{v}\|_{H^s}.$$

This gives the statement of the lemma. □

Remark 3.2. *Since the forcing function $\mathbf{F}$ has been included as an enhancement of the original model, we have some freedom in the hypotheses regarding it. We shall however attempt a minimal set of hypotheses. These are:*

1. *The growth hypothesis of Proposition* 2.6.
2. *An estimate for $F(t, \mathbf{u})$ somewhat analogous to that given in Lemma* 3.7. *Specifically, we assume the following. The mapping $(t, \mathbf{w}) \mapsto \mathbf{F}(t, \mathbf{w})$ is Lipschitz continuous in the topology of H^τ for $0 \le \tau \le s$:*

$$\|\mathbf{F}(t, \mathbf{w}) - \mathbf{F}(t', \mathbf{w}')\|_{H^\tau} \le C[|t - t'| + \|\mathbf{w} - \mathbf{w}'\|_{H^\tau}], \quad \mathbf{w}, \mathbf{w}' \in \bar{W}.$$

Item two is implied by a Lipschitz assumption on $\mathbf{F}$ [5, p. 283]):

$$\mathbf{F} \in Lip([0, T]; C^{[s]+2}(R^m; R^{m+2})),$$

where $[s]$ is the largest integer in s; if s is an integer, we may replace $[s] + 2$ by $[s] + 1$.

3.7 Consolidation of the Semigroup Estimates

We begin by examining functional behavior necessary to correlate the size of the initial datum and the radius r of the ball in which solutions may occur for the semidiscrete map. There is an interplay between the size of the initial datum and the terminal time T.

We shall interpret the constants $\sigma, M, \rho, \bar{\omega}$ and T of (2.7) in the light of §3.5, so that we may apply Propositions 2.6 and 2.7. For consistency of notation, we define

$$M = \max(M_X, M_Y), \quad \bar{\omega} = \max(\omega_X, \omega_Y),$$

where $M_X, M_Y, \omega_X, \omega_Y$ have the meaning of §3.5. Initially, we shall determine the dependence of M and $\bar{\omega}$ on the radius r of the ball in Y. Since ω, β_1, and β_2, as discussed in §3.3 and §3.4, are affine functions of r, we may write,

$$\bar{\omega} = a + br,$$

for positive constants a, b. M is a more complicated expression. If we use (3.11) and (3.14) as a starting point, we may write

$$M(r, T) = \frac{(c + dr)e^{(1+1/\rho_0)\bar{\omega}(r)T}}{\bar{\omega}(r)},$$

for appropriate constants, c, d.

Local Assumption on $\|\mathbf{u}_0\|$ and T

We require:

$$\|\mathbf{u}_0\|_Y < \frac{r}{2M(r,T)} e^{-(1+\bar{\omega}(r))T} := H(r, T). \tag{3.16}$$

(3.16) is the general inequality which must be satisfied by $\|\mathbf{u}_0\|_Y, T$, and the radius r of the admissible ball in Y. However, we consider in detail two important special cases.

1. **T is given**

   If we write,

   $$H(r, T) = \frac{1}{2} e^{-(1+\alpha\bar{\omega}(r))T} h(r), \quad h(r) = \frac{r(a + br)}{c + dr},$$

   where $\alpha = 2 + 1/\rho_0$, then we may maximize $H(\cdot\,, T)$ as a function of r. We find that r is determined by

   $$\frac{h'(r)}{h(r)} = \alpha bT. \tag{3.17}$$

   By direct computation,

   $$\frac{h'(r)}{h(r)} = \frac{c(a + br) + br(c + dr)}{r(a + br)(c + dr)}.$$

The latter function is strictly decreasing on $(0, \infty)$, and satisfies

$$\frac{h'(r)}{h(r)} \to \infty, \ r \to 0+; \ \frac{h'(r)}{h(r)} \to 0, \ r \to \infty. \tag{3.18}$$

It follows that (3.17) has a unique solution, $r(T)$. In this case, (3.16) reduces to: $\|\mathbf{u}_0\|_Y < H(r(T), T)$.

2. $\|\mathbf{u}_0\|$ **is given**

   Since $h(r)$ is a strictly increasing mapping of $(0, \infty)$ onto itself, there is a unique r_0 such that
   $$\|\mathbf{u}_0\|_Y = \frac{1}{2}h(r_0) = H(r_0, 0).$$
   It follows that, for each $r > r_0$, there is a $T_0 = T_0(r)$ such that
   $$\|\mathbf{u}_0\|_Y < H(r, T), \text{ for } T < T_0, \ \ r \text{ fixed}.$$
   This gives an admissible range of r and T which satisfy (3.16).

3.8 The Major Fixed Point Theorem

We now define numbers δ and ρ which allow us to connect (3.16) with the theoretical analysis of Propositions 2.6 and 2.7. Set $\gamma = 1 + \bar{\omega}$, and select ρ satisfying

$$2M(r, T)\|\mathbf{u}_0\|_Y e^{(1+1/\rho)\gamma(r)T} < r,$$

which is possible by (3.16). Define:

$$\delta = re^{-(1+1/\rho)\gamma(r)T}/(2M\|\mathbf{u}_0\|_Y) - 1.$$

It is immediate that

$$2(1+\delta)Me^{(1+1/\rho)\gamma(r)T}\|\mathbf{u}_0\|_Y = r.$$

We further define:

$$\sigma = 2(1+\delta)M(r, T)e^{(1+1/\rho)\gamma(r)T}.$$

These definitions then describe the framework investigated in Propositions 2.6 and 2.7. In particular,

$$\sigma\|\mathbf{u}_0\|_Y = r.$$

We then have the following.

Theorem 3.1. *If* (3.16) *holds, and N is sufficiently large, then the mapping Q, with Lipschitz constant C_Q given by* (2.15)*, is a strict contraction on $\bar{W}_0$. In this case, Q has a unique fixed point, denoted $\mathbf{u}_k^N$. The implicit relation* (1.16) *is satisfied.*

4 Analysis on the Space-Time Domain

We begin by defining the relevant sequences which make use of the semidiscrete solutions.

Definition 4.1. *For N_0 given according to Theorem* 3.1*, and $N \geq N_0$, define the piecewise linear and step function sequences as follows. For $\Delta t = T/N$, $t_k = k\Delta t$, and $0 \leq t \leq T$, set*

$$\theta_k^N(t) = \begin{cases} 1, \; t_{k-1} \leq t < t_k, \; k = 1, \ldots, N, \\ 0, \; \text{otherwise}. \end{cases}$$

Then for $x \in R^m$, define:

$$\begin{aligned}
\mathbf{u}_{PL}^N(x,t) &= \mathbf{u}_k^N(x) + \frac{t - t_k}{\Delta t}(\mathbf{u}_k^N(x) - \mathbf{u}_{k-1}^N(x)), \; t_{k-1} \leq t < t_k, \; k = 1, \ldots, N, && (4.1) \\
\mathbf{u}_S^N(x,t) &= \sum_{k=1}^N \mathbf{u}_k^N(x)\theta_k^N(t), && (4.2) \\
\mathbf{F}_S^N(x,t) &= \sum_{k=1}^N \mathbf{F}(t_k, \mathbf{u}_k^N(x))\theta_k^N(t), && (4.3) \\
a_j^N(x,t) &= a_j(\mathbf{u}_S^N(x,t)), && (4.4) \\
b^N(x,t) &= b(\mathbf{u}_S^N(x,t)). && (4.5)
\end{aligned}$$

We also require function space notation:

$$\mathcal{Y} = W_\infty^1((0,T); H^{s-2}(R^m; R^{m+2})) \cap L_\infty((0,T); Y).$$

Lemma 4.1. *The sequence $\{\mathbf{u}_S^N\}$ is bounded in norm in $L_\infty((0,T);Y)$. The sequence $\{\mathbf{u}_{PL}^N\}$ is bounded in norm in $\mathcal{Y}$.*

Proof. The boundedness of both sequences in $L_\infty((0,T);Y)$ reflects the construction of the sequences via the fixed points of the mapping Q. To establish the boundedness of

$$\partial \mathbf{u}_{PL}^N / \partial t = \frac{\mathbf{u}_k^N - \mathbf{u}_{k-1}^N}{\Delta t}$$

in $L_\infty((0,T); H^{s-2}(R^m))$, we directly take the norm in the equation (2.2). For the term, $A(\mathbf{u}_k^N)\mathbf{u}_k^N$, we use the Y bound for $\mathbf{u}_k^N$, together with the uniform estimate on $A(\mathbf{u}_k^N)$ from $H^s(R^m; R^{m+2})$ to $H^{s-2}(R^m; R^{m+2})$, which can be deduced from Lemma 3.7. For the term, $\mathbf{F}(t_k, \mathbf{u}_k^N)$, we use the estimate, $\|\mathbf{F}(t_k, \mathbf{u}_k^N)\|_Y \leq \sigma \|\mathbf{u}_0\|_Y$, which is also satisfied by $\mathbf{u}_k^N$. $\square$

Remark 4.1. *By the Aubin lemma* ([7, p. 197]),

$$\mathcal{Y} \hookrightarrow^{\text{compactly}} L_{2,\text{loc}}(\mathcal{D}).$$

Here, we have used the notation 'loc' *to indicate convergence on bounded subsets of $\mathcal{D} = (0,T) \times R^m$.*

This fact will be used in the following proposition.

Proposition 4.1. *There are subsequences, denoted* $\mathbf{u}_{PL}^{N_j}$, $\mathbf{u}_S^{N_j}$, *and a function* $\mathbf{u} \in \mathcal{Y}$, *satisfying the constraint* (1.16)*, such that:*

1. $$\mathbf{u}_{PL}^{N_j} \rightharpoonup \mathbf{u} \text{ weakly in } L_2((0,T);Y),$$

2. $$\mathbf{u}_{PL}^{N_j} \rightharpoonup^* \mathbf{u} \text{ weak-* in } W^1_\infty((0,T);H^{s-2}(R^m)) \cap L_\infty((0,T);Y),$$

3. $$\mathbf{u}_{PL}^{N_j} \rightarrow \mathbf{u} \text{ in } L_{2,\mathrm{loc}}(\mathcal{D}).$$

4. $$\mathbf{u}_S^{N_j} \rightharpoonup \mathbf{u} \text{ weakly in } L_2((0,T);Y),$$

5. $$\mathbf{u}_S^{N_j} \rightarrow \mathbf{u} \text{ in } L_{2,\mathrm{loc}}(\mathcal{D}).$$

6. $$a_j^{N_j} \rightarrow a(\cdot\,,\mathbf{u}) \text{ in } L_{2,\mathrm{loc}}(\mathcal{D}).$$

7. $$b^{N_j} \rightarrow b(\cdot\,,\mathbf{u}) \text{ in } L_{2,\mathrm{loc}}(\mathcal{D}).$$

Proof. The Aubin lemma shows that the limits in (1) and (2), which exist by weak compactness, coincide and lead to (3). In this connection, recall that a compact mapping (injection) maps weakly convergent sequences onto strongly convergent sequences. That the limit in (4) may be taken to be $\mathbf{u}$ follows from (1) and Lemma 4.1; in particular, from the uniform H^{s-2} bound for

$$\frac{\mathbf{u}_k^N - \mathbf{u}_{k-1}^N}{\Delta t}$$

(see [7, Lemma 5.2.6]). This bound also implies that the limits in (3) and (5) coincide. The limits in (6) and (7), and the constraint (1.16), follow from the definitions and the assumed properties of the mapping Φ. □

4.1 The Weak Solution

It is now standard (see [6, pp. 201–202]) that the function $\mathbf{u}$ satisfies a weak solution formulation. We shall summarize the key facts in this more general situation.. The weak solution is an important transition because it utilizes the convergence properties we have described in the preceding proposition. We have the following.

Theorem 4.1. *The function* $\mathbf{u}$ *is a weak solution of the Cauchy problem. In particular, if* $\psi \in C^\infty([0,T]; \mathrm{P}C_0^\infty(R^m, R^m) \otimes C_0^\infty(R^m, R^2))$, *and* $T' \leq T$, *then, for* $\mathcal{D}_{T'} = R^m \times (0, T')$,

$$\int_{D_{T'}} \{\mathbf{u}\psi_t - A(\cdot\,, \mathbf{u})\mathbf{u}\psi + \mathbf{F}(\cdot\,, \mathbf{u})\psi\}\ dxdt + \int_{R^m \times \{0\}} \mathbf{u}_0 \psi\ dx - \int_{R^m \times \{T'\}} \mathbf{u}\psi\ dx = 0. \quad (4.6)$$

Proof. Define $\psi_k^N = 1/\Delta t \int_{t_{k-1}}^{t_k} \psi(x,t)\ dt, k = 1, \dots, N$, and $\psi^N = \sum_{k=1}^N \psi_k^N \theta_k^N$. We dot multiply (2.2) by ψ_k^N, sum on $k, k = 1, \dots, N$, and integrate over R^m to obtain:

$$\sum_{k=1}^{N-1} \left(\mathbf{u}_k^N, \frac{\psi_k^N - \psi_{k+1}^N}{\Delta t}\right)_{L_2} \Delta t + (\mathbf{u}_N^N, \psi_N^N)_{L_2} - (\mathbf{u}_0, \psi_1^N)_{L_2}$$

$$+ \sum_{k=1}^{N} \left(A(\mathbf{u}_k^N)\mathbf{u}_k^N, \psi_k^N\right)_{L_2} \Delta t = \sum_{k=1}^{N} (\mathbf{F}(t_k, \mathbf{u}_k^N), \psi_k^N)_{L_2} \Delta t.$$

If one rewrites this expression, it becomes, with $\zeta^N = \sum_{k=1}^{N-1} (\psi_k^N - \psi_{k+1}^N)\theta_k^N / \Delta t$,

$$\int_0^{(N-1)\Delta t} (\mathbf{u}_S^N, \zeta^N)_{L_2}\ dt + (\mathbf{u}_N^N, \psi_N^N)_{L_2} - (\mathbf{u}_0, \psi_1^N)_{L_2}$$

$$+ \int_0^T (A(\mathbf{u}_S^N)\mathbf{u}_S^N - \mathbf{F}_S^N, \psi^N)_{L_2}\ dt = 0.$$

By [7, Lemma 5.2.5], it follows that

$$\psi^N \to \psi \text{ in } L_2(\mathcal{D}),\ \zeta^N \to -\frac{\partial \psi}{\partial t} \text{ in } L_2(\mathcal{D}).$$

We now allow $N = N_j \to \infty$. The terms involving A and $\mathbf{F}$ are analyzed by Lemma 3.7 and Remark 3.2. Further, $\int_{(N_j-1)\Delta t}^T (\mathbf{u}_S^{N_j}, \zeta^{N_j})_{L_2}\ dt \to 0$, by the pointwise boundedness of the integrated sequences. In order to analyze $(\mathbf{u}_0, \psi_1^{N_j})_{L_2}$, we use the mean value theorem of integral calculus to deduce that $\psi_1^{N_j}(x) = \psi(x, t(x))$, for some $0 < t(x) < \Delta t$. Uniform continuity then leads to

$$(\mathbf{u}_0, \psi_1^{N_j})_{L_2} \to \int_{R^m \times \{0\}} \mathbf{u}_0 \psi\ dx.$$

One may then use the fundamental theorem of calculus in reflexive Banach spaces [16] to deduce that $(\mathbf{u}_{N_j}^{N_j}, \psi_{N_j}^{N_j})_{L_2} \to \int_{R^m \times \{T\}} \mathbf{u}\psi\ dx$. To see this, one argues as follows. The intermediate representation,

$$(\mathbf{u}_{N_j}^{N_j}, \psi)_{L_2} - (\mathbf{u}_0, \psi)_{L_2} = \int_0^T [(\mathbf{u}_{PL}^{N_j}, \psi)_{L_2}]_t\ dt,$$

and its limit are used to deduce that $(\mathbf{u}_{N_j}^{N_j}, \psi)_{L_2} \to \int_{R^m \times \{T\}} \mathbf{u}\psi\ dx$. Use of the uniform convergence of the ψ-averages and the triangle inequality completes the argument. We thus have the limit rendered by (4.6) for $T = T'$. The general case is similar. □

4.2 Existence and Uniqueness of Strong Solutions

Before stating the result on strong solutions extending (4.6), we require a technical estimation lemma, to be used in the regularity argument for solutions.

Lemma 4.2. *The estimate,*

$$\|\mathbf{u}_k^N\|_{H^s}^2 \leq \|\mathbf{u}_0\|_{H^s}^2 + C\left(\sum_{j=1}^{k}\|\mathbf{u}_j^N\|_{H^s}^2 + \sum_{j=1}^{k}\|F(t_k^N, \mathbf{u}_j^N)\|_{H^s}^2\right)\Delta t, \tag{4.7}$$

holds for some constant C independent of N. If

$$\alpha^N(t) = \left\{\sum_{k=1}^{N}\|\mathbf{u}_k^N\|_{H^s}^2\theta_k^N(t)\right\},\quad \alpha(t) = \sup_N\{\alpha^N(t)\},$$

and

$$F^N(t) = \left\{\sum_{k=1}^{N}\|\mathbf{F}(t_k, \mathbf{u}_k^N)\|_{H^s}^2\theta_k^N(t)\right\},\ F(t) = \sup_N\{F^N(t)\},$$

then $\alpha \in L_1$, $F \in L_1$, and it follows that

$$\|\mathbf{u}(t)\|_{H^s}^2 \leq \|\mathbf{u}_0\|_{H^s}^2 + C\left(\int_0^t (\alpha(\tau) + F(\tau))\, d\tau\right). \tag{4.8}$$

Proof. We begin with (2.2), indexed by j, and take the H^s-inner product with $\mathbf{u}_j^N$, for $j = 1, \dots, k$. Upon addition, and use of the cancellation induced by the inequality,

$$(\mathbf{u}_j^N, \mathbf{u}_{j-1}^N)_{H^s} \geq -\frac{1}{2}(\|\mathbf{u}_j^N\|_{H^s}^2 + \|\mathbf{u}_{j-1}^N\|_{H^s}^2),$$

we obtain:

$$\frac{1}{2}\|\mathbf{u}_k^N\|_{H^s}^2 + \sum_{j=1}^{k}(A(\mathbf{u}_j^N)\mathbf{u}_j^N, \mathbf{u}_j^N)_{H^s}\Delta t \leq \frac{1}{2}\|\mathbf{u}_0\|_{H^s}^2 + \frac{1}{2}\left(\sum_{j=1}^{k}\|\mathbf{u}_j^N\|_{H^s}^2 + \sum_{j=1}^{k}\|F(t_k^N, \mathbf{u}_j^N)\|_{H^s}^2\right)\Delta t. \tag{4.9}$$

We now use the uniform semigroup generation property on the smooth space, as applied to each of the operators A_{11}, A_{12}, A_{22}, to deduce that (see Proposition 2.1)

$$\sum_{j=1}^{k}(A(\mathbf{u}_j^N)\mathbf{u}_j^N, \mathbf{u}_j^N)_{H^s}\Delta t \geq -c\sum_{j=1}^{k}\|\mathbf{u}_j^N\|_{H^s}^2\Delta t,$$

for some $c > 0$. If the negative of the right hand side of this inequality is added to both sides of (4.9), we obtain (4.7), for some constant C. In order to obtain (4.8), we rewrite (4.7) and use the definitions to obtain the inequality,

$$\alpha^N(t) \leq \|\mathbf{u}_0\|_{H^s}^2 + C\left(\int_0^{t+\Delta t}\alpha(\tau)\, d\tau + \int_0^{t+\Delta t}F(\tau)\, d\tau\right). \tag{4.10}$$

If $\Sigma \subset (0,T)$ is any set of positive measure, then, by item 2 of Proposition 4.1, we conclude:

$$\mathbf{u}_S^{N_j} \rightharpoonup^* \mathbf{u} \text{ weak-* in } L_\infty(\Sigma; Y).$$

We have used uniqueness of limits (see [7, Proposition 5.2.6]) and standard subsequential arguments. By using the lower semicontinuity of the norm with respect to weak-* convergence [20, Theorem 9, p. 125], we have the inequality,

$$\int_\Sigma \|\mathbf{u}(t)\|_{H^s}^2 \leq \liminf_N \int_\Sigma \alpha^N(t)\, dt \leq C \int_\Sigma \left(\int_0^t (\alpha(\tau) + F(\tau)\, d\tau \right)\, dt + \|\mathbf{u}_0\|_{H^s}^2.$$

Since Σ is an arbitrary measurable subset of $(0,T)$, we have the pointwise inequality (4.8). $\square$

Corollary 4.1. *The solution* $\mathbf{u}$ *of* (4.6) *and* (1.16) *is a strong solution. Specifically,*

$$\mathbf{u}_t \in C([0,T]; H^{s-2}(R^m)), \tag{4.11}$$

and the equation (1.18) *holds in the strong sense described by* (4.11)*. Moreover,*

$$\mathbf{u} \in C([0,T]; H^s(R^m)). \tag{4.12}$$

Proof. The regularity (4.11) follows from item 2 of Proposition 4.1 and the fundamental theorem of calculus in reflexive Banach spaces. This also validates an integration by parts, and hence the strong form of the evolution equation. Note that each of the terms in (1.18) is in the class (4.11) (see Remark 3.2 and Lemma 3.7). The regularity (4.12) is more subtle and can be deduced from

$$\mathbf{u} \in L_\infty((0,T); H^s(R^m)),$$

established in Proposition 4.1, in a manner similar to that employed in [17, pp. 44–46], where it is noted that right continuity at zero suffices to establish the continuity on $[0,T]$. The technique to establish right continuity at zero relies on establishing an estimate of the form,

$$\|\mathbf{u}(t)\|_{H^s}^2 \leq \|\mathbf{u}(0)\|_{H^s}^2 + \int_0^t f(\tau)\, d\tau,$$

where f is an L_1 function. This is precisely what was done in Lemma 4.2; we identify f with $\alpha + F$. We may now proceed as in [17]. This concludes the proof. $\square$

Proposition 4.2. *The strong solution of* (1.18) *described by Corollary* 4.1 *is unique.*

Proof. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are solutions, then one obtains in the standard way,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_1 - \mathbf{u}_2\|_{L_2}^2 - (D\Delta(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2)_{L_2} &= -(\mathcal{P}E(\mathbf{u}_1)(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2)_{L_2} \\ &\quad -(\mathcal{P}(E(\mathbf{u}_1) - E(\mathbf{u}_2))\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_{L_2} + (\mathbf{F}(\mathbf{u}_1) - \mathbf{F}(\mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2)_{L_2}.\end{aligned}$$

Here, we have used the notation of (1.18) and $\mathbf{F}(\mathbf{u}) = \mathbf{F}(t, \mathbf{u})$. Noting that both terms on the left hand side of this equation are nonnegative, we have, via Lemma 3.7, the inequality,

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{L_2}^2(t) \leq C \int_0^t \|\mathbf{u}_1 - \mathbf{u}_2\|_{L_2}^2(\tau)\, d\tau,$$

for $0 \leq t \leq T$. Here, C is a positive constant depending only on the smooth norms of $\mathbf{u}_1, \mathbf{u}_2$. Use of the Gronwall inequality concludes the proof. $\square$

4.3 Stability Under the Inviscid Limit

An important feature of the semigroup-based theory which we have presented in this paper is its ability to permit the passage to the case of the incompressible charged inviscid fluid; i. e., the passage under the limit $\nu \to 0$. A result of this type was obtained for the case of the Euler system in [6], where the core of the theory presented in this paper was first developed. It was noted in [6], and will be repeated here, that Kato developed his theory to cover (among other applications) both the Navier-Stokes and the Euler system. The case distinctions are discussed in [12, p. 55] for the semigroup generation on $\mathrm{P}L_2$. We shall only sketch the modifications. In fact, there is only one essential modification: the domain of $-A_{11}$ is larger, with regularity index decreased by one. The rigorous argument appears in the proof of Lemma 3.1. However, this has no impact upon the invariance results and fixed point arguments for Q, nor upon the arguments in the space-time domain. The following result is a natural consequence of our arguments. It is a generalization of [6, Lemma 5.1].

Proposition 4.3. *There is a strong solution, in the regularity classes defined by* (4.11), (4.12)*, of the Euler/Poisson-Nernst-Planck system* (1.18)/ (1.16)*, where D is defined by* (1.17)*, with $\nu = 0$. Moreover, the solution interval is stable under the inviscid limit $\nu \to 0$. More precisely, if $\mathbf{u}_1, \mathbf{u}_2$, are solutions of* (1.18) *for values $\nu_1 \geq \nu_2 \geq 0$, then there is a constant C such that*

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{C([0,T];L_2(R^m))} \leq C(\nu_1 - \nu_2).$$

The terminal time T is independent of ν.

Proof. A generalization of the identity of the previous proposition yields:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{L_2} - (D_1\Delta(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2)_{L_2} = ((D_1 - D_2)\Delta\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_{L_2} -$$

$$(\mathcal{P}E(\mathbf{u}_1)(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2)_{L_2} - (\mathcal{P}(E(\mathbf{u}_1) - E(\mathbf{u}_2))\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_{L_2} + (\mathbf{F}(\mathbf{u}_1) - \mathbf{F}(\mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2)_{L_2}.$$

The use of the regularity classes and previous estimates yields the inequality, for $0 \leq t \leq T$,

$$\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{L_2}(t) \leq C\{(\nu_1 - \nu_2) + \int_0^t \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{L_2}(\tau)\, d\tau\},$$

for some constant C. Use of the Gronwall inequality concludes the proof. □

4.4 Summation

We can now complete the analysis of the well-posedness of the model by proving the nonnegativity of n, p.

Theorem 4.2. *If n_0, p_0 are given satisfying* (1.12)*, and $\mathbf{u}$ is the unique solution on $[0, T]$ satisfying Corollary* 4.1 *and Proposition* 4.2*, then there is a maximum subinterval $[0, T']$ of $[0, T]$ such that $n \geq 0, p \geq 0$ on this subinterval.*

Proof. The proof is immediate because of the regularity class (4.11) and the Sobolev embedding theorem, which together guarantee that $\mathbf{u}(x,t)$ is bounded and uniformly continuous as a function of x,t. Thus, given a unique strong solution $\mathbf{u}$ of (1.18) on $[0,T]$, one defines:

$$T' = \max\{t' : u_{m+1}(\cdot\,,t) \geq 0,\ 0 \leq t \leq t',\ u_{m+2}(\cdot\,,t) \geq 0,\ 0 \leq t \leq t'\}$$

□

Remark 4.2. *As noted in the introduction, the use of the fully implicit method of horizontal lines has allowed for the formulation of a precise condition (see* (3.16)*) for the local assumption. By using semigroup methods, rather than parabolic methods, we are able to pass to the inviscid Euler limit. An abstract semidiscrete method for the general Cauchy problem has been carried out in* [3]*. The first use of semidiscrete methods, however, in the context of the Kato semigroup framework appears to be* [6] *(see also* [7, Section 7.5]*).*

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