A Trapping Principle for Discontinuous Elliptic Systems of Mixed Monotone Type

Joseph W. Jerome*

April, 2001

Abstract

We consider discontinuous semilinear elliptic systems, with boundary conditions on the individual components of Dirichlet/Neumann type. The system is a divergence form generalization of $\Delta \mathbf{u} = \mathbf{f}(\cdot, \mathbf{u})$. The components of $\mathbf{f}$ are required to satisfy monotonicity conditions associated with competitive or cooperative species. The latter model defines a system of mixed monotone type. We also illustrate the theory via higher order mixed monotone systems which combine competitive and cooperative subunits. We seek solutions on special intervals defined by lower and upper solutions associated with outward pointing vector fields. It had been shown by Heikkilä and Lakshmikantham that the general discontinuous mixed monotone system does not necessarily admit solutions on an interval defined by lower and upper solutions. Our result, obtained via the Tarski fixed point theorem, shows that solutions exist for the models described above in the sense of a measurable selection (in the principal arguments) from a maximal monotone multi-valued mapping. We use intermediate variational inequalities in the proof. Applications involving quantum confinement and chemically reacting systems with change of phase are discussed. These are natural examples of discontinuous systems.

Key words: Mixed monotone type, discontinuous semilinear elliptic systems, Tarski fixed point theorem, variational inequalities, competing and cooperating systems.
AMS(MOS) Subject Classification:35J50, 35J65
Acknowledgements: The author is supported by the National Science Foundation under grant DMS-9704458.

*Department of Mathematics, Northwestern University, Evanston, Il 60208

1 Introduction.

Steady-state models of cooperative species provide examples of elliptic systems of mixed monotonicity. In this case, the vector field of the semilinear system has components, each of which is increasing in one species, while decreasing in the other. In their treatise [13] on monotone iterative methods, Heikkilä and Lakshmikantham provide an example of a discontinuous elliptic system of mixed monotonicity, which does not possess solutions on an 'interval' defined by lower and upper solutions. In this example, the monotonicity structure differs from that assigned to cooperative species. One of the goals of this paper is to provide a natural category of discontinuous systems of mixed monotonicity for which solutions do occur on natural 'intervals' for the models described above. However, even for cooperative species, as we indicate later in the introduction, solutions do not exist in the conventional sense. Thus, it will be necessary for us to interpret 'solution' in the sense of a measurable selection in the principal arguments from a maximal monotone multi-valued mapping. By a principal argument here, we mean the $i\ th$ argument of f_i. In particular, for these arguments, range values may lie on the interval defined by right and left hand limits. This incorporates ideas of Brézis [5], in an extended setting. Other authors have also developed the earlier ideas of Brézis (see [8, 18, 23, 7]). We shall also examine other discontinuous monotone systems, which, although not mixed, differ from the usual systems studied in [13]. Models of competitive species provide an example of such systems. We also indicate in §4.2 how higher order discontinuous mixed monotone systems can be consistently constructed by using competitive and cooperative species as subunits. In all cases, our solutions require a measurable selection in the principal arguments.

In this work, we employ a framework, based upon (possibly discontinuous) isotone mappings in conjunction with convex analysis. A prominent role is played by an outward pointing vector field on the boundary of the trapping region containing the solutions of the steady-state mixed boundary value problem. This idea was introduced in [18] for evolution systems, and was implicit in [20] as well. For the case of Carathéodory vector fields, even without monotonicity, results for both elliptic and parabolic systems were obtained by Carl and Grossman in [6]. In the present work, we continue the idea of [20] as applied to steady-state systems. It allows for the consideration of variational inequalities as a bridge to the system format.

Consider such a model, given in simplest steady-state form by

$$\Delta \mathbf{u} = \mathbf{f}(\cdot\,, \mathbf{u}),\ \mathbf{u} = (u_1, u_2)^T,\ \mathbf{f} = (f_1, f_2)^T, \tag{1.1}$$

where $u_i, i = 1, 2$ denote the concentrations of species, on a bounded domain $\Omega \subset \mathbf{R}^N$, and where Δ denotes the Laplacian. We shall actually study a divergence form generalization of (1.1), and give a precise interpretation in terms of measurable selection.

The structure of the vector field $\mathbf{f}$ is significant. Its monotonicity properties define the classifications of models, and the boundary behavior of the vector field is related to stability. In addition, in its principal arguments, each $f_i, i = 1, 2$, is required to satisfy certain conditions relative to its right and left hand limits (minimum modulus condition). It will later be shown how these are related to maximal monotone mappings of convex analysis, and permit a measurable selection. Specifically, one can identify a slab in dependent variable space, $\mathbf{R}^2$, given by $Q = [a_1, b_1] \times [a_2, b_2]$. Furthermore, $\mathbf{f}$ satisfies the following conditions:

1. **Measurability and Minimum Modulus (Property MMM)** 1. For $i = 1, 2$, and arbitrary bounded measurable functions u_i such that $[u_1, u_2]$ has range in Q, the composition functions $f_i = f_i(\cdot\,, u_1(\cdot\,), u_2(\cdot\,))$ are measurable on Ω.

   2. For all x and $\eta_2 \in [a_2, b_2]$, the function $f_1(x, \cdot\,, \eta_2)$ assumes its minimum modulus at each $\eta_1 \in [a_1, b_1]$, relative to the interval of right and left hand limits at η_1:

   $$|f_1(x, \eta_1, \eta_2)| = \min\{|y| : y \in [f_1(x, \eta_1^-, \eta_2), f_1(x, \eta_1^+, \eta_2)]\}.$$

   The same assumption applies to $f_2(x, \eta_1, \cdot\,)$ as a function of its second dependent variable argument. Note that we have anticipated here the fact (see item 3 below) that f_i is increasing in its *i th* argument.

2. **Outward Pointing on the Boundary of Q (Property OP).**

   $$\text{For } i = 1, \text{ or } i = 2, \ \limsup_{u_i \to a_i} f_i(\cdot\,, u_1, u_2) \leq 0;$$

   $$\text{For } i = 1, \text{ or } i = 2, \ \liminf_{u_i \to b_i} f_i(\cdot\,, u_1, u_2) \geq 0.$$

3. **Model Classification.** We distinguish two classifications. Examples are given in §4 involving quantum confinement and change of phase. The monotonicity properties described in these classifications are assumed to hold for almost all $x \in \Omega$.

   a. Competing Species.

   f_1 is separately increasing in u_1, u_2; f_2 is separately increasing in u_1, u_2.

   b. Cooperating Species.

   f_1 is increasing in u_1, and decreasing in u_2;

   f_2 is decreasing in u_1, and increasing in u_2.

We describe the mixed boundary conditions as follows.

i. Dirichlet Boundary. There is a (nonempty relatively open) boundary component Σ_D such that the restriction of $\mathbf{u}$ to Σ_D agrees with a smooth function $\hat{\mathbf{u}} \in C^\infty(\bar{\Omega})$, with range in Q:

$$\Gamma(\mathbf{u} - \hat{\mathbf{u}})|_{\Sigma_D} = 0. \qquad (1.2)$$

Here, Γ denotes the trace operator.

ii. Neumann Boundary. The normal derivative of $\mathbf{u}$ vanishes in a weak sense on the complement of Σ_D with respect to $\partial\Omega$. This is a natural boundary condition subsumed in the weak formulation.

Before proceeding to a presentation of our results, we summarize the counterexample presented in [13, p. 301]. In the notation of (1.1) (opposite in sign to that of [13]), we define the discontinuous vector field,

$$f_1(u_1, u_2) = \begin{cases} -1, & u_2 \le 0, \\ 0, & u_2 > 0, \end{cases}$$

$$f_2(u_1, u_2) = -u_1.$$

The boundary condition is a pure homogeneous Dirichlet condition on the boundary sphere of the ball of radius ρ, which serves as the set Ω. The authors of [13] construct lower and upper solutions,

$$(0,0)^T; \; ((\rho^2 - |x|^2)/(2N), \rho^2(\rho^2 - |x|^2)/(4N^2))^T;$$

but show that there is no solution in the 'interval' defined by these quasi-solutions. The reader will note that this system does not fit into either of the categories of cooperating or competing species. Although we do not use lower and upper solutions as the basis for our results, it is instructive to note that the constant vectors, $(a_1, a_2)^T$ and $(b_1, b_2)^T$, play the role of lower and upper solutions for the boundary value problem studied in this paper. A slight adaptation of the example above shows that existence in the conventional sense cannot occur for cooperating species either. We define the discontinuous vector field,

$$f_1(u_1, u_2) = \begin{cases} -1, & u_1 \le 0, \\ 0, & u_1 > 0, \end{cases}$$

$$f_2(u_1, u_2) = 0.$$

The boundary condition is a pure homogeneous Dirichlet condition on the same domain as above. We choose $Q = [0, 0] \times [b, b]$ for any $b > 0$. It can be shown that, if a weak solution exists, then it must coincide with $(0, 0)$; however, the latter is not a solution. This shows that the notion of solution must be extended. The reader will note that the counter-example just furnished does *not* satisfy the minimum modulus condition (MMM–2) we have imposed on f_1 in its first argument.

We shall now state the elliptic system in the form in which we shall study it. Given bounded measurable functions p_1 and p_2 on Ω, satisfying $p_1 \ge c_1 > 0, \; p_2 \ge c_2 > 0$, we formulate the extension of (1.1) as

$$\nabla \cdot p_1(x) \nabla u_1 = f_1(\cdot, u_1, u_2), \; \nabla \cdot p_2(x) \nabla u_2 = f_2(\cdot, u_1, u_2). \tag{1.3}$$

We now describe the notion of measurable selection.

Measurable Selection for f_1 and f_2 Let u_2 (resp., u_1) be a bounded measurable function with range in $[a_2, b_2]$ (resp., $[a_1, b_1]$). For x fixed, we denote by $\bar{f}_1(x, \eta_1, u_2(x))$ the interval-valued function, with range values given by $[f_1(x, \eta_1^-, u_2(x)), f_1(x, \eta_1^+, u_2(x))]$. If, for almost all x, the single-valued measurable function, $\bar{g}_1(\cdot, u_1(\cdot), u_2(\cdot))$, satisfies $\bar{g}_1(x, u_1(x), u_2(x)) \in \bar{f}_1(x, u_1(x), u_2(x))$, we call $\bar{g}_1$ a measurable selection of $\bar{f}_1$. $\bar{f}_2(x, u_1(x), \eta_2)$ and $\bar{g}_2(x, u_1(x), u_2(x))$ are defined similarly.

The trapping principle for $\mathbf{u}$ is given in the following result.

Theorem 1.1. *Suppose* $\mathbf{f}$ *satisfies properties* MMM *and* OP*, and suppose that there are* $L_p(\Omega)$ *functions* h_i*, such that* $|h_i|$ *dominates each* $|\bar{f}_i(\cdot, \mathbf{v})|$ *whenever* $\mathbf{v}$ *has range in* Q*. Here,* $p = \frac{2N}{N+2}, N \geq 3$*;* $1 < p \leq 2, N = 2$*; and* $p = 1, N = 1$*. Then* (1.3), (1.2) *has a weak solution* $\mathbf{u}$*, with range in* Q *for each of the two classifications* (2a, 2b) *described above, if* $\mathbf{f}(\cdot, \mathbf{u})$ *is interpreted as a measurable selection* $\mathbf{g}(\cdot, \mathbf{u})$ *of* $\bar{\mathbf{f}}(\cdot, \mathbf{u})$*.*

In order to define what is meant by weak solution, we introduce the inner product on $Y = \prod_1^2 H^1(\Omega)$ as

$$(\mathbf{v}, \mathbf{w})_Y = \sum_1^2 \int_\Omega p_i(x) \nabla v_i \cdot \nabla w_i \, dx + \sum_1^2 \int_{\Sigma_D} \Gamma v_i \Gamma w_i \, d\sigma. \tag{1.4}$$

It is a direct consequence of the open mapping theorem that the norm induced by (1.4) is equivalent to the standard norm. We identify the zero trace subspace of Y:

$$Y_0 = \{\mathbf{v} \in Y : \Gamma \mathbf{v}|_{\Sigma_D} = \mathbf{0}\}.$$

Then $\mathbf{u}$ is a weak solution of (1.3), (1.2) if $\mathbf{u}$ satisfies (1.2) and the relation,

$$(\mathbf{u}, \phi)_Y + \langle \mathbf{g}(\cdot, \mathbf{u}), \phi \rangle = 0, \ \forall \phi \in Y_0. \tag{1.5}$$

Here, the duality relation $\langle \cdot, \cdot \rangle$ is used with the L_p components of $\mathbf{g}(\cdot, \mathbf{u})$ acting as continuous linear functionals on L_q, where the index q satisfies $1/p + 1/q = 1$. By the Sobolev embedding theorem [1, Theorem 5.4], which implies the continuous embedding of H^1 into L_q, we may thus identify the components of $\mathbf{g}(\cdot, \mathbf{u})$ with continuous linear functionals on H^1. In this identification, it is necessary to use the equivalent norms on $Y_1 = H^1$, $Y_2 = H^1$, given by

$$(v, w)_{Y_i} = \int_\Omega p_i(x) \nabla v \cdot \nabla w \, dx + \int_{\Sigma_D} \Gamma v \Gamma w \, d\sigma, \ i = 1, 2. \tag{1.6}$$

The first step in the study of (1.3, 1.2) is the analysis of an associated variational inequality in §2, followed by the fundamental equivalence theorem in §3, which employs the *outward pointing* hypothesis. A prominent role is played in the sequel by the Tarski fixed point theorem, which does not require mapping continuity. We are thus able to minimize the traditional approach of convex analysis, employed in [19], and are able to eliminate direct continuity hypotheses. We have only assumed the Lebesgue measurability of vector field composition with bounded measurable functions and the existence of dominating L_p functions. These hypotheses are in the spirit of [13]. Of necessity, we have adjoined the notion of measurable selection. It is consistent with applications to free boundary problems, where so-called mushy regions occur at a phase change. The variational inequality formalism is very natural in tandem with the Tarski theorem, since the latter requires lattice 'interval' endpoint squeezing by the isotone map, which is enforced via the inequality. An early published version of an isotone fixed point theorem appeared in [15]. There, the fixed point theorem is stated and proved for isotone mappings which have a type of continuity. The theorem of Kantorovich [15] allows one to conclude the existence of the fixed point(s) by the convergence of the iteration sequences. This set of ideas has been fruitfully developed in [9]. We prefer to employ the stronger theorem of Tarski, which is valid without continuity. This was used by the author in [18] and is consistent with recent studies in ordered spaces [7]. Proofs of the Tarski theorem are now available which do not require Zorn's lemma [25, Theorem 11.E, p. 507]. We take up the necessary ideas now.

2 The Variational Inequality

We begin by establishing some notation. We set

$$\mathcal{K}_0 = \{\mathbf{v} \in Y : \Gamma(\mathbf{v} - \hat{\mathbf{u}})|_{\Sigma_D} = \mathbf{0},\ \mathbf{v}(x) \in Q \text{ for almost all } x \in \Omega\}. \tag{2.1}$$

The variational inequality can be formulated as: Determine $\mathbf{u} \in \mathcal{K}_0$ and $F(\mathbf{u}) = \mathbf{g}(\cdot\,, \mathbf{u})$, where $\mathbf{g}(\cdot\,, \mathbf{u})$ is a measurable selection of $\mathbf{f}(\cdot\,, \mathbf{u})$, such that

$$\langle F(\mathbf{u}), \mathbf{v} - \mathbf{u}\rangle + (\mathbf{u}, \mathbf{v} - \mathbf{u})_Y \geq 0,\ \forall \mathbf{v} \in \mathcal{K}_0. \tag{2.2}$$

Parallel to Theorem 1.1 is the following proposition.

Proposition 2.1. *Suppose that* $\mathbf{f}$ *is in one of the classifications,* §1, 2a or 2b, *and suppose that its multi-valued extension* $\bar{\mathbf{f}}$ *is* $L_p(\Omega)$*-dominated, where* $p = \frac{2N}{N+2}, N \geq 3$*;* $1 < p \leq 2, N = 2$*; and* $p = 1, N = 1$*. Then the variational inequality* (2.2) *has a solution* $\mathbf{u} \in \mathcal{K}_0$*.*

We shall deduce Proposition 2.1 from the Tarski fixed point theorem in the following subsections. In §3, we show that the outward pointing hypothesis, adjoined to (2.2), yields the essential conclusions of Theorem 1.1 in terms of weak solutions.

2.1 The Lattice and Tarski's Fixed Point Theorem

In this subsection, we introduce a lattice structure. This first involves a partial ordering for functions in L_q, where q is conjugate to p in Proposition 2.1, defined by:

$$v \leq w \text{ if } v(x) \leq w(x), \text{ for almost all } x \in \Omega.$$

This is equivalent to the use of the cone of nonnegative functions to define a partial ordering. L_q is a lattice (see [3]) since the elements $\min(v, w)$ and $\max(v, w)$, defined by real variable operations, provide greatest lower, and least upper bounds, respectively. Now, for $i = 1, 2$, set

$$K_i = \{v \in L_q : a_i \leq v(x) \leq b_i \text{ for almost all } x \in \Omega\}. \tag{2.3}$$

Recently, the authors of [7] have provided an elegant framework for the systematic use of Tarski's theorem in the context of ordered normed spaces. In fact, Proposition 1.1.1 of [7] allows one to introduce our partial ordering, and proceed directly to the existence of a smallest and largest fixed point for an isotone mapping on the set K_i. However, for readers not yet familiar with this framework, we sketch a few 'bridge' ideas related to the use of Tarski's theorem, as well as some background facts.

A chain $\mathcal{C} \subset K_i$ is a subset such that either $v \leq w$ or $w \leq v$ for any pair $v, w \in \mathcal{C}$. It follows from results of [13, Theorem 5.8.1, p. 478] that K_i is an *inductive* lattice in the language of [3]: every chain $\mathcal{C} \subset K_i$ has a greatest lower bound and least upper bound in K_i.

We shall require the notion of isotone mappings. Suppose a lattice is given.

Isotone Lattice Mapping. A mapping A is isotone on its lattice subdomain if

$$u \leq v \Rightarrow Au \leq Av.$$

We shall now state a strong form of the Tarski fixed point theorem (see [24], [3, p. 115], [25, p. 507]). Note that continuity of A *is not assumed*.

Proposition 2.2. *Let* $\mathbf{E}$ *be a partially ordered set, and suppose* $u_0 \leq v_0$ *are elements of* $\mathbf{E}$ *with the property that the interval* $\mathbf{I} = \{v \in \mathbf{E} : u_0 \leq v \leq v_0\}$ *is an inductive lattice. Suppose that* A *is an isotone mapping of* $\mathbf{I}$ *into* $\mathbf{E}$ *such that* $u_0 \leq A(u_0)$ *and* $A(v_0) \leq v_0$. *Then the set of fixed points* u *of* A *satisfying* $u \in \mathbf{I}$ *is nonempty and possesses a smallest element* $\underline{u}$ *and a largest element* $\bar{u}$.

Application of this theorem received much attention during the 1970s by certain authors (see, for example [22]) who were studying two-phase Stefan problems. These authors were proving the existence of solutions of quasi-variational inequalities, equivalent to two-phase Stefan problems. In this context, the isotone property is evident, but continuity is not evident from the strongly implicit nature of the fixed point mapping. The theorem is referred to as Birkhoff's theorem in [22]. The application is described in [18], where Tarski's theorem is proved via Zorn's lemma.

2.2 Outline of the Proof of Proposition 2.1

We shall require the following closed convex subsets of H^1 for $i = 1, 2$:

$$\mathcal{K}_i = \{v \in Y_i : a_i \leq v(x) \leq b_i \text{ for almost all } x \in \Omega,\ v|_{\Sigma_D} = \hat{u}_i|_{\Sigma_D}\}. \tag{2.4}$$

The proof of Proposition 2.1 proceeds by defining a fixed point mapping V defined on the interval K_1 with range in $\mathcal{K}_1 \subset K_1$. The latter injection follows from the conjugacy of p in Proposition 2.1 and q in (2.3).

1. Given $w \in K_1$, an intermediate mapping,

$$T : K_1 \mapsto \mathcal{K}_2 \subset K_2,\ Tw = u,$$

is introduced, where $u \in \mathcal{K}_2$ is the solution of the variational inequality,

$$(u, v - u)_{Y_2} + \langle F_2(\cdot\,, w, u), v - u\rangle \geq 0,\ \forall v \in \mathcal{K}_2. \tag{2.5}$$

2. Vw is then defined as UTw, where

$$U : K_2 \mapsto \mathcal{K}_1 \subset K_1,\ Uu = u_1,$$

and where $u_1 \in \mathcal{K}_1$ is the solution of the variational inequality,

$$(u_1, v - u_1)_{Y_1} + \langle F_1(\cdot\,, u_1, u), v - u_1\rangle \geq 0,\ \forall v \in \mathcal{K}_1. \tag{2.6}$$

In (2.5, 2.6) above, we maintain the definitions of the inner product in Y_i given in (1.6). The elements $F_i, i = 1, 2$, represent measurable selections in the principal arguments. The mapping V will be shown to be isotone in the two cases of competitive and cooperative species, and will possess a fixed point by the Tarski theorem. It is precisely this fixed point which satisfies the proposition.

2.3 Single Inequality Convex Analysis

We shall use traditional convex analysis for the study of a single inequality, with appropriate monotonicity structure. In particular, this will include existence and uniqueness for inequalities (2.5) and (2.6), which together establish the well-posedness of the mappings T and U. The Tarski theorem will be used for the other parts of the proof. We shall first state the abstract result, and then the application to the specific inequality class. The result as stated here is contained in [10, Chapter II, Section 3] (cf. [4] and [18, Proposition 3.1.5] for related results of a more general character). The result is stated here in a duality pairing formulation.

Lemma 2.1. *Let $\mathcal{J}_0$ be a closed, convex, bounded subset of a reflexive Banach space $\mathcal{X}$, suppose that Φ is a proper, convex, lower semicontinuous functional defined on $\mathcal{J}_0$, and that $B : \mathcal{J}_0 \to \mathcal{X}^*$ is a monotone, weakly continuous mapping. Thus, we assume that*

$$\langle Bu - Bv, u - v\rangle \geq 0,\ \forall u, v \in \mathcal{J}_0, \tag{2.7}$$

and

$$\langle Bv_k - Bv, \psi\rangle \to 0,\ \textit{if}\, v_k \rightharpoonup v,\ \forall \psi \in \mathcal{X}. \tag{2.8}$$

Then the variational inequality,

$$\langle B(y) - b, z - y\rangle + \Phi(z) - \Phi(y) \geq 0,\ \forall z \in \mathcal{J}_0, \tag{2.9}$$

possesses a solution $y \in \mathcal{J}_0$ for each $b \in \mathcal{X}^*$.

The preceding lemma implies the following result. We include a proof for completeness.

Proposition 2.3. *Let Z be one of the Hilbert spaces Y_1, Y_2, and, for a prescribed $\mathbf{v}$ with range in Q, let the measurable selection $G(\cdot\,, u)$ denote either the function $F_1(\cdot\,, u, v_2)$ or the function $F_2(\cdot\,, v_1, u)$, correlated with the choice of Z. Then the variational inequality,*

$$(u, v-u)_Z + \langle G(\cdot\,, u), v-u\rangle \geq 0,\ \forall v \in \mathcal{J}, \tag{2.10}$$

has a unique solution pair $u, G(\cdot\,, u)$ in $\mathcal{J}$. Here, $\mathcal{J}$ is one of the corresponding closed convex sets $\mathcal{K}_1, \mathcal{K}_2$.

Proof. We first establish an 'a priori' bound for any solution u of (2.10). This is necessary because we have presented Lemma 2.1 with the hypothesis that $\mathcal{J}_0$ is bounded. Thus, in the application of the lemma, we shall identify $\mathcal{J}_0$ with the intersection of $\mathcal{J}$ and a ball, defined by the 'a priori' bound. Begin by determining a unique element u_0, via the Riesz representation theorem, such that

$$\langle G(\cdot\,, u), \psi\rangle = -(u_0, \psi)_Z,\ \forall \psi \in Z. \tag{2.11}$$

Here, $u \in \mathcal{J}$ is arbitrary, and $G(\cdot\,, u)$ is an associated measurable selection. By earlier hypotheses, u_0 is in a ball $\mathcal{B}$ of radius $C\rho$, where C is the norm of the embedding of Z into L_q, and ρ is the L_p norm of the fixed dominating function $h \in L_p$. By standard results concerning quadratic minimization over closed convex sets, the solution u of (2.10), with $G(\cdot\,, u)$ given, may be characterized as the unique element $v = u$ minimizing the functional,

$$\Psi(v) = \|v-u_0\|_Z^2 - \|u_0\|_Z^2 = \|v\|_Z^2 + 2\langle G(\cdot\,, u), v\rangle,$$

over $\mathcal{J}$. Define $\sigma = 2\max\{\|\hat{u} - v\|_Z : v \in \mathcal{B}\}$. Here, $\hat{u}$ is the appropriate component of the Dirichlet boundary data function, $\hat{\mathbf{u}}$.

'a priori' estimate in terms of σ: u satisfies the estimate,

$$\|u - \hat{u}\|_Z \leq \sigma.$$

Indeed, let u_0 be defined by (2.11), and note that

$$\Psi(u) \leq \Psi(\hat{u}),$$

so that

$$\|u - u_0\|_Z \leq \|\hat{u} - u_0\|_Z,$$

and

$$\|u - \hat{u}\|_Z \leq \|u - u_0\|_Z + \|u_0 - \hat{u}\|_Z \leq \sigma.$$

This concludes the verification of the 'a priori' estimate.

We now use Lemma 2.1 to prove Proposition 2.3. There are two steps in establishing the existence of the solution of the variational inequality (2.10). They are:

1. The identification of the elements in Lemma 2.1;

2. The verification of the hypotheses of this lemma, and, hence the existence of a solution of (2.10).

For the appropriate identifications, set $b = 0$, $\mathcal{X} = Z$, $\mathcal{J}_0 = \mathcal{J} \cap \{v : \|v - \hat{u}\|_Z \leq \sigma\}$. To continue the identifications, we suppose for concreteness that $Z = Y_1$, $\mathcal{J} = \mathcal{K}_1$, and that $f_1(\cdot\,, w, v_2(\cdot))$ is given as a function of w, with v_2 already prescribed. Here, f_1 is the originally given first component of the vector field, satisfying property (MMM–2). We then define the proper, convex, lower semicontinuous functional on $\mathcal{J}_0$,

$$\Phi(v) = \int_\Omega \phi(\cdot\,, v(\cdot)), \tag{2.12}$$

where $\phi(\cdot\,, w)$ is the absolutely continuous and convex (in w) primitive of $f_1(\cdot\,, w, v_2(\cdot))$, vanishing at $w = 0$. We make note of the following critical characterization.

Subdifferential characterization

$$\partial_w \phi(\cdot\,, w) = \bar{f}_1(\cdot\,, w, v_2(\cdot)).$$

Here, the subdifferential (cf. [10, pp. 20–28]) is taken with respect to the indicated variable.

This result is due originally to Brézis [4, 5], who made extensive use of the Yosida approximation, and the fact that a function which satisfies property (MMM–2) in its principal argument coincides with the limit of the Yosida approximation. We refer the reader especially to the concrete example 2.8.1 in [5, Chapter 2, p. 43]. These facts were later extensively developed by Chang [8], and are carefully presented by Carl and Heikkilä [7, pp. 194–197] in their study of differential inclusions of hemivariational inequality type. The author also used such a characterization in his study of the Stefan problem [17, 18]. Now $B : \mathcal{J}_0 \to Z^*$ is defined by:

$$\langle B(u), v\rangle = (u, v)_Z. \tag{2.13}$$

Here, $v \in Z = Y_1$. The verification of the hypotheses of Lemma 2.1 is standard, and we may conclude the existence of a solution $y = u$ of (2.9) with the stated identifications. This permits us to conclude that

$$-Bu \in \partial\Phi(u).$$

To infer that u satisfies (2.10), use the subdifferential characterization of ϕ referred to earlier. If we agree to identify functionals with representers, this means that $-Bu$ can be identified with a measurable selection. This concludes the existence argument for the stated choices. The argument in the remaining case is identical. The uniqueness for the pair $u, G(\cdot\,, u)$ follows from the strict monotonicity property. $\square$

2.4 The Proof of Proposition 2.1

We begin with an analysis of the mapping T.

Lemma 2.2. *Let $w \in K_1$ be given. There is a unique solution $u = Tw$ of the variational inequality* (2.5)*. In addition,*

a. competing species. *T is antitone from K_1 to K_2:*

$$w_1 \leq w_2 \Rightarrow Tw_1 \geq Tw_2.$$

b. cooperating species. *T is isotone from K_1 to K_2.*

Proof. The existence of a unique solution of (2.5) follows from Proposition 2.3. We now establish the antitone/isotone properties of T. Suppose that $w_1 \leq w_2$, and set $u^* = Tw_1, u^{**} = Tw_2$. In the case of competing species, define the admissible functions in $\mathcal{K}_2$,

$$v_1 = u^* + (u^{**} - u^*)^+, \; v_2 = u^{**} - (u^{**} - u^*)^+,$$

and, in the case of cooperating species, define

$$v_1 = u^* + (u^{**} - u^*)^-, \; v_2 = u^{**} - (u^{**} - u^*)^-.$$

Here, $v^+ = \max(0, v) \geq 0$, and $v^- = \min(0, v) \leq 0$. By direct substitution in (2.5), we have the following inequalities.

competing species

$$(u^*, (u^{**} - u^*)^+)_{Y_2} \geq -\langle F_2(\cdot\,, w_1, u^*), (u^{**} - u^*)^+\rangle, \tag{2.14}$$

$$-(u^{**}, (u^{**} - u^*)^+)_{Y_2} \geq \langle F_2(\cdot\,, w_2, u^{**}), (u^{**} - u^*)^+\rangle. \tag{2.15}$$

cooperating species

$$(u^*, (u^{**} - u^*)^-)_{Y_2} \geq -\langle F_2(\cdot\,, w_1, u^*), (u^{**} - u^*)^-\rangle, \tag{2.16}$$

$$-(u^{**}, (u^{**} - u^*)^-)_{Y_2} \geq \langle F_2(\cdot\,, w_2, u^{**}), (u^{**} - u^*)^-\rangle. \tag{2.17}$$

Upon adding the negatives of these two inequalities, and simplifying, we have:

competing species

$$0 \leq (u^{**} - u^*, (u^{**} - u^*)^+)_{Y_2} \leq$$
$$-\langle F_2(\cdot\,, w_2, u^{**}) - F_2(\cdot\,, w_1, u^{**}), (u^{**} - u^*)^+\rangle - \langle F_2(\cdot\,, w_1, u^{**}) - F_2(\cdot\,, w_1, u^*), (u^{**} - u^*)^+\rangle \leq 0.$$

cooperating species

$$0 \leq (u^{**} - u^*, (u^{**} - u^*)^-)_{Y_2} \leq$$
$$\langle F_2(\cdot\,, w_2, u^{**}) - F_2(\cdot\,, w_1, u^{**}), |(u^{**} - u^*)^-|\rangle + \langle F_2(\cdot\,, w_1, u^{**}) - F_2(\cdot\,, w_1, u^*), |(u^{**} - u^*)^-|\rangle \leq 0.$$

For the second set of inequalities, we have used the decrease of the given vector component f_2 in its first dependent variable argument to conclude that the first difference is nonpositive, and the monotone selection property in the second dependent variable argument to conclude that the second difference is nonpositive. We conclude that $(u^{**} - u^*)^- = 0$. For the first set, we use the corresponding increase of f_2. In this case, we conclude that $(u^{**} - u^*)^+ = 0$. This concludes the proof of the lemma. □

We now analyze (2.6) and the mapping U.

Lemma 2.3. *Let $u \in K_2$ be given. The variational inequality* (2.6) *has a unique solution $u_1 = Uu$. In addition,*

a. competing species. *U is antitone from K_2 to K_1.*

b. cooperating species. *U is isotone from K_2 to K_1.*

Proof. The existence of a unique solution of (2.6) follows from Proposition 2.1. We now prove the antitone/isotone properties. Suppose that $u^* \leq u^{**}$, and set $u_1^* = Uu^*, u_1^{**} = Uu^{**}$. In the case of competing species, define the admissible functions in $\mathcal{K}_1$,

$$v_1 = u_1^* + (u_1^{**} - u_1^*)^+,\ v_2 = u_1^{**} - (u_1^{**} - u_1^*)^+.$$

and, in the case of cooperating species, define

$$v_1 = u_1^* + (u_1^{**} - u_1^*)^-,\ v_2 = u_1^{**} - (u_1^{**} - u_1^*)^-.$$

By direct substitution in (2.6), we have the two inequalities,

competing species

$$(u_1^*, (u_1^{**} - u_1^*)^+)_{Y_1} \geq -\langle F_1(\cdot\,, u_1^*, u^*), (u_1^{**} - u_1^*)^+\rangle, \tag{2.18}$$

$$-(u_1^{**}, (u_1^{**} - u_1^*)^+)_{Y_1} \geq \langle F_1(\cdot\,, u_1^{**}, u^{**}), (u_1^{**} - u_1^*)^+\rangle. \tag{2.19}$$

cooperating species

$$(u_1^*, (u_1^{**} - u_1^*)^-)_{Y_1} \geq -\langle F_1(\cdot\,, u_1^*, u^*), (u_1^{**} - u_1^*)^- \rangle, \tag{2.20}$$

$$-(u_1^{**}, (u_1^{**} - u_1^*)^-)_{Y_1} \geq \langle F_1(\cdot\,, u_1^{**}, u^{**}), (u_1^{**} - u_1^*)^- \rangle. \tag{2.21}$$

Upon adding the negatives of these two inequalities, and simplifying, we have:

competing species

$$0 \leq (u_1^{**} - u_1^*, (u_1^{**} - u_1^*)^+)_{Y_1} \leq$$
$$-\langle F_1(\cdot\,, u_1^{**}, u^{**}) - F_1(\cdot\,, u_1^*, u^{**}), (u_1^{**} - u_1^*)^+ \rangle - \langle F_1(\cdot\,, u_1^*, u^{**}) - F_1(\cdot\,, u_1^*, u^*), (u_1^{**} - u_1^*)^+ \rangle \leq 0.$$

cooperating species

$$0 \leq (u_1^{**} - u_1^*, (u_1^{**} - u_1^*)^-)_{Y_1} \leq$$
$$\langle F_1(\cdot\,, u_1^{**}, u^{**}) - F_1(\cdot\,, u_1^*, u^{**}), |(u_1^{**} - u_1^*)^-| \rangle + \langle F_1(\cdot\,, u_1^*, u^{**}) - F_1(\cdot\,, u_1^*, u^*), |(u_1^{**} - u_1^*)^-| \rangle \leq 0.$$

For the first set of these inequalities, we have used the monotone selection property in the first dependent variable argument to conclude that the first difference is nonpositive, and the the increase of the given vector component f_1 in its second dependent variable argument to conclude that the second difference is nonpositive. This implies that $(u_1^{**} - u_1^*)^+ = 0$. For the second set, we use the decrease of f_1 in its second dependent variable argument. Thus, $(u_1^{**} - u_1^*)^- = 0$ in this case. This concludes the proof of the lemma. □

It is now immediate that V is well-defined, and has range contained in $\mathcal{K}_1 \subset K_1$ through the definition of the variational inequality defining U. By use of the preceding lemmas, we conclude that the mapping V satisfies: V is isotone on the lattice interval K_1 in both cases of competing and cooperating species.

Thus, by Proposition 2.2, V has a fixed point, $Vu_1 = u_1$. If now we define $u_2 = Tu_1$, the pair $(u_1, u_2)^T$ is a solution of (2.2): this follows directly from $u_1 = Vu_1 = UTu_1 = Uu_2$, and the individual definitions of T and U in terms of (2.5) and (2.6).

3 Major Equivalence Theorem

In this section, we use the property (OP) satisfied by the vector field $\mathbf{f}$ to prove that a solution of the variational inequality (2.2) is a weak solution of the system (1.3, 1.2), i. e. , satisfies (1.5).

Theorem 3.1. *Let* $(u_1, u_2)^T$ *be a solution of* (2.2)*, with* $g_1(\cdot\,, u_1, u_2), g_2(\cdot\,, u_1, u_2)$ *determined to be measurable selections in their principal arguments. Then, under the hypothesis* (OP) *on* $\mathbf{f}$*,* $(u_1, u_2)^T$ *is a solution of* (1.5)*. In particular, Theorem* 1.1 *holds.*

Proof. An important preliminary observation is that, by the property (OP), $\mathbf{g}$ satisfies the following:

$$\text{If } u_i = a_i,\ i = 1,\ \text{or } i = 2,\ \text{ then } g_i(\cdot\,, u_1, u_2) \leq 0;$$
$$\text{if } u_i = b_i,\ i = 1,\ \text{or } i = 2,\ \text{ then } g_i(\cdot\,, u_1, u_2) \geq 0.$$

We now proceed directly to the proof of the theorem. In order to simplify the argument, we deal componentwise with the system. Thus, we select the test functions in $\mathcal{K}_0$,

$$\mathbf{v} = (v_\pm, u_2)^T,\ \ v_\pm = (u_1 \pm \epsilon_k \phi - a_1)^+ + a_1 + (b_1 - u_1 \mp \epsilon_k \phi)^-.$$

Here, ϕ is the first component of a vector test function, with zero trace on Σ_D, to be used in the weak formulation (1.5). Without loss of generality, we may assume that ϕ is smooth, and is pointwise bounded by unity and has Y_1 norm also bounded by unity. In verifying that $v_\pm \in \mathcal{K}_1$, we observe that $u_1 \pm \epsilon_k \phi - a_1$ and $b_1 - u_1 \mp \epsilon_k \phi$ cannot be simultaneously negative, which reduces the number of possible sign combinations to three: positive/positive, positive/negative, and negative/positive. Each of these is easily seen to lead to $a_1 \leq v_\pm \leq b_1$; the satisfaction of the boundary condition on Σ_D is immediate from the zero trace of ϕ. We shall now define the numbers ϵ_k. Given $\{\eta_k\}$, $\eta_k > 0$, $\eta_k \to 0$, we select sequences $\{\alpha_k\}$ and $\{\beta_k\}$ such that each of the following three conditions is satisfied:

1. $\alpha_k - a_1 := \epsilon_k = b_1 - \beta_k < \eta_k$.
2. For $\mathcal{A}_k = \{a_1 < u_1 < \alpha_k\} \subset \mathbf{R}^N$, $\ \mathcal{B}_k = \{\beta_k < u_1 < b_1\} \subset \mathbf{R}^N$,
$$\int_{\mathcal{A}_k \cup \mathcal{B}_k} p_1(x) |\nabla u_1|^2 \, dx < \left(\frac{\eta_k}{2}\right)^2.$$
3. $\int_{\mathcal{A}_k \cup \mathcal{B}_k} |g_1(x, u_1(x), u_2(x))| \, dx < \frac{\eta_k}{2}$.

The fact that conditions two and three are possible is a property of measure and integration theory [14, Theorem 10.15 and Theorem 12.34]. We observe that

$$v_\pm - u_1 = \pm \epsilon_k \phi \text{ on } \{\alpha_k \leq u_1 \leq \beta_k\}, \tag{3.1}$$

$$v_+ - u_1 = \epsilon_k \phi \text{ on } \{u_1 = a_1\} \cap \{\phi \geq 0\}, \tag{3.2}$$

$$v_- - u_1 = -\epsilon_k \phi \text{ on } \{u_1 = a_1\} \cap \{\phi \leq 0\}, \tag{3.3}$$

$$v_+ - u_1 = \epsilon_k \phi \text{ on } \{u_1 = b_1\} \cap \{\phi \leq 0\}, \tag{3.4}$$

$$v_- - u_1 = -\epsilon_k \phi \text{ on } \{u_1 = b_1\} \cap \{\phi \geq 0\}. \tag{3.5}$$

We require some notation for sets. Thus, make the substitutions,

$$\mathcal{D}_k = \{\alpha_k \leq u_1 \leq \beta_k\},\ \ \mathcal{E}_k = \mathcal{D}_k \cup \{u_1 = a_1\} \cup \{u_1 = b_1\},$$

$$\mathcal{F}_k = \mathcal{D}_k \cup (\{u_1 = a_1\} \cap \{\phi \geq 0\}) \cup (\{u_1 = b_1\} \cap \{\phi \leq 0\}).$$

Note that ∇u_1 vanishes for almost all x on

$$\{u_1 = a_1\} \cup \{u_1 = b_1\}.$$

This follows from [11, Lemma 7.7]. Substitution of (v_+, u_2) into (2.2) yields,

$$\int_{\mathcal{E}_k} p_1 \nabla u_1 \cdot \nabla \phi + \int_{\mathcal{F}_k} g_1 \phi \geq -\int_{\mathcal{A}_k \cup \mathcal{B}_k} p_1 |\nabla u_1 \cdot \nabla \phi| - \epsilon_k^{-1} \int_{\mathcal{A}_k \cup \mathcal{B}_k} |g_1(v_+ - u_1)|. \tag{3.6}$$

In computing the set over which the second integral on the left hand side of this inequality is taken, we have used the fact that $v_+ - u_1$ vanishes on the two sets, $\{u_1 = a_1\} \cap \{\phi \leq 0\}$ and $\{u_1 = b_1\} \cap \{\phi \geq 0\}$. We have also made use of (3.2) and (3.4). We now employ the critical properties that $g_1 \leq 0$ on $\{u_1 = a_1\}$ and $g_1 \geq 0$ on $\{u_1 = b_1\}$, as implied by (OP). This permits us to add two key terms involving integrals of $g_1\phi$, without changing the sense of the above inequality. These are terms involving integration over $\{u_1 = a_1\} \cap \{\phi \leq 0\}$ and $\{u_1 = b_1\} \cap \{\phi \geq 0\}$. When these integrals are added, and the domain of integration for the second left hand side integral is consolidated, we may rewrite the left hand side of the inequality as

$$\int_{\mathcal{E}_k} p_1 \nabla u_1 \cdot \nabla \phi + \int_{\mathcal{E}_k} g_1 \phi.$$

The right hand side of (3.6) is greater than $-\eta_k$, by the restrictions imposed on α_k, β_k, and ϵ_k. Note that $|v_+ - u_1| \leq \epsilon_k$ on $\mathcal{A}_k \cup \mathcal{B}_k$ has been used here. We have obtained the inequality,

$$\int_{\mathcal{E}_k} p_1 \nabla u_1 \cdot \nabla \phi + \int_{\mathcal{E}_k} g_1 \phi > -\eta_k. \tag{3.7}$$

The companion inequality, whereby the left hand side is shown to be bounded above by η_k, is obtained by substitution of v_-. In fact, the inequality,

$$\int_{\mathcal{E}_k} p_1 \nabla u_1 \cdot \nabla \phi + \int_{\mathcal{E}_k} g_1 \phi < \eta_k, \tag{3.8}$$

is obtained as a result of this substitution, and use of arguments parallel to those above. In these arguments, (3.2, 3.4) are replaced by (3.3, 3.5), respectively. It follows that the common left hand side of (3.7) and (3.8) has zero limit as $k \to \infty$. The three inequalities satisfied by $\{\eta_k\}$ above then demonstrate that this zero limit is

$$\int_{\Omega} p_1 \nabla u_1 \cdot \nabla \phi + \int_{\Omega} g_1 \phi.$$

The argument for the second equation is identical. The final result is obtained by addition. □

4 The Models and Extensions of the Systems

The competing species classification includes the semiconductor transport subsystem, when Slotboom variables are employed. The reader may consult [20] for a complete development. Here, we summarize the subsystem. It is given by

$$\nabla\cdot \exp(u(x))\nabla V = G(x)(VW-1),\ \nabla\cdot \exp(-u(x))\nabla W = H(x)(VW-1). \tag{4.1}$$

The variables $V = \exp(-v)$, $W = \exp(w)$ are called the Slotboom variables, corresponding to quasi-Fermi levels v, w; u is the electrostatic potential. In the notation of (1.3), $p_1 = \exp(u)$ and $p_2 = \exp(-u)$; u is a bounded measurable function. The vector field is defined by the electron/hole recombination term, when a particular strategy is employed for decoupling the transport subsystem from the Poisson equation within the full drift-diffusion system. Also, G and H are strictly positive, bounded measurable functions. It is clear that the vector field is outward pointing on

$$Q = [\exp(-B), \exp(-A)] \times [\exp(A), \exp(B)].$$

Here, $A < B$ are prescribed common minimum and maximum values of the quasi-Fermi levels. The typical boundary conditions for this model are mixed conditions on a multi-dimensional polyhedral region. Thus, the trapping principle holds in this case. It is possible to consider a refinement of this result, compatible with the theory of this paper.

4.1 Quantum Confinement

We consider the case suggested by quantum confinement. This model can be thought of as a subsystem of the quantum hydrodynamic model (two carrier version) described in [21]. In this case, one or both of the variables V, W, appearing in the recombination term, cannot assume certain values corresponding to bound energy states for x in the quantum well $\mathcal{W} \subset \Omega$. If $\mathcal{E}_V, \mathcal{E}_W$ are the closed forbidden interval ranges for the variables V, W, with Cartesian product *large*, i. e., $VW - 1$ is outward pointing on its boundary, and χ denotes the characteristic function, then the vector field modification involves the respective redefinitions,

$$F_1(\cdot, V, W) = (1 - \chi(\mathcal{E}_V)\chi(\mathcal{W}))(1 - \chi(\mathcal{E}_W)\chi(\mathcal{W}))G(x)(VW - 1), \tag{4.2}$$
$$F_2(\cdot, V, W) = (1 - \chi(\mathcal{E}_V)\chi(\mathcal{W}))(1 - \chi(\mathcal{E}_W)\chi(\mathcal{W}))H(x)(VW - 1). \tag{4.3}$$

In other words, only free, not bound, carriers, are able to recombine according to the model. The new vector fields are discontinuous, but retain the necessary properties to apply our results to competing systems. Theorem 1.1 thus holds for the system:

$$\nabla\cdot \exp(u(x))\nabla V = F_1(x, V, W), \tag{4.4}$$
$$\nabla\cdot \exp(-u(x))\nabla W = F_2(x, V, W). \tag{4.5}$$

The region Q remains the one given above.

4.2 Higher Order Systems

Various extensions to higher order systems are possible. We indicate this in the following result, involving a system of mixed monotone type, in the terminology of [13].

Theorem 4.1. *Consider a system of three species, X, Y, and Z. We suppose that any one of the species is in cooperation with the other two. Specifically, we consider the system,*

$$\Delta X = F(\cdot\,, X, Y, Z),\ \Delta Y = G(\cdot\,, X, Y, Z),\ \Delta Z = H(\cdot\,, X, Y, Z),$$

where $F$is increasing in X but decreasing in Y, Z; G is increasing in Y but decreasing in X, Z; and H is increasing in Z, but decreasing in X, Y. In this case,

$$Q = \prod_{1}^{3} [a_i, b_i].$$

We assume the obvious extensions of the properties (MMM) and (OP). We also define the dual indices p, q as before, and assume the existence of dominating L_p functions $h_i, i = 1, 2, 3$. Then, a solution exists with range in Q in the sense of a measurable selection in the principal arguments.

Proof. The reader who has followed the analysis of the order-two systems will conjecture, correctly, that the basic map now should operate on the lattice K_{12} of vector functions in $L_q(\Omega) \times L_q(\Omega)$, with range in $[a_1, b_1] \times [a_2, b_2]$, and should proceed as follows.

Let $\mathbf{w} = (w_1, w_2)^T$ be given in K_{12}, and consider the weak form of the decoupled boundary value problem for $u \in \mathcal{K}_3 = \{v \in H^1 : a_3 \leq v(x) \leq b_3, [\Gamma u - \Gamma \hat{u}_3]|_{\Sigma_D} = 0\}$,

$$(u, \phi)_{H^1} + \langle H(\cdot\,, w_1, w_2, u), \phi \rangle = 0,\ \forall \phi \in H^1, \Gamma\phi|_{\Sigma_D} = 0. \tag{4.6}$$

Here, the inner product on H^1 is the equivalent inner product defined in analogy with (1.6). A measurable selection solution exists by Proposition 2.3 and the theory of §3. One writes $u = T\mathbf{w}$, and shows that T is isotone from K_{12} to $\mathcal{K}_3$. If $\tilde{\mathbf{w}} \geq \mathbf{w}$, one can make use of the identity,

$$H(\cdot\,, \tilde{w}_1, \tilde{w}_2, \tilde{u}) - H(\cdot\,, w_1, w_2, u) = [H(\cdot\,, \tilde{w}_1, \tilde{w}_2, \tilde{u}) - H(\cdot\,, w_1, \tilde{w}_2, \tilde{u})] +$$
$$[H(\cdot\,, w_1, \tilde{w}_2, \tilde{u}) - H(\cdot\,, w_1, w_2, \tilde{u})] + [H(\cdot\,, w_1, w_2, \tilde{u}) - H(\cdot\,, w_1, w_2, u)].$$

When each of these three terms is multiplied by $(\tilde{u} - u)^-$, the corresponding product is nonnegative. This permits the subtraction of the equations associated with $\tilde{u}$ and u, and the corresponding conclusion, when the choice $\phi = (\tilde{u} - u)^-$ is made, that $\tilde{u}(\tilde{\mathbf{w}}) \geq u(\mathbf{w})$.

For $\mathbf{w}$ and $u = T\mathbf{w}$ given, a solution $(u_1, u_2)^T$ exists in K_{12} for the decoupled system,

$$\Delta u_1 = F(\cdot\,, u_1, w_2, u),\ \Delta u_2 = G(\cdot\,, w_1, u_2, u), \tag{4.7}$$

satisfying the prescribed boundary conditions, by an application of Theorem 1.1. We write $(u_1, u_2)^T = \mathbf{u} = V\mathbf{w}$. By using the test functions,

$$(\tilde{u}_1 - u_1)^-,\ \ (\tilde{u}_2 - u_2)^-,$$

in the respective weak formulations of each of the equations in (4.7) we see that each of these test functions is zero, and hence that V is isotone on K_{12}. Proposition 2.2 implies a fixed point. When this fixed point is coupled to the component obtained by the action of T, a solution triple is defined for the boundary value problem. □

4.3 A Chemically Diffusing System with Change of Phase

We illustrate the higher order system of §4.2 by means of an idealized model of two chemically reacting species, diffusing in a medium which is defined as having two phases. This example is motivated by the theory described in [2](see also [18, Example 1.3.1]) for a single-phase medium. Here, the physics is adapted to that of a two-phase medium. The system may thus be viewed as a steady-state Stefan problem, coupled to a reaction-diffusion system. The two-phase Stefan problem has been been analyzed in [18]. One can define a generalized temperature, via the Kirchhoff transformation,

$$u = \int_0^\theta k(\xi)\, d\xi,$$

where θ is the usual temperature and k is the thermal conductivity, which may be phase dependent. The phases are then characterized by

$$\{x \in \Omega : u(x) < 0\};\ \{x \in \Omega : u(x) \geq 0\}.$$

We shall think of them as the liquid and vapor phases, respectively. Our convention includes the 'free boundary' between phases in the vapor phase. In the liquid phase, the medium is expected to be a heat source with respect to the chemistry, and this accounts for the sign in the first equation. The source is abruptly shut off in the vapor phase. This causes a discontinuity in the vector field. Similarly, we assume that there is no net concentration flux through the boundary of any closed region in the vapor phase. This also accounts for a vector field discontinuity. To express this mathematically, let H denote the Heaviside-type function,

$$H(u) = \begin{cases} 0, & u < 0, \\ 1, & u \geq 0. \end{cases}$$

Then the system for u and the concentrations C_i of the diffusing species may be written:

$$\Delta u = -(1 - H(u))(h_1 r_1(C_1) + h_2 r_2(C_2)),$$
$$D_1 \Delta C_1 = L(1 - H(u))(r_1(C_1) - r_2(C_2)),$$
$$D_2 \Delta C_2 = L(1 - H(u))(r_2(C_2) - r_1(C_1)).$$

Here, we assume that D_1, D_2 are diffusion constants, and that r_1, r_2 are reaction rates, with h_1, h_2 the corresponding heats of reaction. All quantities are assumed positive, with the rates assumed to be monotone increasing in their respective arguments. L is a positive integer related to the stoichiometry of the reaction. We are also assuming chemical neutrality: the sum of the vector fields for the two reactants is zero.

One sees that $a_1 < 0$ and $b_1 > 0$ are arbitrary choices for the temperature interval, in order to ensure an outward pointing vector field. The minimum modulus property is ensured by the definition of H. The intervals for C_1 and C_2 are not arbitrary, but depend upon the rates. For example, if $r_1(C_1) = C_1^m$ and $r_2(C_2) = C_2^n$, then $a_2 = a_3 = 0$ and $b_2 = b_3 = 1$ are permissible choices, such that the vector field is outward pointing on the boundary of Q. Theorem 4.1 yields a solution to the boundary value problem.

Acknowledgment The author expresses his appreciation to the referee for an exceptional in-depth reading of the manuscript. Constructive suggestions have been incorporated, including a counter-example furnished by the referee.

References

[1] R. Adams. *Sobolev Spaces.* Academic Press, 1975.

[2] R. Aris. *The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts*, Vol. 1. Oxford University Press, 1975.

[3] G. Birkhoff. *Lattice Theory.* (3rd ed.) American Mathematical Society Colloq. Publ. 25, Providence, RI, 1973.

[4] H. Brézis. Équations et inéquations non linéaires dans les espaces vectoriels en dualité. *Ann. Inst. Fourier (Grenoble)*, **18**:115–175, 1968.

[5] H. Brézis. *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert.* North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[6] S. Carl and S. Grossman. Monotone enclosure for elliptic and parabolic systems with nonmonotone nonlinearities. *J. Math. Anal. Appl.*, **151**:190–202, 1990.

[7] S. Carl and S. Heikkilä. *Nonlinear Differential Equations in Ordered Spaces.* CRC Press, 2000.

[8] K.C. Chang. Variational methods for non-differentiable functionals and their applications to partial differential equations. *J. Math. Anal. Appl.*, **80**:102–129, 1981.

[9] L. Collatz. *Functional Analysis and Numerical Mathematics.* Academic Press, 1966.

[10] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems.* North Holland and American Elsevier, 1976.

[11] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, New York, 1977.

[12] R. Haberman. *Mathematical Models.* Prentice-Hall, 1977.

[13] S. Heikkilä and V. Lakshmikantham. *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations.* Vol. 181, Pure and Applied Mathematics Series, Marcel Dekker, 1994.

[14] E. Hewitt and K. Stromberg. *Real and Abstract Analysis.* Springer-Verlag, New York, 1965.

[15] L. Kantorovich. The method of successive approximations. *Acta Math.*, **71**:63–97, 1939.

[16] J.L. Kelley. *General Topology.* Van Nostrand-Reinhold, 1955.

[17] J.W. Jerome. Existence and approximation of weak solutions of nonlinear Dirichlet problems with discontinuous coefficients. *SIAM J. Math. Anal.*, **9**:730–742, 1978.

[18] J.W. Jerome. *Approximation of Nonlinear Evolution Systems.* Academic Press, 1983.

[19] J.W. Jerome. Numerical approximation of PDE system fixed point maps via Newton's method. *Journal of Applied and Computational Mathematics*, **38**:211–230, 1991.

[20] J.W. Jerome. *Analysis of Charge Transport: A Mathematical Study of Semiconductor Devices.* Springer, Berlin, 1996.

[21] J.W. Jerome. Analytical and computational advances for hydrodynamic models of classical and quantum charge transport. *VLSI DESIGN*, **10**:453–466, 2000.

[22] J.L. Lions. *Sur Quelques Questions d'Analyse de Méchanique et de Contrôle Optimal.* Univ. Montreal Press, Montral, 1976.

[23] P.D. Panagiotopoulos. *Hemivariational Inequalities and Applications in Mechanics and Engineering.* Springer-Verlag, 1993.

[24] A. Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific J. Math.*, **5**:285–309, 1955.

[25] E. Zeidler. *Nonlinear Functional Analysis and its Applications*, Vol. I, *Fixed-Point Theorems*, Springer-Verlag, 1986.