TRAPPING REGIONS FOR ELLIPTIC SYSTEMS WITH DISCONTINUOUS COUPLING VECTOR FIELDS

SIEGFRIED CARL[†] **and JOSEPH W. JEROME**[‡]

[†] *Martin-Luther-Universität Halle-Wittenberg, Fachbereich Mathematik und Informatik, Institut für Analysis, 06099 Halle, Germany*

[‡] *Department of Mathematics, Northwestern University, Evanston, IL 60208, USA*

1 SUMMARY

We consider boundary value problems for systems of the form $A_k u_k + f_k(\cdot, u_1, u_2) = 0$, $k = 1, 2$, in a bounded domain $\Omega \subset \mathbb{R}^N$, where the A_k are elliptic operators in divergence form. The boundary conditions are of mixed Dirichlet-Robin type, and given by $\partial u_k/\partial\nu + g_k(\cdot, u_1, u_2) = 0$ on Γ, and $u_k = 0$ on $\partial\Omega \setminus \Gamma$. The coupling vector fields $f = (f_1, f_2) : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ and $g = (g_1, g_2) : \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ may be discontinuous with respect to all their arguments. The main goal is to provide conditions on the vector fields f and g that allow the identification of regions of existence of solutions (so called trapping regions). To this end the problem is transformed to a discontinuously coupled system of variational inequalities. Assuming a generalized outward pointing vector field on the boundary of a rectangle of the dependent variable space, the system of variational inequalities can be solved via a fixed point problem for some increasing operator in an appropriate ordered Banach space. The main tools used in the proof are variational inequalities, truncation and comparison techniques, and fixed point results in ordered Banach spaces.

2 INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^1-boundary $\partial\Omega$, and $\Gamma \subset \partial\Omega$ be such that $\partial\Omega \setminus \Gamma$ is a relatively open C^1-portion of $\partial\Omega$ with positive surface measure. Only for the sake of simplifying our presentation and in order to emphasize the main idea we consider the the following model problem ($k = 1, 2$):

$$\begin{gathered} -\Delta u_k + f_k(u_1, u_2) = 0, \quad \text{in} \quad \Omega, \\ u_k = 0 \quad \text{on } \partial\Omega \setminus \Gamma, \qquad \frac{\partial u_k}{\partial \nu} + g_k(u_1, u_2) = 0 \quad \text{on} \quad \Gamma, \end{gathered} \tag{1}$$

where Δ is the Laplacian and ν denotes the outward normal at Γ. Problem (1) differs from the general case in that the elliptic operators A_k are replaced by $-\Delta$ and the governing vector fields $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ and $g = (g_1, g_2) : \mathbb{R}^2 \to \mathbb{R}^2$ do not depend, in addition, on the space variable x. However, it should be noted that the results obtained for (1) hold true also in the general case.

The novelty of problem (1) is that the vector fields f and g may be discontinuous in all their arguments. More precisely, we assume the following measurability and growth condition for f and g.

(H) The component functions $f_k : \mathbb{R}^2 \to \mathbb{R}$ and $g_k : \mathbb{R}^2 \to \mathbb{R}$ are Baire-measurable and satisfy a growth condition of the form

$$|f_k(s_1, s_2)| \le c\,(1 + |s_1| + |s_2|), \ \forall\ (s_1, s_2) \in \mathbb{R}^2,$$
$$|g_k(s_1, s_2)| \le c\,(1 + |s_1| + |s_2|), \ \forall\ (s_1, s_2) \in \mathbb{R}^2,$$

where c is some positive generic constant.

Remark 2.1. Hypothesis(H) ensures that the components f_k and g_k are superpositionally measurable, i.e., whenever $u, v : \Omega \to \mathbb{R}$ (resp. $u, v : \Gamma \to \mathbb{R}$) are measurable, then also the superposition $x \mapsto f_k(u(x), v(x))$ (resp. $x \mapsto g_k(u(x), v(x))$) is a measurable function in Ω (resp. Γ). The growth conditions imply that the Nemytskij operators F_k (resp. G_k) associated with f_k (resp. g_k) map $L^2(\Omega)$ (resp. $L^2(\Gamma)$) into itself, but may be discontinuous, since we allow the components f_k and g_k to be discontinuous in all their arguments. Due to the latter the structure of the vector fields f and g is significant. We shall impose mixed monotonicity properties on f and g that include models of cooperative and competing species.

Definition 2.1. A vector field $h = (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be of *competitive type* if the component functions $h_1(s_1, s_2)$ and $h_2(s_1, s_2)$ are both separately increasing in $s_1,\ s_2$; and it is of *cooperative type* if $h_1(s_1, s_2)$ is increasing in s_1 and decreasing in s_2, and $h_2(s_1, s_2)$ is decreasing in s_1 and increasing in s_2.

Since problem (1) with f and g of cooperative type can easily be transformed by a simple linear transformation into a system of competitive type and vice versa, these two cases are qualitatively equivalent. Thus in what follows we will assume the following monotonicity condition.

(M) The vector fields f and g are assumed to be of competitive type

Throughout the rest of this paper we assume the hypotheses (H) and (M) to be satisfied.

3 NOTATIONS AND AUXILIARIES

Let $V := W^{1,2}(\Omega)$ denote the usual (real) Sobolev space, and let $V_0 \subset V$ be the subspace of V defined by

$$V_0 = \{u \in V \mid \gamma u = 0 \ \text{ on } \ \partial\Omega \setminus \Gamma\},$$

where $\gamma : V \to L^2(\partial\Omega)$ is the trace operator which is linear and continuous (and even compact). We introduce the natural partial ordering in $L^2(\Omega)$, that is $u \le w$ if and only if $w - u$ belongs to the positive cone $L^2_+(\Omega)$ of all nonnegative elements of $L^2(\Omega)$, which induces also a partial ordering in the Sobolev space V. If $u, w \in V$ and $u \le w$ then $[u, w] = \{v \in V \mid u \le v \le w\}$ denotes the order interval formed by u and w. Further, if $(B, \le)$ is any ordered Banach space, then we furnish the Cartesian product $B \times B$ with the componentwise partial ordering, i.e., $x = (x_1, x_2) \le (y_1, y_2) = y$ iff $x_k \le y_k,\ k = 1, 2$, and thus the order interval $[x, y] \subset B \times B$ corresponds with the rectangle $Q = [x_1, y_1] \times [x_2, y_2] \subset B \times B$.

We use throughout this paper the following Cartesian products: $X := V \times V$, $X_0 := V_0 \times V_0$, $Y := L^2(\Omega) \times L^2(\Omega)$, and $Z := L^2(\Gamma) \times L^2(\Gamma)$. To reformulate problem (1) in a functional analytic setting denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V_0^* and V_0, and consider $-\Delta$, and the Nemytskij operators F_k, G_k as mappings defined as follows:

$$-\Delta : V \to V_0^*;\ \langle -\Delta u, \varphi\rangle := \int_\Omega \nabla u \,\nabla \varphi \, dx,\ \varphi \in V_0,$$
$$F_k : Y \to V_0^*;\ \ \langle F_k(u_1,u_2),\varphi\rangle := \int_\Omega f_k(u_1,u_2)\varphi\, dx,\ \varphi\in V_0,$$
$$G_k : Z \to V_0^*;\ \ \langle G_k(u_1,u_2),\varphi\rangle := \int_\Gamma g_k(u_1,u_2)\gamma\varphi\, d\Gamma,\ \varphi\in V_0,$$

With $u=(u_1,u_2)$, $Du := (-\Delta u_1, -\Delta u_2)$, $\gamma u := (\gamma u_1, \gamma u_2)$, $Fu := (F_1(u), F_2(u))$, and $Gu := (G_1(u), G_2(u))$ the weak formulation of problem (1) reads as follows.

Definition 3.1. The function $u \in X_0$ is a weak solution of (1) if the following vector equation holds:

$$Du + F(u) + G\circ\gamma(u) = 0 \quad \text{in} \quad X_0^*, \tag{2}$$

which means that $\langle -\Delta u_k + F_k(u) + G_k\circ\gamma(u), \varphi\rangle = 0$ is satisfied for all $\varphi \in V_0$, and $k = 1, 2$.

Let $Q = [\underline{u}, \bar{u}] \subset X$ be the rectangle formed by the ordered pair $(\underline{u}, \bar{u})$. By means of the operator vector field $D + F + G\circ\gamma$ that appears on the left-hand side of (2), and which is well defined on X we introduce the following notion.

Definition 3.2. The vector field $(D + F + G\circ\gamma)u$ is called a *generalized outward pointing vector on the boundary ∂Q of the rectangle Q if for all $\varphi \in V_0 \cap L^2_+(\Omega)$ the following inequalities hold:*

$$\langle -\Delta \underline{u}_1 + F_1(\underline{u}_1, v) + G_1\circ\gamma(\underline{u}_1, v), \varphi\rangle \le 0, \quad \forall\ v \in [\underline{u}_2, \bar{u}_2];$$
$$\langle -\Delta \underline{u}_2 + F_2(v, \underline{u}_2) + G_2\circ\gamma(v, \underline{u}_2), \varphi\rangle \le 0, \quad \forall\ v \in [\underline{u}_1, \bar{u}_1];$$
$$\langle -\Delta \bar{u}_1 + F_1(\bar{u}_1, v) + G_1\circ\gamma(\bar{u}_1, v), \varphi\rangle \ge 0, \quad \forall\ v \in [\underline{u}_2, \bar{u}_2];$$
$$\langle -\Delta \bar{u}_2 + F_2(v, \bar{u}_2) + G_2\circ\gamma(v, \bar{u}_2), \varphi\rangle \ge 0, \quad \forall\ v \in [\underline{u}_1, \bar{u}_1].$$

Definition 3.3. Let $\underline{u},\ \bar{u} \in X$ satisfy $\underline{u} \le \bar{u}$, and $\gamma\underline{u} \le 0 \le \gamma\bar{u}$ on $\partial\Omega\setminus\Gamma$. Then the rectangle $Q = [\underline{u}, \bar{u}]$ is called a *trapping region* for the problem (1) if $(D + F + G\circ\gamma)u$ is a generalized outward pointing vector on the boundary ∂Q of the rectangle Q.

Our main goal is to show that each trapping region for problem (1) contains a solution of (1). This result holds for continuous vector fields f and g under certain growth conditions without assuming any monotonicity assumptions, cf., e.g., [2]. However, for discontinuous vector fields as considered here the existence of solutions may fail even under the monotonicity assumption (M) as can be seen by simple examples, cf., e.g., [5]. This shows that the notion of solution as given by Definition 3.1 is too restrictive. Therefore we extend the notion of solution by introducing multivalued vector fields generated by the discontinuous vector fields f and g, and prove the existence of solutions within a trapping region for problem (1) in its corresponding multivalued setting.

Let $\alpha = (\alpha_1, \alpha_2)$ be the multivalued vector field related with f by

$$\alpha_1(s_1, s_2) = [f_1(s_1{-}0, s_2), f_1(s_1{+}0, s_2)],\ \ \alpha_2(s_1, s_2) = [f_2(s_1, s_2{-}0), f_2(s_1, s_2{+}0)], \tag{3}$$

and let $\beta = (\beta_1, \beta_2)$ denote the analogous multivalued vector field related with g, which one gets from (3) by replacing α by β and f by g. Here $f_1(s_1 \pm 0, s_2)$ and $f_2(s_1, s_2 \pm 0)$

denote the one-sided limits of f_1 and f_2 with respect to their *principal argument*, which is the argument s_k of f_k. Thus α_k and β_k are the maximal monotone graphs with respect to their principal arguments, and instead of problem (1) we consider the multivalued version of it given by

$$\begin{aligned} -\Delta u_k + \alpha_k(u_1,u_2) \ni 0, \quad \text{in} \;\; \Omega, \\ u_k = 0 \;\; \text{on} \;\; \partial\Omega \setminus \Gamma, \quad \frac{\partial u_k}{\partial \nu} + \beta_k(u_1,u_2) \ni 0 \;\; \text{on} \;\; \Gamma, \end{aligned} \tag{4}$$

and define the following notion of solution.

Definition 3.4. The vector function $u \in X_0$ is a solution of (4) if there is an $\xi \in Y$ and an $\eta \in Z$ such that $(k = 1, 2)$

(i) $\xi_k(x) \in \alpha_k(u_1(x), u_2(x))$ for a.e. $x \in \Omega$,

(ii) $\eta_k(x) \in \beta_k(\gamma u_1(x), \gamma u_2(x))$ for a.e. $x \in \partial\Omega$, and

(iii) $\langle -\Delta u_k + \xi_k + \gamma^* \eta_k, \varphi \rangle = 0, \;\; \forall\, \varphi \in V_0$,

hold, where $\gamma^* : L^2(\Gamma) \to V_0^*$ denotes the adjoint operator to γ given by

$$\langle \gamma^* \eta_k, \varphi \rangle := \int_\Gamma \eta_k(x)\, \gamma\varphi(x)\, d\Gamma, \;\; \varphi \in V_0.$$

Next we associate with (4) the following coupled system of discontinuous variational inequalities:

$$\begin{aligned} &\langle -\Delta u_1, \varphi - u_1 \rangle + J_1(\varphi, u_2) - J_1(u_1,u_2) + \Phi_1 \circ \gamma(\varphi, u_2) - \Phi_1 \circ \gamma(u_1,u_2) \geq 0, \\ &\langle -\Delta u_2, \varphi - u_2 \rangle + J_2(u_1, \varphi) - J_2(u_1,u_2) + \Phi_2 \circ \gamma(u_1, \varphi) - \Phi_2 \circ \gamma(u_1,u_2) \geq 0, \end{aligned} \tag{5}$$

where the functionals J_k and Φ_k are defined by

$$\begin{aligned} J_1(u_1,u_2) &:= \int_\Omega \Big(\int_0^{u_1(x)} f_1(s, u_2(x))\, ds \Big)\, dx, \quad u \in Y; \\ J_2(u_1,u_2) &:= \int_\Omega \Big(\int_0^{u_2(x)} f_2(u_1(x), s)\, ds \Big)\, dx, \quad u \in Y; \\ \Phi_1(v_1,v_2) &:= \int_\Gamma \Big(\int_0^{v_1(x)} g_1(s, v_2(x))\, ds \Big)\, d\Gamma, \quad v \in Z; \\ \Phi_2(v_1,v_2) &:= \int_\Gamma \Big(\int_0^{v_2(x)} g_2(v_1(x), s)\, ds \Big)\, d\Gamma, \quad v \in Z. \end{aligned}$$

In view of the growth condition (H) and the monotonicity assumption (M) we get from convex analysis calculus (see e.g. [4]) the following result.

Lemma 3.1. *The functionals $J_k : Y \to \mathbb{R}$ and $\Phi_k : Z \to \mathbb{R}$ are convex and locally Lipschitz continuous with respect to their principal argument. If $\partial_k J_k$ and $\partial_k \Phi_k$ denote the subdifferential with respect to the principal argument, then the following holds:*

$$\begin{aligned} \partial_k J_k(u_1,u_2)(x) = \alpha_k(u_1(x), u_2(x)), \quad u \in Y; \\ \partial_k \Phi_k(v_1,v_2)(x) = \beta_k(v_1(x), v_2(x)), \quad v \in Z. \end{aligned} \tag{6}$$

Moreover, by applying the sum rule for subgradients (cf., e.g., [6, Theorem 47.B]) and the chain rule (cf., e.g.,[5, p. 403]) we obtain for the subdifferential of the sum $J_k + \Phi_k \circ \gamma$ with respect to its principal argument the following result: If $u \in X_0$ then

$$\begin{aligned} \partial_k \Big(J_k(u_1,u_2) + \Phi_k \circ \gamma(u_1,u_2) \Big) &= \partial_k J_k(u_1,u_2) + \partial_k \Phi_k \circ \gamma(u_1,u_2) \\ &= \partial_k J_k(u_1,u_2) + \gamma^* \circ \partial_k \Phi_k(\gamma u_1, \gamma u_2) = \alpha_k(u_1,u_2) + \beta_k(\gamma u_1, \gamma u_2). \end{aligned} \tag{7}$$

The following lemma states the equivalence of (4) and (5).

Lemma 3.2. *The vector function $u \in X_0$ is a solution of the system of inclusions (4) if and only if it is a solution of the system of variational inequalities (5).*

Proof. Let $u \in X_0$ be a solution of (5), which by definition of the subdifferential means that for $k = 1, 2$, we have

$$\Delta u_k \in \partial_k \Big(J_k(u_1, u_2) + \Phi_k \circ \gamma(u_1, u_2) \Big) \ \text{ in } \ V_0^*.$$

By using (6) and (7) there is an $\xi_k \in L^2(\Omega) \subset V_0^*$ with $\xi_k(x) \in \alpha_k(u_1(x), u_2(x))$ and an $\eta_k \in L^2(\Gamma)$ with $\eta_k(x) \in \beta(\gamma u_1(x), \gamma u_2(x))$ and $\gamma^* \eta_k \in V_0^*$ such that $-\Delta u_k + \xi_k + \gamma^* \eta_k = 0$, which shows that u is a solution of (4) according to Definition 3.4. Conversely, if u is a solution of (4) then by Definition 3.4 there exists an $\xi \in Y$ and an $\eta \in Z$ such that $\langle -\Delta u_k + \xi_k + \gamma^* \eta_k, \varphi \rangle = 0$, $\forall\ \varphi \in V_0$, where $\xi_k(x) \in \alpha_k(u_1(x), u_2(x))$, $\eta_k(x) \in \beta_k(\gamma u_1(x), \gamma u_2(x))$. For any $\varphi \in V$ (and thus also for $\varphi \in V_0$) the latter inclusions imply the inequalities (for e.g. $k = 1$)

$$\xi_1(x)(\varphi(x) - u_1(x)) \le \int_{u_1(x)}^{\varphi(x)} f_1(s, u_2(x)\, ds,$$
$$\eta_1(x)(\gamma\varphi(x) - \gamma u_1(x)) \le \int_{\gamma u_1(x)}^{\gamma\varphi(x)} g_1(s, \gamma u_2(x))\, ds,$$

which yields after integrating the first inequality over Ω and the second one over Γ the inequalities

$$\begin{aligned} \int_\Omega \xi_1(\varphi - u_1)\, dx &\le J_1(\varphi, u_2) - J_1(u_1, u_2), \\ \int_\Gamma \eta_1(\gamma\varphi - \gamma u_1)\, d\Gamma &\le \Phi_1(\gamma\varphi, \gamma u_2) - \Phi_1(\gamma u_1, \gamma u_2). \end{aligned} \tag{8}$$

Since inequalities analogous to (8) hold also for ξ_2, η_2 we get from $\langle -\Delta u_k + \xi_k + \gamma^* \eta_k, \varphi \rangle = 0$ with the test functions $\varphi - u_k$ the system of variational inequalities (5). □

In view of Lemma 3.2 we may likewise take the system of variational inequalities (5) as an equivalent substitute for the multivalued version (4) of our original problem (1), of which we will take advantage in the proof of our main result. It should be noted that due to the discontinuous vector fields f and g the coupling functionals in (5) depend discontinuously on their nonprincipal argument, which does not allow the application of standard methods for its solution.

4 MAIN RESULT

In the proof of our main result we make use of the following fixed point theorem in ordered normed spaces, see [3, Proposition 1.1.1].

Lemma 4.1. *Let $[\underline{u}, \bar{u}]$ be a nonempty order interval in an ordered normed space $(N, \le)$, and let $P : [\underline{u}, \bar{u}] \to [\underline{u}, \bar{u}]$ be an increasing mapping, i.e., $v \le w$ implies $Pv \le Pw$. If monotone sequences of $P[\underline{u}, \bar{u}]$ converge weakly or strongly in N, then P has the least fixed point u_* and the greatest fixed point u^* in $[\underline{u}, \bar{u}]$.*

Our main result reads as follows.

Theorem 4.1. *Let $Q = [\underline{u}, \bar{u}] \subset X$ be a trapping region in the sense of Definition 3.3. Then the system of variational inequalities (5) possesses solutions within Q.*

Proof. Let us recall system (5): Find $u = (u_1, u_2) \in X_0$ such that for all $\varphi \in V_0$

$$\begin{aligned} &\langle -\Delta u_1, \varphi - u_1\rangle + J_1(\varphi, u_2) - J_1(u_1, u_2) + \Phi_1 \circ \gamma(\varphi, u_2) - \Phi_1 \circ \gamma(u_1, u_2) \geq 0, \\ &\langle -\Delta u_2, \varphi - u_2\rangle + J_2(u_1, \varphi) - J_2(u_1, u_2) + \Phi_2 \circ \gamma(u_1, \varphi) - \Phi_2 \circ \gamma(u_1, u_2) \geq 0. \end{aligned}$$

We are going to show first that the existence of a solution of (5) in Q is equivalent with the existence of a fixed point of some increasing mapping P defined on the order interval $[\underline{u}_1, \bar{u}_1]$, where $\underline{u}_1$ and $\bar{u}_1$ are the first components of $\underline{u} = (\underline{u}_1, \underline{u}_2)$ and $\bar{u} = (\bar{u}_1, \bar{u}_2)$, respectively. We define an operator $R : [\underline{u}_1, \bar{u}_1] \to V_0$ as follows: $v_1 \mapsto Rv_1 = z$, where z is the uniquely defined solution of the second inequality of (5), i.e.,

$$\langle -\Delta z, \varphi - z\rangle + J_2(v_1, \varphi) - J_2(v_1, z) + \Phi_2 \circ \gamma(v_1, \varphi) - \Phi_2 \circ \gamma(v_1, z) \geq 0. \tag{9}$$

The existence of a unique solution of (9) is an immediate consequence of the strong monotonicity of the operator $-\Delta : V_0 \to V_0^*$ in the sense of Brezis/Browder, and the fact that for fixed v_1 the functional $z \mapsto J_2(v_1, z) + \Phi_2 \circ \gamma(v_1, z)$ is convex and continuous. By using the property of the trapping region and hypothesis (M) we will show that R is decreasing and its range is in $[\underline{u}_2, \bar{u}_2]$. To show $R : [\underline{u}_1, \bar{u}_1] \to [\underline{u}_2, \bar{u}_2]$, let us prove first that $Rv_1 \leq \bar{u}_2$. According to Definitions 3.2 and 3.3 we have for $\bar{u}_2$: $\gamma\bar{u}_2 \geq 0$ on $\partial\Omega \setminus \Gamma$ and

$$\langle -\Delta \bar{u}_2, \varphi\rangle + \int_\Omega f_2(v_1, \bar{u}_2)\varphi\, dx + \int_\Gamma g_2(\gamma v_1, \gamma\bar{u}_2)\gamma\varphi\, d\Gamma \geq 0, \ \forall\, \varphi \in V_0 \cap L^2_+(\Omega). \tag{10}$$

Since $f_2(v_1, \bar{u}_2) \in \alpha_2(v_1, \bar{u}_2)$ and $g_2(\gamma v_1, \gamma\bar{u}_2) \in \beta_2(\gamma v_1, \gamma\bar{u}_2)$ we get similar to (8) for any $\varphi \in V$ the inequalities:

$$\begin{aligned} \int_\Omega f_2(v_1, \bar{u}_2)(\varphi - \bar{u}_2)\, dx &\leq J_2(v_1, \varphi) - J_2(v_1, \bar{u}_2), \\ \int_\Gamma g_2(\gamma v_1, \gamma\bar{u}_2)(\gamma\varphi - \gamma\bar{u}_2)\, d\Gamma &\leq \Phi_2 \circ \gamma(v_1, \varphi) - \Phi_2 \circ \gamma(v_1, \bar{u}_2). \end{aligned} \tag{11}$$

Taking in (11), in particular, $\varphi = \bar{u}_2 + (z - \bar{u}_2)^+$, where $w^+ := \max(w, 0)$, and for (10) the special test function $(z - \bar{u}_2)^+ \in V_0 \cap L^2_+(\Omega)$, we obtain from (10) and (11)

$$\begin{aligned} &\langle -\Delta \bar{u}_2, (z - \bar{u}_2)^+\rangle + J_2(v_1, \bar{u}_2 + (z - \bar{u}_2)^+) - J_2(v_1, \bar{u}_2) \\ &\qquad + \Phi_2 \circ \gamma(v_1, \bar{u}_2 + (z - \bar{u}_2)^+) - \Phi_2 \circ \gamma(v_1, \bar{u}_2) \geq 0. \end{aligned} \tag{12}$$

Finally, taking in (9) the special test function $\varphi = z - (z - \bar{u}_2)^+$ and adding inequalities (9) and (12) yield

$$\begin{aligned} \langle -\Delta(z - \bar{u}_2), (z - \bar{u}_2)^+\rangle &\leq J_2(v_1, \bar{u}_2 + (z - \bar{u}_2)^+) - J_2(v_1, \bar{u}_2) \\ &\quad + J_2(v_1, z - (z - \bar{u}_2)^+) - J_2(v_1, z) \\ &\quad + \Phi_2 \circ \gamma(v_1, \bar{u}_2 + (z - \bar{u}_2)^+) - \Phi_2 \circ \gamma(v_1, \bar{u}_2) \\ &\quad + \Phi_2 \circ \gamma(v_1, z - (z - \bar{u}_2)^+) - \Phi_2 \circ \gamma(v_1, z). \end{aligned} \tag{13}$$

An elementary discussion shows that the right-hand side of (13) is zero, which implies

$$\|\nabla(z-\bar{u}_2)^+\|^2_{L^2(\Omega)} = \langle -\Delta(z-\bar{u}_2), (z-\bar{u}_2)^+\rangle \le 0,$$

and thus $(z-\bar{u}_2)^+ = 0$, i.e., $z \le \bar{u}_2$, since $\|w\|^2_{V_0} = \int_\Omega |\nabla w|^2\,dx$ is an equivalent norm in V_0. The proof for $\underline{u}_2 \le z$ is analogous, which ensures that $R : [\underline{u}_1, \bar{u}_1] \to [\underline{u}_2, \bar{u}_2]$. Based on the variational inequality and using similar special test function arguments one can show that R is also decreasing. On the range of R we define next an operator $S : R([\underline{u}_1, \bar{u}_1]) \to V_0$ as follows: $Rv_1 = z \mapsto Sz = u_1$ where u_1 is the uniquely defined solution of the first variational inequality of (5), i.e.,

$$\langle -\Delta u_1, \varphi - u_1\rangle + J_1(\varphi, z) - J_1(u_1, z) + \Phi_1 \circ \gamma(\varphi, z) - \Phi_1 \circ \gamma(u_1, z) \ge 0. \qquad (14)$$

One can show by similar arguments as above that $S : R([\underline{u}_1, \bar{u}_1]) \to [\underline{u}_1, \bar{u}_1]$ is decreasing. Hence it follows that the composed operator $P = S \circ R : [\underline{u}_1, \bar{u}_1] \to [\underline{u}_1, \bar{u}_1]$ is increasing. To apply the abstract fixed point result given in Lemma 4.1 let $(u_{1,n})$ be a monotone sequence of the image $P([\underline{u}_1, \bar{u}_1])$. Since $u_{1,n} \in [\underline{u}_1, \bar{u}_1]$, then $(u_{1,n})$ and the sequence $(\gamma u_{1,n})$ are bounded in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. Furthermore, by definition the $u_{1,n}$ are solutions of the variational inequality:

$$\langle -\Delta u_{1,n}, \varphi - u_{1,n}\rangle + J_1(\varphi, z_n) - J_1(u_{1,n}, z_n) + \Phi_1 \circ \gamma(\varphi, z_n) - \Phi_1 \circ \gamma(u_{1,n}, z_n) \ge 0, \qquad (15)$$

for some $z_n \in [\underline{u}_2, \bar{u}_2]$, which are also bounded in $L^2(\Omega)$ and its traces are bounded in $L^2(\Gamma)$. With $\varphi = 0$ we get from (15) by using the growth condition (H) that $\|u_{1,n}\|_{V_0} \le c,\ \forall n$, which implies the weak convergence in V_0 of some subsequence. However, by the monotonicity of $(u_{1,n})$ and by Lebesgue's dominated convergence theorem we get its convergence in $L^2(\Omega)$, which finally implies the weak convergence of the entire sequence in V_0 and which ensures the existence of extremal fixed points of P in $[\underline{u}_1, \bar{u}_1]$ by Lemma 4.1. Let u_1 be any fixed point of P, i.e., $u_1 = Pu_1 = S(Ru_1)$, and denote $u_2 = Ru_1 \in [\underline{u}_2, \bar{u}_2]$; then one easily can see that $u = (u_1, u_2) \in Q$ is a solution of the coupled system (5), which completes our proof. □

Remark 4.1. In [5] elliptic systems have been considered without coupling nonlinearities on the boundary, which permitted the application of Tarski's fixed point theorem in their treatment. However, due to the discontinuous coupling vector field g on the boundary considered here Tarski's fixed point theorem is no longer applicable, since the domain of definition of the fixed point operator P, which is the interval $[\underline{u}_1, \bar{u}_1]$ taken as a subset of V is not a complete lattice.

5 APPLICATION

We consider a non-isothermal steady-state model of fluid contaminant transfer, with passive advection induced by a conservative velocity field. The template is a river, flowing with velocity $\vec{v}$, into which a hot contaminant is released by one or more service facilities. The facilities must discharge material when temperature and density exceed threshold values. This constitutes a set of discontinuous flux boundary conditions on a portion Γ of the boundary. Since the inward temperature flux beyond threshold varies according to the product of temperature and density, and the density flux varies according to the density, these boundary conditions are monotone in both arguments. The remaining portion of the boundary, $\partial\Omega \setminus \Gamma$, is assumed neutral, with specified boundary values for temperature and density which vary only with the fluid velocity potential. These are specified later.

Environmental probes are positioned in such a way that control sinks can be activated for certain temperature and density ranges, so as to reduce both the fluid temperature T and the contaminant density ρ. These control activations depend discontinuously upon T and ρ, and act as competitive species vector fields.

If $\vec{v}$ is divergence free, we have the equations (see [1, Sections 1.4–1.5]):

$$\nabla\cdot[-\kappa\nabla T + \vec{v}T] = f(T,\rho),$$
$$\nabla\cdot[-K\nabla\rho + \vec{v}\rho] = g(T,\rho).$$

Here, we have used as constitutive relations Fourier's law for the heat flux and Fick's law for the concentration flux; κ and K are the corresponding thermal and contaminant species diffusivities, assumed constant, and without loss of generality, assumed equal to one. Since the fluid density is a contributory term to κ, we are assuming that variations in ρ do not essentially affect κ. The control source terms are designated by f, g. We assume that $\vec{v} = -\nabla u$. If new variables, v, w are introduced so that $T = \exp(v-u)$ and $\rho = \exp(w-u)$, and the further change of variable $V = \exp v, W = \exp w$ is made, we obtain the system,

$$-\nabla\cdot[e^{-u}\nabla V] + F = 0,$$
$$-\nabla\cdot[e^{-u}\nabla W] + G = 0,$$

where $F = -f, G = -g$. The physical problem dictates a vector field of competitive species, discontinuous in its arguments. The control mechanism is not activated until V, W reach threshold values, and F might typically be taken proportional to the product VW and G proportional to W, resp., after that. The boundary conditions on Γ retain proper monotonicity in V, W. In terms of the variables V, W, we have the Dirichlet boundary conditions on $\partial\Omega\setminus\Gamma$: $V = V_0, W = W_0$, where V_0, W_0 are assumed constant for simplicity. The system is not quite in the form of the theory of section 3. Thus, we make the final change of variable,

$$Y = V - V_0,\ Z = W - W_0.$$

By utilizing the minimum and maximum threshold values for activation of the boundary conditions and vector field controls, we can easily determine a trapping region.

6 REFERENCES

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