

PARTICLE HYDRODYNAMIC MOMENT MODELS IN BIOLOGY AND MICROELECTRONICS: SINGULAR RELAXATION LIMITS

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1. INTRODUCTION

The hydrodynamic model has become important in the simulation of semiconductor devices during the last decade, as an alternative to the more specialized drift-diffusion model. The more comprehensive hydrodynamic model includes features of general carrier heating, velocity overshoot, and various small device features, and is based upon moments of the Boltzmann equation. It employs a macroscopic relaxation time approximation, which incorporates averaged collision mechanisms. Unlike the drift-diffusion model, the hydrodynamic model contains hyperbolic modes related to the momentum subsystem. It is therefore much more complex to analyze and simulate than the corresponding drift-diffusion model. A detailed introduction to both models may be found in the book [9]. In particular, it is demonstrated how an averaging process over group velocity space, coupled with identification of particle density as one such average, and velocity as a first moment, leads to a system whose macroscopic characterization is a conservation law system, much in the tradition of classical gas dynamics, with forcing terms due to electrical, mechanical, and heating effects, as well as frictional dissipation due to collisions. When the microscopic and macroscopic derivations are correlated, the moment closure assumptions emerge more clearly. For example, isotropy leads to the ideal gas law for the carriers. Readers may wish to consult the reference [2] for an in depth survey of some of these ideas. The reader interested in a closer explanation of the engineering aspects of semiconductor devices can consult the book [15].

In recent years, there has been an attempt to understand the relationship between the two classes of models. In the biological context of the open ionic channel, it was noticed phenomenologically that small variations of the saturation velocity in the energy relaxation term led to significant variation in the computed temperature (cf. [3]). Indeed, this interpretation suggests that the temperature tends to that of the isothermal drift-diffusion regime, which is seen as a high friction regime. Some mathematical studies suggest this also (cf. [14]). Another way of viewing high friction is in the framework of the inertial approximation, which here would assert that the drift-diffusion regime (nonisothermal) is the limit of increasingly small *momentum* rate of change, with respect to frequency of collisions. This approach is discussed in [10].

In this paper, we shall study the reduced hydrodynamic model, sometimes referred to as the perturbed isentropic model, with adiabatic pressure density relationship. In fact, for the most general initial conditions, rigorous mathematical results exist only in this case (see [18] for the initial-boundary value problem), though recent studies have extended the models to include multi-species and geometric structure, thus permitting multi-dimensional transport with symmetry to be analyzed (cf. [5]). The paper [5] includes comprehensive simulations based on the shock capturing algorithm, ENO (cf. [16] for description of the algorithm). In particular, [5] analyzes in depth the two-valley

Gallium Arsenide Gunn diode, with stable oscillations for current and voltage; the symmetries of the two-dimensional MESFET are also explored, and the bias is characterized as a symmetry-breaking parameter.

The reduced hydrodynamic model in the case of one carrier type (e.g., electrons) can be written in one spatial dimension as:

$$\rho_t + (\rho v)_x = 0, \quad (1.1)$$

$$(\rho v)_t + (\rho v^2 + p(\rho))_x = \rho \phi_x - \rho v / \tau, \quad (1.2)$$

$$\phi_{xx} = \rho - D(x), \quad (1.3)$$

where ρ is the particle density, v is the velocity, $\tau > 0$ is the momentum relaxation time, and the pressure relation $p = p(\rho)$ has the property that $\rho^2 p'(\rho)$ is a strictly increasing function from $[0, \infty)$ onto $[0, \infty)$. A commonly-used hypothesis [7] is:

$$p(\rho) = k\rho^\gamma, \quad \gamma > 1, \quad k > 0,$$

where γ and k are constants. Equations (1.1) and (1.2) describe conservation of particle number and momentum, respectively, while (1.3) is the Poisson electrostatic equation, with electric potential ϕ and doping D . Moreover, for readers familiar with the physics, we have selected units in which the the charge-mass ratio has been absorbed into the units of ρ , the dielectric constant has been absorbed into the units of ϕ , and the charge modulus has been absorbed into the units of D .

We consider the following initial values with impermeable insulated boundaries for (1.1)–(1.3), respectively:

$$(\rho, v)|_{t=0} = (\rho_0(x), v_0(x)), \quad 0 \leq x \leq 1, \quad (1.4)$$

$$(v, \phi)|_{x=i} = 0, \quad i = 0, 1, \quad t \geq 0. \quad (1.5)$$

The boundary conditions can be generalized routinely by following our approach to construct global smooth solutions.

In Section 2, we establish the existence, uniqueness, and asymptotic decay of global smooth solutions of the problem (1.1)–(1.5) by use of the energy method. Such a method has been successfully applied to the global existence, uniqueness and asymptotic stability of classical smooth solutions for nonlinear wave equations in [11], nonlinear hyperbolic Volterra integrodifferential equations in [8], nonlinear thermoelasticity in [17], as well as for the compressible Navier-Stokes equations in [13] and [4] (see also the references cited in [11,8,17,13,4]). Our result indicates that the relaxation effect prevents the development of shock waves for the case of smooth initial data with small oscillation. In Section 3, the singular behavior of solutions is investigated when the relaxation time tends to zero; the drift-diffusion limit is proved by direct use of the energy estimates. The main difficulty in our energy estimates is that the energy norms must be chosen suitably and estimated uniformly with respect to the relaxation time $\tau > 0$ so that the singular relaxation scaling sequence is compact in a certain sense. Such a limit was also studied in [12], via the compensated compactness method.

Then, in Sections 4 and 5, we extend our analysis to the hydrodynamic energy model, and discuss the existence of global smooth solutions and their singular relaxation limit, in terms of the drift-diffusion equations, when the momentum relaxation time τ_p and the energy relaxation time τ_w tend to zero in a controlled manner. In this limit, the relation of τ_p and τ_w is consistent with the physical temperature dependent expressions derived in [1]. These results are stated without proof in Theorems 4.1 and 5.1, respectively, although we provide the framework for the underlying transformations used in the arguments. Complete proofs and complementary results in this case will appear in [6].

2. GLOBAL EXISTENCE, UNIQUENESS, AND ASYMPTOTIC DECAY

In this section we establish the global existence, uniqueness, and asymptotic decay of classical smooth solutions of the problem (1.1)–(1.5). For concreteness, we assume $\int_0^1 \rho_0(x) dy = 1$ without loss of generality.

THEOREM 2.1. Assume that

$$0 < \delta_0 \leq \rho_0(x) \leq M, \quad v_0(i) = 0, \quad i = 0, 1.$$

Also assume that the following norm,

$$\|(\rho_0 - 1, v_0, D - 1)\|_{H^2(0,1)},$$

and τ are sufficiently small. Then there exists a unique global smooth solution of the problem (1.1)–(1.5) satisfying

$$\|(\rho - 1, \rho_x, \rho_t, v, v_x, v_t)\|_{C[0,1]}(t) \rightarrow 0, \quad \text{when } t \rightarrow \infty.$$

Proof By the Lagrangian coordinates $y(x, t) = \int_0^x \rho(\xi, t) d\xi$, as used in [4], we obtain the following equations in Lagrangian coordinates from (1.1)–(1.3):

$$(1/\rho)_t - v_y = 0, \tag{2.1}$$

$$v_t + p(\rho)_y = \rho \phi_y - v/\tau, \tag{2.2}$$

$$(\rho \phi_y)_y = 1 - \tilde{D}/\rho, \tag{2.3}$$

with the following initial and boundary conditions:

$$(\rho, v)|_{t=0} = (\rho_0(x(y, 0)), v_0(x(y, 0))), \quad 0 \leq y \leq 1, \tag{2.4}$$

$$(v, \phi)|_{x=i} = 0, \quad i = 0, 1, \quad t \geq 0, \tag{2.5}$$

where $\tilde{D}(y, t) = D(x(y, t))$.

Let

$$w_t = v, \quad w_y = 1/\rho - 1. \tag{2.6}$$

Then, the problem (2.1)–(2.5) is transformed into the following form:

$$w_{tt} - \frac{k\gamma w_{yy}}{(1+w_y)^{\gamma+1}} + \frac{w_t}{\tau} + w - \int_0^1 w dy = \int_0^1 (w_y + 1) \left(\int_\eta^y (1 - \tilde{D})(1 + w_y) d\xi \right) d\eta, \tag{2.7}$$

$$w(y, 0) = \int_0^y \left(\frac{1}{\rho_0(x(\eta, 0))} - 1 \right) d\eta, \quad w_t(y, 0) = v_0(x(y, 0)), \tag{2.8}$$

$$w(0, t) = w(1, t) = 0. \tag{2.9}$$

The existence of a local smooth solution $w(x, t) \in C^3([0, 1] \times [0, t_0))$, $t_0 > 0$, of (2.7)–(2.9) has been obtained in [18]. In order to obtain the global smooth solution, it suffices to make *a priori* arguments and estimates so that the local solution can be extended to the global solution.

We first assume that

$$\|(w, w_y)\|_{C^1([0,1] \times [0, t_0))} \leq \varepsilon \tag{2.10}$$

for certain small $\varepsilon \leq 1/2$, which implies

$$1/2 \leq |w_y + 1| \leq 3/2 \quad \iff \quad 2/3 \leq \rho(x, t) \leq 2.$$

Then we prove that there exists an $\epsilon_0 > 0$ depending only on ϵ such that, when

$$\|(\rho_0 - 1, v_0, D - 1)\|_{H^2(0,1)} + \tau \leq \epsilon_0,$$

the estimate (2.10) can be achieved.

With this *a priori* argument, $A = k\gamma/(1 + w_y)^{\gamma+1}$ is a C^∞ function of one variable w_y such that, when $|w_y| \leq \epsilon \leq 1/2$,

$$(2/3)^{\gamma+1}k\gamma \leq A \leq 2^{\gamma+1}k\gamma.$$

We rewrite above equation (2.7) as

$$w_{tt} + \frac{w_t}{\tau} + w - \int_0^1 w dy - Aw_{yy} = f, \quad (2.11)$$

or,

$$w_{tt} + \frac{w_t}{\tau} + w = Aw_{yy} + f + \int_0^1 w dy, \quad (2.12)$$

where

$$f(y, t) = \int_0^1 (w_y + 1) \left(\int_\eta^y (1 - \tilde{D})(1 + w_y) d\xi \right) d\eta. \quad (2.13)$$

Then we have from (2.12),

$$(Aw_{yy} + f + \int_0^1 w dy)(i, t) = 0, \quad i = 0, 1. \quad (2.14)$$

We now make energy estimates step by step. We first perform the following calculations: $(2.11) \times (\tau K w)$, $(2.11) \times (K w_t)$, $(2.11) \times (-\tau w_{yy})$, $(2.11)_t \times (\tau^2 K w_{tt})$, $(2.11)_y \times (K w_{yt})$, $(2.11)_{tt} \times (\tau^4 w_{ttt})$, $(2.11)_{yt} \times (\tau^2 w_{ytt})$, $(2.11)_{yy} \times (\tau(Aw_{yy} + f + \int_0^1 w dy))$, and $(2.11)_{yy} \times (Aw_{yy} + f + \int_0^1 w dy)_t$, and integrate over $[0, 1] \times [0, t]$, where $K = K(\gamma, k) \geq 1$ is determined later. Then we use integration by parts with the aid of the boundary conditions (2.9) and (2.14), and use some elementary inequalities such as

$$ab \leq \delta a^2 + \frac{b^2}{4\delta}, \quad |w| = \int_0^1 |w_y| dy \leq \left(\int_0^1 w_y^2 dy \right)^{1/2}. \quad (2.15)$$

Finally we add and re-group all the resulting inequalities from previous calculations and estimates together, and use $\tau \leq 1/2$ to obtain

$$\begin{aligned} & \int_0^1 [Kw^2 + ((A+1)K+1)w_y^2 + (1+\tau^2)Kw_t^2 + A(K+2)w_{yy}^2 + ((1+\tau^2A)K+\tau^2)w_{yt}^2 + \tau^2Kw_{tt}^2 \\ & \quad + (Aw_{yy}+f)_y^2 + \tau^2Aw_{yyt}^2 + \tau^2(1+\tau^2A)w_{ytt}^2 + \tau^4w_{ttt}^2 + A(w_{yy} + \frac{f + \int_0^1 w dy}{A})_t^2] dy \\ & + \int_0^t \int_0^1 [\tau AKw_y^2 + (1-\tau^2)K\frac{w_t^2}{\tau} + \tau Aw_{yy}^2 + (K-\tau^2)\frac{w_{yt}^2}{\tau} + \tau Kw_{tt}^2 \\ & \quad + \tau(Aw_{yy}+f)_y^2] dy ds + A(1-\tau^2)\frac{w_{yyt}^2}{\tau} + \tau w_{ytt}^2 + \tau^3 w_{ttt}^2 \\ & \leq \int_0^1 [2Kw_0^2 + 2\tau Kw_0 w_{0t} + ((A_0+1)K+1)w_{0y}^2 + (1+\tau^2)Kw_{0t}^2 + A_0(K+2)w_{0yy}^2 \\ & \quad + ((1+\tau^2A_0)K+\tau^2)w_{0yt}^2 + \tau^2(K+\tau^2)w_{0tt}^2 + (A_0w_{0yy}+f_0)_y^2 + \tau^2A_0w_{0yyt}^2 \\ & \quad + \tau^2(1+\tau^2A_0)w_{0ytt}^2 + \tau^4w_{0ttt}^2 + A_0(w_{0yy} + \frac{f_0}{A_0})_t^2 - \tau w_{0t}w_{0yy} + 2\tau^2A_{0t}w_{0yy}w_{0yyt}] dy \end{aligned}$$

$$\begin{aligned}
& +2\tau^2 f_{0t} w_{0yyt} + 2\tau w_{0yyt} (A_0 w_{0yy} + f_0 + \int_0^1 w_0 dy) \\
& + 2\frac{A_{0t}}{A_0^2} (A_0 w_{0yy} + f_0 + \int_0^1 w_0 dy) (A_0 w_{0yyt} + f_{0t} + \int_0^1 w_{0t} dy - \frac{A_{0t}}{A_0} (f_0 + \int_0^1 w_0 dy)) dy \\
& + C \int_0^1 [\tau^2 K w_t^2 + \tau^2 w_{yt}^2 + \tau^2 w_{yy}^2 w_{yt}^2 + \tau^2 f_t^2 + \tau^2 w_{yyt}^2 + \tau |w_{yyt} f| + w_{yt}^2 (A w_{yy} + f + \int_0^1 w dy)^2] dy \\
& + C \int_0^t \int_0^1 [K(\tau w_{yy}^4 + \tau w^2 w_{yy}^2 + w_y^2 |w_{yt}| + \tau w_y^2 w_{yy}^2 + \tau^2 |w_{yt}|^3 + \tau^3 w_{yt}^2 w_{tt}^2 + \tau^2 w_t^2 + \tau f^2 + |f w_{yyt}|) \\
& \quad + \tau w^2 + \frac{1}{\tau} w_t^2 + \tau w_{tt}^2 + \tau w_{yt}^2 + \frac{1}{\tau} (1 + \tau^4 K) w_{yy}^2 w_{yt}^2 + \tau f_{tt}^2 + \tau^3 f_{yt}^2 \\
& \quad + \frac{1}{\tau} (1 + \tau^4 K) f_t^2 + w_{yyt}^2 + |f_t w_{yy}| + \tau^4 w_{yy}^2 w_{ytt}^2 + \tau^5 w_{yy}^2 w_{yt}^4 + \tau w_{yt}^2 w_{yyt}^2 + \tau^3 w_{yyt}^2 w_{yy}^2 \\
& \quad + |w_{yt}| (w_{yy}^2 + \tau^4 w_{ytt}^2 + w_{yyt}^2 + w_{yt}^2 w_{yy}^2 + f_t^2 + \int_0^1 |w_t|^2 dy) \\
& \quad + |w_{yyt}| \int_0^1 |w| dy + |w_{yy}| \int_0^1 |w_t| dy + (f + \int_0^1 w dy)^2 (\tau w_{yt}^4 + \tau w_{ytt}^2 + \frac{1}{\tau} w_{yt}^2) \\
& \quad + |f + \int_0^1 w dy| (\tau w_{ytt}^2 + \tau w_{yt}^4 + \frac{1}{\tau} (w_{yyt}^2 + w_{yt}^2 w_{yy}^2 + f_t^2 + \int_0^1 w_t^2 dy)) \\
& \quad + \frac{1}{\tau} |w_{yy}| (w_{yyt}^2 + w_{yt}^2 w_{yy}^2 + \tau w_{ytt}^2 + \tau w_{yt}^4 + f_t^2 + \int_0^1 w_t^2 dy)] dy ds, \tag{2.16}
\end{aligned}$$

where C is a universal constant independent of τ and t .

Using the elementary inequalities (2.15), taking $K \geq K_0(C, \gamma)$ and $\tau \leq \tau_0(C, \gamma, K)$ for sufficiently large $K_0 \geq 1$ and small $\tau_0 \leq 1/2$, and making tedious calculations, we have from (2.16):

$$\begin{aligned}
& \int_0^1 [K(w^2 + w_t^2 + w_y^2 + \tau^2 w_{tt} + w_{yt}^2 + w_{yy}^2) + \tau^4 w_{ttt}^2 + \tau^2 w_{ytt}^2 + (A w_{yy} + f)_y^2 \\
& \quad + (w_{yy} + \frac{f + \int_0^1 w dy}{A})_t^2] dy \\
& + \int_0^t \int_0^1 [K(\tau w^2 + \tau w_y^2 + \frac{w_t^2}{\tau} + \tau w_{tt}^2 + \frac{w_{yt}^2}{\tau}) + \tau w_{yy}^2 + \tau^3 w_{ttt}^2 + \tau w_{ytt}^2 + \frac{1}{\tau} w_{yyt}^2 \\
& \quad + \tau (A w_{yy} + f)_y^2] dy ds \\
& \leq C(K \|w_0\|_{H^2(0,1)}^2 + \|(\rho_0 - 1, v_0, D - 1)\|_{H^2(0,1)}^2) \\
& \quad + C \int_0^1 [w_{yy}^2 w_{yt}^2 + \tau f^2 + \tau^2 f_t^2 + w_{yt}^2 (f + \int_0^1 w dy)^2] dy \\
& \quad + C \int_0^t \int_0^1 [K(\tau w^2 w_{yy}^2 + w_y^2 |w_{yt}| + \tau w_y^2 w_{yy}^2 + \tau w_{yy}^4 + \tau^2 |w_{yt}|^3 + \tau^3 w_{yt}^2 w_{tt}^2 + \tau f^2) \\
& \quad + w_{yt}^2 (\tau w_{yyt}^2 + |w_{yt}| w_{yy}^2) + |w_{yt}| (f_t^2 + w_{yyt}^2 + \tau^4 w_{ytt}^2) \\
& \quad + w_{yy}^2 (\frac{1}{\tau} w_{yt}^2 + \tau^3 w_{yyt}^2 + \tau^5 w_{ytt}^2 + \tau^5 w_{yt}^4) + |w_{yy}| (|w_{yt} w_{yy}| + \tau w_{ytt}^2 + \tau w_{yt}^4) \\
& \quad + \frac{1}{\tau} |w_{yy}| (w_{yyt}^2 + w_{yt}^2 w_{yy}^2 + f_t^2 + \int_0^1 w_t^2 dy)] dy ds, \tag{2.17}
\end{aligned}$$

where we have used the following identities:

$$\tilde{D}_t = -w_t D_x, \quad \tilde{D}_y = D_x / \rho, \quad \tilde{D}_{tt} = -w_t^2 D_{xx} - w_{tt} D_x,$$

and the following inequalities:

$$\begin{aligned}
|f_y| &\leq |1 - \tilde{D}| + |(\tilde{D} - 1)w_y|, & |f_{yt}| &\leq |w_{yt}(\tilde{D} - 1)| + |(w_y + 1)w_t D_x|, \\
|f_t| &\leq \int_0^1 |f_{yt}| dy + \int_0^1 |w_{yt}| dy \int_0^1 |f_y| dy + \int_0^1 |w_y + 1| dy \int_0^1 |f_{yt}| dy, \\
|f_{tt}| &\leq (1 + \int_0^1 |w_y + 1| dy) \int_0^1 |(w_y + 1)w_t^2 \tilde{D}_{xx} - (2w_{yt}w_t + (w_y + 1)w_{tt})\tilde{D}_x + (\tilde{D} - 1)w_{ytt}| d\eta \\
&\quad + \int_0^1 |w_{ytt}| \int_0^1 |f_y| dy + 2 \int_0^1 |w_{yt}| \int_0^1 |f_{yt}| dy.
\end{aligned}$$

With the assumption (2.10), if ϵ and τ are sufficiently small, we have

$$\begin{aligned}
E(t) + \int_0^t \int_0^1 (\tau w^2 + \tau w_y^2 + \frac{1}{\tau} w_t^2 + \tau w_{yy}^2 + \tau w_{tt}^2 + \frac{1}{\tau} w_{yt}^2 + \tau^3 w_{ttt}^2 + \tau w_{ytt}^2 + \frac{1}{\tau} w_{yyt}^2 + \tau w_{yyy}^2) dy ds \\
\leq C \|(\rho_0 - 1, v_0, D - 1)\|_{H^2(0,1)}^2,
\end{aligned} \tag{2.18}$$

where we have used the *a priori* assumption (2.10) and

$$E(t) \equiv \int_0^1 (w^2 + w_t^2 + w_y^2 + \tau^2 w_{tt}^2 + w_{yt}^2 + w_{yy}^2 + \tau^4 w_{ttt}^2 + \tau^2 w_{ytt}^2 + w_{yyy}^2 + w_{yyt}^2)(y, t) dy.$$

From the local existence and uniqueness theorem, there exists an interval $[0, t_0)$ such that (2.9) has a unique smooth solution. It follows from the Sobolev inequality that

$$\|(w, w_y)\|_{C^1([0,1] \times [0, t_0])} \leq \Gamma \sup_{0 \leq t \leq t_0} \sqrt{E(t)},$$

where Γ is a positive constant. Hence, if

$$\|(\rho_0 - 1, v_0, D - 1)\|_{H^2(0,1)} \leq \frac{\epsilon}{\Gamma \sqrt{C}}, \tag{2.19}$$

then (2.10) holds on $0 \leq t \leq t_0$, $0 \leq y \leq 1$. By *a priori* estimate (2.17), we conclude that $E(t_0) \leq \epsilon/\Gamma$. Then we can apply the above arguments again, and continue the local solution to the global solution in $[0, \infty)$ by use of the estimate (2.17).

The solution constructed by the above process satisfies *a priori* estimate (2.17). It thus follows from the Sobolev inequality that

$$\sup_{\substack{0 \leq t < \infty \\ 0 \leq y \leq 1}} |w_y| \leq \Gamma \sqrt{C} \|(\rho_0 - 1, v_0, D - 1)\|_{H^2(0,1)} \leq \epsilon.$$

Hence,

$$A = \frac{k\gamma}{(1 + w_y)^{\gamma+1}} \geq \left(\frac{3}{2}\right)^{\gamma+1} k\gamma$$

if (2.16) holds. Therefore, we have proved the existence of the global smooth solution satisfying

$$w \in L^2([0, \infty); H^3(0, 1)); \quad \|w\|_{C^2[0,1]}(t) \rightarrow 0, \quad \text{when } t \rightarrow \infty.$$

By (2.6), we obtain

$$\|(\rho - 1, \rho_x, \rho_t, v, v_x, v_t)\|_{C[0,1]}(t) \rightarrow 0, \quad \text{when } t \rightarrow \infty.$$

This completes the proof.

3. SINGULAR RELAXATION APPROACH TO DRIFT-DIFFUSION EQUATIONS

Let

$$\rho^\tau(x, s) = \rho\left(x, \frac{s}{\tau}\right), \quad v^\tau(x, s) = \frac{1}{\tau}v\left(x, \frac{s}{\tau}\right), \quad \phi^\tau(x, s) = \phi\left(x, \frac{s}{\tau}\right), \quad D^\tau(x) = D(x).$$

Then (1.1)–(1.3) is transformed as follows:

$$\rho_s^\tau + (\rho^\tau v^\tau)_x = 0, \tag{3.1}$$

$$\tau^2 v_s^\tau + \tau^2 v^\tau v_x^\tau + \frac{1}{\rho^\tau} p(\rho^\tau)_x + v^\tau = \phi_x^\tau, \tag{3.2}$$

$$\phi_{xx}^\tau = \rho^\tau - D^\tau, \tag{3.3}$$

with the following initial and boundary data:

$$(v^\tau, \phi^\tau)(i, s) = 0, \quad i = 0, 1, \quad s \geq 0, \tag{3.4}$$

$$(\rho^\tau, v^\tau)(x, 0) = (\rho_0(x), v_0(x)/\tau). \tag{3.5}$$

Assume that initial data $(\rho_0(x), v_0(x))$ are independent of τ .

THEOREM 3.1. Assume that the conditions of Theorem 2.1 hold. Let $(\rho^\tau, v^\tau, \phi^\tau)$ be the sequence of solutions of (3.1)–(3.5). Then there exists $N(x, s)$ such that

$$(\rho^\tau, \phi_x^\tau) \rightarrow (N, \int_0^1 \int_\eta^x (N(\xi, s) - D(\xi)) d\xi d\eta) \quad \text{a.e. as } \tau \rightarrow 0.$$

The limit function N satisfies the drift-diffusion equation

$$N_s + \left(N \int_0^1 \int_\eta^x (N(\xi, s) - D(\xi)) d\xi d\eta - p(N)_x \right)_x = 0$$

in the sense of distributions.

Proof To show the convergence of subsequence $\{\rho^\tau, \phi_x^\tau\}$, we need uniform estimates which are independent of τ .

Observe the following relations of the scaling sequence and w :

$$\begin{aligned} \rho^\tau(x, s) &= \frac{1}{w_y(y(x, \frac{s}{\tau}), \frac{s}{\tau}) + 1}, & v^\tau(x, s) &= \frac{1}{\tau} w_t(y(x, \frac{s}{\tau}), \frac{s}{\tau}), \\ \rho_x^\tau(x, s) &= -(\rho^\tau)^2 w_{yy}(y(x, \frac{s}{\tau}), \frac{s}{\tau}), & v_x^\tau(x, s) &= \frac{1}{\tau} \rho^\tau w_{yt}(y(x, \frac{s}{\tau}), \frac{s}{\tau}), \\ \rho_{xx}^\tau(x, s) &= -(\rho^\tau)^3 w_{yyy}(y(x, \frac{s}{\tau}), \frac{s}{\tau}) + 2(\rho^\tau)^3 (w_{yy}(y(x, \frac{s}{\tau}), \frac{s}{\tau}))^2, \\ \phi^\tau(x, s)_x &= \int_0^1 (w_y + 1) \left(\int_\eta^{y(x, \frac{s}{\tau})} (1 - (w_y + 1) \tilde{D}) \left(\xi, \frac{s}{\tau} \right) d\xi \right) d\eta, \\ \phi^\tau(x, s)_{xx} &= \rho^\tau(x, s) - D(x), \\ \phi^\tau(x, s)_{xs} &= -(\rho^\tau - D)v^\tau(x, s) - \int_0^1 \frac{w_{yt}}{\tau} \left(\int_0^y (1 - (w_y + 1) \tilde{D}) \left(\eta, \frac{s}{\tau} \right) d\eta \right) dy \\ &\quad - \int_0^1 (w_y + 1) \left(\int_\eta^y \frac{1}{\tau} (w_{yt} \tilde{D} - \tilde{D}_x (w_y + 1) w_t) \left(\xi, \frac{s}{\tau} \right) d\xi \right) d\eta. \end{aligned}$$

Therefore,

$$\|(\rho^\tau, \phi^\tau, \phi_x^\tau)\|_{L^\infty} + \int_0^\infty \int_0^1 [(v^\tau)^2 + (\rho_s^\tau)^2 + (\rho_x^\tau)^2 + (\phi_{xx}^\tau)^2 + (\phi_{xs}^\tau)^2] dy ds \leq C,$$

where $C > 0$ is independent of τ .

Thus, there exist $N(x, s)$ such that

$$(\rho^\tau, \phi_x^\tau) \rightarrow (N, \Phi_x) \quad \text{a.e. as } \tau \rightarrow 0,$$

where $\Phi_x(x, s) = \int_0^1 \int_\eta^x (N(\xi, s) - D(\xi)) d\xi d\eta$.

From (3.1)–(3.2),

$$\begin{aligned} \int \int (\rho^\tau \psi_s + \rho^\tau v^\tau \psi_x) dx ds &= 0, \\ \tau^2 \int \int (\rho^\tau v^\tau \psi_{xs} + \rho^\tau (v^\tau)^2 \psi_{xx}) dx ds + k \int \int (\rho^\tau)^\gamma \psi_{xx} dx ds &= \int \int (\rho^\tau v^\tau - \rho^\tau \phi_x^\tau) \psi_x dx ds, \end{aligned}$$

for all $\psi \in C_0^\infty(R \times R_+)$. That is, for any test function $\psi(x, t) \in C_0^\infty((0, 1) \times (0, \infty))$,

$$\int \int (\rho^\tau \psi_s + \rho^\tau \phi_x^\tau \psi_x + k(\rho^\tau)^\gamma \psi_{xx}) dx ds = - \int \int (\tau^2 \rho^\tau v^\tau \psi_{xs} + \tau^2 \rho^\tau (v^\tau)^2 \psi_{xx}) dx ds.$$

Let $\tau \rightarrow 0$. Since

$$\|\rho^\tau\|_{L^\infty((0,1) \times (0,\infty))} + \|v^\tau\|_{L^2((0,1) \times (0,\infty))} \leq C,$$

where C is independent of τ and t , then

$$\int \int (N \psi_s + N \Phi_x \psi_x + k N^\gamma \psi_{xx}) dx ds = 0.$$

This completes the proof of Theorem 3.1.

4. GENERALIZATION TO THE HYDRODYNAMIC ENERGY MODEL

In this section we discuss the inviscid hydrodynamic energy model:

$$\rho_t + (\rho v)_x = 0, \tag{4.1}$$

$$(\bar{m} \rho v)_t + (\bar{m} \rho v^2 + \rho T)_x = \bar{e} \rho \phi_x - \bar{m} \rho v / \tau_p, \tag{4.2}$$

$$E_t + (v E + v \rho T)_x - (\kappa T_x)_x = \bar{e} \rho v \phi_x - (E - \frac{3}{2} \rho \bar{T}) / \tau_w, \tag{4.3}$$

$$\phi_{xx} = \bar{e}(\rho - d(x, t)), \tag{4.4}$$

where $\rho(x, t) > 0$, $v(x, t)$, $T(x, t)$, $\phi(x, t)$, $\kappa > 0$, $D(x) \geq D_0 > 0$, $\bar{e} > 0$, \bar{T} , and \bar{m} respectively denote *electron density*, *velocity*, *temperature* in energy units (the Boltzmann constant k_B has been set to 1), *electrostatic potential*, *thermal conductivity*, *doping profile*, *electron charge*, *ambient device temperature*, and *effective electron mass*, and

$$E = \frac{3}{2} \rho T + \frac{1}{2} \bar{m} \rho v^2,$$

is *energy density* for parabolic energy bands. The collision terms here have been treated classically through momentum and energy relaxation times $\tau_p > 0$ and $\tau_w > 0$.

We consider the following initial-boundary conditions:

$$\begin{aligned} (\rho, v, T)|_{t=0} &= (\rho_0(x), \tau_p v_0(x), \bar{T} + \tau_w(T_0(x) - \bar{T})), \quad 0 \leq x \leq 1, \\ (v, T_x, \phi)|_{x=i} &= (0, 0, 0), \quad i = 0, 1, \quad t \geq 0, \end{aligned} \quad (4.5)$$

Such an initial condition is natural for the drift-diffusion limit (see Section 5). For the existence problem, this initial condition can be relaxed by modifying our energy estimates.

Using arguments and estimates as in Sections 2 and 3, we can establish the global existence, uniqueness, and asymptotic decay of solutions of (4.1)–(4.5). We describe here the transformation of [6]. Details of proof are given in this reference.

Similar to Section 2, the full hydrodynamic model in Lagrangian coordinates is of the form:

$$(1/\rho)_t - v_y = 0, \quad (4.6)$$

$$v_t + \frac{1}{\bar{m}}(\rho T)_y = \frac{\bar{e}}{\bar{m}}\rho\phi_y - \frac{v}{\tau_p}, \quad (4.7)$$

$$T_t - \frac{2}{3}(\kappa\rho T_y)_y + \frac{2}{3}\rho T v_y = \frac{\bar{m}(2\tau_w - \tau_p)}{3\tau_p\tau_w}v^2 - \frac{T - \bar{T}}{\tau_w}, \quad (4.8)$$

$$(\rho\phi_y)_y = \bar{e}(1 - \tilde{D}/\rho). \quad (4.9)$$

with the initial and boundary conditions:

$$\begin{aligned} (\rho, v, T)|_{t=0} &= (\rho_0(x(y, 0)), \tau_p v_0(x(y, 0)), \bar{T} + \tau_w(T_0(x(y, 0)) - \bar{T})), \quad 0 \leq y \leq 1, \\ (v, T_y, \phi)|_{x=i} &= (0, 0, 0), \quad i = 0, 1, \quad t \geq 0, \end{aligned} \quad (4.10)$$

where $\tilde{D}(y, t) = D(x(y, t))$ and we have assume for concreteness that $\int_0^1 \rho_0(\xi)d\xi = 1$.

Let

$$w_t = v, \quad w_y = 1/\rho - 1, \quad \theta = T - \bar{T}. \quad (4.11)$$

Then, the system (4.6)–(4.9) is transformed into the following form:

$$\begin{aligned} w_{tt} - \frac{\theta + \bar{T}}{\bar{m}(1 + w_y)^2}w_{yy} + \frac{1}{\bar{m}(1 + w_y)}\theta_y + \frac{w_t}{\tau_p} + (w - \int_0^1 w dy) \\ = \int_0^1 (w_y + 1) \left(\int_\eta^y (1 - \tilde{D})(1 + w_y)d\xi \right) d\eta, \end{aligned} \quad (4.12)$$

$$\theta_t - \frac{2\kappa}{3((1 + w_y))}\theta_{yy} + \frac{2\kappa\theta_y}{3(1 + w_y)^2}w_{yy} + \frac{2(\theta + \bar{T})}{3(1 + w_y)}w_{yt} + \frac{\theta}{\tau_w} - \frac{\bar{m}(2\tau_w - \tau_p)w_t^2}{3\tau_p\tau_w} = 0, \quad (4.13)$$

with the following initial-boundary conditions:

$$(w, w_t, \theta)(y, 0) = \left(\int_0^y \left(\frac{1}{\rho_0(x(\eta, 0))} - 1 \right) d\eta, \tau_p v_0(x(y, 0)), \tau_w(T_0(x(y, 0)) - \bar{T}) \right), \quad 0 \leq y \leq 1, \quad (4.14)$$

$$(w, \theta_y)(i, t) = (0, 0), \quad i = 0, 1, \quad t \geq 0. \quad (4.15)$$

Similar to the arguments of Section 2, we can first apply the contraction mapping theorem to (4.12)–(4.15) to obtain the local smooth solution $(\rho(x, t), v(x, t), T(x, t)), t < t_*$, and then make *a priori* arguments and estimates to extend the local solution to the global smooth solution (see [6]). We then obtain the following theorem.

THEOREM 4.1. Assume that

$$0 < \delta_0 \leq \rho_0(x) \leq M, \quad (v_0, \theta_0)(i) = (0, 0), \text{ for } i = 0, 1, \quad |2\tau_w - \tau_p| \leq M\sqrt{\tau_p\tau_w}.$$

Also assume that the following norms,

$$\|(\rho_0 - 1, v_0, D - 1)\|_{H^2(0,1)}, \quad \|\theta_0\|_{H^3(0,1)},$$

and relaxation times are sufficiently small. Then, (4.1)–(4.4) has a unique global smooth solution $(\rho(x, t), v(x, t), T(x, t))$ satisfying

$$\|(\rho - 1, \rho_x, \rho_t, v, v_x, v_t, T - \bar{T}, T_x, T_t)\|_{C[0,1]}(t) \rightarrow 0 \quad \text{when } t \rightarrow \infty.$$

5. DRIFT-DIFFUSION LIMIT OF THE HYDRODYNAMIC ENERGY MODEL

In this section we study the singular limit to the full model (4.1)–(4.4) under the assumption for relaxation times:

$$0 \leq 2\tau_w - \tau_p \leq M\sqrt{\tau_p\tau_w}.$$

As in Section 3, we first scale the variables (ρ, v, T, ϕ) , and then show that the limit functions, as $\tau_p, \tau_w \rightarrow 0$, satisfy the drift-diffusion equations (see [6]).

Let

$$\begin{aligned} \rho^\tau(x, s) &= \rho\left(x, \frac{s}{\tau_p}\right), \quad v^\tau(x, s) = \frac{1}{\tau_p}v\left(x, \frac{s}{\tau_p}\right), \quad \theta^\tau(x, s) = \frac{1}{\tau_w}\left(T\left(x, \frac{s}{\tau_w}\right) - \bar{T}\right), \\ \phi^\tau(x, s) &= \phi\left(x, \frac{s}{\tau_p}\right), \quad D^\tau(x) = D(x). \end{aligned}$$

Then (4.1)–(4.4) is transformed as follows.

$$\rho_s^\tau + (\rho^\tau v^\tau)_x = 0, \tag{5.1}$$

$$\tau_p^2 v_s^\tau + \tau_p^2 v^\tau v_x^\tau + \frac{1}{\bar{m}\rho^\tau} \left((\tau_w \theta^\tau + \bar{T}) \rho^\tau \right)_x + v^\tau = \frac{\bar{e}}{\bar{m}} \phi_x^\tau, \tag{5.2}$$

$$\tau_w^2 \theta_s^\tau - \frac{2\kappa\tau_w}{3\rho^\tau} \theta_{xx}^\tau + \tau_p \tau_w v^\tau \theta_x^\tau + \frac{2\tau_p}{3} (\tau_w \theta^\tau + \bar{T}) v_x^\tau - \frac{\bar{m}\tau_p(2\tau_w - \tau_p)}{3\tau_w} (v^\tau)^2 + \theta^\tau = 0, \tag{5.3}$$

$$\phi_{xx}^\tau = \bar{e}(\rho^\tau - D^\tau), \tag{5.4}$$

with the following initial and boundary data:

$$(v^\tau, \theta_x^\tau, \phi^\tau)(i, s) = (0, 0, 0), \quad i = 0, 1, \quad s \geq 0, \tag{5.5}$$

$$(\rho^\tau, v^\tau, \theta^\tau)(x, 0) = (\rho_0(x), v_0(x), T_0(x) - \bar{T}), \tag{5.6}$$

We assume that initial data $(\rho_0(x), v_0(x), T_0(x))$ are independent of τ_p and τ_w .

THEOREM 5.1. Assume that the conditions of Theorem 4.1 hold. Let $(\rho^\tau, v^\tau, \theta^\tau, \phi^\tau)$ be the sequence of solutions of (5.1)–(5.6). Then, there exists $N(x, s)$ such that

$$(\rho^\tau, \phi_x^\tau) \rightarrow (N, \int_0^1 \int_\eta^x (N(\xi, s) - D(\xi)) d\xi d\eta) \quad \text{a.e. as } \tau_p, \tau_w \rightarrow 0.$$

The limit function $N(x, t)$ satisfies the drift-diffusion equation

$$N_s + \left(\frac{\bar{e}}{m} N \int_0^1 \int_{\eta}^x (N(\xi, s) - D(\xi)) d\xi d\eta - \frac{\bar{T}}{m} N_x \right)_x = 0$$

in the sense of distributions.

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