WAVE PROPAGATION ON ROTATING COSMIC STRING SPACETIMES

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ABSTRACT. A rotating cosmic string spacetime has a singularity along a timelike curve corresponding to a one-dimensional source of angular momentum. Such spacetimes are not globally hyperbolic: they admit closed timelike curves near the string. This presents challenges to studying the existence of solutions to the wave equation via conventional energy methods. In this work, we show that semi-global forward solutions to the wave equation do nonetheless exist, but only in a microlocal sense. The main ingredient in this existence theorem is a propagation of singularities theorem that relates energy entering the string to energy leaving the string. The propagation theorem is localized in the fibers of a certain fibration of the blown-up string, but global in time, which means that energy entering the string at one time may emerge previously.

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1. Introduction

1.1. **Rotating cosmic string metrics and wave propagation.** Cosmic string spacetimes are cosmological models that feature singularities along timelike curves ("strings"[1]). Starting with work of Kibble [7], physicists have speculated on their formation in the early universe, and detection of these structures, or bounds on their prevalence, remain subjects of active current experimental research [17]. In work of Deser–Jackiw–'t Hooft [2], the simplest cosmic string solutions are viewed as solutions to the Einstein equations in $2+1$ dimensions, with a third spatial dimension along the string quotiented out. Such solutions are, of necessity, flat away from the singularity. The solution corresponding to a static string is then simply the product spacetime given by $\mathbb{R}$ (time variable) times a flat 2d cone. The simplest *rotating string* solution, however, is of a more interesting Lorentzian character, given in cylindrical coordinates by the metric

$$g = (dr^2 + r^2 d\varphi^2) - (dt^2 - 2\mathsf{A}\,dt\,d\varphi + \mathsf{A}^2 d\varphi^2). \tag{1}$$

Here $\mathsf{A} = -4GJ$ where G is the gravitational constant and J is the angular momentum of the string. This metric has two features of unusual interest from the point of view of wave propagation: it is singular at $r=0$, and it admits closed timelike curves, hence is not globally hyperbolic.

The corresponding wave operator $\Box_g$ is given by

$$\begin{aligned} \Box_g &= -\left(1 - \frac{\mathsf{A}^2}{r^2}\right)\partial_t^2 + \Delta + \frac{2\mathsf{A}}{r^2}\partial_t\partial_\varphi \\ &= -\partial_t^2 + r^{-2}(r\partial_r)^2 + r^{-2}(\mathsf{A}\partial_t + \partial_\varphi)^2 \end{aligned} \tag{2}$$

Owing to the absence of global hyperbolicity, we are unable to prove existence of solutions to the wave equation by conventional energy methods. In this paper, we thus resort to proving *microlocal* energy estimates in order to deal with the propagation along rays passing through the string at $r=0$. We then use these estimates to prove the existence of *microlocally forward* solutions to the inhomogeneous equation $\Box_g u = f$ (and perturbations thereof). Such solutions have singularities only along the (asymptotically) forward flowout of the singularities of f, together with the forward flowout of the string itself.

[1] Not to be confused with the superstrings of high energy particle physics.

1.2. **Propagation of singularities.** The singularities of the metric and wave operator give an associated fibration of $\{r = 0\}$ (best regarded as the front face of a real blowup of the string): the helical fibers are integral curves of $\mathsf{A}\partial_t + \partial_\varphi$, or, equivalently, are level sets of $\varphi - t/\mathsf{A} \bmod 2\pi\mathbb{Z}$.

In this paper we study lower-order perturbations of $\Box_g$, with a class including real-valued potentials and Klein–Gordon mass parameters as well as certain magnetic potentials: let Υ be a first order differential operator of the form

$$\Upsilon = f_1\partial_t + f_2\partial_\varphi + f_3 r\partial_r + f_4 \tag{3}$$

where $f_\bullet = f_\bullet(r, t, \varphi)$ are complex-valued smooth functions, with uniform derivative bounds in $\mathbb{R} \times [0, \infty) \times S^1$. Let

$$P = \Box_g + \Upsilon.$$

Standard propagation of singularities results hold along all geodesics not hitting the $r = 0$ singularity of the metric. At $r = 0$, however, new techniques are required to obtain such propagation results, or, equivalently, microlocalized energy estimates. Thus our first result concerns the propagation of regularity through $r = 0$. Since, away from $r = 0$, wavefront set propagates along null bicharacteristics (a.k.a. lightlike geodesics), we are concerned with those null bicharacteristics that reach $r = 0$. Along such bicharacteristics, the t variable is in fact monotone, with $dr/dt = \pm 1$, $d\varphi/dt = 0$ (see Section 2.1 for details). We call such curves "incoming" or "outgoing" according to whether $dr/dt = \mp 1$ (the choice of $-$ corresponds to incoming). Different incoming and outgoing curves hit $r = 0$ at different angles φ, different times t, and with different energies τ (dual variable to t in the cotangent bundle). We let

$$\mathscr{F}_{I/O,\varphi_0,\tau_0}$$

denote the union of these incoming resp. outgoing null bicharacteristics, into or out of a specified fiber with $\varphi_0 \equiv \varphi - t/\mathsf{A}$, and with $\tau = \tau_0$. We denote the union of all incoming/outgoing null bicharacteristics by

$$\mathscr{F}_{I/O} \equiv \bigcup_{\varphi_0,\tau_0} \mathscr{F}_{I/O,\varphi_0,\tau_0}.$$

We define a Sobolev space closely associated with the Dirichlet form of $\Box_g$: let the norm on $\mathcal{H}$ be defined as

$$\|u\|^2_{\mathcal{H}} = \|\partial_r u\|^2_{L^2(X)} + \|r^{-1}(\mathsf{A}\partial_t + \partial_\varphi)u\|^2_{L^2(X)} + \|\partial_t u\|^2_{L^2(X)} + \|\partial_\varphi u\|^2_{L^2(X)} + \|u\|^2_{L^2(X)}.$$

In its simplest form, our propagation of singularities theorem is as follows:

Theorem 1.1. *Let $u \in \mathcal{H}$ near $r = 0$ and let $Pu = 0$. If the wavefront set of u is disjoint from $\mathscr{F}_{I,\varphi_0,\tau_0}$ uniformly in t then u has no wavefront set at $\mathscr{F}_{O,\varphi_0,\tau_0}$, uniformly in t.*

The notion of uniformity used here will be elucidated below; moreover the full statement of the theorem, which is Theorem 8.2 below, involves a notion of wavefront set that is appropriately defined down to $r = 0$ (b-wavefront set relative to $\mathcal{H}$). The full statement of the theorem also includes the inhomogeneous equation $Pu = f$ and deals with a family of (b-)Sobolev-based wavefront sets, with orders of regularity measured relative to $\mathcal{H}$. We can also relax the hypotheses on u to allow for a range of regularities, albeit always measured relative to $\mathcal{H}$.

The theorem can be regarded as an energy estimate: it says that estimates along incoming rays down to $r = 0$ can be propagated outward from $r = 0$ in a manner that preserves the *fibration structure* of the boundary (i.e. is local in φ_0 above) and that preserves the sign of τ, the dual variable to t.

1.3. **Existence of microlocally forward solutions.** Forward solvability of an equation such as $\Box_g u = f$ is a thorny problem. As noted in [15], the single-mode version of this equation is elliptic in the region $\{r < \mathsf{A}\}$. Suppose we could solve, e.g., $\Box_g u = \delta_{t_0,r_0} e^{ik\varphi}$ with $r_0 > \mathsf{A}$ and with u supported in $t > T_0$ for some $T_0 \in \mathbb{R}$. Then by unique continuation for elliptic equations, u would vanish identically in $\{r < \mathsf{A}\}$. By the proof of Lemma 6 of [15] (with the zero RHS of the equation used there replaced by δ), u would then further vanish identically for $r < r_0$, contradicting propagation of singularities in the region $\{r > \mathsf{A}\}$, where the equation is hyperbolic.

Thus we should not, in general, seek solutions supported in any set of the form $\{t > T_0\}$: we can at best hope for a weaker notion of forward solution than that obtained by constraining the support in t.

To put the difficulty differently, we do not have a natural global energy estimate for solutions to $Pu = f$, as the conserved energy associated with the Killing vector field ∂_t has mixed sign once we allow the support of u to overlap $r < \mathsf{A}$. Thus, the microlocal estimate of Theorem 1.1 above is our only available energy estimate near $r = 0$. As energy estimates for adjoint operators usually result in existence results, this does allow us to prove an existence theorem for forward solutions to $Pu = f$, provided we interpret the forward character *microlocally*: we can characterize the wavefront set of the solution u as the forward-in-time flowout of the wavefront set of f within the characteristic set, *together with the forward flowout of the string.* Here "forward" and "backward" still make sense, despite the fact that t is not monotone along the null bicharacteristics which do not reach the string, owing to the fact that asymptotically all bicharacteristics that do not arrive at $r = 0$ escape to the region $r > \mathsf{A}$ where t becomes monotone along the flow (see Section 2.1 for details).

In order to make the hypotheses on the inhomogeneity work uniformly down to $r = 0$, regularity statements here involve the *b-Sobolev spaces* H_b^m, where for m a positive integer, membership in this space means that applying up to m-fold products of the vector fields $r\partial_r, \partial_\varphi, \partial_t$ (with uniformly bounded coefficients) leaves a function in L^2. Let Φ_+ denote the (asymptotically) forward-in-time bicharacteristic flow, over $\{r > 0\}$, and let Σ denote the characteristic set of $\Box_g$.

Theorem 1.2. *Assume that the perturbation* Υ*, given by* (3)*, is analytic or else commutes with both* ∂_t *and* ∂_φ*. Given compact sets* $K_0 \subset K \subset X$ *with* $K_0 \subset K^\circ$*, if* $f \in H_b^m(X)$ *with* $\operatorname{supp} f \subset K_0$*, there exists* $u \in H_b^{m+1}(K^\circ)$ *with*

$$Pu = f \quad \text{on } K^\circ,$$

such that over $K^\circ \cap \{r > 0\}$*,*

$$\mathrm{WF}(u) \setminus \mathrm{WF}(f) \subset \mathscr{F}_O \cup \Phi_+(\mathrm{WF}(f) \cap \Sigma). \tag{4}$$

The solution u *is unique modulo a distribution* w *with* $\mathrm{WF}(w) \subset \mathscr{F}_O$*.*

A more general version of this result that does not entail such strong hypotheses on Υ appears below as Theorem 9.1. Theorem 9.1 also specifies the microlocal regularity of the solution at $r = 0$ and allows for an extension to non-integer Sobolev regularity.

Thus, "forward" solutions exist semi-globally (that is to say, over any desired compact set) for any inhomogeneity f, but the wavefront set of the resulting solutions may contain wavefront set propagating forward in time that emanates from the string (i.e., from $r = 0$) *at times prior to the support of* f. The string thus may emit information about disturbances that are yet to occur.

1.4. **Prior work.** The rotating cosmic string metric was introduced by Deser–Jackiw–'t Hooft in [2], but the literature on the behavior of waves on this background seems to have been little studied. Our investigation of solvability of wave equations on this non-causal background owes a considerable debt to the pioneering work of Bachelot [1], who has obtained a number of results about existence and uniqueness of solutions to the wave equation as well as addressing problems in scattering theory. Bachelot's results do not apply to the metric here owing to the singularity; our focus on forward solutions is likewise a different direction of investigation pursued in [1]. Our emphasis on a *microlocally* causal solution rather than one whose support lies forward of the inhomogeneity is partly inspired by the celebrated discussion of global parametrices in [4, Chapter 6].

The authors' previous work [15] address the problem of obtaining single-mode solutions to $\Box_g u = f$, i.e., solutions of the form $u(t, r)e^{ik\varphi}$. The mode-by-mode equation changes type across the cylinder $r = \mathsf{A}$, and turns out to be of *Tricomi type* and hence amenable to some known microlocal tools, following previous work of Payne [16] as well as the methods of Bachelot [1]. In this reduced equation the singularity at $r = 0$ occurs in the elliptic region, hence we were able to deal with it using relatively standard methods. Here, by contrast, singularities can propagate down to (and through) $r = 0$, and the more sophisticated tools of the b-pseudodifferential calculus are needed.

1.5. **Methods of proof.** The necessary tools for showing microlocal energy estimates that propagate through the string at $\partial X \equiv \{r = 0\}$ are positive commutator methods in an appropriately adapted pseudodifferential calculus. Here we use a version of Melrose's *b-calculus* [12], but with some important modifications. First, the noncompactness of the fibration of ∂X means that we need a calculus with uniform estimates in the noncompact directions. Such a calculus is, fortunately, essentially described already in [6], and in an appendix below we describe the necessary changes and the translation of the results of [6] to our setting; the main properties of the calculus are summarized in Section 3. More seriously, though, the wave operator $\Box_g$ does not lie in this calculus: it has singular terms that we will identify below as squares of *singular edge vector fields*. The relationship between these singular vector fields, characterized below in terms of their tangency to the boundary fibration, and the b-calculus, is discussed in Sections 3 and 4. In the latter section, we introduce a further sub-calculus of the b-calculus, the *fiber-invariant* operators, which have improved commutator properties with the singular edge operators composing $\Box_g$. This is the calculus from which we choose test operators, and with respect to which we define our wavefront set over ∂X.

The estimates needed for the propagation of singularities—or, equivalently, propagation of regularity—are then split into the elliptic estimates at ∂X (Section 7) and the propagation estimate itself (Section 8). These suffice to establish propagation of regularity, globally in the fibers of ∂X. These arguments are parallel to those employed in [11] to establish propagation of singularities for the wave equation on manifolds with edge singularities, but the setup must be modified here owing to the noncompactness of the fibers of the boundary fibration.

In Section 9 we show existence of microlocally forward solutions to the wave equation. Here the essential tool is a variant of the argument used by Duistermaat–Hörmander [4] in solving equations of real principal type: the fact that singularities propagate along rays that escape any compact region allows us to get lower bounds for the adjoint operator acting on compactly supported test functions, which then translates into an existence theorem for the distributional equation. We ensure that the solution is a *forward* one by replacing our original operator by one with a complex absorbing potential, which implies that no singularities can be arriving in the (arbitrary) compact region in which we are trying to solve the equation.

2. Geometric Setting

Let $\mathsf{S} \subset \mathbb{R}^3_{t,x_1,x_2}$ denote the subset $\{x_1 = x_2 = 0,\ t \in \mathbb{R}\}$. Let $X = [\mathbb{R}^3; \mathsf{S}]$ be the 3-dimensional manifold obtained by *blowing up* S, i.e. simply by replacing $x \in \mathbb{R}^2$ with the polar coordinates $(r, \varphi) \in [0, \infty) \times S^1$. We equip the interior X° with the Lorentizan metric given by

$$
\begin{aligned}
g &= -dt^2 + (r^2 - \mathsf{A}^2)d\varphi^2 + 2\mathsf{A}dtd\varphi + dr^2 \\
&= -(dt - \mathsf{A}d\varphi)^2 + r^2 d\varphi^2 + dr^2.
\end{aligned}
\tag{5}
$$

Note that ∂X is naturally equipped with a fibration compatible with the metric: define the (complex) vector field

$$\mathsf{F} \equiv i^{-1}(\mathsf{A}\partial_t + \partial_\varphi),$$

and that the integral curves of the real vector field $i\mathsf{F}$ at $r = 0$ are the (helical) fibers of a fibration of the cylinder

$$\partial X = \{r = 0, \varphi \in S^1, t \in \mathbb{R}.\}$$

Let π_0 denote the projection map $\partial X \to S^1$ given (somewhat non-canonically) by

$$\pi_0 \colon (r = 0, \varphi, t) \mapsto \varphi - t/\mathsf{A} \bmod 2\pi\mathbb{Z}. \tag{6}$$

Thus π_0 maps each point in ∂X to the point in the same fiber over $\{t = 0\}$. The rotating cosmic string metric then takes the special form $g = \pi_0^* g_0 + r^2 h$ where h is a symmetric two-form in t, φ. In particular the only nontrival components in the fiber directions are $O(r^2)$.

2.1. **Geometry of bicharacteristics.** The metric g is not globally hyperbolic: the parametrized closed curve

$$\big\{(r = r_0, \varphi = s, t = t_0) : s \in [0, 2\pi]\big\}$$

is timelike if $r_0 < \mathsf{A}$. There are, however, no closed causal geodesics, as will become clear below.

Using usual dual variables in the cotangent bundle (which will be replaced later on with the better-adapted fiber variables in the b-cotangent bundle), the symbol of $\Box_g$ is

$$p = \tau^2 - \xi^2 - \frac{(\mathsf{A}\tau + \eta)^2}{r^2},$$

and has Hamilton vector field

$$\big(2\tau - 2r^{-2}\mathsf{A}(\mathsf{A}\tau + \eta)\big)\partial_t - 2\xi\partial_r - 2r^{-2}(\mathsf{A}\tau + \eta)\partial_\varphi - 2r^{-3}(\mathsf{A}\tau + \eta)^2\partial_\xi.$$

On the characteristic set $\{p = 0\}$, the coefficient of ∂_t has fixed sign as τ as long as $r > |\mathsf{A}|$. Thus the t variable is monotone on the null bicharacteristics as long as they remain in $r > |\mathsf{A}|$.

Only special bicharacteristics reach $r = 0$, just as would be the case for the Minkowski metric in cylindrical coordinates. In Minkowski space, the necessary and sufficient condition would be vanishing of angular momentum, but here it is the condition

$$\mathsf{A}\tau + \eta = 0,$$

which is manifestly conserved along the flow. The time variable is monotone along each of these curves, with $\dot{t} = 2\tau$; moreover $dr/dt = -\xi/\tau = \pm 1$ on the characteristic set.

That the metric is in fact flat away from $r = 0$ is easily seen via the (local in φ) change of variables

$$t' \equiv t - \mathsf{A}\varphi,$$

which reduces the metric to the Minkowski metric

$$g = dr^2 + r^2 d\varphi^2 - dt'^2;$$

projections of null bicharacteristics then become (forward and backward) Minkowski geodesics, expressed in cylindrical coordinates

$$(r\cos\varphi, r\sin\varphi, t') = (x_0 + v_x s, y_0 + v_y s, t_0 \pm s), \tag{7}$$

with $v_x^2 + v_y^2 = 1$. (Note that such a bicharacteristic stays within the coordinate patch we have introduced here, effectively by introducing a branch cut in the xy-plane, since φ asymptotically increments by $\pm\pi$ under the flow along a Minkowski null-bicharacteristic.) In these coordinates, then, it is trivial to see that every null bicharacteristic (except those hitting $r = 0$, which we exclude from discussion for now) escapes the region $\{r \leq |\mathsf{A}|\}$, hence, returning to our original coordinate system, we see that our original t variable is eventually monotone at both ends of every null bicharacteristic curve, with $\dot{t}$ having consistent sign at both ends. We can thus orient each bicharacteristic curve in a direction that makes $\dot{t}$ positive on both ends.

The non-monotocity of t is moreover limited: the coordinate t' is monotone owing to the Minkowski geometry (7). Since $t = t' + \mathsf{A}\varphi$, letting $t(s)$, $t'(s)$ and $\varphi(s)$ denote the values along the bicharacteristic shows that

$$t(s) - t(0) = t'(s) - t'(0) + \mathsf{A}(\varphi(s) - \varphi(0));$$

since, as noted above, the variation in φ along a null geodesic is $\pm\pi$, if we choose signs such that $\dot{t} = +1$ asymptotically (i.e., $t(s) \to +\infty$ as $s \to +\infty$), then $t(s) - t(0)$ can never be less than $-|\mathsf{A}|\pi$.

A geodesic aimed nearly at the string ($r = 0$) shows the non-monotonicity of t most strikingly. As before we can choose the sign of our parametrization to arrange $t'(s) - t'(0) = s$. If the geodesic passes very close to $r = 0$ then consideration of lines in $\mathbb{R}^2$ shows that φ is approximately constant except at the moment when it passes by $r = 0$—without loss of generality, at time $s = 0$—when it rapidly increments or decrements (depending on whether it leaves the string to the left or to the right) by $\pi - \epsilon$ in a short interval $s \in [-\delta, \delta]$. Thus,

$$\begin{aligned} t(\delta) - t(-\delta) &= t'(\delta) - t'(-\delta) + \mathsf{A}(\varphi(\delta) - \varphi(-\delta)) \\ &= 2\delta \pm \mathsf{A}(\pi - \epsilon) \\ &\approx \pm\mathsf{A}\pi. \end{aligned}$$

Hence in the limit in which such geodesics pass through the spatial origin $r = 0$, the time variable instantaneously increments or decrements by $\mathsf{A}\pi$ at the moment of interaction with the string, and is otherwise continuous and monotone increasing.

3. b-Geometry and the b-Calculus

In this section, we describe the geometric setting of the "b-category" and its associated analytic objects as espoused by Melrose in, e.g., [12] (but with a slight complication in its application in the case at hand involving uniformity as $|t| \to \infty$).

The set of b-vectors fields on X is:

$$\mathcal{V}_b(X) = \{\text{smooth vector fields tangent to } \partial X\}.$$

Thus,

$$\mathcal{V}_b(X) = \mathcal{C}^\infty(X)\text{-span}(r\partial_r, \partial_t, \partial_\varphi).$$

We then define the b-differential operators as sums of products of these vector fields:

$$\mathrm{Diff}_b^k(X) = \left\{ \sum_{l,k'(l)\le k} a_l(r,t,\varphi) V_{1,l} \dots V_{k'(l),l}, \quad a_l \in \mathcal{C}^\infty, V_\bullet \in \mathcal{V}_b(X) \right\}.$$

Owing to the noncompactness of X, however, we will employ versions of these spaces involving uniform estimates: let $\mathcal{C}_u^\infty(X)$ denote the space of $\mathcal{C}^\infty$ functions of r, t, φ with uniform derivative bounds:

$$f \in \mathcal{C}_u^\infty \iff \partial_r^i \partial_t^j \partial_\varphi^k f \in L^\infty(X) \text{ for all } i, j, k \in \mathbb{N}.$$

Let

$$\mathcal{V}_{bu}(X) = \mathcal{C}_u^\infty(X)\text{-span}(r\partial_r, \partial_t, \partial_\varphi),$$

and

$$\mathrm{Diff}_{bu}^k(X) = \left\{ \sum_{l,k'(l)\le k} a_l(r,t,\varphi) V_{1,l} \dots V_{k'(l),l}, \quad a_l \in \mathcal{C}_u^\infty, V_\bullet \in \mathcal{V}_{bu}(X) \right\}.$$

We also require a weighted version of this space, denoted $r^\ell \,\mathrm{Diff}_{bu}^m(M)$.

The vector fields in $\mathcal{V}_{bu}(X)$ are sections of a vector bundle, denoted ${}^bTX$. The dual bundle, denoted ${}^bT^*X$, is the bundle whose local sections are 1-forms spanned over $\mathcal{C}^\infty(X)$ by dr/r, dt, $d\varphi$. The *principal symbol* of a b-differential operator is a polynomial function on ${}^bT^*X$: if

$$A = \sum_{i+j+k\le m} a_{ijk}(r,t,\varphi)(rD_r)^i D_t^j D_\varphi^k,$$

$$\sigma_b^m(A) = \sum_{i+j+k=m} a_{ijk}(r,t,\varphi)\xi^i \tau^j \eta^k$$

where the variables ξ, τ, η are defined by writing the canonical one-form on ${}^bT^*X$ as

$$\xi\frac{dr}{r} + \tau dt + \eta d\varphi.$$

Here and throughout the rest of the paper we use ξ, τ, and η to be fiber variables in the b-cotangent bundle. We are also employing the usual convention in microlocal analysis that

$$D_z = i^{-1}\partial_z$$

for any coordinate z.

Note that there is a symplectic form on ${}^bT^*X^\circ$ defined by the differential of the canonical one-form

$$\omega \equiv d\xi \wedge \frac{dr}{r} + d\tau \wedge dt + d\eta \wedge d\varphi. \tag{8}$$

hence there is an associated notion of Hamilton vector fields.

We will have occasion to employ homogeneous coordinates on ${}^bT^*X$ in making symbol constructions; to this end, set

$$\hat{\xi} = \frac{\xi}{|\tau|},\ \hat{\eta} = \frac{\eta}{|\tau|}, \hat{\tau} = \frac{\tau}{|\tau|} \tag{9}$$

The coordinates $\hat{\xi}$, $\hat{\eta}$, together with τ itself, are a coordinate system in the fibers except at $\tau = 0$; as we will see below, this set is disjoint from the characteristic set of $\Box_g$. The function $\hat{\tau}$ is of course just ± 1-valued, but is useful notation nonetheless.

We can write $\Box_g$ with respect to the b-vector fields as

$$\Box_g = -\left(1 - \frac{\mathsf{A}^2}{r^2}\right)\partial_t^2 + \frac{1}{r^2}\big((r\partial_r)^2 + \partial_\varphi^2\big) + \frac{2\mathsf{A}}{r^2}\partial_t\partial_\varphi \in r^{-2}\operatorname{Diff}^2_{bu}(X)$$

so that the principal symbol associated to $\Box_g$ is given by

$$p(t, r, \varphi, \tau, \xi, \eta) = \tau^2 - r^{-2}\xi^2 - r^{-2}\left(\mathsf{A}\tau + \eta\right)^2. \tag{10}$$

Thus the characteristic set is $\Sigma = \{r^2\tau^2 = \xi^2 + (\mathsf{A}\tau + \eta)^2\}$. Over the boundary $\{r = 0\}$ we have $\Sigma|_{\{r=0\}} = \{\xi = \mathsf{A}\tau + \eta = 0\}$.

The b-Hamilton vector field of p is given by

$$\begin{aligned} {}^bH_p &= \partial_\tau p\, \partial_t + r\partial_\xi p\, \partial_r + \partial_\eta p\, \partial_\varphi - \partial_t p\, \partial_\tau - r\partial_r p\, \partial_\xi - \partial_\varphi p\, \partial_\eta \\ &= \big(2\tau - \frac{2\mathsf{A}}{r^2}(\mathsf{A}\tau + \eta)\big)\partial_t - \frac{2\xi}{r}\partial_r - \frac{2}{r^2}(\mathsf{A}\tau + \eta)\partial_\varphi - 2\left(\frac{\xi^2 + (\mathsf{A}\tau + \eta)^2}{r^2}\right)\partial_\xi, \end{aligned}$$

which yields the (rescaled) b-Hamilton vector field, itself in $\mathcal{V}_b({}^bT^*X)$, given by

$$\frac{r^2}{2}\ {}^bH_p = (r^2\tau - \mathsf{A}(\mathsf{A}\tau + \eta))\partial_t - \xi r\partial_r - (\mathsf{A}\tau + \eta)\partial_\varphi - (\xi^2 + (\mathsf{A}\tau + \eta)^2)\partial_\xi. \tag{11}$$

Let

$${}^b\Phi(s) \equiv \exp(s(r^2/2){}^bH_p)$$

denote the flow generated by this rescaled vector field.

Looking at $\frac{r^2}{2}\ {}^bH_p$ over the characteristic set at $r = 0$, we see $\frac{r^2}{2}{}^bH_p|_{\Sigma|_{\{r=0\}}} = 0$ so we cannot expect to obtain any propagation at $r = 0$ by invocation of standard propagation of singularities results.

Remark 3.1. If a null bicharacteristic, i.e., an integral curve of $(r^2/2)^bH_p$ inside the characteristic set, parametrized by s, approaches $r = 0$ as $s \to \pm\infty$ then certainly $\mathsf{A}\tau + \eta = 0$ since p, τ, η are all conserved. Conversely, if $\mathsf{A}\tau + \eta = 0$ then solving in r, ξ shows that in fact $r \to 0$ as $s \to \pm\infty$ (with the $\pm$ determined by the signs of ξ and τ). Thus $\mathsf{A}\tau + \eta = 0$ exactly defines the flowout of $r = 0$ inside Σ.

3.1. **Uniform b-calculus.** The b-calculus of pseudodifferential operators is a microlocalization of the b-differential operators. Here we exclusively employ a *uniform* version of the b-calculus adapted to the noncompact but quasi-periodic setting under study.

A function a on ${}^bT^*X$ is in the symbol class $S^m_u({}^bT^*X)$ if for all multi-indices α, β and integers $N \geq 0$ it satisfies estimates

$$\left|\partial^\alpha_{r,t,\varphi}\partial^\beta_{\xi,\tau,\eta} a\right| \leq C\langle(\xi,\tau,\eta)\rangle^{m-|\beta|}(1+r)^{-N}, \tag{12}$$

where the estimate is assumed to hold over all of ${}^bT^*X$, noncompactness of X notwithstanding. In particular, the estimate is to be uniform as $|t| \to \infty$ (the radial variable in practice will only range over a compact set).

In this section we outline the development of the b-pseudodifferential calculus built by quantizing symbols in $S^m_u({}^bT^*X)$. A more detailed account can be found in Appendix A, where we follow an alternate development of the calculus from the slightly different point of view used in [6], which is in fact equivalent to the account given here (as proved in Proposition A.6).

We may quantize symbols in $S^m_u({}^bT^*X)$ to obtain operators with the following Schwartz kernel (Schwartz kernel of an operator is denoted $\kappa(\bullet)$):

$$\begin{aligned} &\kappa(\widetilde{\mathrm{Op}}_{\mathrm{b}}(a)) \\ &\quad = \int e^{i[(r-r')\xi/r + (t-t')\tau + (\varphi-\varphi')\eta]}\chi a(r,t,\varphi,\xi,\tau,\eta)\frac{r'}{r}\, d\xi d\tau d\eta |dt'\, d\varphi'\, dr'/r'|. \end{aligned} \tag{13}$$

Here we have included a half density factor that is both appropriate to the geometry of the half-line and also which arises naturally from the construction of the b-calculus by quantization of certain singular symbols in [6] discussed below; the function χ is a cutoff in the radial variables: $\chi = \chi(r'/r)$, where $\chi(s)$ is supported near $s = 1$. The convention here is that functions to which this operator is applied should be viewed as 2π-periodic functions in the φ variable, and the symbol a is likewise periodic in φ. The application of the operator to a function is then integration over $\mathbb{R}_\varphi$, with the result being again periodic (cf. [20, Section 5.3.1]).

We have employed the notation $\widetilde{\mathrm{Op}}_{\mathrm{b}}$ to distinguish the quantization used here from that in Appendix A, which may only be applied to a certain subclass class of "lacunary" symbols.

Remark 3.2. We recall from [12, Chapter 4] that it is illuminating to view the Schwartz kernels of these operators in the "b-double-space" obtained from real blowup $X^2_b = [X \times X; (\partial X)^2]$, which here corresponds to just replacing r, r' variables in $[0,\infty)^2$ with polar coordinates in the quarter-plane; equivalently we may use the simpler substitutes for polar coordinates given by

$$(\rho, \theta) \equiv \big(r + r', \frac{r - r'}{r + r'}\big)$$

with $\rho \in [0, \infty)$ and $\theta \in [-1, 1]$. It will often be valuable to use the even simpler coordinates r and $s \equiv r'/r$, but then we must recall that as $\theta \to -1$, $s \to +\infty$, hence these coordinates are not quite global. The set $\{\rho = 0\}$ is the "front face" of the blowup, the new boundary face that we have introduced to replace the corner $r = r' = 0$; the "side faces" are now $\theta = \pm 1$, a.k.a., $s = 0$ resp. $s = +\infty$. The operators in the usual b-calculus are those whose Schwartz kernels are conormal to the lift of the diagonal to this space, smooth up to the front face, and rapidly decaying at the side faces. The operators considered here have the further feature of enjoying uniform estimates in the noncompact boundary variables t, t'.

Operators of the form (13) are not quite a rich enough class to form a calculus closed under composition, owing to the presence of the cutoff χ. In general, composition causes the support of Schwartz kernels to spread transverse to the diagonal. Thus in order to make the calculus closed under composition, we need to allow residual operators that have Schwartz behavior in this transverse direction.

Definition 3.3. *We say an operator R is residual if its Schwartz kernel satisfies the following estimate in coordinates on X_b^2 given by $\rho = r + r'$, $\theta = (r - r')/(r + r')$: for all α, β, γ, N,*

$$\left|\partial^{\alpha}_{t,\varphi,t',\varphi'} \partial^{\beta}_{\rho} \partial^{\gamma}_{\theta} \rho \kappa(R)\right| \leq C_{\alpha,\beta,\gamma,N} (1 + |t - t'| + \rho)^{-N} (1 - \theta^2)^N.$$

Note that the $(1 - \theta^2)^N$ factor has the effect of enforcing rapid decay of the kernel on the side-faces ($\theta = \pm 1$) of the resolved space X_b^2.

With our definition of residual operators in hand, we may define our uniform b-pseudodifferential operators.

Definition 3.4. *An operator A is in Ψ^m_{bu} if $A = A_0 + R$ for some $A_0 = \widetilde{\mathrm{Op}}_{\mathrm{b}}(a)$ with $a \in S^m_u({}^bT^*X)$ and R a residual operator.*

That this is equivalent to the different definition introduced in Appendix A is the content of Proposition A.6.

Proposition 3.5. *The space $\Psi^{\star}_{bu}(X)$ is a calculus, i.e., a filtered $\star$-algebra, and $\Psi^0_{bu}(X)$ is bounded on $L^2(X)$. There is a principal symbol map $\sigma^s_b : \Psi^s_{bu}(X) \to S^s_u(X)/S^{s-1}_u(X)$ that yields a short exact sequence*

$$0 \to \Psi^{s-1}_{bu}(X) \overset{incl.}{\to} \Psi^s_{bu}(X) \overset{\sigma^s_b}{\to} S^s_u(X) \to 0.$$

Furthermore if $A \in \Psi^s_{bu}(X)$, $B \in \Psi^k_{bu}(X)$, then $[A, B] \in \Psi^{s+k-1}_{bu}$ satisfies

$$\sigma^{s+k-1}_b([A, B]) = -i \left\{\sigma^s_b(A), \sigma^k_b(B)\right\}$$

with $\{\cdot, \cdot\}$ the Poisson bracket defined using the symplectic form (8).

The proof of this proposition is the content of Appendix A; essentially the whole result is obtainable from results in [6, Chapter 18.3].

The principal symbol map of the b-calculus is the obstruction to an operator being lower order, but not the complete obstruction to compactness, which is also governed by behavior at $r = 0$. for a b-differential operator

$$A = \sum a_{ijk}(r, t, \varphi)(rD_r)^i D_t^j D_\varphi^k$$

we can create a scaling (in r)-invariant operator by freezing coefficients at $r = 0$; this is the *indicial operator*

$$I(A) \equiv \sum a_{ijk}(0, t, \varphi)(rD_r)^i D_t^j D_\varphi^k.$$

In [12, Section 4.15], the extension of this operator to b-pseudodifferential is discussed, and this same discussion applies with no change here: there is a map

$$I : \Psi_{bu}^s(X) \to \Psi_{bu,I}^s(X)$$

with the latter being the space of operators in the calculus that commute with scaling in the r variable. The essential features here are as follows:

Proposition 3.6. *I is an algebra homomorphism, with the property that for $A \in \Psi_{bu}^s(X)$,*

$$I(A) = 0 \iff A \in r\Psi_{bu}^s(X) \iff A \in \Psi_{bu}^s(X)r,$$

We record some useful consequences involving commutators with r and rD_r.

Lemma 3.7. *Let $A \in \Psi_{bu}^s(X)$. Then*

$$[r, A] \in r\Psi_{bu}^{s-1}(X) = \Psi_{bu}^{s-1}(X)r, \tag{14}$$

$$r^{-1}Ar,\ rAr^{-1} \in \Psi_{bu}^s(X), \quad \sigma_b^s(r^{-1}Ar) = \sigma_b^s(rAr^{-1}) = \sigma_b^s(A). \tag{15}$$

$$[rD_r, A] \in r\Psi_{bu}^{s-1}(X). \tag{16}$$

Proof. Since $I(r) = 0$, (14) follows immediately. Then (15) follows since, e.g., $r^{-1}Ar = A - r^{-1}[r, A]$. Finally, (16) holds since $I(rD_r) = rD_r$ commutes with $I(A)$ by scaling invariance of the latter. □

Remark 3.8. We will need notions of operator wavefront set (a.k.a. microsupport) and the related wavefront set of distributions associated with the b-calculus. The uniform versions of these notions that we need, however, are slighly unsatisfactory, as uniform estimates on a symbol (with rapid fiber decay) in a *noncompact* open conic set $\Omega \subset {}^bT^*X$ are of course not equivalent to the validity of the estimate locally in a conic neighborhood of each point in Ω. Without the addition of some notion of "microsupport at infinity," therefore, it would not suffice for us to define the microsupport as a set. We therefore postpone discussions of wavefront sets to the discussion of the *fiber-invariant calculus* below, where taking fiber quotients will supply the desired uniform estimates in a more satisfying way.

3.2. **Edge structure.** A manifold M with boundary is endowed with an edge structure if the boundary ∂M admits a fibration:

$$Z \to \partial M \xrightarrow{\pi_0} Y$$

with fiber Z and base Y (see [8]). The *edge vector fields* are defined to be those vector fields that are tangent to ∂M and additionally to the fibers within ∂M. In coordinates adapted to the fibration, with x a boundary defining function and (y, z) coordinates on ∂M associated to a local trivialization of the fibration so that the y variables are constant on fibers, we have

$$\mathcal{V}_e(M) = \mathcal{C}^\infty(M)\text{-span}(x\partial_x, x\partial_y, \partial_z)$$

The edge vector fields are the smooth sections of the edge tangent bundle, which we denote ${}^eTM$. The edge cotangent bundle, denoted ${}^eT^*M$, is then defined to be dual to ${}^eTM$, with dual one forms $\frac{dx}{x}$, $\frac{dy}{x}$, and dz.

Our setting has an edge structure given by the fibration tangent to the vector field F; here the leaves are all diffeomorphic to $\mathbb{R}$ and the leaf space is smooth and may be identified with S^1 by the map π_0 in (6). This map is of course somewhat non-canonical: it could be regarded as taking the unique point in the circle in each leaf at time $t = 0$. When possible we will employ the more invariant terminology

$$p \sim q$$

to mean that $p, q \in \partial X$ are in the same leaf.

In coordinates, we then obtain edge forms and vector fields as follows. Set $x = r, y = t - \mathsf{A}\varphi$, and $z = \varphi$. Then the sections of ${}^eTX$ are the $\mathcal{C}^\infty(X)$ span of

$$r\partial_r, \quad r\partial_t, \quad \text{and } \mathsf{A}\partial_t + \partial_\varphi.$$

The canonical one form on ${}^eT^*X$ is then

$$\sigma = \xi_e \frac{dr}{r} + \tau_e \frac{dt - \mathsf{A}d\varphi}{r} + \eta_e d\varphi. \tag{17}$$

We remark that we could alternatively have used

$$x = r, \ y = t - \mathsf{A}\varphi, \ z = t$$

as our coordinates, i.e., we could have used t as a fiber coordinate rather than φ. (This has the virtue that the noncompactness of the helical fibers is more immediately apparent, but doesn't make much difference.) With these choices, the edge vector fields become

$$r\partial_r, \ -\frac{r}{\mathsf{A}}\partial_\varphi, \ \partial_t + \frac{1}{\mathsf{A}}\partial\varphi,$$

which are easily seen to give an alternative basis for the edge vector fields defined above.

In this paper the edge structure is important for understanding the elliptic set over the boundary for $\Box_g$, which can be regarded as a weighted edge differential operator, but the "edge calculus" of Mazzeo [8] plays no direct role here.

3.3. **Edge/b relationship.** It is of considerable motivational and practical interest to understand the relationship between the b and edge cotangent bundles and characteristic sets.

We define the map $\pi : {}^eT^*X \to {}^bT^*X$ via the inclusion map $r^eT^*X \hookrightarrow {}^bT^*X$. That is, for $q \in {}^eT^*X$ with coordinates given by the canonical one-form (17), we have $rq = \xi_e dr + \tau_e(dt - \mathsf{A}d\varphi) + r\eta_e d\varphi$ which can be written as an element of ${}^bT^*X$ as $(r\xi_e)\frac{dr}{r} + \tau_e dt + (r\eta_e - \tau_e\mathsf{A})d\varphi$. Thus

$$\pi : (r, t, \varphi, \xi_e, \tau_e, \eta_e) \mapsto (r, t, \varphi, r\xi_e, \tau_e, r\eta_e - \mathsf{A}\tau_e) = (r, t, \varphi, \xi, \tau, \eta).$$

Note that for $q \in \pi({}^eT^*X)$ we have

$$\xi = \eta + \mathsf{A}\tau = 0 \tag{18}$$

at the boundary $\{r = 0\}$.

We define a further map which combines π with the quotient by fibers. First, note that the screw-displacement flow along the vector field $i\mathsf{F}$, extended trivially to the interior of X (the choice of extension turns out to be irrelevant) preserves the fibration of ∂X, hence gives an identification of ${}^eT^*_pX$ and ${}^eT^*_qX$ whenever $p, q \in \partial X$ with $p \sim q$; it also identifies ${}^bT^*_pX$ and ${}^bT^*_qX$, and commutes with the map π, so that we gain an equivalence relation (also denoted $\sim$) on $\pi({}^eT^*_{\partial X}X)$.

Concretely, we may employ (t, φ, τ) as coordinates on $\pi({}^eT^*_{\partial X}X)$, since $\xi = 0$ there and η is determined by (18); then

$$(t, \varphi, \tau) \sim (t', \varphi', \tau') \iff \tau = \tau',\ \varphi - t/\mathsf{A} = \varphi' - t'/\mathsf{A} \bmod 2\pi\mathbb{Z}.$$

Finally, we let the *compressed b-cotangent bundle* be the quotient

$${}^b\dot{T}^*X = \pi({}^eT^*X)/{\sim};$$

Since π commutes with the flow along $i\mathsf{F}$, we may (and will) view this as a subset of ${}^bT^*X/{\sim}$, and give it the subspace topology of that space (which itself has a quotient topology). There is a natural map

$$\dot{\pi} \colon {}^eT^*_{\partial X}X \to {}^b\dot{T}^*_{\partial X}X$$

given by mapping a point to the equivalence class of its image under π (i.e. $\dot{\pi}(q) = [\pi(q)]_\sim$). In coordinates, a point in ${}^b\dot{T}^*_{\partial X}X$ is simply given by the equivalence class (over varying s) of

$$(t = \mathsf{A}s, r = 0, \varphi = \varphi_0 + s \bmod 2\pi\mathbb{Z}, \tau = \tau_0, \xi = 0, \eta = -\mathsf{A}\tau_0),$$

with $\varphi_0 \in S^1, \tau_0 \in \mathbb{R}\backslash\{0\}$ thus providing coordinates for ${}^b\dot{T}^*_{\partial X}X$. We will also have occasion to use the usual notation

$${}^b\dot{S}^*X$$

for the $\mathbb{R}_+$-quotient of this bundle, which over the boundary is just $S^1_\varphi \times S^0_\tau$.

Given $\varrho = (\varphi_0, \tau_0) \in {}^b\dot{T}^*X$ we let

$$\mathscr{F}_{I/O,\varrho} = \{(t, r, \varphi, \tau, \xi, \eta) \in \Sigma : \mathsf{A}\tau + \eta = 0, \operatorname{sgn}(\xi/\tau) = \pm 1, \quad \tau = \tau_0, \lim_{s\to\pm\infty} (\varphi - t/\mathsf{A}) \circ {}^b\Phi(s) = \varphi_0\} \tag{19}$$

with I corresponding to $+1$ and O to -1 in the sign of ξ/τ, and the direction of the limit in s being chosen with $\pm = -\operatorname{sgn}\xi$. (Recall that ${}^b\Phi(s)$ denotes the flow along the rescaled b-Hamilton vector field.) These are the points "incoming" toward or "outgoing" from the fiber indexed by (φ_0, τ_0), according to whether $dr/dt = -\xi/r\tau$ is negative resp. positive.

4. Twisted H^1 and the fiber-invariant b-calculus

Recall that we define a Sobolev-type norm on the space $\mathcal{H}$ by

$$\|u\|^2_{\mathcal{H}} = \|D_r u\|^2_{L^2(X)} + \|r^{-1}\mathsf{F}u\|^2_{L^2(X)} + \|D_t u\|^2_{L^2(X)} + \|D_\varphi u\|^2_{L^2(X)} + \|u\|^2_{L^2(X)}.$$

Note that owing to the presence of the singular $r^{-1}\mathsf{F} \equiv r^{-1}(\mathsf{A}D_t + D_\varphi)$ term, we could dispense with either the D_t or the D_φ term (but not both). We define the space $\mathcal{H}$ to be the closure of $\mathcal{C}^\infty_c(X^\circ)$ (i.e. vanishing at S is imposed). Thus $\mathcal{H}$ agrees with the usual Sobolev space H^1 away from $r = 0$, but is an appropriately twisted version of $H^1(\mathbb{R}^3)$ at the cosmic string itself.

Let $\mathcal{H}^* \subset \mathcal{C}^{-\infty}(X)$ denote the dual space of $\mathcal{H}$ with respect to the L^2 inner product. Thus

$$P : \mathcal{H} \to \mathcal{H}^*$$

is bounded.

Remark 4.1. The uniform b-calculus is not bounded on $\mathcal{H}$, since even a multiplication operator M_ψ by a cutoff function $\psi(t)$ has the defect that

$$\|M_\psi u\|^2_{\mathcal{H}} \geq \|r^{-1}\mathsf{F}(\psi(t)u)\|^2,$$

and this yields $r^{-1}u$ terms when F acts on $\psi(t)$ which are not, in general, bounded by $\|u\|^2_{\mathcal{H}}$. Consequently we will specialize further to certain operators with better commutation properties with F.

Definition 4.2. *An operator $A \in \Psi^s_{bu}(X)$ is said to be* fiber-invariant *if*

$$[A, \mathsf{F}] = 0.$$

and we write $\Psi^s_{b\mathsf{F}}(X)$ to denote the space of the fiber-invariant b-pseudodifferential operators of order s. We let $\mathrm{Diff}_{b\mathsf{F}}$ denote the subalgebra of fiber-invariant differential operators.

Let $\mathcal{C}^\infty_{\mathsf{F}}$ denote the space of fiber-invariant smooth functions, i.e. those annihilated by F.

Remark 4.3. Operators satisfying weaker notions of fiber-invariance, such as the *very basic* operators introduced in [11, Section 10], would suffice to obtain boundedness on $\mathcal{H}$. However some difficulties involving iterated commutators with F, e.g. in the proof of microlocal elliptic regularity, seem to make these less rigid operators more difficult to use in the setting under consideration here.

Note that fiber-invariance is preserved under composition, by the derivation property of commutation, and is also preserved under multiplication by fiber-invariant smooth functions.

Definition 4.4. *Let $S^m_{\mathsf{F}}({}^bT^*X)$ denote the space of symbols $a \in S^m_u({}^bT^*X)$ that additionally satisfy*

$$\mathsf{F}(a) = 0.$$

Recall that a convenient way of quantizing arbitrary symbols to operators in our calculus is given by the map $\widetilde{\mathrm{Op}}_{\mathrm{b}}$ defined in (13).

Lemma 4.5. *Let $A = \widetilde{\mathrm{Op}}_{\mathrm{b}}(a) \in \Psi^s_{bu}(X)$. Then $A \in \Psi^s_{b\mathsf{F}}(X)$ iff $a \in S^m_{\mathsf{F}}({}^bT^*X)$.*

Proof. We simply note that the Schwartz kernel of $[\mathsf{F}, A]$ is $\widetilde{\mathrm{Op}}_{\mathrm{b}}(\mathsf{F}(a))$. □

Lemma 4.6. *Let $A \in \Psi^s_{bu}(X)$ with $\sigma^s_b(A) = a$. Then*

$$[D_r, A] = E + FD_r = E' + D_rF' \quad \text{and} \quad [r^{-1}, A] = r^{-1}G = G'r^{-1}$$

for $E, E' \in \Psi^s_{bu}(X)$ and $F, F', G, G' \in \Psi^{s-1}_{bu}(X)$, with

$$\sigma^s_b(E) = \sigma^s_b(E') = -i\partial_r(a),$$
$$\sigma^{s-1}_b(F) = \sigma^{s-1}_b(F') = \sigma^{s-1}_b(G) = \sigma^{s-1}_b(G') = -i\partial_\xi(a).$$

If A is additionally fiber-invariant, then all the other operators are as well.

Proof. First we recall that

$$[r, A] \in r\Psi^{s-1}_{bu}(X) = \Psi^{s-1}_{bu}(X)r$$

by Lemma 3.7. Moreover,

$$[r, A] = \tilde{G}r$$

with

$$\sigma^{s-1}_b(\tilde{G}) = -ir^{-1}({}^bH_r)(a) = i\partial_\xi(a).$$

Thus

$$[r^{-1}, A] = -r^{-1}[r, A]r^{-1} = -r^{-1}\tilde{G} = r^{-1}G \tag{20}$$

where

$$\sigma^{s-1}_b(G) = -i\partial_\xi(a).$$

An analogous version of the computation with $[r, A] = r\tilde{G}'$ proves that we we may also write this operator as $G'r^{-1}$.

Next we write

$$[D_r, A] = r^{-1}[rD_r, A] + [r^{-1}, A]rD_r = E + FD_r$$

where $E = r^{-1}[rD_r, A]$ and $F = [r^{-1}, A]r$. By (20) we see $F = r^{-1}Gr$ so that $\sigma^{s-1}_b(F) = \sigma^{s-1}_b(G)$, as desired. The desired properties of E follow from the observation $[rD_r, A] \in r\Psi^s_b$ by Lemma 3.7.

The alternate second form, $E'+D_rF'$, follows using the above, now writing $D_r = rD_rr^{-1} + r^{-1}$. Indeed,

$$\begin{aligned}[D_r, A] &= [rD_rr^{-1}, A] - i[r^{-1}, A] \\ &= (rD_r - i)[r^{-1}, A] + [rD_r, A]r^{-1} \\ &= D_rF' + E'\end{aligned}$$

where $F' = r[r^{-1}, A]$ and $E' = [rD_r, A]r^{-1}$.

□

Finally, we show that 0-th order fiber-invariant pseudodifferential operators map $\mathcal{H}$ continuously into itself.

Lemma 4.7. *Let $A \in \Psi^0_{\mathsf{bF}}$ and $u \in \mathcal{H}$. Then*

$$\|Au\|_{\mathcal{H}} \leq C\|u\|_{\mathcal{H}}$$

and C depends only on the seminorms of the symbol of A.

Proof. Let $A \in \Psi^0_{\mathsf{bF}}$ and $u \in \mathcal{H}$. By boundedness of the uniform calculus, all terms in $\|Au\|^2_{\mathcal{H}}$ are bounded except for $\|D_rAu\|^2$ and $\|r^{-1}\mathsf{F}Au\|^2$. Since $[\mathsf{F}, A] = 0$, boundedness of these two terms additionally follows from Lemma 4.6. □

Since $A^* \in \Psi^0_{\mathsf{bF}}$ when $A \in \Psi^0_{\mathsf{bF}}$, Lemma 4.7 yields the following corollary:

Corollary 4.8. *Let $A \in \Psi^0_{\mathsf{bF}}$ and $u \in \mathcal{H}^*$. Then*

$$\|Au\|_{\mathcal{H}^*} \leq C\|u\|_{\mathcal{H}^*}$$

and C depends only on the seminorms of the symbol of A.

5. Sobolev spaces

First, we recall the definition of b-Sobolev spaces, adapted to our uniform context.

Definition 5.1. *Let $s \geq 0$. A distribution u is in $H^s_{\mathsf{bF}}(X)$ if for all $A \in \Psi^s_{\mathsf{bF}}(X)$, $Au \in L^2$.*

The negative order spaces are defined dually: for $s > 0$, $u \in H^{-s}_{\mathsf{bF}}(X)$ if $u \in \Psi^s_{\mathsf{bF}}(X)(L^2)$.

Remark 5.2. By elliptic estimates of the sort discussed below, we would obtain an equivalent scale of Sobolev spaces by using the algebra $\Psi_{bu}(X)$ rather than $\Psi_{\mathsf{bF}}(X)$: the distinction is irrelevant here. In particular, we could use constant coefficient test operators, so that for $m \in \mathbb{N}$,

$$u \in H^m_{\mathsf{bF}}(X) \Longleftrightarrow (r\partial_r)^i\partial_t^j\partial_\varphi^k u \in L^2(X), \quad i + j + k \leq m.$$

We nonetheless keep the F in the notation to remind the reader that these are in almost every case the test operators under consideration (and the distinction is more important in the $\mathcal{H}$-based spaces defined below).

Note that over compact sets, the distinctions between uniform and ordinary b-Sobolev spaces are also moot, and $H^m_{\mathsf{bF}} \cap \mathcal{E}' = H^m_b \cap \mathcal{E}'$.

We additionally define b-Sobolev regularity with background spaces given by $\mathcal{H}$ or $\mathcal{H}^*$:

Definition 5.3. *Let $s \geq 0$. A distribution u is in $H^s_{\mathsf{bF},\mathcal{H}}(X)$ if for all $A \in \Psi^s_{\mathsf{bF}}(X)$, $Au \in \mathcal{H}$. Likewise $u \in H^s_{\mathsf{bF},\mathcal{H}^*}(X)$ if for all $A \in \Psi^s_{\mathsf{bF}}(X)$, $Au \in \mathcal{H}^*$.*

The negative order spaces are defined dually: for $s > 0$, $u \in H^{-s}_{\mathsf{bF},\mathcal{H}}(X)$ if $u \in \Psi^s_{\mathsf{bF}}(X)(\mathcal{H})$ and likewise for $H^s_{\mathsf{bF},\mathcal{H}^*}(X)$.

As weighted b-Sobolev spaces also arise frequently, we introduce the double-index notation

$$H^{s,l}_{\mathsf{bF}}(X) = r^l H^s_{\mathsf{bF}}(X); \tag{21}$$

thus $H^{s,0}_{\mathsf{bF}} = H^s_{\mathsf{bF}}$, and we will continue to use the latter notation when convenient. When there is no possibility of confusion (in particular, in the solvability argument in Section 9 below) we will write the norm on $H^{s,l}_{\mathsf{bF}}$ as $\|\bullet\|_{s,l}$.

We will partially[2] adopt the notation of [6, Section B.2]: let $\Omega \subset X$ be an open set obtained as $\Omega = \beta^{-1}\Omega_0$ where β is the blowdown map and $\Omega_0 \subset \mathbb{R}^3$ is an open set with Lipschitz boundary.

For $\mathcal{Z}$ any of the Hilbert spaces of distributions on X introduced thus far, let

$$\dot{\mathcal{Z}}(\overline{\Omega}),\ \mathcal{Z}(\Omega)$$

denote respectively the subspace of $\mathcal{Z}$ consisting of elements that are supported on $\overline{\Omega} \subset X$, and the quotient space of $\mathcal{Z}$ consisting of restrictions from X to Ω of elements of $\mathcal{Z}$.

Note that since X is a manifold with boundary, the open set $\Omega \subset X$ may contain points in $\partial X = \{r = 0\}$ (but as soon as it contains one point in $(r = 0, t = t_0, \varphi = \varphi_0)$ it must contain all points $(r = 0, t = t_0, \varphi \in S^1)$, as it is the preimage of a set in $\mathbb{R}^3$). The distinctions about boundary behavior that we are drawing among Sobolev spaces are at other parts of the boundary, not at $r = 0$.

Lemma 5.4. *The following continuous inclusions of subspaces of $\mathcal{C}^\infty_c(X^\circ)'$ hold over any $\Omega, \overline{\Omega} \subset \mathbb{R} \times X$ as above, provided the function r is bounded on Ω:*

$$\dot{H}^{1,1}_b(\overline{\Omega}) \subset \dot{\mathcal{H}}(\overline{\Omega}) \subset \dot{H}^1_b(\overline{\Omega}),$$
$$H^{-1}_b(\Omega) \subset \mathcal{H}^*(\Omega) \subset H^{-1,-1}_b(\Omega)$$

Proof. We write the squared norm on $\mathcal{H}$ as

$$\|r^{-1}(rD_r)u\|^2_{L^2(X)} + \|r^{-1}\mathsf{F}u\|^2_{L^2(X)} + \|D_t u\|^2_{L^2(X)} + \|D_\varphi u\|^2_{L^2(X)} + \|u\|^2_{L^2(X)};$$

[2] We are omitting the "bar" used for spaces of restrictions, as this seems to be the more common usage in other literature.

since the vector fields $r\partial_r$ and F are in $\mathcal{V}_b$ this is certainly dominated by a multiple of the $rH^1_{b\mathsf{F}}$ squared norm, given by

$$\|r^{-1}(rD_r)u\|^2_{L^2(X)} + \|r^{-1}D_t u\|^2_{L^2(X)} + \|r^{-1}D_\varphi u\|^2_{L^2(X)} + \|r^{-1}u\|^2_{L^2(X)};$$

(Here we have used boundedness of r on Ω.) Thus $r\dot{H}^1_b(\overline{\Omega}) \subset \dot{\mathcal{H}}(\overline{\Omega})$. Likewise, the squared $\mathcal{H}$ norm dominates

$$\|(rD_r)u\|^2_{L^2(X)} + \|D_t u\|^2_{L^2(X)} + \|D_\varphi u\|^2_{L^2(X)} + \|u\|^2_{L^2(X)},$$

which is the H^1_b norm, hence $\dot{\mathcal{H}}(\overline{\Omega}) \subset \dot{H}^1_b(\overline{\Omega})$. The remaining inclusions follow by duality, using the usual duality between spaces of extendible and supported distributions as discussed in [6, Section B.2] (and for which the extensions to the hypothesis of merely Lipschitz boundary can be found in [9, Theorems 3.29, 3.30]). □

Corollary 5.5. *Operators in $\Psi^s_{bu}(X)$ are bounded from $H^{s-1}_{b\mathsf{F},\mathcal{H}}(X)$ to $L^2(X)$.*

Proof. Since $\mathcal{H}(X) \subset H^1_b(X)$, we additionally have

$$H^{s-1}_{b\mathsf{F},\mathcal{H}}(X) \subset H^s_b(X),$$

and the result follows from boundedness of the uniform calculus. □

6. Ellipticity, microsupport, and wavefront set

Since our test operators are required to be fiber invariant, we cannot hope to use them to distinguish microlocal behavior at different points in a fiber. Hence we will define, for $s \in \mathbb{R} \cup \{\infty\}$,

$$\mathrm{WF}^s_{b\mathsf{F},\mathcal{H}}\, u,\ \mathrm{WF}^s_{b\mathsf{F},\mathcal{H}^*}\, u \subset {}^bS^*X/\!\sim$$

as subsets of the quotient space of the sphere bundle where we identify fibers in the base space X at $r = 0$. (Recall that $\sim$ lifts from ∂X to give a fibration of ${}^bS^*_{\partial X}X$.) We equip this space with the quotient topology.

As we will deal very frequently in the sequel with objects living on the quotients of ${}^bS^*X$, ${}^bT^*X$, and other spaces by $\sim$, we henceforth adopt the F subscript notation

$${}^bT^*X_{\mathsf{F}} \equiv {}^bT^*X/\!\sim,\ {}^bS^*X_{\mathsf{F}} \equiv {}^bS^*X/\!\sim$$

to denote fiber quotients of these and other spaces. Our previous notation for ${}^b\dot{T}^*X$, matching that used in [11], is admittedly slightly inconsistent with this one, as it also involves a fiber quotient but has no subscript. Note, in any event, that

$${}^b\dot{T}^*X \subset {}^bT^*X_{\mathsf{F}},\ {}^b\dot{S}^*X \subset {}^bS^*X_{\mathsf{F}}.$$

We begin with the usual definitions of ellipticity and microsupport, transported to the quotient space ${}^bT^*X_{\mathsf{F}}$, and with uniformity in the noncompact fibers and fiber invariance built in via the uniform estimates on our symbols.

In the following definition we use the quantization Op defined in Appendix A which is, by definition, a surjective map from an appropriate space of total symbols to the operator calculus (rather than requiring the manual addition of the residual operators, as we must do when using $\widetilde{\mathrm{Op}}_{\mathrm{b}}$). The wavefront sets are defined as subsets of the fiber-quotiented cosphere bundle, but as usual we may equally well view it as a positive conic subset of the relevant cotangent bundle.

Definition 6.1. *An operator $A = \mathrm{Op}_{\mathrm{b}}(a) \in \Psi^s_{b\mathsf{F}}(X)$ is elliptic at $q \in {}^bS^*X_{\mathsf{F}}$ if there exists $c \in S^{-s}_{\mathsf{F}}$ such that $ca = 1$ in a positive conic neighborhood of q in ${}^bT^*X_{\mathsf{F}}$. Let $\mathrm{ell}\, A$ denote the set of points at which A is elliptic.*

A point $q \in {}^bS^*X_{\mathsf{F}}$ is in the complement of the microsupport of $A = \mathrm{Op}_{\mathrm{b}}(a)$ (denoted $\mathrm{WF}'_{b\mathsf{F}}(A)$) if there exists a positive conic neighborhood U of q in ${}^bT^*X_{\mathsf{F}}$ on which for all $N \in \mathbb{N}$, and multiindex α,

$$|\partial^\alpha a| \leq C_{N,\alpha}\langle \tau, \xi, \eta\rangle^{-N}. \tag{22}$$

We use the same definition of microsupport for elements of $\Psi_{bu}(X)$, requiring the estimate (22) *to hold on the $\sim$-equivalence class of the neighborhood U.*

Note then that ellipticity of an element of $\Psi^0_{b\mathsf{F}}(X)$ at an equivalence class of points over $r = 0$ given by $\{(t + \mathsf{A}s, \varphi_0 + s, \xi_0, \tau_0, \eta_0) : s \in \mathbb{R}\}$ implies uniform lower bounds on the symbol on a set of the form

$$\begin{aligned}\{(t + \mathsf{A}s, \varphi + s, \xi, \tau, \eta) : |\varphi - \varphi_0| < \epsilon,\\ \big|(\xi,\tau,\eta)/|(\xi,\tau,\eta)| - (\xi_0,\tau_0,\eta_0)/|(\xi_0,\tau_0,\eta_0)|\big| < \epsilon,\ |(\xi,\tau,\eta)| > \epsilon^{-1}\}.\end{aligned}$$

As usual, we have

$$\mathrm{WF}'_{b\mathsf{F}} A = \varnothing \iff A \in \Psi^{-\infty}_{b\mathsf{F}}(X).$$

In order to regularize our commutator arguments below, we will work with *families* of fiber-invariant operators which are uniformly bounded in some space of uniform b-pseudodifferential operators. Thus we also require a notion of microsupport in this context. In the following definition and subsequent discussion, we use the notion of bounded operator families in $\Psi^k_{b\mathsf{F}}(X)$: this means parametrized families of operators that are quantizations of symbols with all seminorms uniformly bounded in the parameter.

Definition 6.2. *Let $\mathcal{B} \subset \Psi^k_{b\mathsf{F}}$ be a bounded family of fiber-invariant operators indexed by ϖ in that there exists an indexing set $\mathcal{I}$ such that $\mathcal{B} = \{B_\varpi : \varpi \in \mathcal{I}\}$. We say $q \in {}^bS^*X_{\mathsf{F}}$ is in the complement of the microsupport of the family (i.e., $q \notin \mathrm{WF}'_{b\mathsf{F}}\,\mathcal{B}$) if there exists a positive conic neighborhood of q in ${}^bT^*X_{\mathsf{F}}$ on which for all $N \in \mathcal{N}$ and any multiindex α, there exists $C_{N,\alpha}$ independent of ϖ such that*

$$|\partial^\alpha b_\varpi| \leq C_{N,\alpha}\langle \tau, \xi, \eta\rangle^{-N}$$

where $B_\varpi = \mathrm{Op}_b(b_\varpi)$.

For a family $\mathcal{B}$, $\mathrm{WF}'_{b\mathsf{F}}\,\mathcal{B} = \varnothing$ iff for all $B \in \mathcal{B}$, $B \in \Psi^{-\infty}_{b\mathsf{F}}(X)$ *with uniform estimates* on seminorms (which we will abbreviate in what follows as "lying uniformly in $\Psi^{-\infty}_{b\mathsf{F}}$").

The following standard result can be read off from the locality in (x,ξ) of the formula for the total symbol of the composition, just as in the usual calculus.

Proposition 6.3. $\mathrm{WF}'_{\mathsf{bF}} AB \subset \mathrm{WF}'_{\mathsf{bF}} A \cap \mathrm{WF}'_{\mathsf{bF}} B$ *and the same inclusion holds for operator families.*

There do exist elliptic operators in our calculus:

Lemma 6.4. *Given* $p \in {}^bT^*_{\partial X} X_{\mathsf{F}}$, $s \in \mathbb{R}$, *and an open conic neighborhood* U *of* p, *there exists* $A \in \Psi^s_{\mathsf{bF}}(X)$ *that is elliptic at* p *with* $\mathrm{WF}'_{\mathsf{bF}} A \subset U$.

Proof. Letting χ be a cutoff function supported near the origin, we use the notation $\alpha \equiv (\xi,\tau,\eta)$ and $\hat\alpha = \alpha/|\alpha|$, and set

$$a = \chi(t - \mathsf{A}\varphi - \mathsf{A}\varphi_0)\chi(r)\chi(|\hat\alpha - \hat\alpha_0|)\chi(|\alpha|^{-1}).$$

Then $A = \widetilde{\mathrm{Op}}_{\mathrm{b}}(a)$ satisfies our criteria, provided the support of χ is sufficiently small. □

Global and microlocal elliptic parametrices exist in this calculus, with the proof being the usual one (relying, as it does, only on the properties of the symbol calculus—see, e.g., [5, Proposition E.32]):

Proposition 6.5. *Let* $A \in \Psi^\ell_{\mathsf{bF}}(X)$ *and* B *in* $\Psi^k_{\mathsf{bF}}(X)$ *satisfy*

$$\mathrm{WF}'_{\mathsf{bF}} A \subset \mathrm{ell}\, B.$$

Then there exists $Q, Q' \in \Psi^{\ell-k}_{\mathsf{bF}}(X)$ *such that*

$$A = BQ + R = Q'B + R'$$

with

$$R, R' \in \Psi^{-\infty}_{\mathsf{bF}}(X)$$

and $\mathrm{WF}'_{\mathsf{bF}} Q \cup \mathrm{WF}'_{\mathsf{bF}} Q' \subset \mathrm{WF}'_{\mathsf{bF}} A$.

The same result holds if $A \in \Psi^\ell_{bu}(X)$ *and* $B \in \Psi^k_{\mathsf{bF}}(X)$, *now with* $R, R' \in R, R' \in \Psi^{-\infty}_{bu}(X)$.

Note in particular, taking $A = \mathrm{Id}$, we can invert a globally elliptic operator modulo a residual operator.

As usual, elliptic elements of the calculus can be used to test for Sobolev regularity:

Proposition 6.6. *Let* $A \in \Psi^s_{\mathsf{bF}}(X)$ *be elliptic. Then*

$$u \in H^s_{\mathsf{bF}}(X) \text{ iff } Au \in L^2$$
$$u \in H^s_{\mathsf{bF},\mathcal{H}}(X) \text{ iff } Au \in \mathcal{H}$$
$$u \in H^s_{\mathsf{bF},\mathcal{H}^*}(X) \text{ iff } Au \in \mathcal{H}^*.$$

The proof follows directly from Proposition 6.5 and from boundedness of nonpositive-order elements of the calculus on L^2, $\mathcal{H}$, $\mathcal{H}^*$.

Definition 6.7. *Let $u \in \mathcal{C}^{-\infty}(X)$ and $q \in {}^bT^*X_{\mathsf{F}}$. We say $q \notin \mathrm{WF}^s_{b\mathsf{F},\mathcal{H}}\, u$ (resp. $q \notin \mathrm{WF}^s_{b\mathsf{F},\mathcal{H}^*}\, u$, $q \notin \mathrm{WF}^s_{b\mathsf{F}}\, u$) if there is an $A \in \Psi^s_{b\mathsf{F}}$, elliptic at q, such that $Au \in \mathcal{H}$ (resp. $Au \in \mathcal{H}^*$, $Au \in L^2$).*

We say $q \notin \mathrm{WF}_{b\mathsf{F},\mathcal{H}}\, u$ (resp. $q \notin \mathrm{WF}_{b\mathsf{F},\mathcal{H}^*}\, u$, $q \notin \mathrm{WF}_{b\mathsf{F}}\, u$) if there is an $A \in \Psi^0_{b\mathsf{F}}$, elliptic at q, such that $Au \in H^\infty_{b\mathsf{F},\mathcal{H}}(X)$ (resp. $Au \in H^\infty_{b\mathsf{F},\mathcal{H}^*}$, $Au \in H^\infty_b$). We occasionally use the convention that $\mathrm{WF}^\infty_\bullet$ means the same thing as $\mathrm{WF}_\bullet$ with no superscript.

Standard ellipticity arguments show that with the definitions above,

$$\mathrm{WF}^\infty_\bullet\, u = \overline{\bigcup_{s\in\mathbb{R}} \mathrm{WF}^s_\bullet\, u},$$

hence we will not need to deal separately with $s = \infty$ in our results below.

We have an inclusion of wavefront sets based on the corresponding inclusions of Sobolev spaces:

Lemma 6.8. *For $u \in \mathcal{H}$, and $m \in \mathbb{R} \cup \{+\infty\}$,*

$$\mathrm{WF}^{m+1}_{b\mathsf{F}}(u) \subset \mathrm{WF}^m_{b\mathsf{F},\mathcal{H}}(u) \subset \mathrm{WF}^{m+1}_{b\mathsf{F}}(r^{-1}u). \tag{23}$$

For $u \in \mathcal{H}^*$, and $m \in \mathbb{R} \cup \{+\infty\}$,

$$\mathrm{WF}^m_{b\mathsf{F}}(ru) \subset \mathrm{WF}^{m+1}_{b\mathsf{F},\mathcal{H}^*}(u) \subset \mathrm{WF}^m_{b\mathsf{F}}(u) \tag{24}$$

Proof. If a point $q \notin \mathrm{WF}^m_{b\mathsf{F},\mathcal{H}}(u)$, there exists $A \in \Psi^m_{b\mathsf{F}}$, elliptic at q, such that $Au \in \mathcal{H}$. Hence by Lemma 5.4, $Au \in H^1_{b\mathsf{F}}$, and we obtain $q \notin \mathrm{WF}^{m+1}_{b\mathsf{F}}\, u$, which proves the first inclusion in (23). If $q \notin \mathrm{WF}^{m+1}_{b\mathsf{F}}(r^{-1}u)$ then there exists $A \in \Psi^{m+1}_{b\mathsf{F}}$ with $r^{-1}Au \in L^2$, hence for $B \in \Psi^m_{b\mathsf{F}}$ microsupported on the elliptic set of A and elliptic at q, we have $Bu \in rH^1_{b\mathsf{F}} \subset \mathcal{H}$ (again by Lemma 5.4) hence $q \notin \mathrm{WF}^m_{b\mathsf{F},\mathcal{H}}(u)$ and the second inclusion follows.

The inclusions in (24) are proved similarly. □

7. Microlocal elliptic regularity

In this section we prove a microlocal elliptic regularity statement for the weighted edge operator P. This says, quantitatively, that away from the wavefront set of the inhomogeneity, the wavefront set of a solution to $Pu = f$ is contained in the compressed b-cotangent bundle defined in Section 3.3.

Recall that

$$P = -\partial_t^2 + r^{-2}(r\partial_r)^2 + r^{-2}(\mathsf{A}\partial_t + \partial_\varphi)^2 + \Upsilon$$

with

$$\Upsilon = f_1 D_t + f_2 D_\varphi + f_3 r D_r + f_4$$

where $f_\bullet \in \mathcal{C}^\infty_u$.

Lemma 7.1. *Take $\Gamma \subset U \subset {}^bS^*X_{\mathsf{F}}$ with Γ closed and U open. Let $\mathcal{B} \subset \Psi^k_{b\mathsf{F}}$ be a family uniformly bounded in Ψ^k_{bu} with $\mathrm{WF}'_{b\mathsf{F}}(\mathcal{B}) \subset \Gamma$. Let $Q \in \Psi^k_{b\mathsf{F}}$ be elliptic on Γ with $\mathrm{WF}'_{b\mathsf{F}}(Q) \subset U$.*

(i) *If* $u \in \mathcal{H}$ *and* $\mathrm{WF}^k_{\mathsf{bF},\mathcal{H}}(u) \cap U = \varnothing$ *then there exists a constant* $C_1 > 0$ *such that*

$$\|Bu\|_{\mathcal{H}} \leq C_1(\|u\|_{\mathcal{H}} + \|Qu\|_{\mathcal{H}})$$

for $B \in \mathcal{B}$*.*

(ii) *If* $u \in \mathcal{H}^*$ *and* $\mathrm{WF}^k_{\mathsf{bF},\mathcal{H}^*}(u) \cap U = \varnothing$ *then there exists a constant* $C_2 > 0$ *such that*

$$\|Bu\|_{\mathcal{H}^*} \leq C_2(\|u\|_{\mathcal{H}^*} + \|Qu\|_{\mathcal{H}^*})$$

Proof. We begin by proving statement (i). Since Γ is closed, it follows from Proposition 6.5 that we can find a microlocal parametrix G for Q so that $GQ = \mathrm{Id} + E$ where $E \in \Psi^0_{\mathsf{bF}}(X)$, $\mathrm{WF}'_{\mathsf{bF}}(E) \cap \Gamma = \varnothing$. For each $B \in \mathcal{B}$, $BE \in \Psi^{-\infty}_{\mathsf{bF}}(X)$ uniformly since $\mathrm{WF}'_{\mathsf{bF}}(B) \cap \mathrm{WF}'_{\mathsf{bF}}(E) = \varnothing$. Thus

$$\begin{aligned}
\|Bu\|_{\mathcal{H}} &= \|B(GQ - E)u\|_{\mathcal{H}} \\
&\leq \|BGQu\|_{\mathcal{H}} + \|BEu\|_{\mathcal{H}} \\
&\leq C_1(\|u\|_{\mathcal{H}} + \|Qu\|_{\mathcal{H}})
\end{aligned}$$

where in the last line we bound $\|BGQu\|_{\mathcal{H}}$ using Lemma 4.7 and the second using the fact that $BE \in \Psi^{-\infty}_{\mathsf{bF}}$ uniformly.

The proof of statement (ii) is identical but we use Corollary 4.8 instead of Lemma 4.7. □

The following lemma relates our $\mathcal{H}$-based wavefront set to L^2 bounds. Note that the test operator A is merely uniform, but the operator G estimating it is fiber-invariant.

Lemma 7.2. *Take* $\Gamma \subset U \subset {}^bS^*X_{\mathsf{F}}$ *with* Γ *closed and* U *open. Let* $u \in \mathcal{H}$ *and let* $A \in \Psi^{m+1}_{bu}(X)$ *for some* $m \in \mathbb{R}$*. Assume* $\mathrm{WF}'_{\mathsf{bF}}(A) \subset \Gamma$ *and* $\mathrm{WF}^m_{\mathsf{bF},\mathcal{H}}\, u \cap U = \varnothing$*. Then for any* $G \in \Psi^m_{\mathsf{bF}}$ *with* $\mathrm{WF}'_{\mathsf{bF}}\, G \subset U$ *and* G *elliptic on* Γ*, there exist* $C > 0$ *and* $R \in \Psi^{-\infty}_{bu}$ *such that*

$$\|Au\|_{L^2} \leq C\|Gu\|_{\mathcal{H}} + \|Ru\|_{L^2}.$$

Proof. Fixing $G \in \Psi^m_{\mathsf{bF}}(X)$ elliptic on Γ and microsupported in U, the elliptic parametrix construction (Proposition 6.5) gives the factorization

$$A = TG + R$$

with $T \in \Psi^1_{bu}$, $R \in \Psi^{-\infty}_{bu}$. We then estimate, using Corollary 5.5,

$$\|TGu\| \leq C\|Gu\|_{\mathcal{H}}$$

and the desired estimate follows.

□

In our proof of estimates on the Dirichlet form below, we will need to control many terms which share similar structures. To streamline the argument, we provide the following lemma which will allow us to immediately bound the necessary terms.

Lemma 7.3. *Take* $\Gamma \subset U \subset {}^bS^*X_{\mathsf{F}}$ *with* Γ *closed and* U *open. Let* $\mathcal{A}, \mathcal{B}$ *be families of operators uniformly bounded in* $\Psi^k_{b\mathsf{F}}$ *and* $\Psi^{k'}_{b\mathsf{F}}$*, respectively. Assume* $\mathrm{WF}'_{b\mathsf{F}}(\mathcal{A}), \mathrm{WF}'_{b\mathsf{F}}(\mathcal{B}) \subset \Gamma$*. Let* E, F *be weighted differential operators of respective orders* $m, m' \in \{0,1\}$ *lying in* $\mathrm{Diff}_{b\mathsf{F}} + \mathcal{C}^\infty_{\mathsf{F}} r^{-1}\mathsf{F} + \mathcal{C}^\infty_{\mathsf{F}} D_r$*. Set*

$$n = \frac{m + m' + k + k' - 2}{2}.$$

Assume $u \in \mathcal{H}$ *and* $\mathrm{WF}^n_{b\mathsf{F},\mathcal{H}}\, u \cap U = \varnothing$*. Then for any* $Q \in \Psi^n_{b\mathsf{F}}$ *elliptic on* Γ *such that* $\mathrm{WF}'_{b\mathsf{F}}(Q) \subset U$*, there exists a constant* $C > 0$ *such that*

$$|\langle EAu, FBu\rangle| \le C\big(\|u\|^2_{\mathcal{H}} + \|Qu\|^2_{\mathcal{H}}\big),$$

for $A \in \mathcal{A}, B \in \mathcal{B}$*.*

Proof. Set

$$\ell = \frac{-m + m' - k + k'}{2}.$$

Let $\Lambda_\ell \in \Psi^\ell_{b\mathsf{F}}(X)$ be elliptic with parametrix $\Lambda_{-\ell} \in \Psi^{-\ell}_{b\mathsf{F}}(X)$ satisfying

$$\Lambda_{-\ell}\Lambda_\ell = \mathrm{Id} + R$$

with $R \in \Psi^{-\infty}_{b\mathsf{F}}(X)$.

Then using Lemma 4.6 to deal with commutators of r^{-1} and D_r with elements of $\Psi_{b\mathsf{F}}(X)$ we calculate

$$\begin{aligned}|\langle EAu, FBu\rangle| &= |\langle(\Lambda_{-\ell}\Lambda_\ell - R)EAu, FBu\rangle| \\ &\le |\langle \Lambda_\ell EAu, \Lambda^*_{-\ell}FBu\rangle| + |\langle REAu, FBu\rangle| \\ &= \sum |\langle \tilde{E}_j\tilde{A}_j u, \tilde{F}_j\tilde{B}_j u\rangle| + |\langle REAu, FBu\rangle|\end{aligned}$$

where the sum is a finite one with $\tilde{E}_j$, $\tilde{F}_j$ having the same form and order as E resp. F, and where $\tilde{A}$, $\tilde{B}$ are members of families of operators $\tilde{\mathcal{A}}$ (resp. $\tilde{\mathcal{B}}$) which are uniformly bounded in $\Psi^{k+\ell}_{b\mathsf{F}}$, respectively $\Psi^{k'-\ell}_{b\mathsf{F}}$.

By Cauchy–Schwarz,

$$|\langle \tilde{E}_j\tilde{A}_j u, \tilde{F}_j\tilde{B}_j u\rangle| \lesssim \|\tilde{E}_j\tilde{A}_j u\|^2_{L^2} + \|\tilde{F}_j\tilde{B}_j u\|^2_{L^2}. \tag{25}$$

For the first term on the RHS of (25), if $m = 1$ we see $n = k + \ell$ and we use Lemma 7.1 to find

$$\|\tilde{E}_j\tilde{A}_j u\|^2_{L^2} \lesssim \|\tilde{A}_j u\|^2_{\mathcal{H}} \lesssim \|u\|^2_{\mathcal{H}} + \|Qu\|^2_{\mathcal{H}}.$$

If $m = 0$ then $n = k + \ell - 1$ and we use Lemma 7.2 along with Lemma 7.1 to likewise obtain

$$\|\tilde{E}_j\tilde{A}_j u\|^2_{L^2} \lesssim \|u\|^2_{\mathcal{H}} + \|Qu\|^2_{\mathcal{H}}.$$

The second term on the right hand side of (25) is handled analogously. □

Next we establish an estimate on the Dirichlet form. Note that this lemma is an analogue of Lemma 8.8 from [11].

Lemma 7.4. *Let* $A_\varpi \in \Psi^{s-1}_{b\mathsf{F}}$ *be a family of operators indexed by* $\varpi \in I$ *that are uniformly bounded in* Ψ^s_{bu} *with* $\mathrm{WF}'_{b\mathsf{F}}(A_\varpi) \subset \Gamma \subset U \subset {}^bS^*X_\mathsf{F}$ *for* Γ *closed and* U *open. Assume* $u \in \mathcal{H}$ *satisfies* $\mathrm{WF}^{s-\frac{1}{2}}_{b,\mathcal{H}}(u) \cap U = \varnothing$. *There exist* $Q \in \Psi^{s-\frac{1}{2}}_{b\mathsf{F}}$, $\widetilde{Q} \in \Psi^{s+\frac{1}{2}}_{b\mathsf{F}}$ *elliptic on* Γ *with* $\mathrm{WF}'_{b\mathsf{F}}(Q) \cup \mathrm{WF}'_{b\mathsf{F}}(\widetilde{Q}) \subset U$ *and a constant* $C > 0$ *such that*

$$\begin{aligned}(26)\qquad &\Big| \|D_t A_\varpi u\|_{L^2} - \|D_r A_\varpi u\|_{L^2} - \|r^{-1}\mathsf{F} A_\varpi u\|_{L^2}\Big| \\ &\qquad\qquad \leq C\left(\|u\|^2_{\mathcal{H}} + \|Qu\|^2_{\mathcal{H}} + \|Pu\|^2_{\mathcal{H}^*} + \|\widetilde{Q}Pu\|^2_{\mathcal{H}^*}\right)\end{aligned}$$

for all $\varpi \in I$.

Proof. Our assumptions on the wavefront set of u imply that for each ϖ, we have $A_\varpi u \in \mathcal{H}$ so that $PA_\varpi u \in \mathcal{H}^*$. It follows that

$$\begin{aligned}\|D_t A_\varpi u\|_{L^2} - \|D_r A_\varpi u\|_{L^2} - \|r^{-1}\mathsf{F} A_\varpi u\|_{L^2} &= \langle PA_\varpi u, A_\varpi u\rangle - \langle \Upsilon A_\varpi u, A_\varpi u\rangle \\ &= \langle [P, A_\varpi]u, A_\varpi u\rangle + \langle A_\varpi Pu, A_\varpi u\rangle - \langle \Upsilon A_\varpi u, A_\varpi u\rangle.\end{aligned}$$

Thus it suffices to show

$$(27)\qquad |\langle [P, A_\varpi]u, A_\varpi u\rangle - \langle \Upsilon A_\varpi u, A_\varpi u\rangle| \leq C\left(\|u\|^2_{\mathcal{H}} + \|Qu\|^2_{\mathcal{H}}\right)$$

and

$$(28)\qquad |\langle A_\varpi Pu, A_\varpi u\rangle| \leq C\left(\|u\|^2_{\mathcal{H}} + \|Qu\|^2_{\mathcal{H}} + \|Pu\|^2_{\mathcal{H}^*} + \|\tilde{Q}Pu\|^2_{\mathcal{H}^*}\right).$$

First we establish 28. Using the notation from the proof of Lemma 7.3 we have

$$|\langle A_\varpi Pu, A_\varpi u\rangle| \leq |\langle \Lambda_{\frac{1}{2}} A_\varpi Pu, \Lambda^*_{-\frac{1}{2}} A_\varpi u\rangle| + |\langle A_\varpi Pu, R^* A_\varpi u\rangle|$$

where $R \in \Psi^{-\infty}_{b\mathsf{F}}$. The second term is readily bounded as in (26). For the first term we use Cauchy–Schwarz and Lemma 7.1 to find

$$\begin{aligned}\Big|\langle \Lambda_{\frac{1}{2}} A_\varpi Pu, \Lambda^*_{-\frac{1}{2}} A_\varpi u\rangle\Big| &\lesssim \|\Lambda_{\frac{1}{2}} A_\varpi Pu\|^2_{\mathcal{H}^*} + \|\Lambda_{-\frac{1}{2}} A_\varpi u\|^2_{\mathcal{H}} \\ &\lesssim \|Pu\|^2_{\mathcal{H}^*} + \|\tilde{Q}Pu\|^2_{\mathcal{H}^*} + \|u\|^2_{\mathcal{H}} + \|Qu\|^2_{\mathcal{H}},\end{aligned}$$

which proves 28.

As we turn our attention to the commutator term in 27, we are primarily interested in the order of the pseudodifferential operators that arise. Since we commute with $A_\varpi \in \mathcal{A}$, these operators will be members of families indexed by $\mathcal{A}$. To concisely track the relevant information we use the notation B_k to indicate a representative of the class of families of operators indexed by ϖ which are elements of $\Psi^{k-1}_{b\mathsf{F}}(X)$ and are uniformly bounded in $\Psi^k_{b\mathsf{F}}(X)$. The precise operator being specified may change at each appearance of B_k.

Now consider $[P, A_\varpi]$. As $D_r^* = D_r - \frac{i}{r}$, $D_r^* D_r = -r^{-2}(r\partial_r)^2$, hence

$$[\Box_g, A_\varpi] = B_{s+1} + r^{-2}\mathsf{F}^2 B_{s-1} - [D_r^* D_r, A_\varpi].$$

Recalling that Υ is a first-order b-differential operator lying in $\Psi^1_{bu}(X)$, we find that $[\Upsilon, A_\varpi]$ is uniformly bounded in $\Psi^s_{bu}(X)$. Thus

$$[P, A_\varpi] = B_{s+1} + B'_s + r^{-2}\mathsf{F}^2 B_{s-1} - [D_r^* D_r, A_\varpi] + D_r B_{s-1},$$

where $B'_s \in \Psi^s_{bu}(X)$ is uniformly bounded and all other families are in $\Psi_{b\mathsf{F}}(X)$. Thus

$$\begin{aligned}
&\langle [P,A_\varpi]u, A_\varpi u\rangle - \langle \Upsilon A_\varpi u, A_\varpi u\rangle \\
&\quad = \langle B_{s+1}u, A_\varpi u\rangle + \langle r^{-1}\mathsf{F}B_{s-1}u, r^{-1}\mathsf{F}A_\varpi u\rangle - \langle [D_r, A_\varpi]u, D_r A_\varpi u\rangle \\
&\qquad + \langle D_r u, [D_r, A^*_\varpi]A_\varpi u\rangle + \langle D_r B_s u, A_\varpi u\rangle + \langle B'_s u, A_\varpi u\rangle \\
&\quad = \langle B_{s+1}u, A_\varpi u\rangle + \langle r^{-1}\mathsf{F}B_{s-1}u, r^{-1}\mathsf{F}A_\varpi u\rangle - \langle B_s u, D_r A_\varpi u\rangle \\
&\qquad + \langle D_r B_{s-1}u, D_r A_\varpi u\rangle + \langle D_r u, B_{2s}u\rangle + \langle D_r u, D_r B_{2s-1}u\rangle \\
&\qquad A_\varpi u\rangle + \langle D_r B_s u, A_\varpi u\rangle + \langle B'_s u, A_\varpi u\rangle.
\end{aligned}$$

Each term but the B'_s term is then controlled by $\|u\|^2_{\mathcal{H}} + \|Qu\|^2_{\mathcal{H}}$, by Lemma 7.3 since, using the notation of Lemma 7.3, in each term of our expression for $\langle [P, A_\varpi]u, A_\varpi u\rangle$ above we see $m + m' + k + k' \leq 2s + 1$. The B'_s term is likewise controlled by Corollary 5.5. This concludes the proof of 27. ☐

We conclude the section with the proof of our microlocal elliptic regularity statement.

Proposition 7.5. *Let $u \in H^{-N}_{b\mathsf{F},\mathcal{H}}$ for some $N \in \mathbb{R}$. Then for any $m \in \mathbb{R} \cup \{+\infty\}$,*

$$\mathrm{WF}^m_{b\mathsf{F},\mathcal{H}}(u) \setminus \mathrm{WF}^m_{b\mathsf{F},\mathcal{H}^*}(Pu) \subset {}^b\dot{S}^*X.$$

Recall that ${}^b\dot{S}^*X$ is the fiber-quotient of the image of the edge-cotangent bundle inside the b-cotangent bundle, defined in Section 3.3.

Proof. Note that $\mathrm{WF}^{-N}_{b\mathsf{F},\mathcal{H}}(u) = \varnothing$ since $u \in H^{-N}_{b\mathsf{F},\mathcal{H}}$. We claim that $q \in {}^bS^*(X) \setminus \dot{\pi}({}^eS^*(X))$, $q \notin \mathrm{WF}^s_{b\mathsf{F},\mathcal{H}}(u)$ and $q \notin \mathrm{WF}^{s+\frac{1}{2}}_{b\mathsf{F},\mathcal{H}^*}(Pu)$ implies $q \notin \mathrm{WF}^{s+\frac{1}{2}}_{b\mathsf{F},\mathcal{H}}(u)$. The proposition then follows by induction. To establish the claim, we first note for $q \in {}^bS^*X_{\mathsf{F}} \setminus {}^b\dot{S}^*X$ we have

$$\tau^2 < \xi^2 + (\eta + \mathsf{A}\tau)^2.$$

Take $A \in \Psi^{s+\frac{1}{2}}_{b\mathsf{F}}$ such that A is elliptic at q and $\mathrm{WF}'_{b\mathsf{F}}(A) \cap \mathrm{WF}^s_{b,\mathcal{H}}(u) = \varnothing$. Let a be the symbol of A and define A_ϖ as the quantization of $\big(1 + \varpi(\tau^2 + \xi^2 + (\eta + \mathsf{A}\tau)^2)\big)^{-1/2} a$. Then $A_\varpi \in \Psi^{s-\frac{1}{2}}_{b\mathsf{F}}$, A_ϖ is uniformly bounded in $\Psi^{s+\frac{1}{2}}_{b\mathsf{F}}$, and $A_\varpi \to A$ as $\varpi \to 0$.

Define $B \in \Psi^1_{b\mathsf{F}}(X)$ as the quantization of $\left(\chi_q\left(\frac{1}{2\delta^2}(\xi^2 + (\eta + \mathsf{A}\tau)^2) - \tau^2\right)\right)^{\frac{1}{2}}$ where χ_q is a cutoff supported near q with χ_q and δ chosen so that B is elliptic near q. Furthermore we assume $\mathrm{WF}'_{b\mathsf{F}}(\hat{\chi}_q - 1) \cap \mathrm{WF}'_{b\mathsf{F}}(A_\varpi) = \varnothing$ where

$\hat{\chi}_q = \mathrm{Op}(\chi_q)$. Then

$$B^*B = \left(\frac{1}{2\delta^2}[(rD_r)^2 + \mathsf{F}^2] - D_t^2\right)[1 + (\hat{\chi}_q - 1)] + G$$

for some $G \in \Psi^1_{b\mathsf{F}}(X)$. Our assumption on the wavefront set of $\hat{\chi}_q - 1$ and A then give

$$B^*BA_\varpi = \left(\frac{1}{2\delta^2}[(rD_r)^2 + \mathsf{F}^2] - D_t^2\right)A_\varpi + GA_\varpi + E_\varpi$$

where $E_\varpi \in \Psi^{-\infty}_{b\mathsf{F}}$.

We may further assume A_ϖ is supported in $r < \delta$ so that

$$\begin{aligned}\|BA_\varpi u\|^2_{L^2} - \int_X GA_\varpi u\overline{A_\varpi u} &\\ &= \frac{1}{2\delta^2}\big(\|rD_rA_\varpi u\|^2_{L^2} + \|\mathsf{F}A_\varpi u\|^2_{L^2}\big) - \|D_tA_\varpi u\|^2_{L^2}\\ &\leq \frac{1}{2}\big(\|D_rA_\varpi u\|^2_{L^2} + \|r^{-1}\mathsf{F}A_\varpi u\|^2_{L^2}\big) - \|D_tA_\varpi u\|^2_{L^2}.\end{aligned}$$

Since $\int_X GA_\varpi u\overline{A_\varpi u}$ is uniformly bounded by the inductive hypothesis, by Lemma 7.4 we see

$$\frac{1}{2}\big(\|D_rA_\varpi u\|^2_{L^2} + \|r^{-1}\mathsf{F}A_\varpi u\|^2_{L^2}\big) + \|BA_\varpi u\|^2_{L^2}$$

is uniformly bounded as $\varpi \to 0$. It follows that $\|A_\varpi u\|_{\mathcal{H}}$ is uniformly bounded as $\varpi \to 0$. Thus there is a subsequence of $A_\varpi u$ which converges weakly in $\mathcal{H}$. Since $A_\varpi \to A$, we see the weak limit is $Au \in \mathcal{H}$ so that $q \notin \mathrm{WF}^{s+\frac{1}{2}}_{b\mathsf{F},\mathcal{H}}$, as desired.

□

We also state a less precise version of this result, involving only b-regularity:

Corollary 7.6. *Let $u \in H^{-N}_{b\mathsf{F},\mathcal{H}}(X)$ for some $N \in \mathbb{R}$. Then for any $m \in \mathbb{R} \cup \{+\infty\}$,*

$$\mathrm{WF}^{m+1}_{b\mathsf{F}}(u) \setminus \mathrm{WF}^{m-1}_{b\mathsf{F}}(Pu) \subset {}^b\dot{S}^*X.$$

Proof. This follows directly from Proposition 7.5 and the wavefront set inclusions in Lemma 6.8. □

8. Law of reflection

In this section we prove our propagation of singularities result using a positive commutator argument. We begin with a lemma giving the explicit form of the relevant commutator.

Lemma 8.1. *Let $A \in \Psi^m_{b\mathsf{F}}(X)$ with real principal symbol. Then*

$$i[\Box_g, A^*A] = D_r^*L_rD_r + (r^{-1}\mathsf{F})L_{\mathsf{F}}(r^{-1}\mathsf{F}) + D_r^*L' + L''D_r + L_0. \tag{29}$$

where

- $L_r, L_{\mathsf{F}} \in \Psi^{2m-1}_{b\mathsf{F}}$, $\sigma_b^{2m-1}(L_\bullet) = 4a\partial_\xi a$;

- $L', L'' \in \Psi^{2m}_{b\mathsf{F}}(X)$, $\sigma_b^{2m}(L') = \sigma_b^{2m}(L'') = 2a\partial_r a$
- $L_0 \in \Psi^{2m+1}_{b\mathsf{F}}(X)$, $\sigma_b^{2m+1}(L_0) = 4\tau a\partial_t a$.

Proof. Writing

$$\Box_g = D_t^2 - D_r^* D_r - r^{-2}(\mathsf{F})^2,$$

we first note simply that $D_t^2 \in \Psi^2_{b\mathsf{F}}(X)$, so that by ordinary properties of the calculus,

$$i[D_t^2, A^*A] \in \Psi^{2m+1}_{b\mathsf{F}}(X)$$

with symbol

$$2\tau\partial_t(a^2),$$

yielding the term L_0.

Writing

$$[D_r^* D_r, A^*A] = D_r^*[D_r, A^*A] - [D_r, A^*A]^* D_r.$$

By Lemma 4.6,

$$[D_r, A^*A] = E + FD_r$$

where $\sigma_b^{2m}(E) = -i\partial_r(a^2)$ and $\sigma_b^{2m-1}(F) = -i\partial_\xi(a^2)$. Since these are purely imaginary,

$$-[D_r, A^*A]^* = E' + D_r^* F'$$

where E' and F' have the same symbols as E and F respectively. Thus

$$[D_r^* D_r, A^*A] = D_r^* L_r D_r + D_r^* L' + L'' D_r.$$

Here $\sigma_b^{2m}(L_r) = -2i\partial_r(a^2),$.

Finally, F commutes with A^*A by fiber-invariance, and by Lemma 4.6

$$[r^{-2}, A^*A] = r^{-1}[r^{-1}, A^*A] + [r^{-1}, A^*A]r^{-1}$$

equals $r^{-1}L_{\mathsf{F}}r^{-1}$ where $L_{\mathsf{F}} \in \Psi^{s-1}_{b\mathsf{F}}$ has principal symbol $-2i\partial_\xi(a^2)$. Note that we have used our freedom to write $r^{-1}W = W'r^{-1}$ where W, W' have the same principal symbols. $\Box$

We now state our main propagation theorem in precise form.

Theorem 8.2. *Let $u \in H^{-\infty}_{b\mathsf{F},\mathcal{H}}(X)$ be a solution to $Pu = f$. Let $\varrho = (\varphi_0, \tau_0) \in {}^b\dot{S}^*_{\partial X} X_{\mathsf{F}}$ and let U be an open neighborhood of ϱ in ${}^bS^*X_{\mathsf{F}}$. Then (for either choice of $\pm$ below),*

$$\begin{aligned} &\mathrm{WF}^s_{b\mathsf{F},\mathcal{H}}\, u \cap U \cap \{\pm\xi < 0\} = \varnothing, \\ &\mathrm{WF}^{s+1}_{b\mathsf{F},\mathcal{H}^*}\, f \cap U = \varnothing \end{aligned} \implies \varrho \notin \mathrm{WF}^s_{b\mathsf{F},\mathcal{H}}\, u.$$

Remark 8.3.

(1) Recall that since $\varrho \in {}^b\dot{S}^*_{\partial X} X_{\mathsf{F}}$, absence of ϱ from the wavefront set is a *global, fiber invariant* statement (as are the hypotheses, with uniformity along the fibers built in).

(2) Say $f=0$. By elliptic regularity there is a priori no wavefront set of u in $\xi \neq 0$ over ∂X, so the hypothesis of the theorem can be viewed as dealing only with the interior of X, where the points with $\pm\xi < 0$ are those that are "incoming" toward the string or "outgoing" from it. In particular, say we assume that there is no wavefront set of u (uniformly) near all rays with $\hat\tau = \tau_0$ striking a single fiber $\varphi - t/\mathsf{A} = \varphi_0$, i.e., near the set $\mathscr{F}_{I,\varphi_0,\tau_0}$; to further clarify signs, let us take $\tau_0 = +1$ so that $\xi > 0$ on $\mathscr{F}_{I,\varphi_0,\tau_0}$. The uniformity assumption means there exist $r_0 > 0$, $\delta > 0$ such that (using the homogeneous "hat" coordinates of (9))

$$\{r \in (0, r_0),\ |\varphi - t/\mathsf{A} - \varphi_0| < \delta,\ \mp\hat\xi \in (0,\delta),\ \operatorname{sgn}\tau = \tau_0\} \cap \operatorname{WF} u = \varnothing,$$

with uniform estimate (i.e. by testing by an element of $\Psi_{b\mathsf{F}}(X)$). In particular, then, for $r < \min(r_0, \delta/2)$, and $|\varphi - t/\mathsf{A} - \varphi_0| < \delta$, $\operatorname{sgn}\tau = \tau_0$, the points with $\hat\xi \in (0,\delta)$ are not in the wavefront set of u; additionally $\{\hat\xi > \delta\} \cap \operatorname{WF} u = \varnothing$ since this set is disjoint from Σ. The set $r=0, \hat\xi < 0$ is also disjoint from the wavefront set by elliptic regularity. Hence we have shown that the hypotheses of Theorem 8.2 are satisfied at (φ_0, τ_0).

Thus the theorem can be interpreted as saying that uniform regularity along $\mathscr{F}_{I,\varrho}$ yields regularity at ϱ itself, and hence along $\mathscr{F}_{O,\varrho}$ as well, by closedness of wavefront set. Hence this is the sought-after propagation of singularities into and out of the fiber.

The theorem includes the converse statement as well: outgoing regularity yields incoming regularity, by backward-in-time propagation. Note, though, that if $\Box_g u = f$, then setting $\tilde u = u(-t,x)$ and $\tilde f = f(-t,x)$ implies

$$\widetilde{\Box_g}\tilde u = \tilde f,$$

where $\widetilde{\Box_g}$ is the the d'Alembertian for the cosmic string with A replaced by $-\mathsf{A}$. Hence it will suffice to show that incoming regularity implies outgoing regularity (for every value of A). It also suffices to fix one sign of τ_0 : since $\Box_g$ has real coefficients, applying a propagation theorem valid for $\tau_0 > 0$ to the equation $\overline{P}\overline{u} = \overline{f}$ proves the corresponding result for $\tau_0 < 0$.

(3) Consider a solution u to $Pu = 0$ that has a single bicharacteristic in the wavefront set arriving at ∂X at a point $\varrho \in {}^bS^*_{\partial X}X$. Applying the theorem in all fibers *except* that containing ϱ tells us that a single point in $\operatorname{WF}_{b\mathsf{F},\mathcal{H}} u$ striking the cosmic string may at most produce $\operatorname{WF}_{b\mathsf{F},\mathcal{H}} u$ leaving the string *everywhere* along the fiber of ϱ, in the same τ-component of the characteristic set; this is to say that $\varphi - t/\mathsf{A}$ and τ are conserved in the interaction, but t is (apparently) not.

(4) When one proves microlocal propagation of regularity, the contrapositive is usually interpreted as yielding propagation of wavefront set along bicharacteristics, perhaps of a generalized sort. Here, the

generalization would have to be quite broad to permit such a statement: a singularity arriving at $r = 0$ along some ray in $\mathscr{F}_{I,\varphi_0,\tau_0}$ must result in a failure of uniform wavefront set estimates along all rays in $\mathscr{F}_{O,\varphi_0,\tau_0}$. It is the *uniformity* that is the difficulty here: rather than produce wavefront set along some particular outgoing ray, the effect might be merely to produce nonuniformity of estimates along this family of rays as $|t| \to \infty$. An appropriately defined notion of wavefront set at timelike infinity might allow us to recover a more traditional propagation statement, but we will not pursue it here.

(5) Our hypotheses on the inhomogeneity f are phrased in terms of regularity with respect to $\mathcal{H}^*$, the dual space to $\mathcal{H}$, which away from $r = 0$ agrees with H^{-1}; thus the hypotheses involve one less derivative on f than the concluded regularity on u, as befits a second-order hyperbolic equation.

Proof. The strategy of proof is descended from the work of Melrose–Sjöstrand [13], [14] and is more directly inspired by the presentations of Vasy [18] and then Melrose–Vasy–Wunsch [11].

As noted in the remarks above, it suffices to consider the case where our wavefront set assumption is in $\hat{\xi} > 0$ and we also have $\tau_0 > 0$.

We construct a test operator A as follows. Define

$$\omega = r^2 + (\varphi - t/\mathsf{A} - \varphi_0)^2$$

(where as always, we view $\varphi - t/\mathsf{A}$ and $\varphi_0 \in \mathbb{R}/2\pi\mathbb{Z}$ as equivalence classes with the distance squared being the shortest of the possible values) and set

$$\phi = -\hat{\xi} + \frac{1}{\beta^2\delta}\omega$$

where β and δ are parameters to be set later. Let χ_0 be smooth, supported in $[0, \infty)$, and with $\chi_0(s) = e^{-1/s}$ for $s > 0$, hence $\chi_0'(s) = s^{-2}\chi_0(s)$. Let χ_1 be supported on $[0, \infty)$ and equal to 1 on $[1, \infty)$ and with $\chi_1' \geq 0$ and supported in $(0, 1)$. Let $\chi_2 \in \mathcal{C}_c^\infty(\mathbb{R})$ be supported in $[-2c_1, 2c_1]$ and equal to 1 on $[-c_1, c_1]$ for some $c_1 > 0$. Let $\chi_3 = \chi_1$. Our test operator A will be defined to have principal symbol

$$a = \chi_0\big(1 - \frac{\phi}{\delta}\big)\chi_1\big(-\frac{\hat{\xi}}{\delta} + 1\big)\chi_2(\hat{\xi}^2 + (\mathsf{A}\hat{\tau} + \hat{\eta})^2)\chi_3(\operatorname{sgn}(\tau_0)\hat{\tau}).$$

This symbol is fiber invariant, as it depends on t, φ only via $\varphi - t/\mathsf{A}$, hence we may quantize it to a fiber invariant operator $A \in \Psi^0_{b\mathsf{F}}(X)$.

Note that on $\operatorname{supp}\chi_1(\bullet)$, $\hat{\xi} \leq \delta$. Meanwhile, on $\operatorname{supp}\chi_0(\bullet)$, $\omega < \beta^2\delta^2 + \beta^2\delta\hat{\xi}$; owing to the support constraints on $\hat{\xi}$, then, $\omega < 2\delta^2\beta^2$ (and thus both r and $|\varphi - t/\mathsf{A} - \varphi_0|$ are bounded above by $\sqrt{2}\delta\beta$). On $\operatorname{supp}\chi_0(\bullet)$, $\hat{\xi} \geq -\delta$, so overall $\hat{\xi} \in [-\delta, \delta]$ on $\operatorname{supp} a$. Note also, for later use, that $|1 - \phi/\delta| < 4$ on $\operatorname{supp} a$.

The parameter β will later be used to obtain sufficient positivity of commutator terms, and may need to be taken large; thus, note that *for any*

$\beta \in (0,\infty)$ *there exists* $\delta > 0$ *such that* $\operatorname{supp} a \cap \{\hat{\xi} > 0\} \subset U$. We alert the reader that at the moment when β is taken large in the following argument, we simultaneously adjust $\delta = \delta(\beta)$ to maintain the desired support properties.

Now applying Lemma 8.1 (as well as recalling that $[\Upsilon, A^*A] \in \Psi^0_{b\mathsf{F}}(X)$ since Υ is a uniform b-operator of order 1) and pairing with u yields

(30)

$$\begin{aligned} 2\operatorname{Im}\langle f, A^*Au\rangle &= \langle i[P, A^*A]u, u\rangle + \langle i(P-P^*)A^*Au, u\rangle \\ &= \langle (D_r^* L_r D_r + (r^{-1}\mathsf{F})L_{\mathsf{F}}(r^{-1}\mathsf{F}) + D_r^*L' + L''D_r + L_0)u, u\rangle \\ &\quad + \langle i[A^*A, \Upsilon]u, u\rangle + \langle i(P-P^*)A^*Au, u\rangle \\ &= \langle (D_r^* L_r D_r + (r^{-1}\mathsf{F})L_{\mathsf{F}}(r^{-1}\mathsf{F}) + D_r^*L' + L''D_r + L_0 + \tilde{\Upsilon}A^*A)u, u\rangle \\ &\quad + \langle R_0' u, u\rangle \end{aligned}$$

where the L operators are defined as in Lemma 8.1, $\tilde{\Upsilon} = i(\Upsilon - \Upsilon^*) \in \operatorname{Diff}^1_{bu}(X)$, and there are further non-fiber-invariant terms $R_0' \in \Psi^0_{bu}(X)$ arising from commuting with Υ.

Now let $B \in \Psi^{-1/2}_{b\mathsf{F}}$ have symbol

$$b = 2|\tau|^{-1/2}\delta^{-1/2}(\chi_0\chi_0')^{1/2}\chi_1\chi_2\chi_3.$$

This is the quantity appearing in commutator terms with derivatives hitting the χ_0^2 factor of a^2; its support is that same as $\operatorname{supp} a$. In particular,

$$\sigma_b^{-1}(L_r) = \sigma_b^{-1}(L_{\mathsf{F}}) = 4a\partial_\xi a = b^2 + e' + e''$$

where $\operatorname{supp} e' \subset \{\hat{\xi} > 0\}$, and $\operatorname{supp} e'' \cap {}^b\dot{T}^*X = \varnothing$: the b^2 term comes, as noted above, from the derivative falling on χ_0, while the term with a derivative falling on χ_1 gives e' and the term with a derivative on χ_2 gives e''. Note, for later use, that owing to the specific form of the cutoff function χ_0, we have arranged that a^2b^{-1} is a smooth F-invariant symbol of order 1/2.

Thus,

(31)

$$\begin{aligned} D_r^* L_r D_r + (r^{-1}\mathsf{F})L_{\mathsf{F}}(r^{-1}\mathsf{F}) = D_r^*B^*BD_r + (r^{-1}\mathsf{F})B^*B(r^{-1}\mathsf{F}) + D_r^*(E' + E'' + R_{-2})D_r \\ + (r^{-1}\mathsf{F})(E' + E'' + R_{-2})(r^{-1}\mathsf{F}) \end{aligned}$$

where

- $E' \in \Psi^{-1}_{b\mathsf{F}}(X)$, $\operatorname{WF}_b'(E') \subset \hat{\xi}^{-1}((0,\infty))$
- $E'' \in \Psi^{-1}_{b\mathsf{F}}(X)$, $\operatorname{WF}_b'(E'') \cap {}^b\dot{T}^*X = \varnothing$.
- $R_{-2} \in \Psi^{-2}_{b\mathsf{F}}(X)$.
- And where E', E'', R_{-2} will now mean potentially *different* operators with the above properties in each occurrence of these symbols.

Now we turn to the D_r, D_r^* terms in (30). We have $\sigma_b^0(L') = \sigma_b^0(L'') = 2a\partial_r a$, and since the only r-dependence of a is in the $\chi_0(1 - \phi/\delta)$ factor and

$\chi_0\chi_0'\chi_1^2\chi_2^2\chi_3^2 = \frac{1}{4}|\tau|\delta b^2$ we may write these symbols as

$$-\frac{1}{2}|\tau|\delta b^2\partial_r\phi = -|\tau|\frac{1}{\beta^2\delta}rb^2.$$

Recall that $r < \sqrt{2}\delta\beta$ on $\operatorname{supp} b$, so that in fact

$$\sigma_b^0(L') = b^2 f_1'$$

for $f_1' \in S^1$ with

$$|f_1'| \leq \sqrt{2}|\tau|\beta^{-1},$$

supported on any desired open neighborhood of $\operatorname{supp} b$. Analogously, $\sigma_b^0(L'') = b^2 f_1''$ with the same properties.

Likewise, the symbol of the L_0 term involves only t derivatives falling on χ_0, hence

$$\sigma_b^1(L_0) = -\tau|\tau|b^2\partial_t\phi = 2\mathsf{A}^{-1}\tau|\tau|\frac{1}{\beta^2\delta}(\varphi - t/\mathsf{A} - \varphi_0)b^2;$$

Again we thus find

$$\sigma_b^1(L_0) = b^2 f_2$$

where f_2 is estimated by

$$|f_2| \leq 2\sqrt{2}\mathsf{A}^{-1}|\tau|^2\beta^{-1},$$

since $|\varphi - t/\mathsf{A} - \varphi_0| < \sqrt{2}\beta\delta$ on $\operatorname{supp} b$.

We now consider the $\tilde{\Upsilon}A^*A$ term, which is also first order. Note that due to the relationship $\chi_0'(s) = s^{-2}\chi_0(s)$ we have

$$a^2 = \frac{1}{2}|\tau|\delta\left(1 - \frac{\phi}{\delta}\right)b^2,$$

and as noted above, $|1 - \phi/\delta| \leq 4$ on the support of b. Then taking $\sigma_b^1(\tilde{\Upsilon}) = \upsilon$ we have

$$\sigma_b^1(\tilde{\Upsilon}A^*A) = \frac{\upsilon}{2}|\tau|\delta\left(1 - \frac{\phi}{\delta}\right)b^2 = \tilde{f}_2 b^2$$

where

$$|\tilde{f}_2| \leq 2|\upsilon\tau|\delta.$$

In the following argument we will pick β large to make f_1' and f_2 above sufficiently small. As noted earlier, there is then a $\delta(\beta) > 0$ for which a has the requisite support properties whenever $\delta \leq \delta(\beta)$. To make $\tilde{f}_2$ sufficiently small, we pick $\delta \leq \delta(\beta)$.

Assembling this information yields

$$\begin{aligned} L' &= B^*BF_1' + R_{-1}, \\ L'' &= B^*BF_1'' + R_{-1}, \\ L_0 + \tilde{\Upsilon}A^*A &= B^*BF_2 + R_0' \end{aligned}$$

where F_1', F_1'' have symbols f_1', f_1'', F_2 has symbol $f_2 + \tilde{f}_2$, where $R_s \in \Psi_{b\mathsf{F}}^s(X)$, and where $R_0' \in \Psi_{bu}^0(X)$ (with the notation recycled to indicate a different remainder term in each case).

Thus, absorbing further lower order commutator terms in the ever-changing $R_\bullet$ terms below and including a $(1/2)B^*B\Upsilon$ term into R_0' gives

$$\begin{aligned}
&2\operatorname{Im}\langle f, A^*Au\rangle\\
&\quad= \Big\langle \big(B^*B(D_r^*D_r + r^{-2}\mathsf{F}^2) + D_r^*(E' + E'' + R_{-2})D_r\\
&\qquad + (r^{-1}\mathsf{F})(E' + E'' + R_{-2})(r^{-1}\mathsf{F}) + R_{-2}r^{-1}\mathsf{F} + D_r^*(B^*BF_1' + R_{-1})\\
&\qquad + B^*BF_2 + R_0'\big)u, u\Big\rangle\\
&\quad= \Big\langle \frac{1}{2}B^*B(D_r^*D_r + r^{-2}\mathsf{F}^2 + D_t^2 - P) + D_r^*(E' + E'' + R_{-2})D_r\\
&\qquad + (r^{-1}\mathsf{F})(E' + E'' + R_{-2})(r^{-1}\mathsf{F}) + R_{-2}r^{-1}\mathsf{F} + D_r^*(B^*BF_1' + R_{-1})\\
&\qquad + B^*BF_2 + R_0'\big)u, u\Big\rangle\\
&\quad= (1/2)\big(\|D_tBu\|^2 + \|r^{-1}\mathsf{F}Bu\|^2 + \|D_rBu\|^2 - \langle f, B^*Bu\rangle\big)\\
&\qquad + \langle (E' + E'' + R_{-2})D_ru, D_ru\rangle + \langle (E' + E'' + R_{-2})(r^{-1}\mathsf{F})u, r^{-1}\mathsf{F}u\rangle\\
&\qquad + \langle BF_1'u, D_rBu\rangle + \langle BF_2u, Bu\rangle + \langle R_{-2}r^{-1}\mathsf{F}u, u\rangle + \langle R_{-1}u, D_ru\rangle + \langle R_0'u, u\rangle
\end{aligned}$$

Note that for brevity, we have dropped terms of the form $(\bullet)D_r$ in the first line favor of terms $D_r^*(\bullet)$, since they have the same imaginary part modulo commutator terms absorbed elsewhere. As before, R_0' refers to a non-fiber-invariant operator.

We now proceed with our inductive argument. Suppose that

$$\varrho \notin \mathrm{WF}^m_{b\mathsf{F},\mathcal{H}}\, u$$

with $m \leq s - 1/2$; it will suffice to show that

$$\varrho \notin \mathrm{WF}^{m+1/2}_{b\mathsf{F},\mathcal{H}}\, u.$$

To this end we shift around the orders in the commutator computation as follows: for $\varpi \in (0, 1)$, fix

$$Q^s_\varpi \in \Psi^s(\mathbb{R})$$

given by left quantization of

$$\langle \tau\rangle^s(1 + \varpi|\tau|^2)^{-s/2},$$

and commuting with $\Box_g$. Thus, Q^s_ϖ is a family that is uniformly bounded in $\Psi^s(\mathbb{R})$, convergent in $\Psi^{s+\epsilon}(\mathbb{R})$ to $\langle D_t\rangle^s$ modulo a fixed, s-dependent smoothing operator. We would like to view Q^s_ϖ as lying in the calculus $\Psi_{b\mathsf{F}}(X)$ but the mild technical snag is that q is not a symbol in (ξ, τ, η). However, it is such a symbol when cut off away from $\xi = \eta = 0$, hence setting

$$A_{m+1} \equiv AQ^{m+1}_\varpi$$

we easily see that if $A = \widetilde{\mathrm{Op}}_{\mathrm{b}}(a)$ then $A_{m+1} = \widetilde{\mathrm{Op}}_{\mathrm{b}}(aq)$ and lies in our calculus, hence we may treat it for all practical purposes as lying in the calculus.[3] We omit the index ϖ from the notation in order to keep it uncluttered, but here and henceforth on we decorate operator families bounded in $\Psi^s_{b\mathsf{F}}(X)$ with the index s as a bookkeeping aid. We will use consistent notation for the operator B letting $B_{m+1/2}$ denote the family $Q^{m+1}_{\varpi}B$ (hence again indexing the family by the order of the operator space in which it is bounded). Note that the operators $F_\bullet$ previously had a subscript referring to the index, and these remain unchanged (indeed, they are not families). The E', E'', R operators below are replaced by families as well, with the same indexing convention employed.

We then have

$$\begin{aligned}
&2\,\mathrm{Im}\langle f, A^*_{m+1}A_{m+1}u\rangle = \\
&\quad= (1/2)\big(\|D_tB_{m+1/2}u\|^2 + \|r^{-1}\mathsf{F}B_{m+1/2}u\|^2 + \|D_rB_{m+1/2}u\|^2 \\
&\qquad - \langle f, B^*_{m+1/2}B_{m+1/2}u\rangle\big) + \langle (E'_{2m+1} + E''_{2m+1} + R_{2m})D_ru, D_ru\rangle \\
&\qquad + \langle (E'_{2m+1} + E''_{2m+1} + R_{2m})(r^{-1}\mathsf{F})u, r^{-1}\mathsf{F}u\rangle \\
&\qquad + \langle B_{m+1/2}F'_1u, D_rB_{m+1/2}u\rangle + \langle B_{m+1/2}F_2u, B_{m+1/2}u\rangle \\
&\qquad + \langle R_{2m+1}u, D_ru\rangle + \langle R_{2m}u, r^{-1}\mathsf{F}u\rangle + \langle R'_{2m+2}u, u\rangle,
\end{aligned}$$

again with $R'_{2m+2} \in \Psi^{2m+2}_{bu}(X)$, now including terms arising from $[\Upsilon, Q_\varpi]$. We will treat the first three terms on the RHS as the main positive terms, either absorbing the rest of the terms into them or else estimating them by induction or regularity hypothesis.

By elliptic regularity, i.e. the quantitative statement (by the closed graph theorem or inspection of the proof) obtained from Proposition 7.5,

$$(1/2)\big(\|D_tB_{m+1/2}u\|^2 + \|r^{-1}\mathsf{F}B_{m+1/2}u\|^2 + \|D_rB_{m+1/2}u\|^2\big) \geq c\|B_{m+1/2}u\|^2_{\mathcal{H}}$$

for some $c > 0$. Owing to the symbol estimates on the F terms, by taking β sufficiently large and δ sufficiently small, we may apply Cauchy–Schwarz to estimate

$$|\langle B_{m+1/2}F'_1u, D_rB_{m+1/2}u\rangle + \langle B_{m+1/2}F_2u, B_{m+1/2}u\rangle| \leq (c/2)\|B_{m+1/2}u\|^2_{\mathcal{H}},$$

thus allowing us to absorb these terms in the main positive term. Meanwhile, since $\sigma_b(A^*_{m+1}A_{m+1})^2/\sigma_b(B_{m+1/2}) \equiv g \in S^{m+3/2}_{\mathsf{F}}$, we may apply Cauchy–Schwarz and the b-symbol calculus to estimate

$$|\langle f, A^*_{m+1}A_{m+1}u\rangle| \leq \|G_{m+3/2}f\|_{\mathcal{H}^*}\|B_{m+1/2}u\|_{\mathcal{H}} + \|R_{m+1}f\|_{\mathcal{H}^*}\|R_mu\|_{\mathcal{H}}$$

with $R_s \in \Psi^s_{b\mathsf{F}}(X)$, microsupported in an arbitrary neighborhood of $\mathrm{WF}'_{b\mathsf{F}}\,B$. Since by assumption, $m + 3/2 \leq s + 1$, the G term above is bounded by the assumption that $\varrho \notin \mathrm{WF}^{s+1}_{b\mathsf{F},\mathcal{H}^*}\,f$; the second term on the RHS is bounded

[3] Similar considerations famously occur in the study of $D_t - \sqrt{\Delta_g}$ on $\mathbb{R} \times M$ with M a compact manifold: $\sqrt{\Delta_g}$is not a pseudodifferential operator on the product, but this is of little importance; see, e.g., [3].

by the same estimate on f and by the inductive assumption $\varrho \notin \mathrm{WF}^m_{\mathsf{bF},\mathcal{H}}\, u$. The lower order term $\langle f, B^*_{m+1/2} B_{m+1/2} u\rangle$ is likewise easily estimated in the same manner.

Turning to other terms, we find that the E' terms are uniformly bounded as $\varpi \downarrow 0$ by elliptic regularity; the E'' terms are uniformly bounded by our wavefront set hypothesis, since they are (uniformly) microsupported in the control region $\xi > 0$; all $R_\bullet$ terms are uniformly bounded by our inductive assumption $\varrho \notin \mathrm{WF}^m_{\mathsf{bF},\mathcal{H}}\, u$; the R'_{2m+2} term is estimated by the inductive assumption and Corollary 5.5.

Putting together the above observations (and lumping the bounded terms described above, with the exception of $\|Gu\|$, into a single constant) yields

$$(c/2)\|B_{m+1/2}u\|^2_{\mathcal{H}} \leq \|G_{m+3/2}f\|_{\mathcal{H}^*}\|B_{m+1/2}u\|_{\mathcal{H}} + C,$$

hence by a further Cauchy–Schwarz, boundedness of $\|G_{m+3/2}f\|_{\mathcal{H}^*}$ yields uniform boundedness of $\|B_{m+1/2}u\|_{\mathcal{H}}$ as $\varpi \downarrow 0$. A standard compactness argument now implies that

$$\varrho \notin \mathrm{WF}^{m+1/2}_{\mathsf{bF},\mathcal{H}}\, u,$$

and the inductive step is complete. □

Before proceeding to employ our propagation result to obtain an existence theorem, we record a corollary that is essentially just a quantitative restatement.

Let $\mathrm{proj} : {}^b\dot{T}^*X \to X_{\mathsf{F}}$ be the projection map from the compressed b-cotangent bundle onto the fiber-quotiented base space.

Corollary 8.4. *Fix any constants $r_1 > r_0 > 0$, and let $Z \in \Psi^0_{\mathsf{bF}}(X)$ be elliptic on $\mathscr{F}_I \cap \mathrm{proj}^{-1}(\{r \in [r_0, r_1]\})$. There exists $B \in \Psi^0_{\mathsf{bF}}(X)$, elliptic on ${}^b\dot{T}^*_{\partial X}X_{\mathsf{F}}$, such that for all N there exists C such that*

$$\|Bu\|_{H^s_{\mathsf{bF},\mathcal{H}}} \leq C\big(\|Zu\|_{H^s_{\mathsf{bF},\mathcal{H}}} + \|u\|_{H^{-N}_{\mathsf{bF},\mathcal{H}}} + \|Pu\|_{H^{s+1}_{\mathsf{bF},\mathcal{H}^*}}\big) \tag{32}$$

This result can of course be localized in fibers and microlocalized in τ, but this global version is all we need below.

Proof. Ordinary propagation of singularities in the interior allows us to take r_0 as small as desired. By the observations in item 2 of Remark 8.3, control of Zu plus elliptic regularity implies that the hypotheses of the propagation theorem are fulfilled (with appropriate choice of signs) at every ϱ, i.e., we have the desired regularity for all values of φ_0 and τ_0, either in the region $\xi > 0$ for $\tau_0 > 0$ or $\xi < 0$ for $\tau_0 < 0$. Thus, the theorem (technically either with the closed graph theorem applied to obtain the quantitative statement from the qualitative one, or else from direct examination of the quantitative estimate that proves the theorem) yields the estimate (32) where B is microsupported near any desired $\varrho \in {}^b\dot{T}^*_{\partial X}X_{\mathsf{F}}$. Since the cosphere bundle of ${}^b\dot{T}^*_{\partial X}X_{\mathsf{F}}$ is compact (with coordinates $\varphi_0 \in S^1$, $\hat{\tau} \in \pm 1$), we can sum up these estimates and assume B is elliptic on all ${}^b\dot{T}^*_{\partial X}X_{\mathsf{F}}$. □

9. Causal solutions to the wave equation

We now consider the problem of finding causal solutions to the wave equation: given an appropriate f, we wish, e.g., to find u such that

$$Pu = f$$

in the sense of distributions and such that the support of u is in the asymptotically forward-in-time Hamilton flowout of the support of f. Recall the notion of forward-in-time flowout is not very well-defined locally, as t is not monotone along some bicharacteristics. However, along every bicharacteristic not reaching $r = 0$, t is *eventually* monotone along the flow. Whether t is aymptotically increasing or decreasing on such bicharacteristics is determined by the sign of τ, a conserved quantity.

Thus for a set $S \subset \Sigma$, let $\Phi_\pm(S)$ denote the maximally extended generalized flowout of S in the asymptotically forward/backward time-direction, over X°:

$$\Phi_\pm(q) = \bigcup_{s \cdot \operatorname{sgn} \tau(q) \gtrless 0} \Phi_s(q)$$

where on incoming/outgoing rays we take $\Phi_s(q)$ to be undefined for parameter values beyond those where it reaches $r = 0$.

We will not trouble to define the broken flow across the string at $r = 0$, since the global appearance of singularities there seems unavoidable.

The goal stated above turns out to be too much to ask: we cannot expect nontrivial solutions supported in $t > T_0$ for any T_0, e.g. since the mode-by-mode equation obtained by separation of variables in $\Box_g$ is *elliptic* for $r < \mathsf{A}$ and hence enjoys unique continuation. (See [15] for a discussion of the angular mode equation.) So we accept instead a solution on a(n arbitrarily large) compact set with the minimal wavefront set that the propagation theorem above would permit: the forward flowout of $\mathrm{WF}\, f$ together with the forward flowout of the string itself, for all t. (Again, cf. [15], where the authors performed similar analysis for angular mode solutions; this simplified setting did not allow the excitation of singularities emerging from the string, however.)

Recalling that

$$P = \Box_g + \Upsilon$$

where $\Upsilon \in \mathrm{Diff}^1_{bu}(X)$ is a first-order perturbation, we now strengthen our hypotheses on Υ in order to obtain some necessary unique continuation results for the perturbed equation. We offer two different auxiliary hypotheses on Υ that yield stronger solvability results.

Hypothesis $\mathcal{C}^\omega$. Υ has analytic coefficients.

Hypothesis (t, φ). $[\partial_t, \Upsilon] = [\partial_\varphi, \Upsilon] = 0$.

Theorem 9.1. *Given compact sets $K_0 \subset K \subset X$ with $K_0 \subset K^\circ$, there exists a finite dimensional space $N \subset H_b^\infty(X) \cap \mathscr{E}'(X)$ such that if $f \in$*

$\dot{H}_b^m(K_0) \cap N^\perp$ *(L^2 orthocomplement) there exists* $u \in H_b^{m+1}(K^\circ)$ *with*

$$Pu = f \quad \text{on } K^\circ.$$

Furthermore, u satisfies

$$\operatorname{WF}_{\mathsf{bF}} u \setminus \operatorname{WF}_{\mathsf{bF}} f \subset \mathscr{F}_O \cup \Phi_+(\operatorname{WF}_{\mathsf{bF}} f \cap \Sigma). \tag{33}$$

The solution u is unique on K° *modulo a distribution w with* $\operatorname{WF}_{\mathsf{bF},\mathcal{H}} w \subset \mathscr{F}_O$.

If Υ *additionally satisfies either Hypothesis* $\mathcal{C}^\omega$ *or* (t,φ)*, then a (unique in the above sense) solution exists for* all $f \in \dot{H}_b^m(K_0)$.

Remark 9.2.

(1) Here in the existence theorem we have dropped the refinement of Sobolev spaces based on $\mathcal{H}$ in favor of the simpler H_b^m spaces. (We do still need the refined notion of wavefront set in employing our propagation statements.) We will use results obtained above in the setting of the H_{bF}^m scale of spaces, and remind the reader (cf. Remark 5.2) that the distinction between H_b^m and H_{bF}^m is moot over compact sets.

(2) More precise statements than the basic "solvability in H_b^{m+1} for data in H_b^m" in fact hold, using $\mathcal{H}$-based spaces. So, for instance, the method used here shows that for every $f \in H_{\mathsf{bF},\mathcal{H}^*}^{-1}$ with $\operatorname{supp} f \subset \{r < R_0\}$, there exists $u \in (H_{\mathsf{bF},\mathcal{H}^*}^2 \cap L^2)^*$ solving the equation as above. Such estimates have the virtue of treating radial and fiber-derivatives differently from arbitrary b-derivatives. Likewise, we note that we can obtain the more precise statement that the set of obstructions N is contained in $H_{\mathsf{bF},\mathcal{H}}^\infty \cap \mathscr{E}'(X)$.

(3) Even in the argument as given below in terms of b-Sobolev spaces, the reader will note that our hypotheses on f are in fact that it lie the dual space of an *intersection* of Sobolev spaces. This would enable us to give alternate hypotheses on f involving higher regularity but a more singular weight at $r = 0$.

(4) A more precise version of the wavefront relation follows from our elliptic regularity statements and interior propagation of singularities:

$$\operatorname{WF}_{\mathsf{bF},\mathcal{H}} u \setminus \operatorname{WF}_{\mathsf{bF},\mathcal{H}^*} f \subset \mathscr{F}_O \cup \Phi_+(\operatorname{WF}_{\mathsf{bF},\mathcal{H}^*} f \cap \Sigma). \tag{34}$$

(5) The assumption that f is a compactly supported distribution is needed to obtain the uniqueness result but not the existence result, which merely requires f be an extendible distribution defined on K_0.

We now prove Theorem 9.1. Assume without loss of generality (expanding and time-translating K if necessary) that

$$K = \{r \le R_0,\ t \in [0,T]\},$$

and moreover that $R_0 > |\mathsf{A}|$. Note that this means that for $r > R_0$, under the bicharacteristic flow (in the b-cotangent variables)

$$\dot{t} = r^2\tau - \mathsf{A}(\mathsf{A}\tau + \eta)$$

has the same sign as τ on the characteristic set, hence t is monotone along the null bicharacteristic flow in this region.

Let $\mathrm{proj} : {}^bT^*X \to X$ be the projection map from the b-cotangent bundle onto the base space.

Lemma 9.3. *There exists $R > R_0$ as above and $T' \gg 0$ so that for each null bicharacteristic $\gamma(s)$ with $\mathrm{proj}(\gamma(0)) \in K$ and $\gamma(0) \notin \mathscr{F}_O$, there exists s_0 with*

1. $R + 1 < r(\gamma(s_0)) < 2R$
2. $-T' < t(\gamma(s_0)) < t(\gamma(0))$
3. $\dfrac{\xi}{r\tau}|_{\gamma(s_0)} > 3/4$
4. *For s between 0 and s_0,* $\mathrm{proj}(\gamma(s)) \in \{|t| < T',\ r < 2R\}$.

Thus any bicharacteristic from K not emerging from the string comes from the region $r > R + 1$ at backward time, and at that time has a large radial component. It is, moreover, oriented in an incoming direction (property (3)), as it must reach K, where $r < R_0$, as time increases. This must occur within some fixed amount of elapsed backwards time T' (which of course depends on the size of K) and within distance $2R$ of the string.

We also note for later use the simpler fact that sufficiently far back in time, the geodesic lies in $\{r > 2R\}$.

Proof. To begin, we employ the Minkowski coordinates $(t' = t - \mathsf{A}\varphi, x)$ of Section 2.1 near any desired bicharacteristic. Recall that the null geodesic flow is then (up to reparametrization) the lift of the Minkowski geodesic flow

$$x = x_0 + sv,\ t' = t'_0 \pm s,$$

where $|v| = 1$. We use the notation $t(s)$, $x(s)$ instead of $t(\gamma(s))$, $x(\gamma(s))$, etc.

Choose R sufficiently large so that

$$\max\{R_0 + R + 1, 2|\mathsf{A}|\pi\} < 2R - R_0. \tag{35}$$

If we choose s_0 with

$$-2R + R_0 < s_0 < -\max\{R_0 + R + 1, 2|\mathsf{A}|\pi\}, \tag{36}$$

then provided $|x_0| < R_0$,

$$r(s_0) = |x(s)| \geq |s_0| - |x_0| > (R_0 + R + 1) - R_0 = R + 1. \tag{37}$$

Likewise,

$$r(s_0) \leq |s_0| + |x_0| < (2R - R_0) + R_0,$$

establishing (1).

Meanwhile, it is always the case that

$$t \in [t' - |\mathsf{A}|\pi, t' + |\mathsf{A}|\pi],$$

hence, since the flow in t' is likewise very simple, subject to (36) we also certainly have

$$t(s_0) \leq t'(s_0) + |\mathsf{A}|\pi < (t_0' - 2|\mathsf{A}|\pi) + |\mathsf{A}|\pi \leq t(0).$$

On the other hand,

$$t(s_0) \geq t'(s_0) - |\mathsf{A}|\pi > (t_0' - (2R - R_0)) - |\mathsf{A}|\pi = -T'$$

where

$$T' \equiv 2R - R_0 + |\mathsf{A}|\pi.$$

this establishes (2). Note that all the estimates above required was for R to satisfy the inequality (35), hence we may freely increase R in what follows. Note also that increasing R increases T'.

Finally, we turn to (3), which requires examination of the cotangent variables. Assume s_0 satisfies (36). Then

$$\begin{aligned} -\dot{r} &= -\frac{\langle x_0 + s_0 v, v\rangle}{|x_0 + s_0 v|} \\ &= -\frac{\langle x_0, v\rangle + s_0}{|x_0 + s_0 v|} \\ &\geq \frac{|s_0| - R_0}{|s_0| + R_0} \\ &> \frac{1 - R_0/(R_0 + R + 1)}{1 + R_0/(R_0 + R + 1)}; \end{aligned}$$

increasing R as needed, we can thus ensure that for s_0 satisfying (36),

$$\dot{r} \leq -0.9.$$

Likewise, since $r(s_0) > R + 1$ and $|\dot{r}| \leq 1$ since $x(s)$ is parametrized at unit speed,

$$\begin{aligned} \left|\frac{d}{ds}\hat{x}\right| &= \left|\frac{d}{ds}\frac{x}{|x|}\right| \\ &= \left| -\frac{\dot{r}}{r^2}x + \frac{v}{r}\right| \\ &\leq 2(R+1)^{-1} \\ &\leq 0.01/|\mathsf{A}| \end{aligned}$$

provided R is sufficiently large. Hence we certainly have $|\dot{\varphi}| < 0.1/|\mathsf{A}|$ for such s_0, and, finally, since $\dot{t}' = 1$,

$$\dot{t} = \dot{t}' + \mathsf{A}\dot{\varphi} < 1.1$$

at these points. Thus

$$\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} < -\frac{0.9}{1.1} < -0.8. \tag{38}$$

Finally, note from (11) that under the rescaled Hamilton flow employed there (since the change of scaling factor cancels out in numerator and denominator)

$$\begin{aligned}\frac{dr}{dt} &= \frac{-\xi r}{r^2\tau - \mathsf{A}(\mathsf{A}\tau + \eta)}\\ &= -\frac{\xi}{r\tau(1 - r^{-2}\mathsf{A}(\mathsf{A} + \hat{\eta}))}\end{aligned}$$

On the characteristic set over K, $|\mathsf{A} + \hat{\eta}| < R_0$, so certainly $|\mathsf{A}(\mathsf{A} + \hat{\eta})|$ is bounded by some number $\digamma$ on the characteristic set over K. Thus by (38)

$$\frac{\xi}{r\tau(1 - r^{-2}\mathsf{A}(\mathsf{A} + \hat{\eta}))} = -\frac{dr}{dt} > 0.8$$

hence when $r > R + 1$,

$$\frac{\xi}{r\tau} > 0.8(1 - r^{-2}\mathsf{A}(\mathsf{A} + \hat{\eta})) > 0.8(1 - (R + 1)^{-2}\digamma)$$

and taking R large enough ensures that (3) holds.

We now turn to condition (4). To clarify the exposition, assume without loss of generality that $s_0 < 0$—the reverse case merely involves overall sign changes. Since $\ddot{r} > 0$ along the null bicharacteristic flow, r must achieve its maximum on the interval $[s_0, 0]$ at s_0, hence cannot have exceeded $2R$.

As for the t variable, as noted in Section 2.1 it is monotone increasing along the flow when $r > |\mathsf{A}|$, and it cannot ever exceed $t(0) + 2|\mathsf{A}|\pi$ for $s \leq 0$. Its maximum along the flow from K with parameter $s \in [s_0, 0]$ can thus be no larger than $T + 2|\mathsf{A}|\pi$; we increase R as needed to ensure that $T' = 2R - R_0 + |A|\pi$ is larger than this quantity.

Now note that r must exceed $R_0 > |\mathsf{A}|$ whenever $s < -2R_0$ (cf. (37) above). Since

$$t'(s) = t'(0) + s = t(0) - \mathsf{A}\varphi(0) + s,$$

on the interval $s \in [-2R_0, 0]$, $t' \geq -2R_0 - |\mathsf{A}|\pi$ hence $t \geq -2R_0 - 2|\mathsf{A}|\pi$. Thus, if t achieves its minimum on $[-s_0, 0]$ for $s \geq -2R_0$, that minimum must be at least $-(2R_0 + 2|\mathsf{A}|\pi)$. Taking R sufficiently large, we ensure that $T' > 2R_0 + 2|\mathsf{A}|\pi$, hence if the minimum of t on $[s_0, 0]$ arises for $s \in [-2R_0, 0]$, we certainly have $t > -T'$ on $[s_0, 0]$. The same holds if the minimum is not achieved on $[-2R_0, 0]$, since then it must arise at $s = s_0$ by monotonicity of t when $r > |\mathsf{A}|$. Thus the t variable stays within $(-T', T')$ on $[-s_0, 0]$ as desired. □

Recall that $K \subset \{t \in [0, T]\}$. Let K' denote the compact set

$$K' = \{r \leq 2R,\ t \in [-T', T']\} \subset X. \tag{39}$$

Now we construct a complex absorbing potential $W \in \Psi^2_{\mathsf{bF}}(X)$, supported away from $r = 0$, such that

(1) W is elliptic on the set
$$\mathcal{I} \equiv \{r > R+1\} \cap \left\{ \frac{\xi}{r\tau} > 3/4 \right\}.$$
(2) $\operatorname{sgn} \sigma_b^2(W) = \operatorname{sgn} \xi = \operatorname{sgn} \tau$ on $\mathcal{I}$.
(3) The Schwartz kernel of W is supported on $\{r > R\}^2$.
(4) $[\partial_t, W] = [\partial_\varphi, W] = 0$.

Note in particular that the elliptic set of W contains the incoming points $\mathscr{F}_I$ sufficiently far from the string, since on this set $(A\tau+\eta) = 0$ and $\xi/(r\tau) = 1$. We now let
$$P_W \equiv P - iW$$
be the wave operator modified by the complex absorbing potential.

We now state a proposition in which we bring our propagation of singularities results to bear in a global fashion. In what follows, we use the notation $P_W^{(*)}$ to denote an operator that may be taken to be *either* P_W *or* P_W^*.

Proposition 9.4. *For any* $s \in \mathbb{R}$, $N \in \mathbb{N}$, *there exists* C *such that for all* $\phi \in \dot{\mathcal{H}}(K')$,

$$\|\phi\|_{H^s_{\mathsf{bF},\mathcal{H}}} \leq C\|\phi\|_{H^{-N}_{\mathsf{bF},\mathcal{H}}} + C\|P_W^{(*)}\phi\|_{H^{s+1}_{\mathsf{bF},\mathcal{H}^*}}. \tag{40}$$

Recall that the conventions on the Sobolev spaces are such that, away from $r = 0$, the norm on the LHS is H^{s+1} and the norm of the $P_W^{(*)}$ term on the RHS is H^s.

Proof. We prove the result for P_W, with the result for P_W^* being analogous, with reversed signs (and reversal in the direction of propagation of singularities on each bicharacteristic).

We claim that for any $q \in {}^bT^*X$ with $\operatorname{proj}(q) \in K'$ there exists $A_q \in \Psi^0_{\mathsf{bF}}(X)$ elliptic at the fiber through q such that

$$\|A_q\phi\|_{H^s_{\mathsf{bF},\mathcal{H}}} \leq C\|\phi\|_{H^{-N}_{\mathsf{bF},\mathcal{H}}} + C\|P_W\phi\|_{H^{s+1}_{\mathsf{bF},\mathcal{H}^*}}. \tag{41}$$

Adding up these regularity estimates, using compactness of the cosphere bundle ${}^bS^*K'_{\mathsf{F}}$, yields the desired global estimate over K'.

Thus it is left to prove the claim. To this end, let $q \in {}^bT^*X$ with $\operatorname{proj}(q) \in K'$ be given.

If $\operatorname{proj}(q) \in \partial X$ and $q \notin {}^b\dot{T}^*X$, then by our elliptic regularity estimate, Proposition 7.5, there exists $A_q \in \Psi^0_{\mathsf{bF}}(X)$ elliptic at q such that
$$\|A_q\phi\|_{H^s_{\mathsf{bF},\mathcal{H}}} \leq C\|\phi\|_{H^{-N}_{\mathsf{bF},\mathcal{H}}} + C\|P_W\phi\|_{H^s_{\mathsf{bF},\mathcal{H}^*}},$$
which being an *elliptic* estimate, is a stronger estimate than needed on Pu, and indeed implies (41). Likewise, if $\operatorname{proj}(q) \in X^\circ$ and $q \notin \Sigma$, an estimate of the same form applies by standard microlocal elliptic regularity.

If by contrast, either $\operatorname{proj}(q) \in X^\circ$ and $x \in \Sigma$ or else $q \in {}^b\dot{T}^*_{\partial X}X$, then we will apply propagation estimates. First consider the case where $q \notin \mathscr{F}_O$ (so

$\text{proj}(q) \in X^\circ$). Let us suppose for the moment that $\tau(q) > 0$. Recall that τ is conserved along the null bicharacteristic flow and asymptotically we have $\operatorname{sgn} \dot{t} = \operatorname{sgn} \tau > 0$ so that the null bicharacteristic satisfying $\gamma(0) = q$ is asymptotically forward-oriented in time. Furthermore, since geodesics not in $\mathscr{F}_O$ escape to infinity as $t \to -\infty$, there is a point $\gamma(s)$ *backwards* along the flow (relative to the curve parameter) where $\gamma(s) \notin {}^bT^*K'$. Due to our choice of sign for the principal symbol of W, regularity propagates *forward* along the flow (and singularities propagate backward) by the results of [19, Section 2.5]. Thus for some $A_q \in \Psi^0_{\mathsf{bF}}$ elliptic near q, we have

$$\|A_q\phi\|_{H^s_{\mathsf{bF},\mathcal{H}}} \leq C\|\phi\|_{H^s_{\mathsf{bF},\mathcal{H}}((K')^c)} + C\|\phi\|_{H^{-N}_{\mathsf{bF},\mathcal{H}}} + C\|P_W\phi\|_{H^{s+1}_{\mathsf{bF},\mathcal{H}^*}}, \tag{42}$$

(cf. [19, Equation 2.18]), and the first term on the right hand side vanishes since $\operatorname{supp}\phi \subset K'$, hence equation (41) holds. The same argument applies, mutatis mutandis, under forward flow if $\tau < 0$: the geodesic still escapes to infinity as $t \to -\infty$ but now this is forward along the flow, hence the same propagation estimates hold by our choice of the sign of $\sigma^2_b(W)$, which has been arranged so propagation of regularity is forward in t, no matter which sign of τ is chosen.

Now we turn to the case $q \in {}^b\dot{T}^*_{\partial X}X$. Corollary 8.4 yields, for an A_q elliptic near q,

$$\|A_q\phi\|_{H^s_{\mathsf{bF},\mathcal{H}}} \leq C\|\phi\|_{H^{-N}_{\mathsf{bF},\mathcal{H}}} + C\|P_W\phi\|_{H^{s+1}_{\mathsf{bF},\mathcal{H}^*}} + C\|Z\phi\|_{H^s_{\mathsf{bF},\mathcal{H}}}. \tag{43}$$

Here Z is microsupported in a region (away from ∂X, and close to incoming, hence not outgoing) in which we have already obtained control by the estimates we have just obtained on $(\mathscr{F}_O)^c$. Hence we may estimate

$$\|Z\phi\|_{H^s_{\mathsf{bF},\mathcal{H}}} \leq C\|\phi\|_{H^{-N}_{\mathsf{bF},\mathcal{H}}} + C\|P_W\phi\|_{H^{s+1}_{\mathsf{bF},\mathcal{H}^*}},$$

i.e., we may drop this term from the estimate (43) to get (41) for $q \in {}^b\dot{T}^*_{\partial X}X$.

Finally, we can treat the case of $q \in \mathscr{F}_O \cap X^\circ$. We note that in the estimate (41) at the compressed cotangent bundle over the boundary, the operator A_q is elliptic on a *neighborhood* of ${}^b\dot{T}^*_{\partial X}X$, and this includes points arbitrarily close to ∂X along every bicharacteristic in $\mathscr{F}_O$. Thus by ordinary propagation of singularities, we control the whole of $\mathscr{F}_O \cap \text{proj}^{-1}(K')$ by the same right-hand side: for any given point $q \in \mathscr{F}_O \cap X^\circ$ there is A_q elliptic at q such that (41) holds. ☐

In what follows we let $\|\bullet\|_{m,l}$ denote the $H^{m,l}_b$ norm.

Corollary 9.5. *For any $s \in \mathbb{R}$, $N \in \mathbb{N}$, there exists C such that for all $\phi \in \dot{\mathcal{H}}(K')$,*

$$\|\phi\|_{s+1,0} \leq C\|\phi\|_{-N,1} + C\|P^{(*)}_W\phi\|_{s,0}. \tag{44}$$

Proof. This follows directly from Proposition 9.4, together with the inclusions of spaces in Lemma 5.4. ☐

Equation (44) is not quite sufficient to use in the solvability argument due to the fact that $H_b^{s+1,0}$ does not embed compactly into $H_b^{-N,1}$. Below we perform additional analysis that ultimately yields an estimate similar to 44 but where the function space on the left hand side embeds compactly to the space in which the remainder on the right hand side is measured.

In what follows, we deal with distributions in $\mathscr{E}'(K')$ by considering the t variables to lie in an interval: if $u \in \mathscr{E}'(K')$ we may instead, by a mild abuse of notation, view it as

$$u \in \mathscr{E}'([-\pi L/2, \pi L/2]_t \times [0,\infty)_r \times S^1_\varphi),$$

with the underlying space equipped with the same volume form as before.

Lemma 9.6. *For all $s > 0$ there exists $\epsilon > 0$ such that the operator F acting on distributions in $\dot{H}^s([-\pi L/2, \pi L/2] \times S^1_\varphi)$ satisfies*

$$\langle \mathsf{F}^2 \phi, \phi \rangle_{H^s} > \epsilon \|\phi\|^2_{H^s}.$$

The result is a kind of Poincaré inequality, and the case $s = 0$ can be proved in the usual manner by writing

$$|\phi(t,\varphi)|^2 = \int_{-\pi L/2}^{t/\mathsf{A}} (d/ds)|\phi(\mathsf{A}s, \varphi - t/\mathsf{A} + s)|^2\, ds.$$

and employing Cauchy–Schwarz. An even shorter alternative, however, is as follows.

Proof. For notational convenience, we shift to work on $t \in [0, \pi L]$ rather than $[-\pi L/2, \pi L/2]$. We may then use the Fourier basis $\phi_{km} \equiv \frac{1}{\sqrt{2\pi}} \sin(kt/L) e^{im\varphi}$ with $k \in \{1,2,3,\dots\}$ $m \in \mathbb{Z}$. Then

$$\sqrt{2\pi}\mathsf{F}^2 \phi_{km} = \left(\frac{\mathsf{A}^2 k^2}{L^2} + m^2\right) \sin(kt/L) e^{im\varphi} - 2\frac{\mathsf{A}kim}{L} \cos(kt/L) e^{im\varphi}$$

hence

$$\langle \mathsf{F}^2 \phi_{km}, \phi_{km} \rangle = \frac{\pi L}{2}\left(\frac{\mathsf{A}^2 k^2}{L^2} + m^2\right) \geq \frac{\pi \mathsf{A}^2}{2L} \|\phi_{km}\|^2.$$

This establishes the result for $s = 0$; the more general result then follows since F^2 commutes with $(\mathrm{Id} + D_t^2 + D_\varphi^2)^{s/2}$. □

We now prove an estimate that allows us to trade derivatives for decay at $r = 0$ in compactly supported solutions to $P_W^{(*)} u = f$.

Lemma 9.7. *Let $m \in \mathbb{R}$ and let $u \in \dot{H}_b^{m+2}(K')$. Then*

$$\|u\|_{m,2} \lesssim \|P^{(*)} u\|_{m,0} + \|u\|_{m+2,0}$$

Proof. The estimate is trivial away from $r = 0$, hence it suffices to prove it for u replaced by χu with χ a cutoff near $r = 0$ and equal to 1 near $r = 0$. As the commutator term $[P^{(*)}, \chi]u$ is an element of $\Psi^1_{b\mathsf{F}}$, the resulting term can be absorbed in the $H_b^{m+2,0}$ term on the RHS, so in fact it will suffice to simply prove the result for u supported in r small.

Treating the D_t^2 and Υ terms in P as an error term,

$$r^{-2}\mathsf{F}^2 u + r^{-2}(rD_r)^2 u = D_t^2 u + \Upsilon u + P^{(*)}u. \tag{45}$$

Since $D_t^2 \in \Psi^2_{b\mathsf{F}}$, $\Upsilon \in \Psi^1_{b\mathsf{F}}$, we can estimate the H_b^m norm of the RHS by $\|u\|_{m+2,0} + \|P^{(*)}u\|_{m,0}$. It thus suffices to show that the H_b^m norm of the LHS controls the $H_b^{m,2}$ norm of u.

We rewrite equation (45) as

$$(\mathsf{F}^2 + (rD_r)^2)u = g \equiv r^2(D_t^2 u + P^{(*)}u + \Upsilon u) \in r^2 H_b^m.$$

We now Mellin transform this equation in r, using the convention

$$\mathscr{M} f(\xi) \equiv \int_0^\infty f(r) r^{-i\xi-1}\, dr.$$

This yields

$$(\mathsf{F}^2 + \xi^2)\mathscr{M} u(\xi, t, \varphi) = \mathscr{M} g(\xi, t, \varphi)$$

where we recall that we are viewing (t, φ) as lying in a sufficiently large set of the form $(-\pi L/2, \pi L/2) \times S^1_\varphi$.

By Lemma 9.6, $\mathsf{F}^2 > \epsilon$ as an operator on $\dot{H}^m$ hence we may invert $\xi^2 + \mathsf{F}^2$ with uniform bounds (and holomorphy) in ξ to get

$$\|\mathscr{M} u(\xi)\|_{H^m} \lesssim \|\mathscr{M} g(\xi)\|_{H^m}, \tag{46}$$

uniformly in ξ, with the norms being Sobolev norms in (t, φ). We now recall the characterization of b-Sobolev spaces by Mellin transform in [12, Section 5.6]. Since $g \in r^2 H_b^m$ its Mellin transform is holomorphic in $\operatorname{Im}\xi > -2$ (with the -2 corresponding to the r^2 weight, which simply shifts the imaginary part of the Mellin transform parameter). The squared weighted Sobolev norm of g is equivalent to

$$\int_{-\infty}^\infty \langle \xi \rangle^m \|\mathscr{M} g(-2i + \xi)\|^2_{H^m}\, d\xi.$$

By (46),

$$\int_{-\infty}^\infty \langle \xi \rangle^m \|\mathscr{M} u(-2i + \xi)\|^2_{H^m}\, d\xi \lesssim \int_{-\infty}^\infty \langle \xi \rangle^m \|\mathscr{M} g(-2i + \xi)\|^2_{H^m}\, d\xi,$$

and this implies a corresponding inequality of b-Sobolev norms:

$$\|u\|_{m,2} \lesssim \|g\|_{m,2} \leq \|D_t^2 + P^{(*)}u + \Upsilon u\|_{m,0},$$

as desired. □

We now upgrade (44) at the cost of using a stronger norm on $P^{(*)}\phi$ on the RHS. For brevity, we define Hilbert spaces

$$\mathcal{X} = \dot{H}_b^{s,0}(K'), \quad \mathcal{Y} = \dot{H}_b^{s+1,0}(K') \cap \dot{H}_b^{s-2,2}(K').$$

Then adding a sufficiently small multiple of the inequality in Lemma 9.7 with $m = s - 2$ to (44) to be able to absorb the $\|\phi\|_{s,0}$ term on the RHS of the former into the LHS of the latter yields

$$\|\phi\|_{\mathcal{Y}} \leq C\|P_W^{(*)}\phi\|_{\mathcal{X}} + C\|\phi\|_{-N,1}, \quad \operatorname{supp}\phi \subset K'. \tag{47}$$

We crucially note the following compact embedding:

Lemma 9.8. *$\mathcal{Y} \hookrightarrow \dot{H}_b^{-N,1}(K')$ is a compact embedding for $N > 2 - s$.*

Proof. By interpolation

$$\mathcal{Y} = \dot{H}_b^{s+1,0}(K') \cap \dot{H}_b^{s-2,2}(K') \subset \dot{H}_b^{s-1/2-3\delta/2,1+\delta}(K'), \quad |\delta| \leq 1.$$

Choosing $\delta \in (0,1]$ then gives a compact embedding

$$\dot{H}_b^{s-1/2-3\delta/2,1+\delta}(K') \hookrightarrow \dot{H}_b^{-N,1}(K'),$$

since the space on the LHS has both greater differentiability when $N > s-2$ and has greater decay at $r = 0$ (see, e.g., [10, Lemma 6.6]). □

Thus equation (47) establishes an inequality with the functional analytic properties that will be necessary for the solvability argument below.

Now we consider the nullspace of P_W^*, which will be the finite dimensional space N in the statement of Theorem 9.1. Let

$$N \equiv N(P_W^*) \equiv \{u \in \dot{\mathcal{H}}(K') : P_W^* u = 0\}.$$

Lemma 9.9. *$N(P_W^*)$ is a finite-dimensional subspace of $H^\infty_{\mathsf{bF},\mathcal{H}}$. If Hypothesis $\mathcal{C}^\omega$ or (t,φ) holds, then the elements of $N(P_W^*)$ are supported in $\{r > R\}$.*

Proof. Let $u \in N(P_W^*)$. By our global propagation of singularities theorem, Proposition 9.4,

$$u \in H^\infty_{\mathsf{bF},\mathcal{H}}(X).$$

Furthermore, we employ (47) to see that by Lemma 9.8, the unit ball in $N(P_W^*)$ is compact in the $H_b^{-N,1}$ topology, so $N(P_W^*)$ is finite dimensional.

We now show that under our stronger hypotheses, N is in fact trivial. Recall that $K' = \{r < 2R,\ t \in [-T',T']\}$, so that $u = 0$ for $t \leq -T'$.

First, consider the case of the $\mathcal{C}^\omega$ assumption. For any $\epsilon > 0$, fix a bump function $\chi_0(r)$ such that

$$\chi_0(r) > 0,\ r \in (\epsilon, |\mathsf{A}| - \epsilon), \quad \operatorname{supp}\chi_0 \subset (\epsilon/2, |\mathsf{A}| - \epsilon/2).$$

For $s \in [0,\infty)$, let Y_s denote the hypersurface

$$Y_s = \{(t,r,\varphi) : t = -T' + s\chi_0(r),\ \varphi \in S^1\}.$$

We compute

$$N^*Y_s = \operatorname{span}\{dt - s\chi_0'(r)\,dr\} = \{\xi = rs\chi_0'(r)\,\tau,\ \eta = 0\}.$$

Thus, on N^*Y,

$$p = \sigma_b^2(P) = \tau^2 - s^2(\chi_0'(r))^2\tau^2 - \frac{\mathsf{A}^2\tau^2}{r^2} < 0 \quad \text{for } \tau \neq 0.$$

Hence the whole family Y_s is noncharacteristic, and by Fritz John's global Holmgren Theorem, subject to Hypothesis $\mathcal{C}^\omega$ (analyticity), we obtain $u = 0$ on Y_s for all s. Taking $s \to \infty$ we thus find that for any $\epsilon > 0$,

$$u = 0 \text{ on } \{\epsilon < r < |\mathsf{A}| - \epsilon\},$$

hence by continuity, $u \equiv 0$ on $\{r \in [0, |\mathsf{A}|]\}$.

Now let $\tilde{\chi}(t) \in \mathcal{C}_c^\infty$ with $\tilde{\chi}(t) = 1$ for all $t \in [-1, 1]$. Let Γ be a smooth nondecreasing function on $[0, \infty)$ with

$$\Gamma(0) = 0,\ \Gamma(s) = R - |\mathsf{A}| \text{ for } s \geq 1.$$

For any $\mu > 0$, set

$$\chi_s(t) = |\mathsf{A}| + \Gamma(s\tilde{\chi}(t/\mu)).$$

Note $\Gamma(s\tilde{\chi}(t/\mu))$ is compactly supported in t, uniformly in $s \geq 0$; if we take $\mu > T'$ then for each $t \in [-T', T']$, $\chi_s(t) = R$ whenever $s \geq 1$. Moreover, for any s, μ

$$|\chi_s'(t)| \leq \sup|\Gamma'| \sup|\tilde{\chi}'| s\mu^{-1};$$

by enlarging μ further we may thus ensure that

$$\sup|\chi_s'(t)| \leq 1, \text{ for all } s \in [0, 1].$$

Now we again apply the global Holmgren theorem, this time for the family of surfaces $\tilde{Y}_s = \{r = \chi_s(t)\}$, $s \in [0, 1]$. Then

$$N^*\tilde{Y}_s = \{\tau = r^{-1}\chi_s'(t)\xi,\ \eta = 0\}$$

hence on this set

$$p = -\frac{\xi^2}{r^2}\big(1 + (\chi_s'(t))^2((\mathsf{A}^2/r^2) - 1)\big).$$

We have $p < 0$ for $\xi \neq 0$ since $r \geq |\mathsf{A}|$ on Y_s and $|\chi_s'| < 1$. Hence again the surfaces are noncharacteristic and the coefficients of P_W^* are analytic on $\tilde{Y}_s$ for all $s \geq 0$ since we have arranged that they all lie in $\{r \leq R\}$. Thus, once again by the global Holmgren theorem we find that $u = 0$ on $\tilde{Y}_s$ for all s, and these surfaces sweep out all of $K' \cap \{r \in [|\mathsf{A}|, R]\}$. Thus we have obtained $u = 0$ on $K' \cap \{r \leq R\}$, as asserted. (Note that similar arguments to this one, using the Holmgren theorem, were employed in [1] in the proof of Theorem 3.5.)

We now turn instead to the case of Hypothesis (t, φ) on the perturbation Υ. Since $[\Upsilon, \partial_\varphi] = 0$, we may decompose $u \in N(P_W^*)$ in angular modes,

$$u = \sum u_k e^{ik\varphi},$$

where u_k solves the mode-by-mode equation

$$\Big(\frac{1}{r^2}(\mathsf{A}\partial_t + ik)^2 - \partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r + \Upsilon_k\Big)u_k(t, r) = 0, \quad \text{for } r \leq R \tag{48}$$

with Υ_k again first-order. We have thus reduced to (a perturbation of) the equation considered in [15], and we proceed as in that paper: First, note that the operator on the LHS of (48) is now *elliptic* in t, r for $t < \mathsf{A}$, hence by unique continuation for elliptic equations, $u_k = 0$ for $r < \mathsf{A}$. Now we Fourier transform in t and use t-invariance of Υ_k to find

$$\big(-\frac{1}{r^2}(\mathsf{A}\tau + k)^2 + \tau^2 + \partial_r^2 + \frac{1}{r}\partial_r + \hat{\Upsilon}_k\big)\hat{u}_k(\tau, r) = 0, \quad \text{for } r \leq R \tag{49}$$

where $\hat{\Upsilon}_k(\tau, r)$ is a first order operator in r obtained as the Fourier conjugate of Υ_k (i.e., D_t turns into τ). By Picard–Lindelöf, since the function $\hat{u}_k(\tau, r)$ vanishes for $r < \mathsf{A}$, it vanishes identically on $r \leq R$. Thus, recovering u by Fourier synthesis, we find that $u = 0$ for $r \leq R$ as well, and we have obtained the result under Hypothesis (t, φ). □

Now we turn to the solvability argument, which is inspired by the work of Duistermaat–Hörmander [4, Theorem 6.3.1]; see also [15] for the analogous argument in the mode-by-mode case.

Assume $\phi \perp N(P_W^*)$. Then we claim that (47) (applied to P_W^*) can be replaced by

$$\|\phi\|_{\mathcal{Y}} \leq C\|P_W^*\phi\|_{\mathcal{X}}, \quad \operatorname{supp}\phi \subset K', \tag{50}$$

i.e., the error term involving ϕ on the RHS can be dropped. To see this, note that if (50) fails then there exists a sequence of

$$\phi_j \in \mathcal{Y} \cap N(P_W^*)^\perp$$

with

$$\|\phi_j\|_{\mathcal{Y}} = 1, \ \|P_W^*\phi_j\|_{\mathcal{X}} \to 0. \tag{51}$$

Extracting a subsequence in $\mathcal{Y}$, converging weakly in that space to $\phi \in \mathcal{Y} \cap N(P_W^*)^\perp$, we then obtain

$$P_W^*\phi_j \to P_W^*\phi$$

in the distributional sense. Thus since $P_W^*\phi_j \to 0$ in $\mathcal{X}$ we obtain $P_W^*\phi = 0$, which implies $\phi = 0$. On the other hand, by Lemma 9.8, ϕ_j is strongly convergent in $\dot{H}_b^{-N,1}(K')$, hence the limit ϕ must have $H_b^{-N,1}(K')$ norm bounded below by 1, by (47) and (51). Thus we obtain a contradiction, and this yields the improved estimate (50).

Now for $f \in \mathcal{Y}^* \cap N(P_W^*)^\perp$, consider the map

$$T : P_W^*\phi \mapsto \langle \phi, f\rangle.$$

By the constraint on f, this map is well-defined on the range of P_W^* on $\mathcal{C}_c^\infty((K')^\circ)$, considered as a subset of $\mathcal{X}$. The estimate (50) then implies

$$|T(P_W^*\phi)| \leq C\|P_W^*\phi\|_{\mathcal{X}}\|f\|_{\mathcal{Y}^*}.$$

By Hahn–Banach, we now extend T to a map defined on all of $\mathcal{X}$, satisfying

$$|T\psi| \leq C\|\psi\|_{\mathcal{X}}\|f\|_{\mathcal{Y}^*}.$$

Thus by the Riesz Lemma, there exists $u \in \mathcal{X}^*$ such that

$$T\psi = \langle \psi, u\rangle,$$

hence for all test functions ϕ we certainly have

$$\langle P_W^* \phi, u\rangle = \langle \phi, f\rangle$$

hence u solves $P_W u = f$. Since $P_W = P$ on K, this certainly implies that $Pu = f$ on K.

In particular, then, given $f \in N^\perp \cap \mathcal{Y}^*$, we can solve $Pu = f$ on K° for $u \in \mathcal{X}^*$. We note that

$$\mathcal{Y}^* = (\dot{H}_b^{s+1,0}(K') \cap \dot{H}_b^{s-2,2}(K'))^* \supset H_b^{-s-1,0}((K')^\circ),$$

while

$$\mathcal{X}^* = H_b^{-s}((K')^\circ);$$

setting $s = -m-1$ proves the existence of a solution in the desired space. By Lemma 9.9, if the stronger hypothesis Hypothesis $\mathcal{C}^\omega$ or (t,φ) holds, then orthogonality to N is no constraint on f (since f is by hypothesis supported in $\{r \leq R\}$).

To prove the wavefront set relation (33) we begin by establishing the stronger statement (34):

$$\mathrm{WF}_{b\mathsf{F},\mathcal{H}}\, u \backslash \mathrm{WF}_{b\mathsf{F},\mathcal{H}^*}\, f \subset \mathscr{F}_O \cup \Phi_+(\mathrm{WF}_{b\mathsf{F},\mathcal{H}^*}\, f \cap \Sigma).$$

At points not in $\Sigma \cup {}^b\dot{T}^*X$ this follows from microlocal elliptic regularity (Proposition 7.5). At interior points in $\Sigma\backslash\mathscr{F}_O$, it follows from interior propagation of singularities. Indeed, by Lemma 9.3, at every point in $\mathrm{proj}^{-1}(K) \setminus \mathcal{F}_O$, the (asymptotically) backwards in time flow through that point eventually hits the elliptic set of W while remaining within K', where u solves $P_W u = f$. Owing to our choice of signs for W, regularity propagates (asymptotically) forward in time. Thus a point in $\Sigma \setminus \mathcal{F}_O$ is only in the LHS if it reaches $\mathrm{WF}_{b\mathsf{F},\mathcal{H}^*}\, f \cap \Sigma$, where there may be wavefront set, backwards in time. The weaker statement (33) then follows from Lemma 6.8

To obtain uniqueness, we note that subtracting two solutions u_0, u_1 yields $w = u_0 - u_1$ such that

$$P_W w = 0 \text{ on } K,$$
$$\mathrm{WF}_{b\mathsf{F},\mathcal{H}}\, w \subset \mathscr{F}_O \cup \Phi_+(\mathrm{WF}_{b\mathsf{F},\mathcal{H}^*}\, f \cap \Sigma).$$

At every point $\rho \notin \mathscr{F}_O$, backward flow eventually reaches $T^*K\backslash T^*K_0$, before leaving K entirely. (Recall that f is supported in K_0.) At such points, neither u_0 nor u_1 has wavefront set, since

$$\mathrm{WF}_{b\mathsf{F},\mathcal{H}}\, u_\bullet \subset \mathscr{F}_O \cup \Phi_+(\mathrm{WF}_{b\mathsf{F}}\, f \cap \Sigma),$$

and these points are neither outgoing nor in the forward flowout of $\mathrm{WF}\, f$. Hence by interior propagation of singularities for the equation $P_W u = 0$, $\rho \notin \mathrm{WF}\, w$, and the asserted uniqueness follows.

Appendix A. The uniform b-calculus

The usual construction of the b-pseudodifferential calculus, e.g. as in [12], is in the context of compact manifolds with boundary. Here we work on $X = [\mathbb{R}^3; \mathsf{S}]$, which is noncompact (recall $\mathsf{S} = \{x_1 = x_2 = 0\}$). The noncompactness in r is of no consequence here, as we always do our estimates in a neighborhood of S (or its lift to the blowup), but the noncompactness in t is a more serious issue and worthy of comment.

Happily, the treatment of the subject by Hörmander in [6, Section 18.3] begins by developing the calculus on a half-space with just the sort of uniform symbols estimates that we require here. While the later passage to manifolds in that work (Definition 18.3.18) is only phrased in terms of local estimates (since Definition 18.2.6, giving the conormal distributions used here, relies on *local* Besov spaces), we may still make use of the half-space construction here in order to work on

$$X \equiv [0, \infty)_r \times \mathbb{R}_t \times \mathbb{R}_\varphi / 2\pi\mathbb{Z}$$

by identifying functions on X with functions on $\mathbb{R}^3_+$ that are $2\pi\mathbb{Z}$-periodic in the φ variable.

Thus, following [6], we may define the Schwartz kernel of the b-quantization of a symbol $a \in S^m_u$ as

(52)
$$\kappa(\mathrm{Op}_b(a)) \equiv \int e^{i[(r-r')\xi/r+(t-t')\tau+(\varphi-\varphi)'\eta]} a(r,t,\varphi,\xi,\tau,\eta)\, d\xi d\tau d\eta |dt\, d\varphi\, dr/r|.$$

Here we have taken the form of the kernel (18.3.4) from [6] with x_n denoted r, and made a change of fiber variable; we include the half-density factor necessary to make the operator act on functions (with the r^{-1} arising naturally from change of fiber variables). The κ denotes the Schwartz kernel of the operator in question.

Recall that the class of symbols a considered in [6] and used here (12) are those satisfying the uniform (in r, t, φ) estimate

$$\left|\partial^\alpha_{r,t,\varphi} \partial^\beta_{\xi,\tau,\eta} a\right| \leq C_{\alpha,\beta,N} \langle(\xi,\tau,\eta)\rangle^{m-|\beta|} \langle r\rangle^{-N}. \tag{53}$$

Owing to the periodicity in φ, we make the additional requirement

$$a(r,t,\varphi+2\pi k,\xi,\tau,\eta) = a(r,t,\varphi,\xi,\tau,\eta), \quad k \in \mathbb{Z}. \tag{54}$$

And following [6], in order that this quantization produce a sensible operator on $r \geq 0$, we also require the "lacunary condition"

$$\mathscr{F}_{\xi\to w} a = 0, \quad \text{for } w \leq -1,\ r \geq 0. \tag{55}$$

Remark A.1. In the language of blowups, we remark that the lacunary condition means that all derivatives of $\kappa(A)$ vanish at $s = 0$ where $s = r'/r$ is a smooth variable along the interior of the front face of the blowup $[X \times X; (\partial X)^2]$. Note that we may view the ξ integration in the quantization (52) as the Fourier transform in ξ, evaluated as $w = s - 1$. The set $\{w = -1\}$, a.k.a. $\{s = 0\}$ is the "right face" of the blowup, at $r' = 0$. Rapid vanishing at the "left face" where $r = 0$ is, by contrast, automatic owing to the rapid decay of the Fourier transform of a symbol in the base variables, since $s \to +\infty$ as we approach this face. Hence the operators obtained in this way are indeed locally (in t) the same as those described in [12, Definition 4.22], except that here we have built a right b-density into the definition of the operator in order to let it act on functions.

Definition A.2. *An operator A is in $\Psi^m_{bu}(X)$ if it can be written as* $\mathrm{Op}_b(a)$ *for some a satisfying* (53)*,* (54)*,* (55)*.*

We further note, as in [20, Section 5.3.1], that quantizing our φ-periodic symbols as in (52) and applying the result to a φ-periodic distribution u results in a φ-periodic distribution, hence the action of $\mathrm{Op}_b(a)$ is well-defined on sufficiently regular and decaying functions on X.

For a satisfying (53) and (55), the boundedness of $\mathrm{Op}_b(a)$ on a half-space is [6, Theorem 18.3.12]; the result then follows on $X = [0, \infty) \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ by applying the proof of Theorem 5.5 of [20] in the t-variable to deal with periodic functions. This of course yields boundedness with respect to

$$L^2(X; dr\, dt\, d\varphi)$$

rather than the metric volume form, however. To obtain boundedness with respect to the metric volume form $rdr\, dt\, d\varphi$, we simply note the following

Lemma A.3. *For all $m, k \in \mathbb{Z}$,*

$$A \in \Psi^m_{bu}(X) \Longleftrightarrow r^{-k} A r^k \in \Psi^m_{bu}(X).$$

Proof. First, let $k \in \mathbb{N}$. We note that, by integration by parts in ξ,

$$\kappa(r^{-k} A r^k) = \mathrm{Op}_b\left((1 - D_\xi)^k a\right).$$

The symbol $(1 - D_\xi)^k a$ satisfies (53), (54), (55) if a does, hence $r^{-k} A r^k \in \Psi_{bu}(X)$ for $k \in \mathbb{N}$. The case of general k now follows by duality; recall that the calculus is closed under adjoints by [6, Theorem 18.3.8]. □

Thus we obtain

Proposition A.4. *An operator $A \in \Psi^m_{bu}(X)$ is bounded $H^{s,l}_{\mathsf{bF}} \to H^{s-m,l}_{\mathsf{bF}}$ for all $m, s, l \in \mathbb{R}$.*

Proof. Since $L^2(X)$ equipped with the metric density equals

$$r^{-1} L^2(r\, dr\, dt\, d\varphi),$$

Lemma A.3 implies that operators of order zero are bounded on $r^l L^2(X)$ for all $l \in \mathbb{Z}$. We can then extend to non-integer l by interpolation, establishing the result for $s = m = 0$. The more general version of the result follows by employing elliptic operators in the b-calculus as in the usual proof on manifolds without boundary. □

We could alternatively omit the lacunary condition (55) on symbols as long as we build a cutoff function into our quantization, as in the presentation of this material in Section 3.1. We must then put the residual operators into the calculus "by hand," however. We begin by recalling the characterization of residual operators, proved in [6, Theorem 18.3.6].

Proposition A.5. *The elements of* $\Psi_{bu}^{-\infty}$ *(a.k.a.* residual operators*) are those operators* R *whose Schwartz kernels satisfy the following estimate in coordinates on* X_b^2 *given by* $\rho = r + r'$, $\theta = (r - r')/(r + r')$*: for all* α, β, γ, N,

$$\left|D_{t,\varphi,t',\varphi'}^{\alpha} D_{\rho}^{\beta} D_{\theta}^{\gamma} \rho \kappa(R)\right| \leq C_{\alpha,\beta,\gamma,N}(1 + |t - t'| + \rho)^{-N}.$$

(Note that the leading factor of ρ is compensating for the ρ^{-1} factor in the b-half-density arising e.g. in (52).)

Now let $\chi(s)$ equal 1 for $s \in (1/2, 2)$ and be supported in $(1/4, 4)$.

Proposition A.6. *For any* $a \in S_u^m$ *the operator*

(56)
$$\kappa(\widetilde{\mathrm{Op}}_{\mathrm{b}}(a)) \equiv \int e^{i[(r-r')\xi/r + (t-t')\tau + (\varphi-\varphi)'\eta]} \chi(r'/r) a(r, t, \varphi, \xi, \tau, \eta)\, d\xi d\tau d\eta |dt\, d\varphi\, dr/r|$$

is in $\Psi_{bu}^m(X)$*. Conversely every element of* $\Psi_{bu}^m(X)$ *differs from an operator of this form by an element of* $\Psi_{bu}^{-\infty}(X)$*.*

This result follows from [6, Lemma 18.3.4].

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