ON NON-DIFFRACTIVE CONES

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Abstract. A subject of recent interest in inverse problems is whether a corner must diffract fixed frequency waves. We generalize this question somewhat and study cones \([0, \infty) \times Y\) which do not diffract high frequency waves. We prove that if \(Y\) is analytic and does not diffract waves at high frequency then every geodesic on \(Y\) is closed with period \(2\pi\). Moreover, we show that if \(\dim Y = 2\), then \(Y\) is isometric to either the sphere of radius 1 or its \(\mathbb{Z}^2\) quotient, \(\mathbb{R}P^2\).

1. Introduction

A subject of recent interest in the study of inverse problems has been the question of whether corners must diffract fixed-frequency solutions of the Helmholtz equation with potential in \(\mathbb{R}^2\); here a corner is the location of a singularity of the potential, which is of the form of a smooth function times the indicator function of a sector. Affirmative answers to this question have been obtained under various conditions by Blåsten–Päivärinta–Sylvester [2] and Päivärinta–Salo–Vesalainen [17] (who treat certain kinds of conic singularities in \(\mathbb{R}^3\) as well). More recently, results on diffraction by partially transparent polygons and polyhedra have been obtained by Elschner–Hu [8].

In this note, we introduce a related problem that seems fundamental to the theory of diffraction. On a cone, perhaps the simplest setting in which diffraction is known to occur, must there be nontrivial diffraction at high frequency? In posing the problem as a high-frequency one, we restate it as a question about singularities of solutions to the wave equation. If we study the half-wave propagator \(e^{-it\Delta}\), we thus ask if there must be singularities to the solution other than those along (the closure of) the geodesics missing the cone tip, i.e., those predicted by geometric optics in its naivist form. Our main theorem, admittedly a very partial result in the desired direction, is that if a real-analytic cone exhibits no diffraction in this sense, then its link must have the property that every geodesic is \(2\pi\)-periodic. In the special case when the link has dimension 2 (and is still analytic) we are further able to show that the link must be \(S^2\) equipped with its standard round metric of circumference \(2\pi\), or else \(\mathbb{R}P^2\), its \(\mathbb{Z}^2\)-quotient.

Some remarks on conjectured stronger results may be found below.

We now state our results more precisely.
Definition 1. A cone $C(Y)$ over a Riemannian manifold $(Y, h)$ of dimension $d - 1$ is the $d$-manifold
\[ X = [0, \infty)_x \times Y \]
whose interior is equipped with the metric
\[ g = dx^2 + x^2h. \]

We say that $C(Y)$ is nondiffractive if
\[ \text{singsupp } \kappa(e^{-it\sqrt{\Delta}}) = \{ p, p' : p, p' \text{ are endpoints of a geodesic of length } |t| \text{ in } X^\circ \}, \]
where $\kappa(A)$ denotes the Schwartz kernel of the operator $A$. Otherwise, we say $C(Y)$ is diffractive. (Here $\Delta$ denotes the Friedrichs extension of the nonnegative Laplace-Beltrami operator from $C^\infty_c(X^\circ)$.)

It is known that in general there are additional “diffracted” singularities of this Schwartz kernel, at
\[ D_t \equiv \{ p, p' : x(p) + x(p') = |t| \}; \]
indeed there is a conormal singularity along this set, degenerating near its intersection with the set of endpoints of geodesics in $X^\circ$, which always carries singularities. We remark that this intersection occurs exactly at the set
\[ \{ p, p' : x(p) + x(p') = |t|, y(p), y(p') \text{ endpoints of a geodesic of length } \pi \text{ in } Y \}. \]

It follows from the work of Cheeger–Taylor [3], [4] (see [9, Corollary 2.3]) that the principal symbol of the diffracted wave on $D_t$ is a nonvanishing multiple of the Schwartz kernel of the operator
\[ \exp \left( -i\pi \sqrt{\Delta_Y + \frac{(d - 2)^2}{4}} \right), \]
where $\Delta_Y$ is the (positive definite) Laplacian on the link $Y$ of the cone, with respect to the metric $h$. Setting
\[ \nu = \sqrt{\Delta_Y + \frac{(d - 2)^2}{4}}, \]
we thus find that a sufficient condition for $C(Y)$ to be diffractive is that for some $y \in Y$, $\kappa(e^{-i\nu \delta_y})$ should have support outside the distance sphere of radius $\pi$ centered at $y$. It is this condition that we exploit in proving the following.

Theorem 1. Let $C(Y)$ be nondiffractive, and $Y$ real analytic. Then every geodesic on $Y$ must be periodic with (not necessarily minimal) period $2\pi$.

Remark 1. The conclusion of the theorem is that $Y$ must be a $P_{2\pi}$ manifold in the terminology of [1, Section 7.8]. Many such manifolds exist: in addition to compact Riemannian symmetric spaces and their quotients, there is a menagerie of so-called Zoll manifolds which enjoy this property—see [1] for detailed discussion.
Conversely, we remark that if $Y$ is a spherical space form, i.e., the quotient of $S^{d-1}$ with the standard metric on the unit sphere by the fixed-point-free action of a subgroup of $G \subset O(d)$, then $C(Y)$ is the quotient of $\mathbb{R}^d$ by the action of $G$, blown-up at the origin (i.e., viewed in polar coordinates). The method of images then shows that $C(Y)$ is nondiffractive, since the Schwartz kernel of $e^{-it\nu}$ on $C(Y)$ may be obtained by averaging over the action of $G$ the corresponding Schwartz kernel on $\mathbb{R}^d$, where ordinary propagation of singularities along geodesics holds true. In the case of $d = 2$, i.e., $\dim Y = 1$, $e^{-i\pi \nu}$ can be calculated explicitly as in the work of Hillairet [15] and it is easy to verify that these are the only non-diffractive links. In fact we conjecture that these are the only examples, even in the smooth category: if $C(Y)$ is nondiffractive and $Y$ merely $C^\infty$, then we conjecture that $Y$ must be a spherical space form. This conjecture seems out of reach for the moment.

Returning to the analytic case, we have been able to verify our conjecture in the case of dimension 2, ruling out Zoll manifolds.

**Theorem 2.** Let $C(Y)$ be nondiffractive with $Y$ analytic and $\dim Y = 2$. Then $Y$ is either $S^2$ or $\mathbb{RP}^2$ equipped with its standard metric.

We emphasize that by “standard metric” on $S^2$ or $\mathbb{RP}^2$ we do not mean “standard metric up to scale,” but rather the metric on the unit sphere in $\mathbb{R}^3$ and its $\mathbb{Z}_2$-quotient respectively; spheres and projective spaces of other sizes do diffract (as our proof shows).

In order to clarify these distinctions, we will use the notation $S^2_a$ and $\mathbb{RP}^2_a$ for the sphere equipped with the round metric of circumference $a$ and its $\mathbb{Z}_2$-quotient, respectively. Hence $S^2_{2\pi}$ and $\mathbb{RP}^2_{2\pi}$ are the standard sphere and projective space, and in general, $S^2_a$ is a $P_a$ surface while $\mathbb{RP}^2_a$ is a $P_{a/2}$ surface; recall that a $P_a$ manifold is one on which all geodesics are periodic with (not necessarily minimal) period $a$.

Let us also now adopt the slightly nonstandard notation that $Y$ is a $\tilde{P}_a$ manifold if $a$ is a $P_a$ manifold but not a $P_{a/k}$ manifold for any integer $k \geq 2$ (i.e., if $a$ is the minimal common period).

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2. Proof of Theorem 1

Let $\Phi_t$ denote geodesic flow for time $t$ on $S^*Y$, i.e., the time-$t$ flow generated by the Hamilton vector field of $(1/2)|\xi|_g^2$, restricted to the unit cotangent bundle. Let $\pi_Y$ denote the projection $S^*Y \to Y$.

Let

$$K \equiv \kappa(e^{-i\pi\nu}).$$

Recall that a necessary condition for $Y$ to be non-diffractive is

$$\text{supp } K \subset \{y, y' : y, y' \text{ are endpoints of a geodesic of length } \pi}\}.$$  

(Standard propagation of singularities results [7] show that the singular support of $K$ lies in the latter set.)

Since $Y$ is analytic, we note that in order to show that all geodesics are periodic with period $2\pi$, it suffices to show that on an open set in $U \subset S^*Y$,

$$\Phi_{2\pi} = \text{Id}.$$  

Hence our strategy is to show that the support condition for $e^{-i\pi\nu}$ implies the existence of these closed geodesics.

Consider first the manifold

$$\Lambda \equiv \text{graph}(\Phi_\pi) \subset S^*Y \times S^*Y$$

and then its projection

$$\pi_{Y \times Y} \Lambda \subset Y \times Y,$$

which is where $\text{supp } K$ lives by hypothesis. We remark that $\pi_{Y \times Y} \Lambda$ is certainly not guaranteed to be a smooth manifold. However, since our hypotheses imply that $\Lambda$ is analytic, certainly $\pi_{Y \times Y} \Lambda$ is subanalytic, by definition. A theorem of Gabrielov [10], later rediscovered by Hironaka [16] and Hardt [14] then implies that $\pi_{Y \times Y} \Lambda$ is a stratified space, and in particular, contains as an open subset a maximal-dimensional embedded submanifold $F \subset Y \times Y$. Let $\tilde{F} = \pi_{Y \times Y}^{-1} F \cap \Lambda$ denote its preimage.

A key observation is now that $\text{WF } K = \Lambda'$ (see, e.g., [6, Theorem 1]) where

$$\Lambda' := \{(x, \xi, y, \eta) \mid (x, -\xi, y, \eta) \in \Lambda\}.$$  

Our hypotheses are that $\text{supp } K \subset \pi_{Y \times Y} \Lambda$, hence in a neighborhood $V$ of any point in $F$, $\text{supp } K \subset F$. Since $F$ is a smooth manifold, we thus known that on $V$, we may express

$$K = \sum \delta^\alpha(u) \phi_u(y)$$

where $u = (u_1, \ldots, u_k)$ are defining functions for $F$, and $y$ complete $u$ to a local coordinate system. Such a distribution has the property that its wavefront set is invariant under the negation map on fibers:

$$(y, \eta, y', \eta') \in \text{WF } K \cap \pi_{Y \times Y}^{-1} (F \cap V) \implies (y, -\eta, y', -\eta') \in \text{WF } K \cap \pi_{Y \times Y}^{-1} (F \cap V).$$
Thus, since $WF K = \Lambda'$,

$$(y, \eta, y', \eta') \in \Lambda \cap \pi_{y'y}^{-1}(F \cap V) \implies (y, -\eta, y', -\eta') \in \Lambda \cap \pi_{y'y}^{-1}(F \cap V).$$

This precisely means that for $(y, \eta) \in \pi L (\Lambda \cap \pi_{y'y}^{-1}(F \cap V))$,

$$\Phi_\pi(y, \eta) = -\Phi_\pi(y, -\eta) = \Phi_{-\pi}(y, \eta)$$

(with negation interpreted as acting on the fibers). Hence

$$\Phi_2 \pi(y, \eta) = (y, \eta).$$

Setting $U = \pi L (\Lambda \cap \pi_{y'y}^{-1}(F \cap V))$, we have thus proved the desired periodicity of geodesics. □

3. Proof of Theorem 2

By Theorem 1, $Y$ is a $P_{2\pi}$ surface. Thus, it is diffeomorphic to either $S^2$ or $\mathbb{RP}^2$—see [1, Section 4.3].

We begin with the case where $Y$ is diffeomorphic to $S^2$. As in the proof of Theorem 1, we consider $\pi_{S^2 \times S^2} \Lambda \subset S^2 \times S^2$, the projection of the graph of time-$\pi$ geodesic flow out in $S^*(S^2)$; we again use crucially that this is a stratified space. Since the dimension of $\Lambda$ itself is 3 and since left and right projections of $\pi \Lambda$ are surjective, the dimension of the maximal stratum of $\pi_{S^2 \times S^2} \Lambda$ may only be 2 or 3. If it is 3, then there is an open set, $F$ in $\pi_{S^2 \times S^2} \Lambda$ that is a submanifold of $S^2 \times S^2$ of codimension-1, so that the Schwartz kernel of the propagator $K$ is locally given by

$$K = \sum \delta^{(\alpha)}(u)\phi_\alpha(y)$$

where now $u \in \mathbb{R}$ is locally a defining function for $\pi_{S^2 \times S^2} \Lambda$.

We will need a slightly stronger consequence of [6, Theorem 1] than that $(WF(K))' = \Lambda$. In particular, we need that

$$WF^{-1}(K)' = \Lambda, \quad WF^{-1-\epsilon}(K) = \emptyset$$

for all $\epsilon > 0$,

where $WF^s$ denotes the $s$-wavefront set, i.e., $\rho \notin WF^s(u)$ if and only there exists $A \in \Psi^0$ so that $Au \in H^s$ and $\sigma(A)(\rho) \neq 0$. Now, let $x_0 \in F$ and $V$ a neighborhood of $x_0$ so that (1) is valid on $V$. Let $\chi \in C^\infty_c(V)$ with $\chi(x_0) = 1$. Then, by (1)

$$\chi K \in \bigcup_{\epsilon > 0} H^{-1/2-j-\epsilon} \setminus H^{-1/2-j},$$

where $j$ is the largest $|\alpha|$ such that the coefficient $\phi_\alpha$ in (1) is nonvanishing on $\text{supp} \chi$. On the other hand,

$$T_{x_0}^*(S^2 \times S^2) \cap WF^{-1}(K) \neq \emptyset, \quad T_{x_0}^*(S^2 \times S^2) \cap WF^{-1-\epsilon}(K) = \emptyset.$$
Either the first or the second of these statements contradicts (3) depending on whether $j = 0$ or $j \geq 1$.

We conclude from this contradiction that in fact the dimension of the maximal stratum is 2. Since this is the minimal possible dimension, $\dim \pi \Lambda = 2$ globally. This means that $\pi_{S^2} \Phi_\pi(S^2(S^2))$ is a single point for each $x \in S^2$; for brevity we denote this set $\Phi_\pi(x)$. We claim that either $\Phi_\pi(x) = x$ for all $x$, or else $\Phi_\pi(x) \neq x$ for all $x$. Indeed it has been shown by Gromoll–Grove [12] that on $Y$ diffeomorphic to $S^2$, the $\tilde{P}_\pi$ condition implies that $Y$ is an SC manifold (again in the terminology of [1, Section 7.8]), which is to say, all geodesics have minimal period exactly $a$ and are without self-intersection (“simple”). If $Y$ is a $\tilde{P}_{2\pi/k}$ manifold for $k$ odd, we thus conclude from [12] that $\Phi_\pi(x) \neq x$ for all $x$, as otherwise this would contradict simplicity of the geodesics. This means that $Y$ is a Blaschke surface, and hence $\tilde{P}_{2\pi/k}$ surface for some even integer $k$, hence that $Y$ is a $P_{\pi/k}$ surface; we will then derive a contradiction.

For each $y \in S^2$, $e^{-i\pi \nu} \delta_y$ is by hypothesis supported at the flowout of $S^*_{\pi y}(S^2)$, which we now know to be the point $y$ itself. Thus $e^{-i\pi \nu} \delta_y$ must equal $\psi(y) \delta_y$ for some function $\psi$; more generally this tells us that

$$e^{-i\pi \nu} f = \psi f$$

for every $f \in L^2$. Applying (4) to $f = \phi_j$, an eigenfunction of $\Delta_Y$ with eigenvalue $\lambda^2$, tells us that

$$e^{-i\pi \sqrt{\lambda_j^2 + 1/4}} = \psi(y).$$

The left side is of course constant, so $\psi$ is in fact constant, and all of these values must agree, i.e., there exists $\beta \in \mathbb{R}$ such that for all $\lambda_j^2$ in the spectrum of $\Delta_Y$,

$$\sqrt{\lambda_j^2 + 1/4} \equiv \beta \mod 2\mathbb{Z};$$

equivalently this is just the statement that the spectrum of $\nu$ lies in $\beta + 2\mathbb{Z}$.

Now in order to derive a contradiction, we turn to the strong results known about spectral asymptotics of Zoll surfaces; this argument is based on the fact that the spectrum of a $P_{\pi/k}$ manifold must closely resemble that of $S^2_{\pi/k}$, which is indeed diffractive. Duistermaat–Guillemin [6], Weinstein [18], and Colin de Verdière [5] have obtained very precise estimates of the clustering of the eigenvalues of such a

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1 There are several equivalent definitions of a Blaschke surface. Among these [1, Theorem 5.43] is that the cut locus is spherical, which is to say the distance to the first cut point is independent of direction at each point. For $Y$ as above, $\Phi_\pi(x)$ has distance $\pi$ from $x$ for all $x$, since otherwise a geodesic from $x$ would pass through $\Phi_\pi(x)$ at time $t_0 \in (0, \pi)$ and then would self-intersect at time $\pi$, contradicting the simplicity of the geodesics from [12]. But then every geodesic must in fact be minimizing up to time-$\pi$, as a failure to be minimizing would allow us to construct a continuous, piecewise smooth curve from $x$ to $\Phi_\pi(x)$ of length shorter than $\pi$. Hence the cut-radius is exactly $\pi$, at every point in every direction, and our surface is indeed Blaschke.
Zoll surface. Thus, e.g., [5, Corollaire 1.2] (see also [13], [19]) shows that there is $M > 0$ so that the spectrum of $\Delta_Y$ is entirely contained in a union of intervals

$$I_n = [4(n + \alpha/4)^2 - M, 4(n + \alpha/4)^2 + M],$$

where $\alpha$ is the Maslov index of all the $\pi$-periodic geodesics; the (crucial) factors of 4 arise since we are dealing with a $P_\pi$ surface rather than a $P_{2\pi}$ surface as in [5].

Thus, the square roots $\lambda_n$ of the eigenvalues live in intervals

$$J_n = [2(n + \alpha/4) - C/n, 2(n + \alpha/4) + C/n],$$

and the constraints (6) implies that each interval $I_n$ can contain at most one eigenvalue, as otherwise differences of $\sqrt{\lambda^2 + 1/4}$ for eigenvalues in the same cluster form a nonzero sequence converging to 0, i.e. must have infinitely many different fractional parts.

We have thus reduced to the situation studied by Zelditch in [20] of a maximally degenerate Laplacian; Zelditch proves [20, Theorem C] that this places a yet stronger constraint on the locations of the eigenvalues and that there is an operator $A$ with spectrum in $\mathbb{N}$ such that

$$\Delta_Y = 4(A + \frac{1}{2})^2 - 1 + S$$

with $S$ a smoothing operator; here again we have rescaled by a factor of 4 since we are dealing with a $P_\pi$ surface (indeed an $SC_\pi$ surface by [12]). Hence the eigenvalues $\sqrt{\lambda^2 + 1/4}$ of $\nu$ are all of the form

$$\sqrt{4(n + 1/2)^2 - 3/4 + O(n^{-\infty})}.$$ 

Recall on the other hand that they are in $\beta + 2\mathbb{Z}$ by (6). Plugging in any value of $n$ in the spectrum of $A$ gives and recalling that $\nu$ is a positive operator,

$$4n^2 + 4n + \frac{1}{4} + O(n^{-\infty}) = 4m^2 + 4m\beta + \beta^2, \quad m \in \mathbb{N}$$

(with $m$ dependent on $n$ of course). Now setting $m = n + \ell$ we expand to find

$$4n(1 - \beta) + \frac{1}{4} + O(n^{-\infty}) = 8n\ell + 4\ell^2 + 4\ell\beta + \beta^2.$$ 

Since $m > 0$ in (7), we must take a root in $\ell$ of (8) with $m + \ell > 0$. Of the two roots of this quadratic equation in $\ell$ only one gives $m = n + \ell > 0$, and it is given by

$$\ell = \frac{1 - \beta}{2} + O(n^{-\infty});$$

taking $n$ large, we see that in fact $(1 - \beta)/2 \in \mathbb{Z}$ and this equation must be satisfied exactly:

$$\ell = \frac{1 - \beta}{2}.$$ 

Inserting this back into (8) gives

$$4n(1 - \beta) + \frac{1}{4} + O(n^{-\infty}) = 8n\frac{1 - \beta}{2} + 4\frac{(1 - \beta)^2}{4} + 4\frac{1 - \beta}{2}\beta + \beta^2,$$
which yields
\[ \frac{1}{4} + O(n^{-\infty}) = 1, \]
a contradiction. Hence we have ruled out all $P_\pi$ manifolds, and completed the case of $Y$ diffeomorphic to $S^2$.

To finish the proof, we now turn to the (easier) case when $Y$ is diffeomorphic to $\mathbb{RP}^2$. In this case, by Green’s proof of Blaschke’s Conjecture [11] (see also footnote 1 above), we know that $Y = \mathbb{RP}^2_{4\pi/k}$ for some $k \in \mathbb{N}$. We now rule out all but $\mathbb{RP}^2_{2\pi}$.

To start, we know by the same argument as in the sphere case that $\Phi_\pi(x)$ is a single point for each $x$. This does not happen unless $k$ is even and at least 2. For $k$ even, $\mathbb{RP}^2_{4\pi/k}$ is a $P_\pi$-manifold so the same argument in the sphere case tells us that the spectrum of $\nu = \sqrt{\Delta_Y} + 1/4$ lies in $\beta + 2\mathbb{Z}$ for $\beta$ fixed. The spectrum of $\mathbb{RP}^2_{4\pi/k}$ is the set
\[ \frac{4}{k^2} 2n(2n+1), \quad n \in \mathbb{N}; \]
thus the spectrum of $\nu$ is
\[ \left( \frac{4}{k^2} 2n(2n+1) + \frac{1}{4} \right)^{1/2} = \frac{2}{k} (2n+1) + \frac{1/4 - 1/k^2}{4(2n+1)/k} + O(n^{-3}); \]
successive differences of these cannot have constant fractional part unless $k = 2$. □

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