RESOLVENT ESTIMATES AND LOCAL DECAY OF WAVES ON CONIC MANIFOLDS

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ABSTRACT. We consider manifolds with conic singularites that are isometric to $\mathbb{R}^n$ outside a compact set. Under natural geometric assumptions on the cone points, we prove the existence of a logarithmic resonance-free region for the cut-off resolvent. The estimate also applies to the exterior domains of non-trapping polygons via a doubling process.

The proof of the resolvent estimate relies on the propagation of singularities theorems of Melrose and the second author [23] to establish a "very weak" Huygens' principle, which may be of independent interest.

As applications of the estimate, we obtain a exponential local energy decay and a resonance wave expansion in odd dimensions, as well as a lossless local smoothing estimate for the Schrödinger equation.

1. INTRODUCTION

In this paper we consider a manifold X of dimension n with conic singularities that is isometric to $\mathbb{R}^n$ outside a compact set. We impose geometric hypotheses (elucidated in Section 2.4 as Assumptions 1–3) that

(1) The flow along "geometric" geodesics is non-trapping. (Geometric geodesics are those that miss the cone points or that are everywhere locally given by limits of families of geodesics missing the cone points.)
(2) No three cone points are collinear.
(3) No two cone points are conjugate to each other.

Our main result is as follows (throughout the paper, Δ denotes the Laplacian with *positive* spectrum):

Theorem 1. *For $\chi \in \mathcal{C}_c^\infty(X)$, there exists $\delta > 0$ such that the cut-off resolvent*

$$\chi(\Delta - \lambda^2)^{-1}\chi$$

can be analytically continued from $\operatorname{Im}\lambda > 0$ *to the region*

$$\operatorname{Im}\lambda > -\delta \log \operatorname{Re}\lambda, \ \operatorname{Re}\lambda > \delta^{-1}$$

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and for some $C, T > 0$ enjoys the estimate

$$\left\|\chi(\Delta-\lambda^2)^{-1}\chi\right\|_{L^2\to L^2} \leqslant C|\lambda|^{-1}e^{T|\operatorname{Im}\lambda|}$$

in this region.

As shown by Lax-Phillips [20] and Vainberg [29] in certain geometric settings and later generalized by Tang-Zworski [28] to "black-box" perturbations, if the dimension n is odd, then Theorem 1 results in a decay estimate for solutions to the wave equation in such a geometry, and indeed in a full resonance-wave expansion for solutions to the wave equation. Let $\mathcal{D}_s$ denote the domain of $\Delta^{s/2}$ (see Section 2 below) and let $\sin t\sqrt{\Delta}/\sqrt{\Delta}$ be the wave propagator. Let χ equal 1 on the set where X is not isometric to $\mathbb{R}^n$.

Corollary 1. *Let n be odd. For all $A > 0$, small $\epsilon > 0$, $t > 0$ sufficiently large, and $f \in \mathcal{D}_1$,*

$$\chi\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}\chi f = \sum_{\substack{\lambda_j\in\operatorname{Res}(\Delta)\\ \operatorname{Im}\lambda>-A}}\sum_{m=0}^{M_j} e^{-it\lambda_j}t^m w_{j,m} + E_A(t)f$$

where the sum is of resonances of Δ, i.e. over the poles of the meromorphic continuation of the resolvent, and the $w_{j,m}$ are the associated resonant states corresponding to λ_j. The error satisfies

$$\|E_A(t)\|_{\mathcal{D}_1\to L^2} \leqslant C_\epsilon e^{-(A-\epsilon)t}.$$

In particular, since the resonances have imaginary part bounded above by a negative constant, $\chi\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}\chi f$ is exponentially decaying.

(We refer the reader to Theorem 1 of [28] for details of the resonance wave expansion.)

Another consequence of our resolvent estimate is a *local smoothing* estimate without loss for the Schrödinger equation. Local smoothing estimates were originally established for the Schrödinger equation on $\mathbb{R}^n$ by Sjölin [25], Vega [30], Constantin–Saut [11], Kato–Yajima [18], and Yajima [32]. Doi [15] showed that on smooth manifolds the absence of trapped geodesics is necessary for the local smoothing estimate to hold without loss. We show that even in the presence of very weak trapping due to the diffractive geodesics, the local smoothing estimate holds without loss.

Corollary 2. *Suppose u satisfies the Schrödinger equation on X:*

$$\begin{aligned} i^{-1}\partial_t u(t,z) + \Delta u(t,z) &= 0 \\ u(0,z) = u_0(z) &\in L^2(X) \end{aligned}$$

Then for all $\chi \in C_c^\infty(X)$, u satisfies the local smoothing estimate without loss:

$$\int_0^T \|\chi u(t)\|^2_{\mathcal{D}^{1/2}}\,dt \leqslant C_T\|u_0\|^2_{L^2(X)}$$

This result follows directly from our Theorem 1 by an argument of Burq [3].

Another application of the resolvent estimate of Theorem 1 is to the damped wave equation. Although we do not pursue it here, under suitable convexity assumptions (e.g., if no geodesic passing through the perturbed region re-enters it; see Datchev–Vasy [12, 13] for more general conditions), it is possible to obtain decay estimates for the damped wave equation on conic manifolds when the only undamped geodesics are diffractive ones. This relies on a gluing construction of Datchev–Vasy to obtain a suitable resolvent estimate and on the recent work of Christianson, Schenck, Vasy and the second author [9] to yield the estimate.

In addition to applying to manifolds with cone points, our results also apply to the more elementary setting of certain exterior domains to polygons in the plane. Let $\Omega \subset \mathbb{R}^2$ be a compact region with piecewise linear boundary. We further suppose that the complement $\mathbb{R}^2 \backslash \Omega$ is connected, that no three vertices of $\overline{\Omega}$ are collinear, and that $\mathbb{R}^2 \backslash \Omega$ is non-trapping, in the sense that all billiard trajectories not passing through the vertices of $\overline{\Omega}$ escape to infinity.[1] Figure 1 illustrates an example of such an exterior domain. For this class of domains, the analogue of Theorem 1 holds.

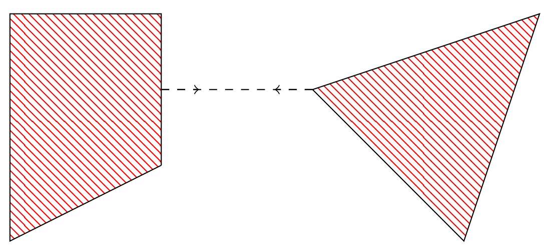

FIGURE 1. An example of a domain to which Corollary 3 applies. The dashed line represents a trapped diffractive orbit.

Corollary 3. *If $X = \mathbb{R}^2 \backslash \Omega$ is the exterior of a non-trapping polygon with no three vertices collinear and Δ is the Dirichlet or Neumann extension of the Laplacian on X, then the result of Theorem 1 holds for the resolvent on X.*

The proof of Corollary 3 relies on reducing the problem to one on a surface with conic singularites. Indeed, such an exterior domain can be *doubled* by gluing together two copies of it across the common boundary; this results in a manifold with cone points, corresponding to the vertices of the initial polygonal domain, and with two ends each isometric to $\mathbb{R}^2$. Solutions to the wave equation in this "doubled" manifold are closely related to solutions to the Dirichlet or Neumann problem on the original exterior domain via the

[1] In fact we require a slightly different condition. We ask that all billiard trajectories that are *locally* approximable by trajectories missing the vertices escape to infinity. This is not quite the same condition but is generically equivalent.

method of images. Our results hold for the exterior problem to such non-trapping polygons as well, although this entails some mild complication in the proof (the introduction of "black-box" methods)—see Section 5 below. In particular, our result affirmatively answers a conjecture of Chandler-Wilde, Graham, Langdon, and Spence [4]. In the case when the obstacle is star-shaped, we remark that the exponential energy decay is a consequence of the classical technique of Morawetz estimates (see e.g. Lemma 3.5 of [5])); we believe that the estimate for general non-trapping polygons is new, however.

We note that stronger estimates than those of Theorem 1 are known to hold in the case of a non-trapping metric or even an appropriately non-trapping "black box" perturbation such as a smooth non-trapping obstacle (see [20], [29], [28]): in these cases there are finitely many resonances above *any* logarithmic curve $\operatorname{Im}\lambda > -N\log\operatorname{Re}\lambda$. That the result here is likely to be sharp can be seen from the explicit computation of Burq [2], who shows that in the case of obstacle scattering by two strictly convex analytic obstacles in $\mathbb{R}^2$, one of which has a corner, the resonances (poles of the analytic continuation of the resolvent $(\Delta-\lambda^2)^{-1}$) are located along curves of the form $\operatorname{Im}\lambda = -C\log\operatorname{Re}\lambda$. Burq's setting is not of course exactly that of manifolds with cone points, but is suggestively close to that of polygonal domains discussed above. (Similar logarithmic strings of resonance poles also appear in Zworski [33] where they are shown to arise from finite order singularities of a one-dimensional potential, substantiating heuristics from Regge [24].)

By contrast, what seems the weakest trapping possible in the setting of smooth manifolds, a single closed hyperbolic geodesic, is known in certain settings to produce strings of resonances along lines of constant imaginary part [10], and hence yields an analytic continuation to a smaller region than that shown here, which does not permit a resonance wave expansion in the strong sense of Corollary 1 except in very special cases [8].

The fact that the estimates demonstrated here are weaker, by only a very small margin, than those for non-trapping situations, reflects that fact that cone points induce a kind of "weak trapping:" there exist geodesics connecting every pair of cone points, and concatenation of such geodesics starting and ending at the same cone point should be considered a legitimate geodesic curve in a conic geometry. In particular, such concatenations of geodesics are known (generically) to propagate singularities of the wave equation on exact cones by results of Cheeger-Taylor [6, 7]; this reflects the *diffraction* of singularities by the cone point. Melrose and the second author [23] subsequently showed that on any manifold with conic singularities, the propagation of singularities is limited to geodesics entering and leaving a given cone point at the same time ("diffractive propagation"). It was further shown in [23] that the fundamental solution of the wave group with initial pole near a cone point was *smoother* along generic geodesics emerging from the cone point than along those that are approximable by geodesics emanating from the initial pole and missing the cone point; this "smoothing effect" in fact holds for any solution that satisfies an appropriate *nonfocusing* condition with respect to the cone point in question (see Section 2.2 below). Thus, colloquially, [23] showed that "diffracted singularities are smoother than geometrically propagated singularities." It also showed that the spherical wavefront of diffracted singularities is a conormal wave. It is the smoothing property and the conormality that play an essential role in the proof of Theorem 1. The proof proceeds via another result which may be of independent interest, a theorem on the *weak non-trapping of singularities* for manifolds with cone points. In the following theorem, $U(t)$ denotes the wave group, and $\mathcal{E}_r$ denotes the Sobolev space of energy data $\mathcal{D}_r \oplus \mathcal{D}_{r-1}$.

Theorem 2. *Let $\chi \in \mathcal{C}_c^\infty(X)$. For any $s \in \mathbb{R}$, there exists $T_s \gg 0$ such that whenever $t > T_s$,*

$$\chi U(t)\chi : \mathcal{E}_r \to \mathcal{E}_{r+s}$$

for all r.

We recall that Huygens' Principle, valid in odd dimensional Euclidean space, says that $\chi U(t)\chi$ is eventually identically zero. More generally, in even dimensional Euclidean space or indeed in any "non-trapping" metric in which all geodesics escape to infinity, $\chi U(t)\chi$ eventually has a Schwartz kernel in $\mathcal{C}^\infty$. Our Theorem 2 is weaker yet: here the cut-off wave kernel is as smooth as one likes, after a sufficiently long time.

Resolvent estimates similar to ours have been previously demonstrated by Duyckaerts [16] for operators of the form $\Delta + V$ where Δ is the Euclidean Laplacian and V has multiple inverse-square singularities: these singularities are analytically similar to (albeit geometrically simpler than) cone points.

2. Geometric set-up

The basic material in this section on conic geometry comes from [23] while the more detailed discussion of the global geometry of geodesics is taken from [31].

Let X be a noncompact manifold with boundary, K a compact subset of X, and let g be a Riemannian metric on X° such that $X\backslash K$ is isometric to the exterior of a Euclidean ball $\mathbb{R}^n \backslash \overline{B^n(0, R_0)}$ and such that g has conic singularities at the boundary of X:

$$g = dx^2 + x^2 h(x, dx, y, dy);$$

here g is assumed to be nondegenerate over X° and $h|_{\partial X}$ induces a metric on ∂X. We let Y_α, $\alpha = 1, \dots N$ denote the components of ∂X; we will refer to these components in what follows as *cone points*, as each boundary component is a single point when viewed in terms of metric geometry.

We further let

$$M = \mathbb{R} \times X$$

denote our space-time manifold.

We recall from Theorem 1.2 of [23] that by judicious choice of coordinates x, y on a collar neighborhood of ∂X, we may reduce g to the normal form

$$g = dx^2 + x^2 h(x, y, dy), \tag{1}$$

where h is now a family (in x) of metrics on Y. Then the curves $y =$ const. are geodesics, with x the length parameter. Indeed, the curves of this form are the *only* geodesics reaching ∂X and they foliate a neighborhood of ∂X. We let ${}^X\mathcal{F}^s_\alpha$ denote the collection of the continuations of forward and backward bicharacteristics in T^*X° which reach the boundary component Y_α in time $|t| \leqslant s$ (with $\mathcal{F}$ denoting "flowout" of the cone point Y_α). Thus for small s, in canonical coordinates ξ, η dual to x, y,

$${}^X\mathcal{F}^s_\alpha = U \cap \{x \in (0, s), y \in Y_\alpha, \xi \in \mathbb{R}, \eta = 0\}$$

where U is a neighborhood of the single boundary component Y_α containing a component of $x < s$. We further refer to points in ${}^X\mathcal{F}^s_\alpha$ as *incoming* or *outgoing* with respect to the cone point according to whether they reach the boundary at positive or negative time respectively under the flow (this separates the manifold into components). We will also be concerned with the corresponding flowout sets in space-time. Letting Σ denote the characteristic set of $\Box = D_t^2 - \Delta$ on T^*M° we define

$$\mathcal{F}^s_\alpha = \{(t, \tau, z, \zeta) \in \Sigma : \ (z, \zeta) \in {}^X\mathcal{F}^s_\alpha\} \subset T^*M^\circ.$$

As discussed in [22] (where the notation $\mathcal{F}$ was first used) the manifolds ${}^X\mathcal{F}^s_\alpha$, $\mathcal{F}^s_\alpha$ are *coisotropic* conic submanifolds of T^*X°, resp. T^*M°.

We let $\mathcal{D}_s$ denote the domain of the $s/2$ power of the Friedrichs extension of the Laplacian on $\mathcal{C}^\infty_c(X^\circ)$. Note that this agrees with the ordinary Sobolev space H^s away from the cone points (and was characterized in [23, Section 3] in terms of the scale of *weighted b-Sobolev spaces*). Let

$$\mathcal{E}_s = \mathcal{D}_s \oplus \mathcal{D}_{s-1}$$

denote the corresponding space of Cauchy data for the wave equation, and let

$$U(t) = \exp it \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

denote the wave propagator, hence

$$U(t) \colon \mathcal{E}_s \to \mathcal{E}_s$$

for each $s \in \mathbb{R}$. We will frequently need to deal with error terms that are residual in the scale of space $\mathcal{E}_s$, so we define

$$\mathcal{R} = \{R : \mathcal{E}_{-\infty} \to \mathcal{E}_{+\infty,c}\}$$

with the additional c subscript denoting compact support in X. In dealing with wave equation solutions as functions in spacetime, it is convenient to think of them lying in the Hilbert space

$$L^2([0, \widetilde{T}]; \mathcal{E}_s)$$

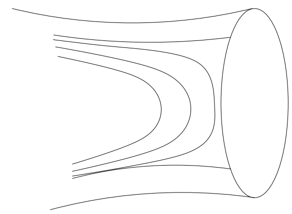

FIGURE 2. A family of geodesics missing the cone point (i.e., boundary component), limiting to a pair of geodesics entering and leaving it normally together with a geodesic of length π in the boundary connecting the two.

with $\widetilde{T} \gg T_s$ taken large enough to encompass all time intervals under consideration. We thus denote this space

$$L^2\,\mathcal{E}_s$$

for brevity. We recall that solutions to the wave equation in $L^2\,\mathcal{E}_s$ have unique restrictions to fixed-time data lying in $\mathcal{E}_s$, and will use this fact freely in what follows.

For convenience, we will equip the cosphere bundle S^*X with a Riemannian metric inducing a distance function, denoted $d(\bullet, \bullet)$.

2.1. **Geometric and diffractive geodesics.** We now recall the different notions of "geometric" and "diffractive" bicharacteristic which enter into the propagation of singularities on manifolds with cone points.

Definition 1. A *diffractive geodesic* on X is a union of a finite number of closed, oriented geodesic segments $\gamma_1, \dots, \gamma_N$ in X such that all end points except possibly the initial point in γ_1 and the final point of γ_N lie in $Y = \partial X$, and γ_i ends at the same boundary component at which γ_{i+1} begins, for $i = 1, \dots, N-1$.

A *geometric geodesic* is a diffractive geodesic such that in addition, the final point of γ_i and the initial point of γ_{i+1} are connected by a geodesic of length π in a boundary component Y_α (w.r.t. the metric $h_0 = h|_{Y_\alpha}$) for $i = 1, \dots, N-1$.

The proof the following proposition was sketched in [23], and yields the equivalence of the above definition of "geometric geodesic" with the more casual one used in the introduction above:

Proposition 1. *The geometric geodesics are those that are* locally *realizable as limits of families of geodesic in X° as they approach a given boundary component.*

Remark 1. We note that while every geometric geodesic is *locally* approximable by smooth geodesics in X°, a geodesic undergoing multiple interactions with cone points may not be *globally* approximable in this sense.

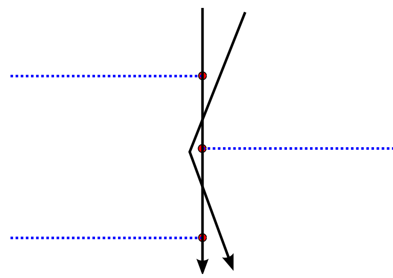

Figure 3. A non-approximable geometric geodesic in the triply-slit-plane (or its branched cover), with an approximating broken geodesic in X°.

Figure 3 shows such a situation. It is most easily interpreted as showing a domain with boundary given by the three slits; geodesics then reflect off smooth parts of the boundary, and geometric geodesics either pass straight through the end of the slits or reflect specularly as if the slits continued. Then the vertical line, while it can be uniformly approximated by broken geodesics as shown, cannot be approximated globally: any approximating geodesic would have to reflect off one or another of the slits. To place this example in the context of the current paper rather than that of domains with boundary, we should instead interpret the picture as showing one sheet of a two-sheeted ramified cover of $\mathbb{R}^2$, with the slits representing branch cuts. This makes the ends of the slits into cone points, with the link of each cone point a circle of circumference 4π. In this situation, any unbroken approximating geodesic would have to move onto the other sheet of the cover by passing through one of the slits, hence could not globally approximate the line shown, which remains on a single sheet of the cover.

This very simple example has three collinear cone points, which we are ruling out by hypothesis; however one can make other examples involving only two interactions with cone points, and this is permitted by our geometric hypotheses. For this reason Assumption 1 is formulated so as to cover geometric geodesics explicitly, rather than just as a uniform statement on geodesics in X° (which might be preferable). Ruling out propagation along non-approximable geometric geodesics, if indeed true, would likely require somewhat delicate second-microlocal arguments.

As we will need to understand the lift of the geodesic flow to the cotangent bundle, it is helpful to see how this can be accomplished uniformly up to ∂X (although in this paper, microlocal considerations will only arise over X°, which is a considerable simplification). We let ${}^bT^*X$ denote the b-cotangent bundle of X, i.e. the dual of the bundle whose sections are smooth vector fields tangent to ∂X. Let ${}^bS^*X$ denote the corresponding sphere bundle. Let $\xi\, dx/x + \eta \cdot dy$ denote the canonical one-form on ${}^bT^*X$. (We refer the reader to Chapter 2 of [21] for a further explanation of "b-geometry," of which we only use the rudiments here.)

Let K_g be the Hamilton vector field (with respect to the symplectic form $d\,(\xi\,dx/x + \eta \cdot dy))$ for $g/2 = (\xi^2 + h(x,y,\eta))/(2x^2)$, the symbol of $\Delta/2$ on ${}^bT^*X$; note that K_g is merely the geodesic spray in ${}^bT^*X$ with velocity $\sqrt{g}$. It is convenient to rescale this vector field so that it is both tangent to the boundary of X and homogeneous of degree zero in the fibers. Near a boundary component Y_α, for a metric in the reduced form (1), we have (see [23])

$$K_g = x^{-2}\Big(H_{Y_\alpha}(x) + \big(\xi^2 + h(x,y,\eta) + \frac{x}{2}\frac{\partial h}{\partial x}\big)\partial_\xi + \xi x \partial_x\Big), \tag{2}$$

where $H_{Y_\alpha}(x)$ is the geodesic spray in Y_α with respect to the family of metrics $h(x,\cdot)$. Hence the desired rescaling is

$$Z = \frac{x}{\sqrt{g}} K_g.$$

(Note here that g refers to the *metric function* on the cotangent bundle and not the determinant of the metric tensor.) By the homogeneity of Z, if we radially compactify the fibers of the cotangent bundle and identify ${}^bS^*X$ with the "sphere at infinity" then Z is tangent to ${}^bS^*X$, and may be restricted to it. Henceforth, then, we let Z denote the restriction of $(x/\sqrt{g})K_g$ to the *compact* manifold ${}^bS^*X$ on which the coordinates ξ, η have been replaced by the (redundant) coordinates

$$(\bar{\xi}, \bar{\eta}) = \Big(\xi/\sqrt{\xi^2 + h(\eta)}, \eta/\sqrt{\xi^2 + h(\eta)}\Big).$$

Z vanishes only at certain points $x = \bar{\eta} = 0$ over ∂X, hence the closures of maximally extended integral curves of this vector field can only begin and end over ∂X. Since Z is tangent to the boundary, such integral curves either lie entirely over ∂X or lie over ∂X only at their limit points. Interior and boundary integral curves can meet only at limit points in $\{x = \bar{\eta} = 0\} \subset {}^bS^*X$.

It is helpful in studying the integral curves of Z to introduce the following way of measuring their lengths: Let γ be an integral curve of Z over X°. Let k denote a Riemannian metric on ${}^bS^*X^\circ$ such that $k(Z,Z) = 1$. Let

$$\omega = xk(\cdot, Z) \in \Omega^1({}^bS^*X). \tag{3}$$

Then

$$\int_\gamma \omega = \int_\gamma xk(d\gamma/ds, Z)\,ds = \int_\gamma \frac{x}{\sqrt{g}} k(K_g, Z)\,ds = \int_\gamma ds = \text{length}(\gamma)$$

where s parametrizes γ as an integral curve of $K_g/\sqrt{g}$, the unit speed geodesic flow. With this motivation in mind, we now define, for each $t \in \mathbb{R}_+$, two relations in ${}^bS^*X$, a "geometric" and a "diffractive" relation. These correspond to the two different possibilities for geodesic flow through the boundary.

Definition 2. Let $p, q \in {}^bS^*X$. We write

$$p \overset{G,t}{\sim} q$$

if there exists a *continuous,* piecewise smooth curve $\gamma : [0,1] \to {}^bS^*X$ with $\gamma(0) = p$, $\gamma(1) = q$, such that $[0,1]$ can be decomposed into a finite union of closed subintervals I_j, intersecting at their endpoints, where

(1) on each I_j°, γ is a (reparametrized) positively oriented integral curve of Z in ${}^bS^*X$,
(2) On successive intervals I_j and I_{j+1}, interior and boundary curves alternate,
(3) $\int_\gamma \omega = t$, with ω as defined in (3).

We write

$$p \overset{D,t}{\sim} q$$

if there exists a piecewise smooth (not necessarily continuous) curve $\gamma : [0,1] \to {}^bS^*X$ with $\gamma(0) = p$, $\gamma(1) = q$, such that $[0,1]$ can be decomposed into a finite union of closed subintervals I_j, intersecting at their endpoints, where

(1) on each I_j°, γ is a (reparametrized) positively oriented integral curve of Z in ${}^bS^*X^\circ$,
(2) the final point of γ on I_j and the initial point of γ on I_{j+1} lie over the same component Y_α of ∂X,
(3) $\int_\gamma \omega = t$.

Integral curves of Z over X° are lifts of geodesics in X°, and it follows from (2) that the maximally extended integral curves of Z in ${}^bS^*_{\partial X}X$ are lifts of geodesics of length π in ∂X (see [23] for details), hence:

Proposition 2. *$p \overset{G,t}{\sim} q$ iff p and q are connected by a (lifted) geometric geodesic of length t.*

$p \overset{D,t}{\sim} q$ iff p and q are connected by a (lifted) diffractive geodesic of length t.

A very important feature of these equivalence relations, proved in [31], is the following:

Proposition 3. ([31], Prop. 4) *The sets $\{(p,q,t) : p \overset{G,t}{\sim} q\}$ and $\{(p,q,t) : p \overset{D,t}{\sim} q\}$ are closed subsets of ${}^bS^*X \times {}^bS^*X \times \mathbb{R}_+$.*

2.2. **Propagation and diffraction of singularities.** We now recall the key propagation results from [23]. In order to do this, we must introduce the notions of *coisotropic regularity* and *nonfocusing* with respect to a conic coisotropic submanifold $\mathcal{I}$ of T^*M°. We will usually take $\mathcal{I} = \mathcal{F}_\alpha \sim \mathcal{F}_\alpha^S$ with S taken larger than all times on which we will be considering propagation of singularities (and hence ignored in our notation).

Let $\mathcal{M}$ denote the space of compactly supported pseudodifferential operators $A \in \Psi^1(M^\circ)$ that are characteristic on $\mathcal{I}$. Let $\mathcal{A}$ be the filtered algebra generated by $\mathcal{M}$ over $\Psi^0(M)$, with $\mathcal{A}^k = \mathcal{A} \cap \Psi^k(M)$. Fix a Sobolev space $\mathcal{H}$. Fix a compact set $K \subset S^*M^\circ$.

We now recall the following definitions from [22].

- We say that u has coisotropic regularity with respect to $\mathcal{I}$ of order k relative to $\mathcal{H}$ in K if there exists $A \in \Psi^0(M)$, elliptic on K and supported over M° such that $\mathcal{A}^k Au \subset \mathcal{H}$. We say that u has coisotropic regularity relative to $\mathcal{H}$ in K if it has coisotropic regularity of order k for all k¿
- We say that u satisfies the nonfocusing condition of degree k with respect to $\mathcal{I}$ relative to $\mathcal{H}$ on K if there exists $A \in \Psi^0(M)$, elliptic on K and supported over M° such that $Au \subset \mathcal{A}^k\mathcal{H}$. We say that u satisfies the nonfocusing condition relative to $\mathcal{H}$ on K if it satisfies the condition to some degree.

If K is omitted (which will only be the case when the distribution is microsupported away from the boundary), the relevant condition is assumed to hold on all of S^*M°.

In the special case when $\mathcal{I} = \mathcal{F}_\alpha$ and we work in a collar neighborhood of the boundary component Y_α in which the metric has the form (1), these definitions simplify considerably, as the module $\mathcal{M}$ is then generated by the operators ∂_{y_j}. A distribution u microsupported over such a neighborhood is coisotropic of order k relative to $\mathcal{H}$ iff $\langle \Delta_Y \rangle^{k/2} u \in \mathcal{H}$, while it satisfies the nonfocusing condition relative to $\mathcal{H}$ iff there exists N with $\langle \Delta_Y \rangle^{-N} u \in \mathcal{H}$. (Here Δ_Y denotes the Laplacian in the y-variables with respect to the family of metrics $h(x)$ on Y_α.)

We are now in a position to recall the main results of [23]. (Our notation however sticks more closely to that of [22], which treats the more general case of edge manifolds.)

Note that here and henceforth we employ the notation $s-0$ to mean $s-\epsilon$ for every $\epsilon > 0$.

Proposition 4. *Let $u \in \mathcal{C}(\mathbb{R}; \mathcal{D}_r)$ be a solution to the wave equation on M. Fix a point $\rho \in \mathcal{F}_\alpha$ outgoing with respect to the cone point Y_α.*

1. *If u is microlocally in H^s on all incoming bicharacteristics in $\mathcal{F}_\alpha$ that are* diffractively related *to ρ then $u \in H^s$ microlocally at ρ.*
2. *Assume that u is nonfocusing with respect to H^s on $\mathcal{F}_\alpha$ and has no wavefront set along incoming bicharacteristics in $\mathcal{F}_\alpha$ that are* geometrically related *to ρ. Then $u \in H^{s-0}$ microlocally at ρ, and enjoys coisotropic regularity relative to H^{s-0} in a neighborhood of ρ.*

Remark 2.

- The coisotropic regularity part of this result is left slightly implicit in the results of [23] but follows easily by interpolating part (iii) of Theorem 8.1 of that paper, which gives coisotropic regularity relative

to *some* Sobolev space, with the overall regularity of the solution microlocally near ρ.
- The following consequence is more germane to what follows (and perhaps easier to digest): Fix s, k with $r < k < s$. Say our solution is nonfocusing with respect to H^s. If a singularity in WF^k (the set where $u \notin H^k$ microlocally) arrives at Y_α along just one ray in $\mathcal{F}_\alpha$, with u microlocally smooth along all other arriving rays, then it may produce milder singularities, at worst in H^{s-0} and coisotropic relative to this space, along rays *diffractively* related to the incoming singularity, but has strong singularities in WF^k along (at least some of) those *geometrically* related to it.

The crucial example is of course the fundamental solution

$$u = \frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}\delta_p$$

with p close to a boundary component Y_α. This solution is overall in $H^{-n/2+1-0}$, but it satisfies the nonfocusing condition relative to $H^{1/2-0}$. When the singularity strikes Y_α is produces strong singularities (no better than $H^{-n/2+1-0}$) along the geometric continuations of the geodesic from p to Y_α but only weaker singularities in $H^{1/2-0}$ along the rest of the points in a spherical wavefront emanating from Y_α, along which the solution in fact enjoys *Lagrangian* regularity.

The same reasoning that applies to the fundamental solution in fact applies to a solution given by a Lagrangian distribution with respect to a Lagrangian manifold intersecting $\mathcal{F}_\alpha$ transversely. In this paper, however, we will be concerned with a slightly different setting, in which our hypotheses on the solution are themselves that of coisotropic regularity, but with respect to a *different* coisotropic manifold, intersecting $\mathcal{F}_\alpha$ transversely (in particular, with respect to $\mathcal{F}_\beta$ for some β). In applying Proposition 4 above, we will need a result in order to verify the nonfocusing hypotheses. This result is discussed in the next section.

2.3. **Intersecting coisotropics.** In this section we prove the following result allowing us to verify the nonfocusing hypotheses of Proposition 4.

Proposition 5. *Let $\mathcal{I}$ denote a conic coisotropic manifold of codimension $n-1$ intersecting $\mathcal{F}_\alpha$ in such a way that the isotropic foliations of the two coisotropic manifolds are transverse. Let u enjoy coisotropic regularity with respect to $\mathcal{I}$, relative to H^s. Then u is nonfocusing with respect to $\mathcal{F}_\alpha$ relative to $H^{s+(n-1)/2-0}$.*

The proof of the proposition relies on two lemmas. We first show it in a model setting:

Lemma 1. *In $\mathbb{R}^{n_1+n_2}$ with coordinates $z \in \mathbb{R}^{n_1}$, $z' \in \mathbb{R}^{n_2}$, let $\mathcal{I}$, $\widetilde{\mathcal{I}}$ denote the two model coisotropic manifolds*

$$\mathcal{I} = \{\zeta = 0\},\ \widetilde{\mathcal{I}} = \{z = 0\}.$$

If a compactly supported distribution enjoys coisotropic regularity with respect to $\widetilde{\mathcal{I}}$ relative to H^s then it is nonfocusing with respect to $\mathcal{I}$ relative to $H^{s+n_1/2-0}$. That is to say, there exists $N \gg 0$ such that

$$\langle \Delta_z \rangle^{-N} u \in H^{s+n_1/2-0}_{loc}(\mathbb{R}^{n_1+n_2}).$$

Note that n_1 is the dimension of the leaves of the isotropic foliation.

Proof. By applying powers of Δ we may reduce to the case $s = 0$. Coisotropic regularity with respect to $\widetilde{\mathcal{I}}$ relative to L^2 means that

$$z^\alpha D^\beta_{z'} D^\gamma_z u \in L^2,\ \text{if } |\alpha| = |\beta| + |\gamma|,$$

as the vector fields $z_i D_{z_j}$ and $z_i D_{z'_k}$ have symbols cutting out $\widetilde{\mathcal{I}}$.

In particular, the iterated regularity under $z_i D_{z_k}$ means that we have

$$z^\alpha D^\gamma_{z'} u \in L^2(\mathbb{R}^{n_2}_z; I^{(0)}(\mathbb{R}^{n_1}; 0)),\ |\alpha| = |\gamma|,$$

where $I^{(s)}(\mathbb{R}^{n_1}; 0)$ denotes the space of functions $v \in H^s(\mathbb{R}^{n_1})$ enjoying iterated regularity under vector fields $z_i D_{z_k}$, i.e., conormal regularity with respect to the origin.

Now we claim that if $\rho < n_1/2$ is positive,

$$|z|^{-\rho} I^{(0)}(\mathbb{R}^{n_1}; 0) \subset H^{-n_1/2-0}(\mathbb{R}^{n_1}).$$

Indeed, interpolation shows that for a compactly supported $u \in I^{(0)}(\mathbb{R}^{n_1}; 0)$,

$$\Delta_z^{s/2} |z|^s u \in L^2$$

for all $s > 0$. Hence Sobolev embedding shows that $I^{(0)}$ is contained in $|z|^{-n_1/2-0} L^\infty$. Thus, for $\rho < n_1/2$, $|z|^{-\rho} I^{(0)}(\mathbb{R}^{n_1}; 0)$ is contained in $|z|^{-n_1+\epsilon} L^\infty$ for some $\epsilon > 0$. This in turn implies that for compactly supported $u \in |z|^{-\rho} I^{(0)}(\mathbb{R}^{n_1}; 0)$, $\check{u}$ is bounded and so $\langle \zeta \rangle^{-n_1/2-\epsilon} \check{u} \in L^2$ for all ϵ. Taking the inverse Fourier transform yields $u \in H^{-n_1/2-\epsilon}(\mathbb{R}^{n_1})$ for all $\epsilon > 0$.

Thus, applying powers of $z^\alpha D^\gamma_{z'} u$ (and again interpolating to deal with fractional powers) we find that

$$u \in H^s(\mathbb{R}^{n_2}; H^{-n_1/2-0}(\mathbb{R}^{n_1})),\ s < n_1/2.$$

Finally, this implies that for $N > n_1/2$,

$$\langle \Delta_z \rangle^{-N} u \in H^s(\mathbb{R}^{n_2}; H^{n_1/2}(\mathbb{R}^{n_1})),\ s < n_1/2.$$

Since $H^{n_1/2-0}(\mathbb{R}^{n_2}; H^{n_1/2}(\mathbb{R}^{n_1})) \subset H^{n_1/2-0}(\mathbb{R}^{n_1+n_2})$ this implies nonfocusing with respect to $n_1/2 - 0$ as desired. $\square$

Lemma 1 is in fact quite general, as the form of the coisotropic distributions employed there is a normal form for intersecting conic coisotropic manifolds with transverse foliations:

Lemma 2. *Let $\mathcal{I},\widetilde{\mathcal{I}}$ be conic coisotropic submanifolds of $T^*(\mathbb{R}^n)$, each of codimension $k < n$, and intersecting at $\rho \in T^*(\mathbb{R}^n)$ in such a way that the isotropic foliation of each is transverse to the other. Then there exist local symplectic coordinates (z, z', ζ, ζ') in which ρ lies at the origin, and*

$$\mathcal{I} = \{\zeta = 0\},\ \widetilde{\mathcal{I}} = \{z = 0\}.$$

Proof. We choose a degree-one homogeneous function f vanishing simply along $\mathcal{I}$. Thus its Hamilton vector field H_f has degree zero and is by definition tangent to $\mathcal{I}$ and transverse to $\widetilde{\mathcal{I}}$. On the flowout of $\widetilde{\mathcal{I}}$ by H_f we let g denote the homogeneous, degree-zero function measuring the time of flowout from $\widetilde{\mathcal{I}}$; we may further extend g to a homogeneous degree-zero function (also denoted g) on $T^*\mathbb{R}^n$ such that

$$\{f, g\} = 1$$

and with

$$g = 0,\ dg \neq 0 \text{ on } \widetilde{\mathcal{I}}$$

(To make this extension, take coordinates, not necessarily symplectic, in which $\widetilde{\mathcal{I}}$ lies along coordinate axes, and define g locally to be the time to flow out to a nearby point from a hyperplane containing $\widetilde{\mathcal{I}}$ and transverse to H_f.)

Now by Theorem 21.1.9 of [19], we may extend the coordinates

$$(z_k, \zeta_k) \equiv (g, f)$$

to a full system of homogeneous symplectic coordinates. Thus, we have locally achieved

$$\mathcal{I} \subset \{\zeta_k = 0\},\ \widetilde{\mathcal{I}} \subset \{z_k = 0\}.$$

Since $\mathsf{H}_{\zeta_k}, \mathsf{H}_{z_k}$ are respectively tangent to $\mathcal{I},\widetilde{\mathcal{I}}$, these manifolds are locally *products* of the form

$$\mathcal{I}' \times \{(z_k \in \mathbb{R}, \zeta_k = 0)\},\ \widetilde{\mathcal{I}}' \times \{(z_k = 0, \zeta_k \in \mathbb{R})\}$$

with $\mathcal{I}',\widetilde{\mathcal{I}}'$ coisotropic submanifolds of $T^*\mathbb{R}^{n-1}$ satisfying the hypotheses of the lemma with n replaced by $n-1$ and k replaced by $k-1$.

The result then follows by induction. □

Proof of Proposition 5. Proposition 5 now follows from the two lemmas above: we can bring the two coisotropic manifolds $\mathcal{F}_\alpha$ and $\mathcal{I}$ into the normal form given by Lemma 2. Lemma 1 then shows that nonfocusing holds, with $n_1 = n-1$, the dimension of the isotropic foliation (which in the case of $\mathcal{F}_\alpha$ has leaves obtained by varying $y \in Y_\alpha$). □

2.4. **Geometric Assumptions.** We are now able to state our geometric assumptions:

Assumption 1. *Let $\Omega \supset K$ be an open set with $X\backslash\Omega \sim \mathbb{R}^n\backslash B^n(0, R_1)$ for some $R_1 \gg 0$. We assume that there exists a time $T_0 > 0$ such that any* geometric *geodesic starting in $S^*_K X^\circ$ leaves Ω in time less than T_0 (and does not come back, by convexity of the ball).*

Assumption 2. *No geometric geodesic passes through three cone points.*

Assumption 3. *No two cone points Y_α, Y_β are conjugate to one another along geodesics in S^*X° of lengths less than T_0 in the sense that whenever $s + t \leqslant T_0$, $\mathcal{F}^s_\alpha$ and $\mathcal{F}^t_\beta$ intersect transversely for each α, β.*

Remark 3. Assumption 1 is a quantitative statement of non-trapping of geodesics that do not hit the cone points.

Assumption 2 is generic at the formal level of dimension counting, as a geodesic arriving at Y_γ from Y_α can be geometrically continued from a family of points in Y_γ of dimension $n - 2$ (those points at distance π in Y_γ from the arrival point); on the other hand there is a (generically) discrete set of departure points in Y_γ for geodesics of bounded length leading to Y_β. So we have an intersection of set of codimension-one and a set of dimension zero dictating the existence of a geometric geodesic through these three points.

The following result (whose proof is contained in Appendix A) justifies our use of the term “conjugate” above. We let $\mathcal{V}_b(X)$ denote the space of “b-vector fields,” meaning those tangent to ∂X.

Proposition 6. *The coisotropic manifolds $\mathcal{F}^s_\alpha$ and $\mathcal{F}^t_\beta$ intersect non-transversely if and only if there exist cone points Y_α, Y_β, a geodesic $\gamma(t)$ with $\gamma(0) \in Y_\alpha$, $\gamma(t') \in Y_\beta$ (with $t' < s + t$) and a normal Jacobi field W along γ with $W \in \mathcal{V}_b(X)$.*

Remark 4.

- This result is equivalent to the same statement for the manifolds ${}^X\mathcal{F}^s_\alpha$, ${}^X\mathcal{F}^t_\alpha$, as the incoming and outgoing components of $\mathcal{F}^s_\alpha$ are each naturally identified with $\mathbb{R}_t \times \mathcal{F}^s_\alpha$.
- The transverse intersection stated here is equivalent to the transverse intersection of the *isotropic fibers* of the coisotropic manifolds, as will be seen in the proof below. In particular, a point of intersection automatically entails that the tangent spaces of both manifolds contain the tangent to a geodesic and the radial vector field; in both manifolds, the space spanned by these two directions is the base of the coisotropic foliation.
- The usual description of conjugate points involves the vanishing of the Jacobi field at the endpoints of the geodesic; here, by contrast, since $W \in \mathcal{V}_b(X)$, we find that the metric length of W vanishes at the endpoints, hence the variation along W should be regarded as

an admissible one for a one-parameter family of geodesics: varying the endpoint in Y_α should be regarded as varying the *direction* of departure from the cone point, with the location of the end fixed at the "point" Y_α.

Finally, we note that it always suffices to prove the results of this paper with χ replaced by $\widetilde{\chi}$ which is 1 on $\operatorname{supp}\chi$; also, by the convexity of the Euclidean ball, the hypotheses of the theorems are still satisfied if we replace our given K by a larger compact set. *Therefore, we will assume without loss of generality that our manifold is Euclidean on* $\operatorname{supp}(1-\chi)$ *and that* $\operatorname{supp}\chi \subset K$. We also use Ω to denote a small neighborhood of the compact set K.

3. Decomposition of the wave propagator

Let
$$L = \min_{\alpha,\beta} d(Y_\alpha, Y_\beta)$$
denote the minimum distance between cone points. Fix
$$\delta_A \ll 1,\ \delta_\psi < \frac{L}{200}.$$
We let ψ_α be cutoff functions, each equal to 1 in a small neighborhood $x < \delta_\psi/4$ of a single cone point Y_α and supported in $x < \delta_\psi$; let $\Upsilon \in \mathcal{C}^\infty(X)$ equal 1 outside Ω and vanish on $\operatorname{supp}\chi$; and finally let A_j $(j = 1, \dots, N)$ be a pseudodifferential partition of $I - \sum \psi_\alpha - \Upsilon$ in which each element is microsupported on a set of diameter less than δ_A (with respect to our arbitrary but fixed metric on S^*X) and in particular is microsupported over K. We can then further arrange that
$$\sum_{j=1}^{N} A_j + \sum_\alpha \psi_\alpha + \Upsilon - I \in \mathcal{R}.$$
(The error term in fact can be taken to lie in $\Psi_c^{-\infty}(X^\circ)$, which is to say it is a smoothing operator with Schwartz kernel compactly supported in X°, and thus in particular vanishing in a neighborhood of all boundary components.)

We will consider sequences of wave propagators sandwiched between various of the A_j's: with $J = (j_0, \dots, j_{k+1})$, set
$$T_J = A_{j_0} U(t_0) A_{j_1} U(t_1) \dots A_{j_k} U(t_k) A_{j_{k+1}}.$$
Associated to each such propagator is a *word* $j_0 j_1 \dots j_{k+1}$ referring to a sequence of elements of the cover, as well as the additional data of a time t_ℓ associate to each pair of successive letters $j_\ell j_{\ell+1}$. We say that such a word is *geometrically realizable* ("GR") if for each pair of successive sets $j_\ell j_{\ell+1}$, in the word, there exist $p_\ell \in \mathrm{WF}' A_{j_\ell}$ and $p_{\ell+1} \in \mathrm{WF}' A_{j_{\ell+1}}$ with
$$p_{\ell+1} \overset{G,t_\ell}{\sim} p_\ell.$$

We call the word *diffractively realizable* ("DR") if instead there are points with
$$p_{\ell+1} \overset{D,t_\ell}{\sim} p_\ell.$$
Note that any GR word is DR, and also that the words are read from right to left in order to conform with the composition of operators. We also describe individual pairs of successive letters in a word as *diffractive* or *geometric interactions* depending on whether these two-letter subwords are GR or DR. We say that such a successive pair of letters *interacts with a cone point* if there exists a diffractive geodesic of length t_ℓ between $p_\ell \in \mathrm{WF}' A_{j_\ell}$ and $p_{\ell+1} \in \mathrm{WF}' A_{j_{\ell+1}}$ that passes through some boundary component Y_α

The following result comes direction from ("diffractive") propagation of singularities, Theorem I.2 of [23]; its content is simply that singularities of solutions to the wave equation propagate along diffractive geodesics:

Proposition 7. *If the word J is not DR then $T_J \in \mathcal{R}$.*

Lemma 3. *If δ_A is chosen small enough and $t_0 + \cdots + t_k > 2T_0$, the word $J = (j_0 \dots j_{k+1})$ cannot be GR.*

Proof. Shrinking $\delta_A = 2^{-m} \downarrow 0$, if there are GR words with fixed time intervals, in the limit there must be a geometric geodesic of length $2T_0$ that remains in Ω, which is ruled out by assumption. □

We also have the following, more granular, reformulation of Assumption 2, which we will need in what follows:

Lemma 4. *If δ_A is sufficiently small, there do not exists words of the form $ijk\ell$, where jk is GR and ij, jk, and $k\ell$ all interact with cone points.*

Proof. If the result fails, then taking $\delta_A = 2^{-m}$ gives a family of broken geodesics given by the concatenation of γ_0^m, γ_1^m, γ_2^m where

- γ_0^m starts and γ_2^m ends at a cone point,
- The end of γ_i^m and the start of γ_{i+1}^m are within distance 2^{-m} of each other in S^*X°,
- γ_1^m undergoes geometric interaction with a cone point.

Then taking $m \to \infty$ would yield (by compactness of ∂X and Proposition 3) a limiting geodesic passing through three cone points, interacting geometrically with the one in the middle; this contradicts Assumption 2. □

The crucial ingredient in the proof of Theorem 2 is the following lemma which shows that the propagator eventually locally smooths data microsupported in $\mathrm{WF}' A_m$ by $(n-1)/2 - 0$ derivatives. We cannot quite just add together all these terms for differing m and conclude the theorem, as that still leaves singularities starting in $\operatorname{supp} \psi_\alpha$ to be dealt with; however, we will see later on that the singularities near the cone points in these latter terms can be moved away from the cone points by applying the propagator for short time.

In order to make iterative arguments, it is convenient to keep track explicitly of singularities that leave Ω never to return: we let $\mathcal{O}$ (for "outgoing") denote the subset

$$\mathcal{O} = \big\{(t, z, \tau, \zeta) \in T^*(M\backslash\Omega) : \langle z, \zeta/\tau\rangle > 0\big\}.$$

(Here we have abused notation by identifying $X\backslash\Omega$ with a subset of Euclidean space to which it is isometric.) The set $\mathcal{O}$ is mapped to itself by positive time geodesic flow; moreover, any bicharacteristic starting in $\operatorname{supp}\chi$ that escapes Ω lies in $\mathcal{O}$ over $X\backslash\Omega$. We let $L^2 H^s(\mathcal{O})$ denote the space of distributions that are microsupported in $\mathcal{O}$ (hence in $\mathcal{D}_\infty$ on K) and lie in $L^2 H^s$ (where as before we use the notation $L^2 H^s$ for $L^2([0, \widetilde{T}]; H^s)$). Let $L^2 \mathcal{E}_s(\mathcal{O}) = L^2 \left(H^s(\mathcal{O}) \oplus H^{s-1}(\mathcal{O})\right)$.

Lemma 5. *There exist δ_A, δ_ψ sufficiently small that for each m and s, for all $t > 5T_0$,*

$$U(t)A_m : \mathcal{E}_s \to L^2 \mathcal{E}_{s+(n-1)/2-0} + L^2 \mathcal{E}_s(\mathcal{O}).$$

Since distributions in $L^2 \mathcal{E}_s(\mathcal{O})$ are smooth on Ω this implies in particular that $\chi U(t)A_m : \mathcal{E}_s \to \mathcal{E}_{s+(n-1)/2-0}$.

Proof. If all diffractive bicharacteristics starting in A_m escape to $\mathcal{O}$ in time less than $5T_0$, then the result holds, by propagation of singularities. If not, some bicharacteristic must hit a cone point within time T_0—none can do so in longer time, as otherwise it would be a (trivially) geometric bicharacteristic remaining in Ω for time T_0, contradicting Assumption 1. We now let s_0 be the time at which the first cone point is reached under the flow. In time $s_0 + 3\delta_\psi$, then this particular bicharacteristic is at least distance $2\delta_\psi$ from the boundary; hence if δ_A is small enough then taking $t_0 = s_0 + 3\delta_\psi$ we find that any singularity starting within distance δ_A of this one is propagated by $U(t_0)$ to lie at distance greater than δ_ψ from the boundary, by propagation of singularities. (Note that we choose δ_A small enough that bicharacteristics of length close to t_0 starting from $\mathrm{WF}' A_m$ interact with only a single cone point.) Thus, *either* $U(5T_0)A_m$ has range in $\mathcal{E}_s(\mathcal{O})$ *or* there exists $t_0 < T_0$ such that:

$$U(t_0)A_m = \sum_\ell A_\ell U(t_0)A_m \bmod \mathcal{R}.$$

Now some of the words ℓm are DR in time t_0 and others (most) are not. Those that are not give smoothing terms, so we discard them. For those that are, we repeat the construction above, twice, starting in $\mathrm{WF}\, A_\ell$ and $\mathrm{WF}\, A_k$ successively instead of in $\mathrm{WF}' A_m$. We may thus write $U(5T_0)A_m$ as a sum of terms

$$U(5T_0)A_m = \sum U(5T_0 - t_0 - t_1 - t_2)A_j U(t_2)A_k U(t_1)A_\ell U(t_0)A_m + E + R$$

where all words $jk\ell m$ are DR, and where

$$E : \mathcal{E}_s \to \mathcal{E}_s(\mathcal{O}),\ \ R \in \mathcal{R}.$$

The choices of t_1 and t_2 in the sum depend on k and ℓ just as our choice of t_0 depended on how long it took a bicharacteristic in starting in $\mathrm{WF}' A_m$ to hit a cone point. Since this dependence is not relevant in what follows, however, we suppress it in the notation. Note that each t_i is less than T_0, as otherwise the bicharacteristics must have escaped to $\mathcal{O}$ rather than interacting with another cone point.

Now in each word $jk\ell m$ associated with an element of the sum there are three interactions with cone points. Some of these interactions are GR, and some are not. We encode this by associating a string of G's and D's to a word, so for instance a diffractive interaction followed by two geometric gives (reading right to left) the string GGD. We break our sum into pieces based on this classification.

For a word containing two successive D's, i.e. GDD, DDG, or DDD, we claim that the propagator maps $\mathcal{E}_s \to \mathcal{E}_{s+(n-1)/2-0}$. The proof is as follows: given initial data in $\mathcal{E}_s$, Proposition 4 tells us that the first diffractive interaction results in a solution $U(t)A_m$ that is coisotropic with respect to $\mathcal{F}_\alpha$ (for the relevant cone point α,) relative to H^{s-0}. This distribution then propagates in T^*M° so as to preserve this coisotropic regularity (see Proposition 12.2 of [23] or Lemma 4.7 of [22] from which it also follows). Now when the singularities arrive at the next cone point (say, Y_β) for the second diffraction, by Assumption 3 (i.e., nonconjugacy) we may apply Proposition 5 to conclude that the solution is nonfocusing with respect to $\mathcal{F}_\beta$ relative to $H^{s+(n-1)/2-0}$. Then the second diffractive interaction puts it in $H^{s+(n-1)/2-0}$, by a second application of Proposition 4.

By contrast a word containing G in the middle, i.e., DGD, GGD, DGG, or GGG cannot be realizable by Lemma 4.

This leaves words of the form GDG as the only remaining summands to treat. A geometric geodesic passing starting in $\mathrm{WF}' A_j$, traveling for time t_2, reaching $\mathrm{WF}' A_i$, and then traveling for time $5T_0 - t_0 - t_1 - t_2$ must reach $\mathcal{O}$ since the only other option is for it to reach another cone point, which again would contradict Lemma 4. Thus $U(5T_0 - t_0 - t_1 - t_2)A_j U(t_2)A_k$ must map singularities to $\mathcal{O}$.

Thus, we have established that every term in the sum representing $U(5T_0)A_m$ either maps singularities to $\mathcal{O}$ or smooths them by $(n-1)/2 - 0$ derivatives. $\square$

4. Weak non-trapping of singularities

In this section we prove Theorem 2, which tells us that weak non-trapping of singularities holds.

The proof consists of two steps. To start, we will prove the theorem when $s = (n-1)/2 - 0$. To accomplish this, we apply Lemma 5 as follows. We decompose

$$\chi U(t)\chi = \sum_j \chi U(t)\chi A_j + \sum_\alpha \chi U(t)\chi\psi_\alpha. \tag{4}$$

Then for $t > 5T_0$ the first sum has the desired mapping property by Lemma 5 since $\mathrm{WF}'\,\chi A_j \subset \mathrm{WF}'\,A_j$. We deal with the second as follows: pick $\tau > 2\delta_\psi$, and smaller than $L/50$ (recall that L is the minimum distance between cone points). By propagation of singularities, if $\mathrm{WF}\,v \subset \mathrm{WF}'\,\psi_\alpha$ then

$$\psi_\beta U(\tau)v \in \mathcal{E}_\infty \text{ for all } \beta,$$

hence

$$U(\tau)v - \sum_j A_j U(\tau)v \in L^2\,\mathcal{E}_\infty + L^2\mathcal{E}_s(\mathcal{O}).$$

Thus, we may rewrite the second sum in (4) as

$$\sum_\alpha \sum_j \chi U(t-\tau)\big(A_j U(\tau)\chi\psi_\alpha\big) \bmod \mathcal{R},$$

and again applying Lemma 5, this time with $t > \tau + 5T_0$, shows that these terms, too, enjoy the desired mapping properties. This concludes the first step in the proof.

To finish the proof, we need to show that further smoothing occurs as time evolves. To this end, note that we may iterate the result obtained above as follows. Given $f = (f_0, f_1) \in \mathcal{E}_s$ we choose $\chi_1 \in \mathcal{C}_c^\infty(X)$ equal to 1 on K and split $U(5T_0)\chi f = \chi_1 U(5T_0)\chi f + (1-\chi_1)U(5T_0)\chi f$ so that for $t > 5T_0$

$$U(t)\chi f = u + v$$

where

$$u = U(t-5T_0)\chi_1 U(5T_0)\chi f,\ \ v = U(t-5T_0)(1-\chi_1)U(5T_0)\chi f.$$

Thus by our previously established results, if χ_1 is chosen with support sufficiently close to K, $u \in L^2([5T_0, \widetilde{T}]; \mathcal{E}_{s+(n-1)/2-0})$ and $v \in L^2([5T_0, \widetilde{T}]; \mathcal{E}_s(\mathcal{O}))$. For $t > 10T_0$ we now employ the smoothing result established above with

$$u(5T_0) = \chi_1 U(5T_0)\chi f \in \mathcal{E}_{s+(n-1)/2-0}$$

now functioning as our initial data (and the previous localizer χ replaced by χ_1) to obtain

$$\begin{aligned} u(t) &= U(t-5T_0)u(5T_0) \\ &\in L^2([10T_0, \widetilde{T}]; \mathcal{E}_{s+2(n-1)/2-0}) + L^2([10T_0, \widetilde{T}]; \mathcal{E}_{s+(n-1)/2-0}(\mathcal{O})). \end{aligned} \tag{5}$$

Hence overall

$$U(t)\chi f \in L^2([10T_0, \widetilde{T}]; \mathcal{E}_{s+2(n-1)/2-0}) + L^2([10T_0, \widetilde{T}]; \mathcal{E}_s(\mathcal{O})).$$

Further iteration of this argument now yields smoothing by $k(n-1)/2 - 0$ derivatives after time $5kT_0$. $\square$

5. Exterior polygonal domains

In this section we show that the very weak Huygens' principle of Theorem 2 also holds for the wave equation exterior to a non-trapping polygonal obstacle with Dirichlet or Neumann boundary conditions. In particular, we suppose that $\Omega \subset \mathbb{R}^2$ is a compact region with piecewise linear boundary. We further suppose that the complement $\mathbb{R}^2 \backslash \Omega$ is connected, that no three vertices of $\overline{\Omega}$ are collinear, and that $\mathbb{R}^2 \backslash \Omega$ is non-trapping in the sense that the doubling described next satisfies Assumption 1. This assumption is generically equivalent to the requirement that all billiard trajectories missing the vertices escape to infinity in some uniform time.

We now form a manifold X by gluing two copies of $\mathbb{R}^2 \backslash \Omega$ along their boundaries. This process yields a Euclidean surface (X, g) with conic singularities satisfying the assumptions of Section 2.4 (but with two Euclidean ends). As its proof goes through verbatim for a manifold with two Euclidean ends rather than one, Theorem 2 then holds for (X, g).

Suppose now that Δ is the Dirichlet or Neumann extension of the Laplacian on $\mathbb{R}^2 \backslash \Omega$.[2] The method of images then shows that Theorem 2 holds for $\mathbb{R}^2 \backslash \Omega$. Indeed, by solving the wave equation on the double X and then summing (respectively, taking the difference) over the two copies one obtains a solution for the wave equation with the Neumann (respectively, Dirichlet) extension of the Laplacian on $\mathbb{R}^2 \backslash \Omega$.

6. From weak non-trapping to exponential decay

In this section, we recapitulate the argument of Vainberg [29] as repackaged by Tang–Zworski [28] in the setting of *weak* non-trapping of singularities in order to deduce our resolvent estimate Theorem 1 and hence exponential energy decay for the wave equation in odd dimensions and the resonance wave expansion (Corollary 1).

Having established that the weak non-trapping of singularities holds both in the settings of manifolds with cone points and of exterior domains to polygons, it will behoove us to adopt a formalism for passing from this property to resolvent estimates that will simultaneously apply in both cases. For this reason we now adopt the "black-box" formalism as used in [28].

6.1. **Preliminaries.** We start by recalling the framework of "black box" scattering from Sjöstrand–Zworski [26], as used by Tang–Zworski [28]. The following presentation follows [28] closely.

We consider a complex Hilbert space $\mathcal{H}$ with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2\left(\mathbb{R}^n \backslash B(0, R_0)\right),$$

[2] By an observation of Blair, Ford, Herr, and Marzuola [1], both the Dirichlet and Neumann Laplacians are taken to the Friedrichs extension of the Laplacian on the conic doubled manifold.

where $R_0 > 0$ is fixed. We assume that P is a self-adjoint operator, $P : \mathcal{H} \to \mathcal{H}$, with domain $\mathcal{D} \subset \mathcal{H}$, satisfying the following conditions:

$$\begin{aligned} \mathbb{1}_{\mathbb{R}^n \backslash B(0,R_0)} \mathcal{D} &= H^2(\mathbb{R}^n \backslash B(0,R_0)), \\ \mathbb{1}_{\mathbb{R}^n \backslash B(0,R_0)} P &= \Delta|_{\mathbb{R}^n \backslash B(0,R_0)} \\ (P+i)^{-1} &\text{ is compact} \\ P &\geqslant -C, \quad C \geqslant 0. \end{aligned}$$

Under the above conditions, it is known that the resolvent $R(\lambda) = (P - \lambda^2)^{-1} : \mathcal{H} \to \mathcal{D}$ meromorphically continues from $\{\lambda : \operatorname{Im} \lambda > 0, \lambda^2 \notin \sigma(P)\}$ to the whole complex plane $\mathbb{C}$ when n is odd or to the logarithmic plane Λ when n is even, as an operator from $\mathcal{H}_{\text{comp}}$ to $\mathcal{D}_{\text{loc}}$ with poles of finite rank. We denote by $\mathcal{D}_s$ the spaces given by $(P+i)^{-s/2}\mathcal{H}$.

The poles (i.e., the resonances of P) wil be denoted $\operatorname{Res}(P)$; we count them with multiplicity, denoted $m(\lambda)$. We require an additional condition on P to guarantee a polynomial bound on the resonance counting function.

To formulate the additional condition, we use P to construct a self-adjoint reference operator $P^\#$ on

$$\mathcal{H}^\# = \mathcal{H}_{R_0} \oplus L^2\left(M \backslash B(0,R_0)\right)$$

as in [26] by gluing the "black box" into a large torus instead of Euclidean space; here $M = (\mathbb{R}/R\mathbb{Z})^n$ for some $R > R_0$. Let $N(P, I)$ denote the number of eigenvalues of $P^\#$ in the interval I. The assumption we need is then

$$N(P^\#, [-C, \lambda]) = O(\lambda^{n^\#/2}), \quad \lambda \geqslant 1 \tag{6}$$

for some number $n^\# \geqslant n$.[3]

In addition we have a polynomial bound for the logarithm of the norm of the cutoff resolvent in the whole complex plane away from the resonance set in odd dimensions and in neighborhoods of the real axis in even dimensions: for any $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi = 1$ near $B(0, R_0)$,

$$\|\chi R(\lambda) \chi\|_{\mathcal{H} \to \mathcal{H}} \leqslant C_\theta e^{C_\theta |\lambda|^{n^\# + \epsilon}}, \quad \lambda \in \{\lambda : \arg \lambda < \theta\} \backslash \bigcup_{\lambda_j \in \operatorname{Res}(p)} B(\lambda_j, \langle \lambda_j \rangle^{-n^\# - \epsilon}).$$

Note that the condition (6) is satisfied for the polygonal exteriors of Section 5 as well as for conic manifolds, with $\mathcal{H}_{R_0}$ taken to be $L^2(K)$ in either case, and $\mathcal{D}$ the domain of the square root of the appropriate Laplace operator (i.e., with boundary conditions in the polygonal case and simply the Friedrichs extension in the conic case).

[3] This implies (see [26] and the references of [28]) that the resonance counting function $N(r)$ satisfies

$$N(r) = \sum_{\substack{\lambda \in \operatorname{Res}(P) \\ |\lambda| \leqslant r, \arg \lambda < \theta}} m(\lambda) \leqslant C_\theta r^{n^\#}.$$

The wave group of a black box perturbation can be defined abstractly as in Proposition 2.1 of Sjöstrand–Zworski [27]:

$$U(t) = \exp it \begin{pmatrix} 0 & I \\ P & 0 \end{pmatrix}.$$

The entries of the matrix representation of $U(t)$ are

$$U(t) = i \begin{pmatrix} D_t\mathcal{U}(t) & \mathcal{U}(t) \\ D_t^2\mathcal{U}(t) & D_t\mathcal{U}(t) \end{pmatrix},$$

where the strongly continuous family of operators $\mathcal{U}(t) : \mathcal{D}_s \to \mathcal{D}_{s+1}$ can be identified as the solution operator of the following initial value problem:

$$\begin{gathered} \left(D_t^2 - P\right)\mathcal{U}(t)g = 0 \text{ for } t \in \mathbb{R} \\ \mathcal{U}(0)g = 0 \\ \partial_t\mathcal{U}(0)g = g \end{gathered}$$

where $g \in \mathcal{H}$. In other words,

$$\mathcal{U}(t) = \frac{\sin t\sqrt{P}}{\sqrt{P}}.$$

Since $\mathbb{1}_{\mathbb{R}^n\backslash B(0,R_0)}\mathcal{U}(t)\mathbb{1}_{\mathbb{R}^n\backslash B(0,R_0)}$ maps Schwartz functions to tempered distributions we can describe it by its Schwartz kernel $\mathcal{U}(t,x,y)$, which is a distribution in on $\mathbb{R} \times (\mathbb{R}^n\backslash B(0,R_0)) \times (\mathbb{R}^n\backslash B(0,R_0))$.

6.2. **The resolvent estimate.** We work in the framework of black-box scattering described in Section 6.1. As above, we use the notation $\mathcal{U}(t)$ to represent the sine wave propagator $\sin\left(t\sqrt{P}\right)/\sqrt{P}$. In what follows we use $\mathcal{F}^{-1}$ to denote the inverse Fourier transform:

$$\mathcal{F}^{-1}_{t\to\lambda}f(\lambda) = \int_{\mathbb{R}} e^{it\lambda}f(t)\,dt$$

We use the usual notation in which $\check{f}$ denotes $\mathcal{F}^{-1}f$.

The main result of this section is an adaptation of an argument of Vainberg [29] as repackaged by Tang–Zworski [28]. It states that the very weak Huygens' principle of Theorem 2 implies the resolvent bounds of Theorem 1.

Proposition 8. *Suppose that P is a black-perturbation for which the very weak Huygens' principle of Theorem 2 holds. Then there exists a $\delta > 0$ such that the cut-off resolvent*

$$\chi\left(P-\lambda^2\right)^{-1}\chi$$

can be analytically continued from $\operatorname{Im}\lambda > 0$ to the region

$$\operatorname{Im}\lambda > -\delta\log\operatorname{Re}\lambda, \quad \operatorname{Re}\lambda > \delta^{-1}$$

and for some $C, T > 0$ enjoys the estimate

$$\left\|\chi\left(P-\lambda^2\right)^{-1}\chi\right\|_{\mathcal{H}\to\mathcal{H}} \leqslant C|\lambda|^{-1}e^{T|\operatorname{Im}\lambda|}$$

in this region.

Lemma 6. *Suppose that H_1 and H_2 are Hilbert spaces, and that $N(t) : H_1 \to H_2$ is a family of bounded operators that have k continuous derivatives in t when $t \in \mathbb{R}$, depend analytically on t when $\operatorname{Re} t > T > 0$, and are equal to zero when $t < 0$. Suppose that there are constants j_0, $k \geqslant j_0 + 2$, and C_j so that for all $0 \leqslant j \leqslant k$,*

$$\left\| \frac{\partial^j}{\partial t^j} N(t) \right\| \leqslant C_j |t|^{j_0 - j}, \quad \text{for } \operatorname{Re} t > T.$$

Then the operator

$$\check{N}(\lambda) = \mathcal{F}^{-1}_{t \to \lambda} N(t) : H_1 \to H_2, \quad \text{for } \operatorname{Im} \lambda > 0$$

can be continued analytically to the domain $-\frac{3\pi}{2} < \arg \lambda < \frac{\pi}{2}$ and when $|\lambda| \geqslant 1$, it satisfies the estimate

$$\left\| \check{N} \right\| \leqslant C_j |\lambda|^{-j} e^{T|\operatorname{Im} \lambda|}, \quad \text{for } j = 0, \dots, k.$$

Proof of Lemma 6. This is a straightforward adaptation of the proof due to Vainberg [29, Lemma 4 on page 346] (see also Lemma 3.1 of [28]). We include it here for completeness.

The operator $\check{N}$ is defined and depends analytically on λ when $\operatorname{Im} \lambda \geqslant 0$ and, for $0 \leqslant j \leqslant k$,

$$\check{N} = (-i\lambda)^{-j} \int_0^\infty e^{i\lambda t} \frac{\partial^j}{\partial t^j} N(t) \, dt.$$

Let $\Gamma_\pm$ denote the contours in the complex t-plane formed by the interval $[0, T]$ and the rays $[T, T \pm i\infty)$. When $j \geqslant j_0 + 2$, we have

$$\check{N} = (-i\lambda)^{-j} \int_{\Gamma_+} e^{i\lambda t} \frac{\partial^j}{\partial t^j} N(t) \, dt \quad 0 < \arg \lambda < \frac{\pi}{2},$$
$$\check{N} = (-i\lambda)^{-j} \int_{\Gamma_-} e^{i\lambda t} \frac{\partial^j}{\partial t^j} N(t) \, dt \quad \frac{\pi}{2} < \arg \lambda < \pi.$$

The first formula allows us to continue $\check{N}$ analytically to the half-plane $\operatorname{Re} \lambda \geqslant 0$ and the second formula to the half-plane $\operatorname{Re} \lambda < 0$. This also yields the estimates for $j \geqslant j_0 + 2$ and therefore for all $0 \leqslant j \leqslant k$. □

Proof of Theorem 1. We start by fixing $\chi \in C_c^\infty(X)$ and s large ($s \geqslant \frac{1-n}{2} + 2$ should suffice) and then fixing $T_0 = T_s$ as in the statement of Theorem 2 so that for all $t > T_0$,

$$\chi U(t) \chi : \mathcal{E}_r \to \mathcal{E}_{r+s}$$

for all r.

In addition to the cutoff function $\chi_1 \equiv \chi$, we introduce two other spatial cutoff functions, χ_2 and χ_3, so that $\chi_i \in C_c^\infty(X)$, $P(1 - \chi_i) = \Delta_0(1 - \chi_i)$, $\chi_1 \chi_2 = \chi_2$, and $\chi_2 \chi_3 = \chi_3$.

We introduce a spacetime cutoff function ζ so that ζ is independent of the spatial variables z on K, $0 \leqslant \zeta \leqslant 1$, and

$$\zeta(t,z) = \begin{cases} 1 & t \leqslant |z| + T_0 \\ 0 & t \geqslant |z| + T_0' \end{cases}$$

for some $T_0' \geqslant T_0$. Finite speed of propagation and our weak non-trapping hypothesis imply that

$$(1-\zeta)\mathcal{U}(t)\chi : L^2 \to \mathcal{D}_s$$

for all t.

For any $g \in L^2 = \mathcal{D}_0$, consider $\zeta\mathcal{U}(t)\chi_1 g$, which trivially satisfies

$$\begin{aligned} \left(D_t^2 - P\right)\zeta\mathcal{U}(t)\chi g &= -\left(D_t^2 - P\right)(1-\zeta)\mathcal{U}(t)\chi g, \\ \mathcal{U}(0)\chi g &= 0, \\ D_t\mathcal{U}(0)\chi g &= \chi g. \end{aligned}$$

We now define $F(t)g = -\left(D_t^2 - P\right)(1-\zeta)\mathcal{U}(t)\chi g$. Our weak non-trapping assumption implies that

$$F(t)g \in C^0\left(\mathbb{R}_t; \mathcal{D}_{s-2}\right) \cap C^{s-2}\left(\mathbb{R}_t; \mathcal{D}_0\right).$$

Note that $F(t)g$ vanishes identically for $t < T_0$ and has compact support in t for each fixed z (though the size of the support depends on z).

We now define $\tilde{R}(\lambda)$ by

$$\tilde{R}(\lambda) = -i\mathcal{F}^{-1}_{t\to\lambda}\left(\zeta H(t)\mathcal{U}(t)\chi\right),$$

where $H(t)$ is the Heaviside function. A simple calculation shows that

$$D_t^2\left[H(t)\mathcal{U}(t)\chi g\right] = H(t)D_t^2\mathcal{U}(t)\chi g - i\delta(t)\chi g,$$

and so $\tilde{R}(\lambda)$ satisfies

$$\begin{aligned} \left(P - \lambda^2\right)\tilde{R}(\lambda)g &= i\mathcal{F}^{-1}\left(\left(D_t^2 - P\right)\zeta H(t)\mathcal{U}(t)\chi g\right) \\ &= i\mathcal{F}^{-1}\left(H(t)\left(D_t^2 - P\right)\mathcal{U}(t)\chi g\right) + \chi g \\ &= i\mathcal{F}^{-1}\left(F(t)g\right)(\lambda) + \chi g, \end{aligned}$$

where the last equality holds because the support of $F(t)$ is contained in $t \geqslant 0$.

We now write $F(t)g = \chi_2 F(t)g + (1-\chi_2)F(t)g$ and solve an inhomogeneous wave equation on a flat background. In particular, if Δ_0 denotes the (flat) Laplacian on $\mathbb{R}^n$, we find $V(t)g$ so that it solves

$$\begin{aligned} \left(D_t^2 - \Delta_0\right)V(t)g &= (1-\chi_2)F(t)g \\ V(0)g = D_tV(0)g &= 0. \end{aligned}$$

Using the cutoff function χ_3 (and the fact that $\chi_3(1-\chi_2)=0$), we observe that

$$\begin{aligned}(1-\chi_2)\,F(t)g &= \left(D_t^2-\Delta_0\right)(\chi_3 V(t)g) + \left(D_t^2-\Delta_0\right)((1-\chi_3)V(t)g)\\ &= -\left[\Delta_0,\chi_3\right]V(t)g + \left(D_t^2-\Delta_0\right)((1-\chi_3)V(t)g)\,.\end{aligned}$$

We now define $R^{\#}(\lambda)$ as follows:

$$R^{\#}(\lambda) = \tilde{R}(\lambda) + i\mathcal{F}^{-1}_{t\to\lambda}\left((1-\chi_3)V(t)\right)$$

Since $P=\Delta_0$ on the support of $1-\chi_3$, we observe that

$$\begin{aligned}\left(P-\lambda^2\right)R^{\#}(\lambda)g &= \chi g + i\mathcal{F}^{-1}\left(F(t)g - (D_t^2-P)(1-\chi_3)V(t)g\right)\\ &= \chi g + i\mathcal{F}^{-1}\left(F(t)g-(1-\chi_3)(1-\chi_2)F(t)g - \left[\Delta_0,\chi_3\right]V(t)g\right)\\ &= \chi g + i\mathcal{F}^{-1}\left(\chi_2 F(t)g - \left[\Delta_0,\chi_3\right]V(t)g\right)\\ &= \chi\Big(\,\mathrm{I} + i\mathcal{F}^{-1}\left(\chi_2F(t)-\left[\Delta_0,\chi_3\right]V(t)\right)\Big)g.\end{aligned}$$

In other words,

$$R^{\#}(\lambda) = R(\lambda)\chi\Big(\,\mathrm{I} + i\mathcal{F}^{-1}\left(\chi_2F(t)-\left[\Delta_0,\chi_3\right]V(t)\right)\Big). \tag{7}$$

We claim that the term $\mathrm{I}+i\mathcal{F}^{-1}\left(\chi_2F(t)-\left[\Delta_0,\chi_3\right]V(t)\right)$ is invertible in a logarithmic region, and so the estimates for $\chi R(\lambda)\chi$ will follow from those of $\chi R^{\#}(\lambda)\chi$ in the same region.

The following four estimates, proved below, will justify the above claim:

$$\left\|\chi\tilde{R}(\lambda)\right\|_{L^2\to\mathcal{D}_j} \leqslant C_j|\lambda|^{j-1}e^{T|\operatorname{Im}\lambda|},\quad j=0,1 \tag{8}$$

$$\left\|\widetilde{\chi F(\bullet)}\right\|_{L^2\to L^2} \leqslant C_j|\lambda|^{-j}e^{T|\operatorname{Im}\lambda|},\quad j=0,1,\dots,\lfloor s-2\rfloor \tag{9}$$

$$\left\|\chi\widetilde{(1-\chi_3)V}(\bullet)\right\|_{L^2\to\mathcal{D}_j} \leqslant C_j|\lambda|^{j-1}e^{T|\operatorname{Im}\lambda|},\quad j=0,1 \tag{10}$$

$$\left\|\widetilde{[\Delta_0,\chi_3]V}(\bullet)\right\|_{L^2\to L^2} \leqslant C_j|\lambda|^{-j}e^{T|\operatorname{Im}\lambda|},\quad j=0,1,\dots,\lfloor s-2\rfloor \tag{11}$$

Given these estimates, the theorem holds with $R(\lambda)$ replaced by $R^{\#}(\lambda)$ by (8), (10). One may now take $\delta<T^{-1}$, and then there is some constant C so that for $\operatorname{Re}\lambda>C$ and $\operatorname{Im}\lambda>-\delta\log\operatorname{Re}\lambda$, one has $C|\lambda|^{-1}e^{T|\operatorname{Im}\lambda|}<1/4$, showing the invertibility of the claimed term in equation (7). Shrinking δ then finishes the proof. We must thus only prove the four estimates.

The first two estimates follow from writing out the Fourier transform and noting that for z in the support of χ, there is some T so that $\zeta(t,z)\equiv 0$ for $t>T$. Estimate (8) for $j=1$ follows directly from the energy estimate, while the estimate for $j=0$ uses the energy estimate and integration by parts, as $\lambda e^{it\lambda}=D_te^{it\lambda}$. The estimate (9) follows by the same sort of integration by parts argument and the observation that $\chi F(t)$ is compactly supported in time. The lack of smoothness in the t variable prevents the estimate from holding for all j (and is one of the main differences of the set-up here from that using in [29]).

The other two estimates are somewhat more subtle and rely on properties of the free wave group. We start by writing

$$V(t)g = (1-\chi_2)\zeta H(t)\mathcal{U}(t)\chi g - H(t)\mathcal{U}_0(t)(1-\chi_2)\chi g + q(t,z), \tag{12}$$

where $\mathcal{U}_0(t)$ is the free sine propagator, i.e., $\mathcal{U}_0(t) = \sin t\sqrt{\Delta_0}/\sqrt{\Delta_0}$. By using the equation for V (and that $\Delta_0 = P$ on $\operatorname{supp}(1-\chi_3)$ and $\left[D_t^2, \zeta\right]$ is order 1), we see that

$$\begin{gathered} \left(D_t^2 - \Delta_0\right) q = -\left[\Delta_0, \chi_2\right] \zeta H(t)\mathcal{U}(t)\chi g, \\ q(0,z) = D_t q(0,z) = 0. \end{gathered} \tag{13}$$

The inhomogeneous term in (13) has compact support in both space and time and vanishes identically for $t < 0$.

We now undertake to show that (10), (11) hold by verifying them for each term on the right-hand-side of (12), starting with q. If E_+ is the forward fundamental solution for the wave equation on $\mathbb{R}^n$, then we may write

$$\chi(z)q(t,z) = -\int_0^t \chi(z)E_+(t-\tilde{t}) \star \left[\Delta_0, \chi_2\right] \zeta \mathcal{U}(\tilde{t})\chi g \, d\tilde{t}.$$

If n is odd, then Huygens' principle implies that χq vanishes identically for large t. Thus estimates (10), (11) hold with V replaced by q in this case.. If n is even, then Huygens' principle no longer applies but if t is large then the inhomogeneous term is disjoint from the singular support of E_+ and therefore for large t,

$$\chi(z)q(t,z) = c_n \chi(z) \int_0^\infty \int_{\mathbb{R}^n} \left((t-\tilde{t})^2 + |z-\tilde{z}|^2\right)^{\frac{1-n}{2}} \left(\left[\Delta_0, \chi_2\right] \zeta \mathcal{U}(\tilde{t})\chi g(\tilde{z})\right) \, d\tilde{z}\, d\tilde{t}.$$

In particular, it is analytic for large t and satisfies the derivative estimates (10), (11) for large t as well.

A similar result holds for the second term in (12), since

$$\mathcal{U}_0(t)(1-\chi_2)\chi g = E_+(t) \star (1-\chi_2)\chi g.$$

Now to see the estimate (10), first observe that the first term in equation (12) does not occur because $\left[\Delta_0, \chi_3\right](1-\chi_2) = 0$. Combining the estimates on χq and $\mathcal{U}_0(t)$ for large t with the known C^{s-1} bounds on $V(t)$ we apply Lemma 6.

The final estimate (11) follows in the same manner, but uses the estimate (8) to bound the first term from equation (12). □

Appendix A. Proof of Proposition 6

In this section we prove Proposition 6. By Remark 4 it suffices to prove it for ${}^X\mathcal{F}_\alpha$ and ${}^X\mathcal{F}_\beta$ rather than $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$.

In what follows, $\phi_t : TX \to TX$ denotes geodesic flow for time t and $\gamma_{(p,v)}$ denotes the geodesic in X such that $\gamma(0) = p$ and $\gamma'(0) = v$. We denote by π the projection $TX \to X$.

The kernel of the pushforward π_* consists of those vectors in $T_{(p,v)}(TX)$ tangent to the fibers of TX and is referred to as the *vertical subspace* of $T_{(p,v)}(TX)$. The Levi-Civita connection on X defines the *connection map* $K : T_{(p,v)}(TX) \to T_pX$, whose kernel defines a horizontal subspace of $T_{(p,v)}(TX)$. The connection map K thus provides an identification of the vertical subspace of $T_{(p,v)}(TX)$ with T_pX. Similarly, the pushforward π_* provides an identification of the horizontal subspace with T_pX.

We now recall the following characterization of Jacobi fields on a Riemannian manifold X from Proposition 1.7 of Eberlein [17].

Lemma 7. *If $\gamma_{(p,v)}$ is a geodesic in X then there is a one-to-one correspondence between Jacobi fields along γ and vectors in $T_{(p,v)}(TX)$. In particular, for $\zeta \in T_{(p,v)}(TX)$, let $Y_\zeta(t)$ be the unique Jacobi field along γ with $Y_\zeta(0) = \pi_*\zeta$ (the horizontal part of ζ) and $Y_\zeta'(0) = K\zeta$ (the vertical part of ζ). Then $Y_\zeta(t)$ is given by the following:*

$$Y_\zeta(t) = \pi_*(\phi_t)_*\zeta, \quad Y_\zeta'(t) = K(\phi_t)_*\zeta.$$

In other words, the Jacobi field and its derivative are, taken together, invariant under the geodesic flow.

Note further that because the metric is flow-invariant, if $\pi_*\zeta$ and $K\zeta$ are both orthogonal to $v \in T_pX$, then the Jacobi field $Y_\zeta(t)$ is everywhere orthogonal to $\gamma'(t)$.

We now abuse notation and consider ${}^X\mathcal{F}^s_\alpha$ and ${}^X\mathcal{F}^t_\beta$ as subsets of TX rather than T^*X. We further abuse notation and use ξ and η to denote coordinates on the *tangent bundle* (with $\xi\partial_x + \eta\cdot\partial_y$ denoting the corresponding tangent vector) rather than the cotangent bundle. We note further that they extend to the boundary and use a bar to denote their completions, i.e., in a neighborhood U of Y_α,

$$\overline{{}^X\mathcal{F}^s_\alpha} = U \cap \{0 \leqslant x < s,\ y \in Y_\alpha,\ \xi \in \mathbb{R},\ \eta = 0\}.$$

Observe that if $s > 0$ and $\xi > 0$, then $\phi_s(0, y, \xi, 0) = (s\xi, y, \xi, 0)$. A similar statement holds for $s, \xi < 0$.

We also require the following lemma, which describes how the pushforward acts on the tangent space to ${}^X\mathcal{F}^s_\alpha$.

Lemma 8. *Suppose $(p, v) = (x, y, \xi, 0) \in {}^X\mathcal{F}^s_\alpha$ lies in a small neighborhood of Y_α and that $\zeta \in T_{(p,v)}({}^X\mathcal{F}^s_\alpha) \subset T_{(p,v)}(TX)$. If $K\zeta$ and $\pi_*\zeta$ are both orthogonal to v then $K\zeta = 0$ and $\pi_*\zeta = \tilde{\zeta}' \cdot \partial_y$.*

Moreover, for small t (i.e., t so that $x + t\xi \geqslant 0$ and $\phi_t(p, v)$ still lies in this small neighborhood), then

$$K(\phi_t)_*\zeta = 0, \quad \pi_*(\phi_t)_*\zeta = \tilde{\zeta}' \cdot \partial_y.$$

The proof of Lemma 8 is a simple calculation and omitted here for brevity.

We also remark that because ${}^X\mathcal{F}^s_\alpha$ is given as a flow-out, if ζ is tangent to this flow-out then so is $(\phi_t)_*\zeta$.

We now turn our attention to the proof of Proposition 6.

Proof of Proposition 6. Observe that if ${}^X\mathcal{F}^s_\alpha$ and ${}^X\mathcal{F}^t_\beta$ intersect then a segment of a geodesic connecting Y_α and Y_β lies in their intersection, i.e., ${}^X\mathcal{F}^s_\alpha \cap {}^X\mathcal{F}^t_\beta$ contains a segment of a geodesic γ with $\gamma(0) \in Y_\alpha$ and $\gamma(t') \in Y_\beta$ with $t' < s + t$.

We start by assuming that ${}^X\mathcal{F}^s_\alpha$ and ${}^X\mathcal{F}^t_\beta$ intersect non-transversely at the point $(p, v) \in TX$. We let γ be the geodesic through (p, v) and observe that γ connects Y_α and Y_β. Both flow-outs have dimension $n + 1$ and the tangent space to their intersection contains both the direction of the flow and the radial vector field. Because the intersection is non-transverse, it also contains a vector ζ linearly independent of the previous two. We may thus assume that both $\pi_*\zeta$ and $K\zeta$ are orthogonal to v. By pushing ζ forward by the flow and applying Lemma 7, we obtain a family of vectors tangent to both flow-outs that corresponds to a normal Jacobi field along γ. Lemma 8 then implies that this family has a limit at the cone points that projects to a b-vector field, since the coefficient of ∂_x vanishes at the boundary. This completes one direction of the proof.

Conversely, suppose that W is a Jacobi field along γ so that $W(0)$ and $W(t')$ are both b-vector fields. By standard arguments we may assume that W is normal to the flow. By using Lemma 7, we may associate to W a vector field ζ along γ in $T(TX)$. Because it is a b-vector field at the endpoints, Lemma 8 implies that it is initially tangent to both flow-outs and so stays tangent under the pushforward. Because $\pi_*\zeta$ and $K\zeta$ are both orthogonal to the flow, the tangent space of the intersection along this geodesic must have dimension at least 3. Each flow-out has dimension $n+1$ and therefore a transverse intersection must have dimension 2, implying that the intersection is non-transverse. □

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