SEMICLASSICAL DIFFRACTION BY CONORMAL POTENTIAL SINGULARITIES

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Abstract. We establish propagation of singularities for the semiclassical Schrödinger equation, where the potential is conormal to a hypersurface. We show that semiclassical wavefront set propagates along generalized broken bicharacteristics, hence reflection of singularities may occur along trajectories reaching the hypersurface transversely. The reflected wavefront set is weaker, however, by a power of h that depends on the regularity of the potential. We also show that for sufficiently regular potentials, wavefront set may not stick to the hypersurface, but rather detaches from it at points of tangency to travel along ordinary bicharacteristics.

1. Introduction

1.1. **Statement of results.** Let (X, g) be a smooth n-dimensional Riemannian manifold, and $Y \subset X$ a hypersurface. We study propagation of semiclassical singularities for the Schrödinger operator

$$P = -h^2 \Delta_g + V, \tag{1.1}$$

where the real-valued potential V is conormal to Y. Semiclassical propagation of singularities theorems constrain the distribution of energy in phase space of a solution to (1.1), asymptotically as $h \to 0$: for V smooth, it is known that the energy concentrates on the classical energy surface and is invariant under the associated classical dynamics. Here, by contrast, the singularities of the potential V play an important role, diffracting energy along *broken* classical trajectories.

The class of potentials V that we consider are real-valued *conormal distributions* with respect to Y, a class of distributions that are smooth functions except at Y. If x is a defining function of Y then x_+^α is an instructive example, with $\alpha > 0$. More generally, we assume throughout that $V \in I^{[-1-\alpha]}(Y)$ for some $\alpha > 0$. This means that V is locally the inverse Fourier transform of a Kohn–Nirenberg symbol of order $-1-\alpha$, transverse to Y. In particular, V is $1+\alpha$ orders more regular than the delta distribution along Y. If $\alpha \geq k + \gamma$ with $k \in \mathbb{N}$ and $\gamma \in (0,1)$, then $V \in \mathcal{C}^{k,\gamma}(X)$, but V is $\mathcal{C}^\infty$ away from Y. (See Section 2.1 below for details.)

Let $p = |\xi|_g^2 + V$ denote the semiclassical principal symbol of P. Let H_p denote its associated Hamilton vector field, e.g., $\mathsf{H}_p = 2\xi \cdot \partial_x - (\partial_x V) \cdot \partial_\xi$ if g is the Euclidean metric. Recall that $\mathrm{WF}_h^s(u)$, the semiclassical wavefront set of order s, measures where, in T^*X, the family u fails to be $\mathcal{O}_{L^2}(h^s)$. If $Pu = 0$, then known results imply that the semiclassical wavefront set $\mathrm{WF}_h^s(u)$ of order s is contained in the characteristic set $\Sigma \equiv \{p = 0\}$, and is invariant under the H_p flow for each $s \in \mathbb{R} \cup \{+\infty\}$, at least away from Y. This result breaks down for singularities striking $T^*_Y X$: the conormal singularity of V causes ray splitting, generating wavefront set along both the reflected and transmitted components.

To make the notion of ray-splitting precise, we introduce a suitable *generalized broken bicharacteristic* (GBB) flow, taking into account both transverse and tangential incidence to Y. Properties of this GBB flow are described in detail in Section 4.3; its main feature is that the allowed trajectories are continuous in space but potentially discontinuous in momentum, with momentum tangent to Y conserved at interactions with this hypersurface, in accordance with the laws of reflection and refraction. The GBB flow is, consequently, not defined on the usual cotangent bundle, where it would be discontinuous. Instead, we introduce an adapted notion of semiclassical wavefront set by using a variant of Melrose's *b-calculus* of pseudodifferential operators. This gives rise to a *semiclassical b-wavefront set* which lives in a rescaling of the usual cotangent bundle, and agrees with the usual semiclassical wavefront set away from Y, but has the combined virtue and defect of not distinguishing different normal momenta over Y itself. The compressed characteristic set employed below is likewise an appropriately rescaled version of the set $\{p = 0\}$, which does not distinguish among different normal momenta over Y. (For details, including the relevant notation, see Section 3.)

Theorem 1 (Propagation of singularities)**.** *Let $\alpha > 0$ and $s \in \mathbb{R} \cup \{+\infty\}$. If u is h-tempered in $H^1_{h,\mathrm{loc}}(X)$, then $\mathrm{WF}^s_{\mathrm{b},h}(u) \setminus \mathrm{WF}^{-1,s+1}_{\mathrm{b},h}(Pu)$ is the union of maximally extended* GBB*s within the compressed characteristic set $\dot{\Sigma}$.*

Suppose $Pu = 0$. Then Theorem 1 tells us that a given point in the wavefront set must give rise to wavefront set along *at least* one maximally extended GBB through it, but does not distinguish among the various possibilities. The theorems that follow draw subtler distinctions among them, and in particular give a special role to GBBs that are in fact ordinary solutions to Hamilton's equations of motion. Thus we now return to the usual cotangent bundle, where we may consider the usual Hamilton flow provided that there is enough regularity for it to make sense. Introduce local coordinates (x, y) such that $Y = \{x = 0\}$, and let (x, y, ξ, η) be the corresponding canonical coordinates on T^*X. Even though Hamilton's equations become singular over Y when $\alpha \le 1$, the integral curves of H_p are well defined near transversally incident points

$$\varpi_\pm = (0, y_0, \pm\xi_0, \eta_0) \in \Sigma$$

where the normal momentum $\pm\xi_0$ does not vanish; see Lemma 4.2. The integral curves $\gamma_\pm$ with $\gamma_\pm(0) = \varpi_\pm$ therefore exist on some interval $(-\varepsilon, \varepsilon)$. To use the terminology of [DHUV], the points $\varpi_\pm$ are said to be related, in the sense of having the same

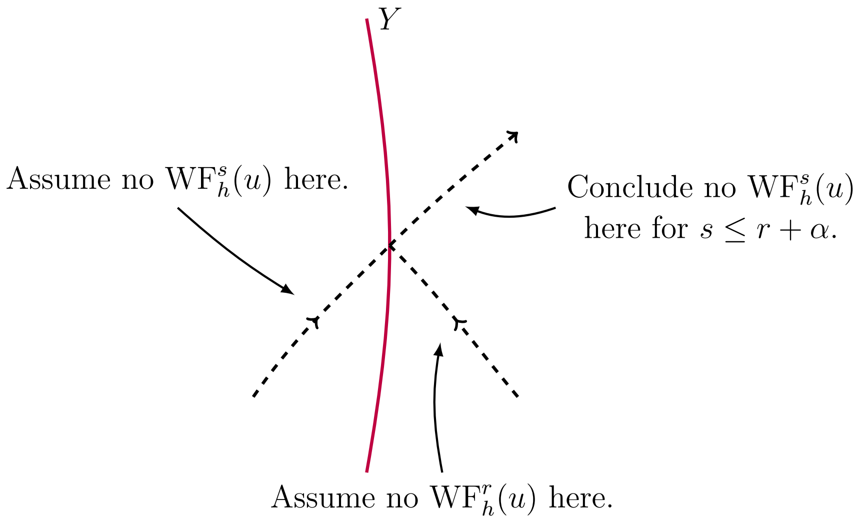

FIGURE 1. Illustration of the diffractive improvement. The trajectory at lower left is $\gamma_+((-\varepsilon,0))$; its continuation across the interface is $\gamma_+((0,\varepsilon))$. The other incident trajectory at lower right is $\gamma_-((-\varepsilon,0))$. The limitation on the propagation of regularity through the interface is $s \le r+\alpha$.

tangential momentum. Since $\mathrm{WF}_{\mathrm{b},h}^s(u) = \mathrm{WF}_h^s(u)$ away from Y, Theorem 1 states the following at transversally incident points: if $\gamma_+((-\varepsilon,0))$ and $\gamma_-((-\varepsilon,0))$ are both disjoint from $\mathrm{WF}_h^s(u)$, then

$$\gamma_+((0,\varepsilon)) \cap \mathrm{WF}_h^s(u) = \emptyset. \tag{1.2}$$

On the other hand, the reflected singularity (namely the contribution of incident wavefront set along $\gamma_-((-\varepsilon,0))$ to outgoing wavefront set along $\gamma_+((\varepsilon,0))$) is expected to be weaker than the original incident singularity along $\gamma_-((-\varepsilon,0))$. In other words, if $\gamma_+((-\varepsilon,0))$ is disjoint from $\mathrm{WF}_h^s(u)$ and $\gamma_-((-\varepsilon,0))$ is disjoint from $\mathrm{WF}_h^r(u)$, then (1.2) should hold for a range of s depending on α and r. We show that at least when $\alpha > 1$, this holds for $s \le r+\alpha$.

Theorem 2 (Diffractive improvement at transverse reflection)**.** *Let $\alpha > 1$ and $s \le r+\alpha$, where $s, r \in \mathbb{R} \cup \{+\infty\}$. Suppose that u is h-tempered in $H^1_{h,\mathrm{loc}}(X)$ with $Pu \in L^2_{\mathrm{loc}}(X)$, and $\mathrm{WF}_h^{s+1}(Pu) = \emptyset$. Let*

$$\varpi_\pm = (0, y_0, \pm\xi_0, \eta_0) \in \Sigma$$

with $\xi_0 \neq 0$, and let $\gamma_\pm$ be as above. If $\varpi_+ \in \mathrm{WF}_h^s(u)$, then there exists $\varepsilon > 0$ such that

$$\gamma_+((-\varepsilon,0)) \subset \mathrm{WF}_h^s(u) \ \textit{or}\ \gamma_-((-\varepsilon,0)) \subset \mathrm{WF}_h^r(u).$$

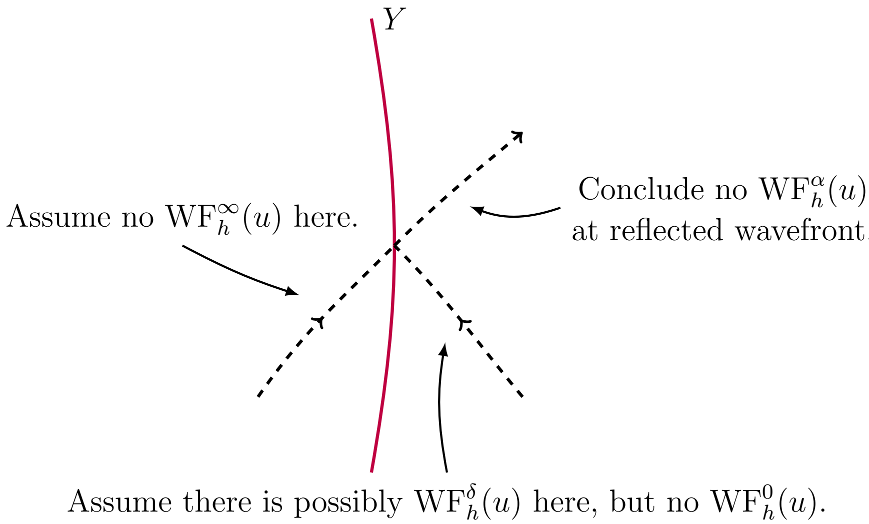

FIGURE 2. Diffractive reflection of a single incident singularity.

For an illustration, see Figure 1. We refer to this result as a "diffractive improvement" as it shows that corrections to the naive geometric optics ansatz (wherein singularities propagate along ordinary bicharacteristics) is in fact a small perturbation. It is perhaps easier to visualize the following reinterpretation in terms of reflection: let $Pu = 0$, where $\mathrm{WF}_h^0(u) = \emptyset$. This of course allows $\gamma_-((-\varepsilon, 0))$ to possibly contain incoming singularities in $\mathrm{WF}_h^\delta(u)$ for $\delta > 0$. On the other hand, assume that $\mathrm{WF}_h^\infty(u)$ is disjoint from $\gamma_+((-\varepsilon, 0))$. Then using the background regularity $r = 0$, the theorem guarantees absence of $\mathrm{WF}_h^\alpha(u)$ along $\gamma_+((0, \varepsilon))$. No matter how small $\delta > 0$, any incident singularity in $\mathrm{WF}_h^\delta(u)$ is partially *reflected* (the sign of ξ has flipped) to produce at most a milder singularity — see Figure 2.

The threshold $s \leq r + \alpha$ is in general sharp, as we show by example in the next section. The same example indicates that Theorem 2 may hold for $\alpha > 0$, rather than just $\alpha > 1$.

One might further ask exactly what happens to semiclassical wavefront set at points tangent to Y; an understanding of diffractive improvements along this set is essential in understanding global propagation phenomena. For instance, propagation along generalized broken bicharacteristics as in Theorem 1 permits singularities to "stick" to the boundary of a convex Y rather than detaching from it. Our final result shows that, at least for slightly more regular V, this sticking phenomenon does not in fact occur.

We consider points in the *glancing set* $\mathcal{G}$ (defined below in (4.2)) which is essentially the points in the characteristic set where rays are tangent to the boundary; as $\mathcal{G}$ is technically a subset of the *compressed* cotangent bundle (a quotient of T^*X, also defined in Section 3), it is actually points in $\pi^{-1}(\mathcal{G}) \subset T^*X$ at which we consider microlocal regularity, where π is the relevant quotient map.

For the moment we continue to assume that $\alpha > 1$, in which case H_p is a $\mathcal{C}^0$ vector field, hence we in general have existence but not uniqueness of bicharacteristics (see Remark 4.4 for an example where uniqueness fails). Thus, given any $\varpi_0 \in \Sigma$, there exists at least one bicharacteristic $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ with $\gamma(0) = \varpi_0$. If $\alpha > 2$, then the Hamilton vector field is Lipschitz and this bicharacteristic is unique.

Theorem 3 (Diffractive improvement at glancing)**.** *Let $\alpha > 1$ and $r \in \mathbb{R}$. Let $\varpi_0 \in \pi^{-1}(\mathcal{G})$. Suppose that u is h-tempered in $H^1_{h,\mathrm{loc}}(X)$ with $Pu \in L^2_{\mathrm{loc}}(X)$, and $\mathrm{WF}_h^{r+1}(Pu) = \emptyset$. If $\varpi_0 \in \mathrm{WF}_h^r(u)$, then there exists $\varepsilon > 0$ and a bicharacteristic γ with $\gamma(0) = \varpi_0$ such that*

$$\gamma((-\varepsilon, 0)) \subset \mathrm{WF}_h^r(u).$$

While this theorem certainly holds for the range $\alpha > 1$, it is considerably more powerful when $\alpha > 2$ since the set

$$\{\gamma((-\varepsilon, 0)) : \gamma \text{ is a bicharacteristic, } \gamma(0) = \varpi_0\}$$

consists of the *unique* solution to Hamilton's equations on $(-\varepsilon, 0]$ with $\gamma(0) = \varpi_0$; in this case, the theorem proves the "non-sticking" alluded to above, as it shows that a singularity in $\mathcal{G}$ propagates along the unique ordinary bicharacteristic through that point rather than along one of the many possible generalized broken bicharacteristics: to see this we use Theorem 3 to obtain absence of $\mathrm{WF}_h^r(u)$ at ϖ_0 based on regularity along the backward bicharacteristic; if the bicharacteristic is, e.g., tangent to Y at the single point ϖ_0 before leaving it, then since $\mathrm{WF}_h^r(u)$ is closed, we obtain this regularity at nearby points, and may propagate it forward over $X \backslash Y$ (by the usual propagation of singularities) to obtain absence of $\mathrm{WF}_h^r(u)$ along the whole bicharacteristic — see Figure 3.

It would be of considerable interest to know in more detail what happens in the range $1 < \alpha < 2$. We at least know that singularities propagate along one or more of the non-unique bicharacteristics; it is possible that bicharacteristics sticking to the interface Y may gain regularity at a fixed rate as they do so.

1.2. **A one-dimensional example.**

On $\mathbb{R}$, consider a compactly supported potential $V \in L^\infty(\mathbb{R})$ with the following properties:

- $V = x_+^\alpha$ on an interval $(-\infty, x_0)$ with $x_0 \in (0, 1)$, where $\alpha > 0$.
- V is $\mathcal{C}^\infty$ away from $x = 0$, and $\sup V < 1$.

Observe that $V \in I^{[-1-\alpha]}(\{x = 0\})$. Consider the operator $P = (hD_x)^2 + V$. Working at energy $E = 1$, away from the support of V solutions to $(P - 1)u = 0$ are linear

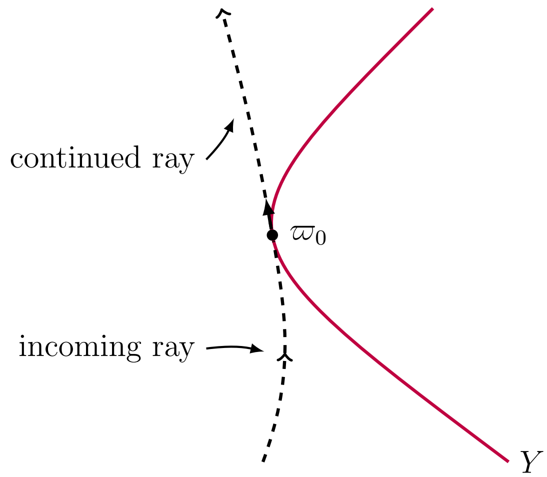

FIGURE 3. A bicharacteristic (dashed line) that is tangent to Y. For any r, absence of $\mathrm{WF}_h^r(u)$ on the part of the bicharacteristic marked "incoming ray" implies absence of $\mathrm{WF}_h^r(u)$ at ϖ_0; since wavefront set is closed, ordinary propagation of singularities then gives absence of $\mathrm{WF}_h^r(u)$ on the part of the bicharacteristic labeled "continued ray," i.e., propagation of regularity along this bicharacteristic. (We are assuming $\alpha > 2$.)

combinations of $e^{\pm ix/h}$. There is a unique solution of the equation $(P-1)u = 0$ such that

$$u = \begin{cases} e^{ix/h} + Re^{-ix/h} & \text{for } x \leq 0, \\ Te^{ix/h} & \text{for } x \gg 1, \end{cases} \tag{1.3}$$

where $R, T \in \mathbb{C}$.

Proposition 1.1. *If $\alpha \in (0,1)$, then $R \sim 2^{-\alpha-2} e^{i\alpha\pi/2}\Gamma(\alpha+1)h^\alpha$ as $h \to 0$.*

Note that to leading order R is independent of the choice of potential satisfying the properties above. Thus reflected waves exist and are exactly order h^α in this simple example. A proof of this result is given in Appendix A.

Proposition 1.1 is almost certainly true for $\alpha \geq 1$ as well; see Figure 4 for a numerical example. An analytic proof would require computing lower order terms in various asymptotic expansions that quickly becomes impractical. For the case of integer $\alpha = k \in \mathbb{N}$ an analysis of this problem can be found in Berry [Ber], where it is shown that if the k'th derivative of the potential is discontinuous, then the reflection coefficients are (to top order) explicit multiples of the jump in $V^{(k)}$ times h^k.

1.3. **Related work.** While there is little literature on semiclassical problems with rough coefficients, the related problem of the wave equation with a rough metric has

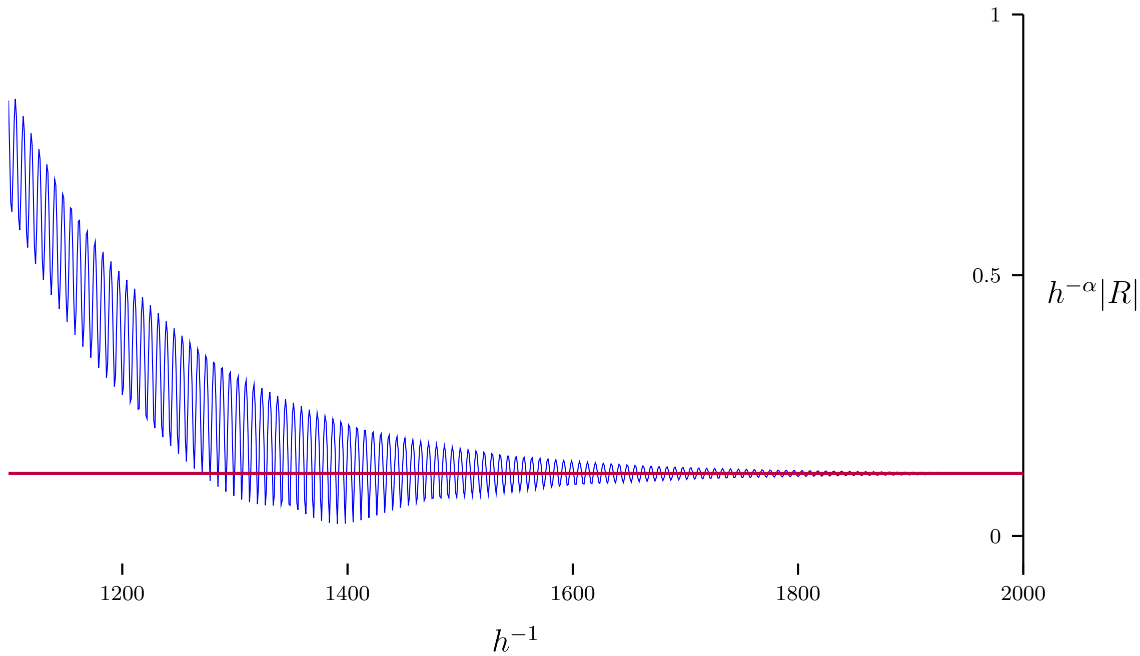

FIGURE 4. The rescaled reflection amplitude corresponding to a potential as in Section 1.2 with $\alpha = 1.2$ plotted against h^{-1}. The horizontal line represents the analytic expression from Proposition 1.1. The limiting asymptotics only emerge for very small values $h \sim 10^{-3}$; this phenomenon was already observed in [Ber].

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attracted considerable attention. In particular, there is a long history of propagation of singularities theorems in the setting of $\mathcal{C}^{k,\alpha}$ coefficients, showing propagation of smoothness along bicharacteristics up to a maximum level of regularity as in our Theorem 2; see Bony [Bon], Beals–Reed [BR], Smith [Smi1, Smi2], Geba–Tataru [GT], Taylor [Tay2].

While the papers listed above are primarily focused on unstructured coefficient singularities, the only prior study on *conormal* singularities appears to be the work of De Hoop–Uhlmann–Vasy [DHUV]. This paper, which deals with the wave equation with coefficients in $I^{[-1-\alpha]}(Y)$ for Y a hypersurface and $\alpha > 1$, was the primary inspiration for our work. The authors are able to show that singularities propagate along generalized broken bicharacteristics and that transversely reflected singularities are weaker, in analogy with our first two theorems, although the regularity obtained for the reflected wave (i.e., the threshold regularity up to which one can obtain propagation results based on a fixed level of background regularity) does not appear to be sharp. Differences in the approach taken here include use of mixed-norm rather than L^2 estimates in the commutator arguments, as well as a precise decomposition of the potential into high and low frequencies.

In the semiclassical case, there are explicit one-dimensional computations due to Berry [Ber]. Semiclassical diffraction effects from potentials with conical singularities have been studied by Fermanian-Kammerer–Gérard–Lasser [FKGL] and Chabu [Cha1], [Cha2]; see also Harris–Lukkarinen–Teufel–Theil [HLTT] for a discussion of potentials with singularities of the form $|x|$. A closely related problem of propagation of semiclassical defect measure across an interface whose width shrinks at an h-dependent speed has also been studied by Nier [Nie] and Miller [Mil].

The principal novelties of this paper, in addition to obtaining in a semiclassical setting results analogous to those of [DHUV], are, first, the sharpness of the regularity of the diffracted wave, and, second the improvement at glancing, which ensures that for $\alpha > 2$ there is no sticking of singularities to the boundary.

1.4. **Strategy of proof.** We follow the same overall strategy as employed in the study of the wave equation in [DHUV]. We obtain Theorem 1 by a commutator argument in a semiclassical version of Melrose's *b-calculus* of pseudodifferential operators. This calculus, which loosely speaking consists of operators

$$A = A(x, y, hxD_x, hD_y)$$

where x is a defining function for Y, are effective at localizing in both position and tangential momentum with respect to Y, but not in the normal momentum, since hD_x is not in the calculus. This makes these operators useful for proving that the tangential momentum is conserved in the interaction of singularities with the boundary, which is the main content of the propagation along GBBs theorem (albeit at glancing the connection to the definition of GBBs is somewhat tricky to untangle). Such a strategy, employing a positive commutator argument, goes back to the original work of Melrose–Sjöstrand on boundary problems [MS1, MS2]; our approach is strongly influenced by Vasy's work on manifolds with corners [Vas3].

The diffractive improvement at transverse reflections is obtained instead via a commutator argument involving a commutant that is an *ordinary* semiclassical pseudodifferential operator, ignoring the singularity of the operator P across Y. The price one pays is that the commutator is then no longer a pseudodifferential operator, but involves operators whose Schwartz kernels are *paired Lagrangian distributions*, which must be estimated separately. It is in the estimates of these terms that we are forced to use assumptions on the background regularity of u, and it is here that limitations are placed on the range of exponents for which we can expect to obtain propagation of regularity directly across the interface.

Paired Lagrangians were introduced in the setting of homogeneous microlocal analysis by Guillemin–Uhlmann [GU3] and Melrose–Uhlmann [MU] and studied by Antoniano–Uhlmann [AU], Greenleaf–Uhlmann [GU1, GU2], and De Hoop–Uhlmann–Vasy [DHUV]. There seems to be very little literature on these objects in the semiclassical setting, however, so we have provided a self-contained presentation of the basic theory here.

One key to obtaining the sharp threshold regularity in the transverse reflection theorem is to estimate certain terms by using *mixed-norm* estimates in the space $L^\infty(\mathbb{R}_x; L^2(Y))$ (where x is a defining function for Y) rather than the L^2 estimates customary in commutator arguments. Our ability to work in this space relies on a simple energy estimate similar to the estimates standard in hyperbolic problems. Another novelty to our approach is the decomposition of the potential V into low- and high-frequency pieces, which simplifies the decomposition of the commutator into paired Lagrangian pieces, one of which is nearly microlocal. This decomposition is readjusted from step to step in the iterative commutator argument to allow for shrinking microsupports necessary in the iteration.

The improvement in the glancing region is obtained much as in the case of transverse interaction, with the important difference that we are able to microlocalize the necessary background regularity more finely: we require only background regularity in a region of specified tangential momentum very close to glancing. In this region, b-regularity and ordinary regularity turn out to be essentially interchangeable, and we are thus able to make a propagation argument that can be *iterated* as in the usual commutator proof, with the necessary background regularity being obtained at each inductive step by the output of the previous one.

The structure of the paper is as follows. In Section 2 we discuss background from microlocal analysis, starting with a description of the properties of the class of conormal distributions from which V is drawn (Section 2.1). We then discuss pseudodifferential operators, starting with the ordinary semiclassical calculus and associated conormal distributions (to set notation and as a point of comparison), also recalling some basic energy estimates. Next, we move on to the *semiclassical b-calculus* (Section 3), which is the essential tool in proving Theorem 1. This section introduces the wavefront sets that we use to measure regularity; we need both the semiclassical b-wavefront set with respect to L^2, and the analogous wavefront set measured with respect to the energy space and its dual. The relationships among these wavefront sets, elucidated in Lemma 3.10, explain the different wavefront sets arising in Theorem 1.

In Section 4 we then discuss the geometry of bicharacteristics, which for our purposes are of two kinds: the *generalized broken bicharacteristics*, the largest set along which singularities may propagate, and the ordinary solutions to Hamilton's equations (well-defined whenever $\alpha > 1$, and for transverse rays even when $\alpha > 0$) which are distinguished by our diffractive improvements at hyperbolic (i.e., transverse) and glancing sets.

Section 5 is devoted to the proof of Theorem 1. This splits into three steps, where we must first treat estimates on the *elliptic set* for the operator and then prove distinct propagation estimates on the *hyperbolic set* (rays transverse to Y) and on the *glancing set* (rays tangent to Y).

We then turn to setting the stage for the proofs of Theorems 2 and 3. We begin in Section 6 by introducing the calculus of *semiclassical paired Lagrangian distributions*, together with associated operator estimates. Finally in Section 7 we prove Theorems 2 and 3.

Contents

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2. Microlocal and semiclassical preliminaries

2.1. **Conormal distributions.** In this section we record Hölder and integrability properties of conormal distributions not discussed in standard references such as [Hör1, Chapter 18.2]. While these facts are well known, we were unable to find a suitable reference in the existing literature.

Let X be an m-dimensional manifold without boundary, and $Y \subset X$ a codimension-k submanifold. Let $\mathcal{C}_c^{-\infty}(X)$ denote the space of compactly supported distributions on X.

Given a closed conic Lagrangian submanifold $\Lambda \subset T^*X$, let $I^m(X;\Lambda)$ denote the space of Lagrangian distributions of order m as defined in [Hör2, Chapter 25.1]. For $\mu \in \mathbb{R}$, we define the conormal distributions of order μ with respect to Y as

$$I^{[\mu]}(Y) = I^{\mu+(2k-m)/4}(X; N^*Y). \tag{2.1}$$

Recall that our standing assumption on the potential V is that $V \in I^{[-1-\alpha]}(Y)$ is real valued, with $\alpha > 0$ and Y a hypersurface.

In elucidating the class of conormal distributions, we first recall the local characterization of $u \in I^{[\mu]}(Y)$ via the Fourier transform. Let $\mathcal{U}$ be a coordinate patch intersecting Y with local coordinates

$$x = (x', x'') = (x_1', \dots, x_k', x_1'', \dots, x_{m-k}'')$$

such that $\mathcal{U} \cap Y = \{x' = 0\}$. Assume that u has compact support in $\mathcal{U}$; since $I^{[\mu]}(Y)$ is a $\mathcal{C}^\infty(X)$-module, one can always reduce to this case by passing to a partition of unity subordinate to a covering of X by coordinate patches. Thus we consider $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^m)$ of the form

$$u(x) = (2\pi)^{-(m+2k)/4} \int e^{i\langle x', \xi' \rangle} a(x, \xi')\, d\xi' \tag{2.2}$$

for a symbol $a \in S^\mu(\mathbb{R}^m_x; \mathbb{R}^k_{\xi'})$.

If $\mu < -k$, then $a \in L^1(\mathbb{R}^m)$, so certainly u is continuous by the Riemann–Lebesgue lemma. In fact, u has much stronger continuity properties; to describe these properly, we must first recall the Zygmund spaces. If $1 = \sum_{j\geq 0} \psi_j$ is a dyadic partition of unity on $\mathbb{R}^k$ with $\psi_j(\xi) = \psi_1(2^{-j}\xi)$ and $\operatorname{supp} \psi_1 \subset \{1/2 \leq |\xi| \leq 2\}$, then the Zygmund space $\mathcal{C}_*^s(\mathbb{R}^k)$ consists of all distributions $v \in \mathcal{S}'(\mathbb{R}^k)$ for which

$$\|v\|_{\mathcal{C}_*^s} = \sup_j 2^{sj} \|\psi_j(D_{x'})v\|_{L^\infty} < \infty.$$

Directly from the Littlewood–Paley characterization of $\mathcal{C}_*^s$ given above, any u of the form (2.2) satisfies

$$u \in \mathcal{C}^\infty(\mathbb{R}^{m-k}_{x''}; \mathcal{C}_*^{-\mu-k}(\mathbb{R}^k_{x'})).$$

We now return to the assumption that $\mu < -k$. It is well known (see e.g. [Tay1, Section 13.8]) that if $s = r + \alpha$ for some $r \in \mathbb{N}$ and $\alpha \in (0,1)$, then $\mathcal{C}_*^s(\mathbb{R}^k)$ agrees with the Hölder space $\mathcal{C}^{r,\alpha}(\mathbb{R}^k)$. From this, we immediately obtain the following lemma.

Lemma 2.1. *If $\mu < -k$, then there exists $\theta \in (0,1]$ depending only on $\mu + k$ such that any $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^m)$ of the form (2.2) satisfies*

$$|u(x) - u(y)| \leq C(|x' - y'|^\theta + |x'' - y''|) \tag{2.3}$$

for each $x, y \in \mathbb{R}^m$.

Proof. If $-k - \mu \in (0,1)$, then we can let $\theta = -k - \mu$. If $-k - \mu > 1$, then actually $u \in \mathcal{C}^1(\mathbb{R}^m)$, and we can take $\theta = 1$. The case $-k - \mu = 1$ is borderline in the sense that $\mathcal{C}_*^1(\mathbb{R}^k)$ functions are not necessarily Lipschitz, although (2.3) is certainly valid for any $\theta \in (0,1)$. □

More concisely, if $\mu_0 > \mu$, then we can take $\theta = \min(1, -\mu_0 - k)$ in (2.3). For general $\mu \in \mathbb{R}$, the distribution u need not be represented by a locally integrable function; on the other hand, we have the following sufficient criterion:

Lemma 2.2. *If $-k < \mu < 0$, then any $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^m)$ of the form (2.2) satisfies $u \in L^1(\mathbb{R}^m)$, and moreover*

$$|u(x)| \leq C|x'|^{-\mu-k}$$

for $x' \neq 0$.

Proof. Since $u \in \mathcal{C}^\infty(\mathbb{R}^m \setminus \{x' = 0\})$, it follows that $u(x) = \sum_{j \geq 0} \psi_j(D_{x'})u(x)$ for $x' \neq 0$. Now $|x'|^{-\mu-k}$ is locally integrable (since $-\mu - k > -k$) so by the dominated convergence theorem it suffices to show that

$$\sum_{j=0}^{N} |\psi_j(D_{x'})u(x)| \leq C|x'|^{-\mu-k} \tag{2.4}$$

for $x' \neq 0$ and every $N \geq 0$, where C does not depend on N. (This will establish that u differs from a locally L^1 function by a distribution supported along $\{x' = 0\}$, and the latter are ruled out since $\mu < 0$.) Integration by parts using the operator $\Delta_{\xi'}^M$ now yields

$$|\psi_j(D_{x'})u(x)| \leq C_M |x'|^{-2M} 2^{j(\mu+k-2M)} \tag{2.5}$$

for each $M \in \mathbb{N}$. Now simply split the sum (2.4) into two pieces, the first where $2^j < |x'|^{-1}$, taking $M = 0$ in (2.5) and using that $\mu + k > 0$, and the second where $2^j \geq |x'|^{-1}$, taking $2M > \mu + k$ in (2.5). □

Under the hypotheses of Lemma 2.2 and applying the mean value theorem in the x'' variables,

$$|u(x', x'') - u(x', y'')| \leq C|x'|^{-\mu-k}|x'' - y''| \tag{2.6}$$

for $x' \neq 0$ and $x'', y'' \in \mathbb{R}^{m-k}$. This estimate will be important when discussing Hamilton's equations in Section 4.2.

Lemma 2.3. *Let $\mu < -k+1$. If u is given by (2.2) and $f \in \mathcal{C}^1(\mathbb{R}^m)$ vanishes along $Y = \{x' = 0\}$, then fu vanishes along Y.*

Proof. We may assume that f is given by one of the coordinate functions $f = x'_j$. Upon splitting $\xi' = (\xi'_j, \xi'')$,

$$(fu)(0,x'') = \int D_{\xi'_j} a(0,x'',\xi)\, d\xi = \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}} D_{\xi'_j} a(0,x'',\xi)\, d\xi'_j\, d\xi'' = 0$$

by Fubini's theorem, since $D_{\xi'_j} a(0,x'',\cdot) \in L^1(\mathbb{R}^k)$. $\square$

Suppose that u and f are as in Lemma 2.3, where $\mu < -k+1$. Combined with the Hölder bound (2.3), we conclude that

$$|(fu)(x)| \leq C|x'|^\theta \tag{2.7}$$

for some $\theta \in (0,1)$ depending only on $\mu + k$.

2.2. **Semiclassical pseudodifferential operators.** Next, we give a brief overview of the semiclassical analysis used in this paper. For a detailed exposition, the reader is referred to [Zwo] and [DZ, Appendix E].

We say that an h-dependent family of symbols $a(x,\theta) = a(x,\theta;h)$ is in $S^m(\mathbb{R}^p_x;\mathbb{R}^q_\theta)$ if the usual symbol bounds

$$|D_x^\alpha D_\theta^\beta a(x,\theta)| \leq C_{\alpha\beta} \langle\theta\rangle^{m-|\beta|}$$

are uniform in $h \in (0,1)$. We also say that $a(x,\theta) \in S^{\mathrm{comp}}(\mathbb{R}^p;\mathbb{R}^q)$ if a is supported in an h-independent compact set, and its $\mathcal{C}_c^\infty(\mathbb{R}^p \times \mathbb{R}^q)$ seminorms are all uniformly bounded in h.

On $\mathbb{R}^n$, we obtain an operator from $a(x,\xi) \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ by the standard left quantization procedure,

$$\mathrm{Op}_h(a)u(x) = (2\pi h)^{-n} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi) u(y)\, dy d\xi. \tag{2.8}$$

This operator acts on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$.

For a manifold X, we similarly define the class of h-dependent symbols on T^*X, which we continue to denote by $S^m(T^*X)$. The space $S^{\mathrm{comp}}(T^*X)$ is defined analogously. We use semiclassical pseudodifferential operators $\Psi_h^m(X)$ with symbols in $S^m(T^*X)$. For simplicity, assume that X is compact; this is only used to avoid issues such as proper supports, and is inessential. The space $\Psi_h^m(X)$ enjoys the following properties:

(I) Each $A \in \Psi_h^m(X)$ maps $\mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ and $\mathcal{C}^{-\infty}(X) \to \mathcal{C}^{-\infty}(X)$.

(II) There is a principal symbol map $\sigma_h : \Psi_h^m(X) \to S^m(T^*X)/hS^{m-1}(T^*X)$ such that the sequence
$$0 \to h\Psi_h^{m-1}(X) \to \Psi_h^m(X) \xrightarrow{\sigma_h} S^m(T^*X)/hS^{m-1}(T^*X) \to 0$$
is exact.

(III) There exists a (non-canonical) quantization map $\mathrm{Op}_h : S^m(T^*X) \to \Psi_h^m(X)$ such that if $a \in S^m(T^*X)$, then
$$\sigma_h(\mathrm{Op}_h(a)) = a$$
in $S^m(T^*X)/hS^{m-1}(T^*X)$.

(IV) If $A \in \Psi_h^m(X)$, then $A^* \in \Psi_h^m(X)$ with principal symbol
$$\sigma_h(A^*) = \overline{\sigma_h(A)}.$$
Here the adjoint is taken with respect to any fixed density on X.

(V) If $A \in \Psi_h^m(X)$ and $B \in \Psi_h^{m'}(X)$, then $[A,B] \in h\Psi_h^{m+m'-1}(X)$ with principal symbol
$$\sigma_h(\tfrac{i}{h}[A,B]) = \{\sigma_h(A), \sigma_h(B)\} = \mathsf{H}_{\sigma_h(A)}\sigma_h(B)$$
where $\{\cdot,\cdot\}$ is the Poisson bracket, and H_f is the Hamilton vector field of a function f on T^*X.

(VI) Each $A \in \Psi_h^m(X)$ extends to a bounded operator $H_h^s(X) \to H_h^{s-m}(X)$. Moreover, if $A \in \Psi_h^0(X)$, then there exists $A' \in \Psi_h^{-\infty}(X)$ such that
$$\|Au\|_{L^2} \leq 2\sup|\sigma_h(A)|\|u\|_{L^2} + \mathcal{O}(h^\infty)\|A'u\|_{L^2} \tag{2.9}$$
for each $u \in L^2(X)$. Here $\sigma_h(A)$ is any representative of the principal symbol in $S^0(T^*X)/hS^{-1}(T^*X)$.

In (2.9), $H_h^s(X)$ refers to the usual Sobolev space $H^s(X)$ but equipped with its semiclassically rescaled Sobolev norm $\|\cdot\|_{H_h^s}$. In particular, given $u \in H_h^1(X)$, we can take
$$\|u\|_{H_h^1} = \int_X |u|^2 + h^2|du|^2\,dg, \tag{2.10}$$
where dg is the volume density for a Riemannian metric g, and the magnitude of du is computed with respect to g.

The negligible operators $h^\infty\Psi_h^{-\infty}(X)$ in this calculus are precisely those with smooth Schwartz kernels, such that each $\mathcal{C}^\infty(X)$ seminorm is of order $\mathcal{O}(h^\infty)$. Given $A \in \Psi_h^m(X)$, there exists $a \in S^m(T^*X)$ such that
$$A = \mathrm{Op}_h(a) + h^\infty\Psi_h^{-\infty}(X). \tag{2.11}$$

The operator wavefront set (also known as the microsupport) $\mathrm{WF}_h(A)$ of $A \in \Psi_h^m(X)$ can be defined as the essential support of its full symbol in any coordinate representation. Here essential support is meant in the semiclassical sense: if $a(x,\theta) \in S^m(\mathbb{R}^p;\mathbb{R}^q)$, then

$$\operatorname{esssupp}(a)^{\complement} = \{(x,\theta) : \, a \in h^\infty S^{-\infty}(\mathbb{R}^p;\mathbb{R}^q) \text{ near } (x,\theta)\}.$$

Note that we are viewing $\operatorname{esssupp}(a)$ as a subset of the radial compactification $\mathbb{R}^p \times \overline{\mathbb{R}^q}$. Thus $\mathrm{WF}_h(A)$ is a subset of the fiber-radially compactified cotangent bundle $\overline{T^*}X$ (see [DZ, Section E.2]). We also write $\mathrm{ell}_h(A)$ for the elliptic set of $A \in \Psi_h^m(X)$, again viewed as a subset of $\overline{T^*}X$: this is the set where the principal symbol is invertible.

The compactly microlocalized operators $\Psi_h^{\mathrm{comp}}(X) \subset \Psi_h^{-\infty}(X)$ are defined to be those with compact operator wavefront set in $T^*X \subset \overline{T^*}X$. Equivalently, $A \in \Psi_h^{\mathrm{comp}}(X)$ if A can be written in the form (2.11) with $a \in S^{\mathrm{comp}}(T^*X)$. If X is not compact, we also assume that the Schwartz kernel of $A \in \Psi_h^{\mathrm{comp}}(X)$ has compact support in $X \times X$.

We need to consider distributions which are h-tempered relative to a fixed order Sobolev space.

Definition 2.4. We say that an h-dependent family $u = u(h) \in \mathcal{C}^{-\infty}(X)$ is h-tempered in $H_h^s(X)$ if there exists $C, N > 0$ such that

$$\|u\|_{H_h^s} \leq C h^{-N}.$$

Thus the usual notion of an h-tempered distribution $u \in \mathcal{C}^{-\infty}(X)$ is that u is h-tempered in some $H_h^{-M}(X)$.

Definition 2.5. Let $r \in \mathbb{R}$. If u is h-tempered in $L^2(X)$ we say that $(x,\xi) \notin \mathrm{WF}_h^r(u)$ if there exists $A \in \Psi_h^0(X)$ elliptic at (x,ξ) such that

$$\|Au\|_{L^2} \leq C h^r.$$

If $r = +\infty$, we write $\mathrm{WF}_h(u)$ for $\mathrm{WF}_h^\infty(u)$.

Recall that ellipticity, and hence wavefront set, is defined at points in the fiber-compactified cotangent bundle.

We will also occasionally employ a wavefront set measured with respect to spaces other than L^2 :

Definition 2.6. Let $r, s \in \mathbb{R}$. If u is h-tempered in $H_h^s(X)$ we say that $(x,\xi) \notin \mathrm{WF}_h^{s,r}(u)$ if there exists $A \in \Psi_h^0(X)$ elliptic at (x,ξ) such that

$$\|Au\|_{H_h^s} \leq C h^r.$$

Lastly, we consider a class of "tangential" pseudodifferential operators on $\mathbb{R}^{d+1}$. Fix a splitting of coordinates $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^d$. Given $k \in \mathbb{N} \cup \{+\infty\}$, we consider operators

$$Q \in \mathcal{C}^k(\mathbb{R}_{x_1}; \Psi_h^m(\mathbb{R}^d_{x'})).$$

Thus we can write $Q = \mathrm{Op}_h(q)$, where $q \in \mathcal{C}^k(\mathbb{R}; S^m(\mathbb{R}^d))$ and Op_h denotes the quantization procedure (2.8) on $\mathbb{R}^d$. However, since q is not necessarily smooth in x_1, the notion of operator wavefront set must be modified. We say that $(x, \xi') \notin \mathrm{esssupp}(q)$ if there is a neighborhood of (x, ξ') in $\mathbb{R}^{d+1} \times \overline{\mathbb{R}^d}$ where

$$D^j_{x_1} D^\alpha_{x'} D^\beta_{\xi'} q(x_1, x', \xi') = \mathcal{O}(h^\infty \langle \xi' \rangle^{-\infty})$$

for $j \leq k$. We then define $\mathrm{WF}_h(Q) = \mathrm{esssupp}(q)$. This definition guarantees that $\mathrm{WF}_h(\partial^k_{x_1}(Q)) \subset \mathrm{WF}_h(Q)$ for $k \geq 1$.

2.3. **Energy estimates.** In this section we prove a microlocal energy estimate that will eventually be applied to the operator P in (1.1). These estimates follow the strategy used in [Hör1, Sections 23.1–23.2] for hyperbolic operators; similar estimates for semiclassical problems have also been obtained in [Chr, Section 3.2].

In what follows we will employ the notation Diff^k_h for the algebra of semiclassical differential operators

$$\sum_{|\alpha| \leq k} a_\alpha(x; h)(hD)^\alpha$$

with $a_\alpha \in \mathcal{C}^\infty$, uniformly in $h \to 0$.

We work on $\mathbb{R}^{d+1}$. Let $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^d$, and consider a differential operator

$$L = (hD_{x_1})^2 - R + hR_0$$

where $R \in \mathcal{C}^1(\mathbb{R}; \mathrm{Diff}^2_h(\mathbb{R}^d))$ and $R_0 \in \mathcal{C}^1(\mathbb{R}; \mathrm{Diff}^1_h(\mathbb{R}^d))$. Writing $r(x, \xi') = \sigma_h(R)$, we make the following microlocal hyperbolicity assumption:

$$r(x, \xi') > 0 \text{ near } (-\varepsilon, \varepsilon) \times U,$$

where $U \subset T^*\mathbb{R}^d$ is open with compact closure. Therefore we can find a self-adjoint tangential operator $\Lambda \in \mathcal{C}^1(\mathbb{R}; \Psi_h^{\mathrm{comp}}(\mathbb{R}^d))$ with $\sigma_h(\Lambda) = r^{1/2}$ near $(-\varepsilon, \varepsilon) \times U$ such that

$$\Lambda^2 = R + R',$$

where $R' \in \mathcal{C}^1(\mathbb{R}; \Psi^2_h(\mathbb{R}^d))$ and $(-\varepsilon, \varepsilon) \times U \cap \mathrm{WF}_h(R') = \emptyset$. Then we have

$$\begin{aligned}(hD_{x_1} \mp \Lambda)(hD_{x_1} \pm \Lambda) &= (hD_{x_1})^2 - \Lambda^2 \pm [hD_{x_1}, \Lambda] \\ &= L + R' \pm hR_1,\end{aligned}$$

where $R_1 = h^{-1}[hD_{x_1}, \Lambda] \pm R_0 \in \mathcal{C}^0(\mathbb{R}; \Psi_h^{\mathrm{comp}}(\mathbb{R}^d)) + \mathcal{C}^1(\mathbb{R}; \mathrm{Diff}^1_h(\mathbb{R}^d))$. Given $u \in \mathcal{C}^\infty(\mathbb{R}^{d+1})$, write $u(x_1)$ for the function $x' \mapsto u(x_1, x')$ on $\mathbb{R}^d$.

Lemma 2.7. *If $A \in \mathcal{C}^\infty(\mathbb{R};\Psi_h^{\mathrm{comp}}(\mathbb{R}^d))$ satisfies $\mathrm{WF}_h(A) \subset (-\varepsilon,\varepsilon)\times U$ and $B \in \mathcal{C}^\infty(\mathbb{R};\Psi_h^{\mathrm{comp}}(\mathbb{R}^d))$ is elliptic on $\mathrm{WF}_h(A)$, then*

$$\begin{aligned}\|Au(x_1)\|_{L^2(\mathbb{R}^d)} \leq C\| \langle hD_{x_1}\rangle Bu\|_{L^2} + Ch^{-1}\int_{-\varepsilon}^{\varepsilon} \|BLu(s)\|_{L^2(\mathbb{R}^d)}\,ds \\ + \mathcal{O}(h^\infty)\|\langle hD_{x_1}\rangle u\|_{L^2}\end{aligned} \tag{2.12}$$

for every $u \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$ and $x_1 \in (-\varepsilon,\varepsilon)$.

Since (2.12) is the first of many estimates of this form, we clarify that the inequality means that there exists C fixed such that for every $M \in \mathbb{N}$, there exist C_M and $h_0 = h_0(M)$ such that for $h \in (0,h_0)$,

$$\begin{aligned}\|Au(x_1)\|_{L^2(\mathbb{R}^d)} \leq C\| \langle hD_{x_1}\rangle Bu\|_{L^2} + Ch^{-1}\int_{-\varepsilon}^{\varepsilon} \|BLu(s)\|_{L^2(\mathbb{R}^d)}\,ds \\ + C_M h^M\|\langle hD_{x_1}\rangle u\|_{L^2}.\end{aligned}$$

Proof. The usual energy inequalities hold for the operators $hD_{x_1}\pm\Lambda$, cf [Hör1, Lemma 23.1.1]: for each $x_1,t\in\mathbb{R}$,

$$\|u(x_1)\|_{L^2(\mathbb{R}^d)} \leq \|u(t)\|_{L^2(\mathbb{R}^d)} + h^{-1}\int_t^{x_1}\|(hD_{x_1}\pm\Lambda)u(s)\|_{L^2(\mathbb{R}^d)}\,ds. \tag{2.13}$$

Given $B_1 \in \mathcal{C}^\infty(\mathbb{R};\Psi_h^{\mathrm{comp}}(\mathbb{R}^d))$, set $v_\pm = B_1(hD_{x_1}\mp\Lambda)u$ and compute

$$(hD_{x_1}\pm\Lambda)v_\pm = B_1Lu + [hD_{x_1}\pm\Lambda, B_1]u \pm hB_1R_1u + B_1R'u$$

Take B_1 elliptic on $\mathrm{WF}_h(A)$ with $\mathrm{WF}_h(B_1)\subset(-\varepsilon,\varepsilon)\times U$ and let B be elliptic on $\mathrm{WF}_h(B_1)$. Then

$$\begin{aligned}\|(hD_{x_1}\pm\Lambda)v_\pm(x_1)\|_{L^2(\mathbb{R}^d)} &\leq C\|BLu(x_1)\|_{L^2(\mathbb{R}^d)} \\ &\quad + Ch\|Bu(x_1)\|_{L^2(\mathbb{R}^d)} + \mathcal{O}(h^\infty)\|u(x_1)\|_{L^2(\mathbb{R}^d)}\end{aligned}$$

for $x_1\in(-\varepsilon,\varepsilon)$. Applying (2.13) to $v_\pm$ yields the estimate

$$\begin{aligned}\|v_\pm(x_1)\|_{L^2(\mathbb{R}^d)} &\leq \|v_\pm(t)\|_{L^2(\mathbb{R}^d)} \\ &\quad + C\int_t^{x_1} h^{-1}\|BLu(s)\|_{L^2(\mathbb{R}^d)} + \|Bu(s)\|_{L^2(\mathbb{R}^d)} + \mathcal{O}(h^\infty)\|u(s)\|_{L^2(\mathbb{R}^d)}\,ds\end{aligned}$$

for $x_1,t\in(-\varepsilon,\varepsilon)$. Furthermore, we can estimate

$$\|v_\pm(t)\|_{L^2(\mathbb{R}^d)} \leq C\|\langle hD_{x_1}\rangle Bu(t)\|_{L^2(\mathbb{R}^d)}.$$

On the other hand, since $\mathrm{WF}_h(A)\subset\mathrm{ell}_h(\Lambda)$,

$$\|Au(x_1)\|_{L^2(\mathbb{R}^d)} \leq C(\|v_+(x_1)\|_{L^2(\mathbb{R}^d)} + \|v_-(x_1)\|_{L^2(\mathbb{R}^d)}) + \mathcal{O}(h^\infty)\|u(x_1)\|_{L^2(\mathbb{R}^d)}.$$

Estimating the $\mathcal{O}(h^\infty)\|u(x_1)\|_{L^2(\mathbb{R}^d)}$ term on the right hand side by (2.13), we conclude that

$$\begin{aligned}\|Au(x_1)\|_{L^2(\mathbb{R}^d)} &\leq C\| \langle hD_{x_1}\rangle Bu(t)\|_{L^2(\mathbb{R}^d)}\\ &+ C\int_t^{x_1} h^{-1}\|BLu(s)\|_{L^2(\mathbb{R}^d)} + \|Bu(s)\|_{L^2(\mathbb{R}^d)} + \mathcal{O}(h^\infty)\| \langle hD_{x_1}\rangle u(s)\|_{L^2(\mathbb{R}^d)}\, ds\end{aligned}$$

for $x_1, t \in (-\varepsilon, \varepsilon)$. Integrating in t finishes the proof. □

2.4. **Semiclassical conormal distributions.** We return to the setting of Section 2.1, adopting the notation there.

Definition 2.8. If $u \in \mathcal{C}_c^{-\infty}(X)$ has compact support in a coordinate patch $\mathcal{U}$ as in Section 2.1, we say that $u \in I_h^p(X; N^*Y)$ if

$$u = (2\pi h)^{-(m+2k)/4} \int e^{\frac{i}{h}\langle x', \xi'\rangle} a(x, \xi')\, d\xi' \tag{2.14}$$

for some $a(x, \xi') = a(x, \xi'; h) \in S^{p+(m-2k)/4}(\mathbb{R}^m_x; \mathbb{R}^k_{\xi'})$.

The general definition of $I^p(X; N^*Y)$ is obtained by localization. If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ is given by (2.14), then u is certainly h-tempered, and

$$\mathrm{WF}_h(u) \subset \{(x, \xi) \in \overline{N^*}Y : (x, \xi') \in \operatorname{esssupp}(a)\}.$$

Here we have written $\overline{N^*}Y \subset \overline{T^*}X$ for the fiber-radially compactified conormal bundle to Y.

We say that $u \in I_h^{\mathrm{comp}}(X; N^*Y)$ if $u \in I_h^{-\infty}(X; N^*Y)$ has compact support, and $\mathrm{WF}_h(u)$ is compact in T^*X. Equivalently, u can locally be written in the form (2.14) with $a \in S^{\mathrm{comp}}(\mathbb{R}^n; \mathbb{R}^k)$, modulo an $h^\infty \mathcal{C}_c^\infty(\mathbb{R}^m)$ remainder.

3. Semiclassical b-pseudodifferential operators

3.1. **b-Tangent and b-cotangent bundles.** Let X be a manifold with boundary. Let $\mathcal{V}(X)$ denote the Lie algebra of smooth vector fields on X, and $\mathcal{V}_{\mathrm{b}}(X)$ the subalgebra of vector fields tangent to ∂X. Let $(x, y) = (x, y_1, \ldots, y_n)$ be local coordinates on a chart $\mathcal{U}$ intersecting ∂X, such that $\mathcal{U} \cap \partial X = \{x = 0\}$. With respect to these coordinates, elements of $\mathcal{V}_{\mathrm{b}}(X)$ are locally of the form

$$f(x, y) x\partial_x + \sum g_i(x, y)\partial_{y_i}. \tag{3.1}$$

Furthermore, $\mathcal{V}_{\mathrm{b}}(X)$ coincides with sections of a bundle, the b-tangent bundle ${}^{\mathrm{b}}TX$. There is also a natural bundle map

$$i : {}^{\mathrm{b}}TX \to TX \tag{3.2}$$

induced by the inclusion $\mathcal{V}_{\mathrm{b}}(X) \hookrightarrow \mathcal{V}(X)$. Over $q \in X^\circ$ (the interior of X) this map is an isomorphism, which gives the identification ${}^{\mathrm{b}}T_{X^\circ}X = TX^\circ$. Here we use the notation ${}^{\mathrm{b}}T_Z X$ for the restriction of ${}^{\mathrm{b}}TX$ to the submanifold Z.

The dual bundle to ${}^{\mathrm{b}}TX$ is the b-cotangent bundle ${}^{\mathrm{b}}T^*X = ({}^{\mathrm{b}}TX)^*$. In coordinates (x, y) near the boundary, sections of ${}^{\mathrm{b}}T^*X$ are of the form

$$\sigma(x,y)\frac{dx}{x} + \sum \eta_i(x,y)dy_i. \tag{3.3}$$

Thus (x, y, σ, η) provide coordinates on ${}^{\mathrm{b}}T^*X$. Let $\pi : T^*X \to {}^{\mathrm{b}}T^*X$ denote the adjoint of (3.2). Over the interior, π induces a dual identification ${}^{\mathrm{b}}T^*_{X^\circ}X = T^*X^\circ$. On the other hand, if (x, y, ξ, η) are the usual coordinates on T^*X induced by (x, y), then

$$\pi(x, y, \xi, \eta) = (x, y, x\xi, \eta).$$

In particular, since it maps to $\sigma = x\xi$, π is not surjective over ∂X. We denote by ${}^{\mathrm{b}}\dot{T}^*X$ the image T^*X under π, referred to as the *compressed cotangent bundle*.

By a slight abuse of notation, we also consider $T^*\partial X$ as a subset of ${}^{\mathrm{b}}T^*_{\partial X}X$. More precisely, i takes ${}^{\mathrm{b}}T_{\partial X}X$ onto $T\partial X$, and the inclusion $T^*\partial X \hookrightarrow {}^{\mathrm{b}}T^*_{\partial X}X$ is the adjoint of this restriction; in local coordinates, it is just the map $(y, \eta) \mapsto (0, y, 0, \eta)$.

While the definitions above apply to a manifold with boundary, for our purposes we need to replace ∂X with an embedded interior hypersurface $Y \subset X$, where X is now boundaryless. In that case we consider the relative b-tangent bundle ${}^{\mathrm{b}}T(X; Y)$. Sections of ${}^{\mathrm{b}}T(X; Y)$ coincide with the subalgebra $\mathcal{V}_{\mathrm{b}}(X; Y) \subset \mathcal{V}(X)$ of vector fields tangent to Y. The discussion above applies verbatim to ${}^{\mathrm{b}}T(X; Y)$ by replacing ∂X with Y, and $X^\circ = X \setminus \partial X$ with $X \setminus Y$.

3.2. **b-Pseudodifferential operators.** We now describe the class of semiclassical b-pseudodifferential operators on a compact manifold X with boundary. This is a variant on the *b-calculus* introduced in the setting of homogeneous microlocal analysis by Melrose [Mel2], [MM] (see also [Mel1] for a detailed treatment). A description of the semiclassical b-calculus employed here can be found in [HV, Appendix A].

We begin by defining the class of residual operators $h^\infty\Psi^{-\infty}_{\mathrm{bc},h}(X)$. Here we resort to a geometric description in terms of a certain *blow-up* of $X \times X$ since this yields the most concise definition. (We refer the reader to [Mel1] for a discussion of real blow-up in the context of the b-calculus and for further references.)

Recall that the b-stretched product $X \times_{\mathrm{b}} X$ is defined by blowing up the corner $\partial X \times \partial X$ in $X \times X$,

$$X \times_{\mathrm{b}} X = [X \times X; \partial X \times \partial X].$$

The blow-down map is denoted by $\beta_{\mathrm{b}} : X \times_{\mathrm{b}} X \to X \times X$. The front face, namely the lift of $\partial X \times \partial X$, is denoted ff, whereas the lifts of $X^\circ \times \partial X$ and $\partial X \times X^\circ$ are denoted lf and rf, respectively.

If M is a manifold with corners, we use the notation $\mathcal{A}(M)$ for the space of L^∞ based conormal distributions on M:

$$\mathcal{A}(M) = \{u \in \mathcal{C}^{-\infty}(M) : \mathcal{V}_{\mathrm{b}}(M)^k u \in L^\infty(M) \text{ for all } k \in \mathbb{N}\}.$$

Returning to the b-stretched product, let ρ_{sf} be a total boundary defining function for the side faces. We then consider operators A with Schwartz kernels in $\rho_{\mathrm{sf}}^\infty \mathcal{A}(X \times_{\mathrm{b}} X)$. Note that this space has a natural family of seminorms.

In what follows $\dot{\mathcal{C}}^\infty(X)$ denotes the set of smooth functions on X vanishing to infinite order at the boundary (cf. [Hör1, Appendix B.2]).

Definition 3.1. A family of operators $A = A(h) : \dot{\mathcal{C}}^\infty(X) \to \dot{\mathcal{C}}^\infty(X)$ belongs to $h^\infty \Psi^{-\infty}_{\mathrm{bc},h}(X)$ if its kernel K_A is the pushforward by β_{b} of an element

$$\widetilde{K} = \widetilde{K}(h) \in \rho_{\mathrm{sf}}^\infty \mathcal{A}(X \times_{\mathrm{b}} X),$$

where each seminorm of $\widetilde{K}$ is of order $\mathcal{O}(h^\infty)$. We say that A belongs to $h^\infty \Psi^{-\infty}_{\mathrm{b},h}(X)$ if $\widetilde{K}$ is in addition smooth up to ff.

In general, semiclassical b-pseudodifferential operators have Schwartz kernels with additional singularities on the diagonal. We choose to give a definition via localization. First we describe the appropriate semiclassical symbol classes. Let us identify

$${}^{\mathrm{b}}T^*\mathbb{R}^n_+ = \mathbb{R}^n_+ \times \mathbb{R}^n,$$

with coordinates $(x, y) \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$ in the first factor, and $(\sigma, \eta) \in \mathbb{R} \times \mathbb{R}^{n-1}$ in the second. In that case, we define h-dependent Kohn–Nirenberg $S^m_{\mathrm{bc}}({}^{\mathrm{b}}T^*\mathbb{R}^n_+)$ corresponding to symbol bounds of the form

$$|(xD_x)^j D_y^\alpha D_\sigma^k D_\eta^\beta a(x, y, \sigma, \eta)| \le C_{kj\alpha\beta} \langle (\sigma, \eta) \rangle^{m-k-|\beta|} \tag{3.4}$$

uniformly in h. Thus a need not be smooth up to the boundary of ${}^{\mathrm{b}}T^*\mathbb{R}^n_+$. If we wish to require smoothness, we can define $S^m_{\mathrm{b}}({}^{\mathrm{b}}T^*\mathbb{R}^n_+)$ by replacing xD_x with D_x in (3.4). In general, $S^m_{\mathrm{bc}}({}^{\mathrm{b}}T^*X)$ is defined by localization, and similarly for $S^m_{\mathrm{b}}({}^{\mathrm{b}}T^*X)$.

We now define a left quantization procedure on $\mathbb{R}^n_+$. For this, fix $\phi \in \mathcal{C}^\infty_c((1/2, 2))$ such that $\phi(s) = 1$ near $s = 1$. Given $a \in S^m_{\mathrm{bc},h}({}^{\mathrm{b}}T^*\mathbb{R}^n_+)$, define $\mathrm{Op}_{\mathrm{b},h}(a)$ by

$$\begin{aligned} &\mathrm{Op}_{\mathrm{b},h}(a)u(x, y) \\ &\qquad = (2\pi h)^{-n} \int e^{\frac{i}{h}((x-\tilde{x})\xi + \langle y-\tilde{y}, \eta\rangle)} \phi(x/\tilde{x}) a(x, y, \eta, x\xi) u(\tilde{x}, \tilde{y}) \, d\xi d\eta d\tilde{x} d\tilde{y}. \end{aligned} \tag{3.5}$$

Semiclassical b-pseudodifferential operators are defined in general by localization:

Definition 3.2. A family of operators $A = A(h) : \dot{\mathcal{C}}^\infty(X) \to \dot{\mathcal{C}}^\infty(X)$ belongs to $\Psi^m_{\mathrm{bc},h}(X)$ if the following properties hold.

(1) If $\varphi, \psi \in \mathcal{C}^\infty(X)$ have disjoint supports, then $\varphi A \psi \in h^\infty \Psi^{-\infty}_{\mathrm{b},h}(X)$.

(2) If $\psi \in \mathcal{C}_c^\infty(O)$ has support in an interior coordinate patch O and $\kappa : O \to O_\kappa \subset \mathbb{R}^n$ is a diffeomorphism, then $(\kappa^*)^{-1}\psi A \psi \kappa^* \in \Psi_h^m(\mathbb{R}^n)$.

(3) If $\psi \in \mathcal{C}_c^\infty(O)$ has support in a boundary coordinate patch O and $\kappa : O \to O_\kappa \subset \mathbb{R}_+^n$ is a diffeomorphism, then

$$(\kappa^{-1})^*\psi A \psi \kappa^* = \mathrm{Op}_{\mathrm{b},h}(a) + R \tag{3.6}$$

for some $a \in S^m_{\mathrm{bc},h}({}^{\mathrm{b}}T^*\mathbb{R}^n_+)$ and $R \in h^\infty \Psi^{-\infty}_{\mathrm{bc},h}(\mathbb{R}^n_+)$

We say that A belongs to $\Psi^m_{\mathrm{b},h}(X)$ if (3.6) holds for some $a \in S^m_{\mathrm{b},h}({}^{\mathrm{b}}T^*\mathbb{R}^n_+)$ and $R \in h^\infty \Psi^{-\infty}_{\mathrm{b},h}(X)$.

The space $\Psi_{\mathrm{bc},h}(X)$ of semiclassical b-pseudodifferential operators with conormal coefficients on a compact manifold X with boundary has the following properties.

(I) Each $A \in \Psi^m_{\mathrm{bc},h}(X)$ maps $\dot{\mathcal{C}}^\infty(X) \to \dot{\mathcal{C}}^\infty(X)$ and $\mathcal{C}^{-\infty}(X) \to \mathcal{C}^{-\infty}(X)$.

(II) There is a principal symbol map $\sigma_{\mathrm{b},h} : \Psi^m_{\mathrm{bc},h}(X) \to S^m_{\mathrm{bc}}({}^{\mathrm{b}}T^*X)/hS^{m-1}_{\mathrm{bc}}({}^{\mathrm{b}}T^*X)$ such that the sequence

$$0 \to h\Psi^{m-1}_{\mathrm{bc},h}(X) \to \Psi^m_{\mathrm{bc},h}(X) \xrightarrow{\sigma_{\mathrm{b},h}} S^m_{\mathrm{bc}}({}^{\mathrm{b}}T^*X)/hS^{m-1}_{\mathrm{bc}}({}^{\mathrm{b}}T^*X) \to 0$$

is exact.

(III) There exists a non-canonical quantization map $\mathrm{Op}_{\mathrm{b},h} : S^m_{\mathrm{bc}}({}^{\mathrm{b}}T^*X) \to \Psi^m_{\mathrm{bc},h}(X)$ such that if $a \in S^m_{\mathrm{bc}}({}^{\mathrm{b}}T^*X)$, then

$$\sigma_{\mathrm{b},h}(\mathrm{Op}_{\mathrm{b},h}(a)) = a$$

in $S^m_{\mathrm{bc}}({}^{\mathrm{b}}T^*X)/hS^{m-1}_{\mathrm{bc}}({}^{\mathrm{b}}T^*X)$.

(IV) If $A \in \Psi^m_{\mathrm{bc},h}(X)$, then $A^* \in \Psi^m_{\mathrm{bc},h}(X)$ with principal symbol

$$\sigma_{\mathrm{b},h}(A^*) = \overline{\sigma_{\mathrm{b},h}(A)}.$$

Here the adjoint is taken with respect to any fixed density on X.

(V) If $A \in \Psi^m_{\mathrm{bc},h}(X)$ and $B \in \Psi^{m'}_{\mathrm{bc},h}(X)$, then $[A, B] \in h\Psi^{m+m'-1}_{\mathrm{bc},h}(X)$ with principal symbol

$$\sigma_{\mathrm{b},h}(\tfrac{i}{h}[A, B]) = \{\sigma_{\mathrm{b},h}(A), \sigma_{\mathrm{b},h}(B)\} = \mathsf{H}^{\mathrm{b}}_{\sigma_{\mathrm{b},h}(A)}\sigma_{\mathrm{b},h}(B)$$

where the Poisson bracket is with respect to the usual symplectic form on $T^*X^\circ = {}^{\mathrm{b}}T^*_{X^\circ}X$ extended by continuity to ${}^{\mathrm{b}}T^*X$, which also defines the b-Hamilton vector $\mathsf{H}^{\mathrm{b}}_f$.

In canonical coordinates given by (3.3), the symplectic form is

$$\omega = \frac{d\sigma \wedge dx}{x} + d\eta \wedge dy$$

while the Hamilton vector field of f is

$$\mathsf{H}^{\mathrm{b}}_f = x(\partial_\sigma f)\partial_x - x(\partial_x f)\partial_\sigma + (\partial_\eta f)\cdot\partial_y - (\partial_y f)\cdot\partial_\eta.$$

(VI) Each $A \in \Psi^0_{\mathrm{bc},h}(X)$ extends to a bounded operator on $L^2(X)$, and moreover there exists $A' \in \Psi^{-\infty}_{\mathrm{bc},h}(X)$ such that

$$\|Au\|_{L^2} \leq 2\sup|\sigma_{\mathrm{b},h}(A)|\|u\|_{L^2} + \mathcal{O}(h^\infty)\|A'u\|_{L^2}$$

for each $u \in L^2(X)$. Here $\sigma_{\mathrm{b},h}(A)$ is any representative of the principal symbol in $S^0_{\mathrm{bc}}({}^{\mathrm{b}}T^*X)/hS^{-1}_{\mathrm{bc}}({}^{\mathrm{b}}T^*X)$.

The subspace of operators with smooth coefficients, $\Psi^m_{\mathrm{b},h}(X) \subset \Psi^m_{\mathrm{bc},h}(X)$, satisfies (I), (II), (III), (IV), (V), (VI) above, simply dropping the subscript c throughout. Moreover, $\Psi^m_{\mathrm{b},h}(X)$ enjoys better mapping properties, namely each element of $\Psi^m_{\mathrm{b},h}(X)$ maps $\mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ and $\dot{\mathcal{C}}^{-\infty}(X) \to \dot{\mathcal{C}}^{-\infty}(X)$.

Suppose that $F \in I^{[-1-\alpha]}(\mathbb{R}^n_+;\partial\mathbb{R}^n_+)$ has compact support, where $\alpha > 0$. Then F is continuous, smooth away from the boundary, and after a semiclassical rescaling the Schwartz kernel of multiplication by F is

$$\delta(x-\tilde{x})\delta(y-\tilde{y})F(x,y) = (2\pi h)^{-n}\int e^{\frac{i}{h}((x-\tilde{x})\sigma+\langle y-\tilde{y},\eta\rangle)}F(x,y)\,d\sigma d\eta. \tag{3.7}$$

We can always insert a cutoff $\phi(x/\tilde{x})$ as in (3.5), since the kernel is supported by the diagonal. In particular, (3.7) can be written in the form (3.5). The reason for introducing the algebra with conormal coeffients is that when viewed as a symbol (independent of σ,η),

$$F \in S^0_{\mathrm{bc},h}({}^{\mathrm{b}}T^*\mathbb{R}^n_+),$$

namely multiplication by F is in $\Psi^0_{\mathrm{bc},h}(X)$ when $\alpha > 0$ (but not $\Psi^0_{\mathrm{b},h}(X)$).

3.3. **Interaction with differential operators.** We will also need to consider the interaction between $\Psi_{\mathrm{b},h}(X)$ and the algebra of semiclassical differential operators $\mathrm{Diff}_h(X)$, which of course is not a subalgebra of $\Psi_{\mathrm{b},h}(X)$. The material in this section is not relevant for the class of conormal coefficient operators $\Psi_{\mathrm{bc},h}(X)$.

The key consideration in what follows is the indicial operator family of $A \in \Psi_{\mathrm{b},h}(X)$, defined for $\sigma \in \mathbb{C}$ and $v \in \mathcal{C}^\infty(\partial X)$ by

$$\widehat{N}(A)(\sigma)v = x^{-i\sigma}A(x^{i\sigma}u)|_{\partial X},$$

where $u \in \mathcal{C}^\infty(X)$ is an arbitrary extension of v; here x is a fixed, global boundary defining function. Thus $\widehat{N}(A) = 0$ for $A \in \Psi^m_{\mathrm{b},h}(X)$ precisely when $A \in x\Psi^m_{\mathrm{b},h}(X)$. Furthermore, the indicial operator map is an algebra homomorphism to σ-dependent families of semiclassical pseudodifferential operators on ∂X :

$$\widehat{N}(AB)(\sigma) = \widehat{N}(A)(\sigma)\circ\widehat{N}(B)(\sigma).$$

Observe that $\widehat{N}(hxD_x)(\sigma)$ is simply multiplication by σ, and $\widehat{N}(x)(\sigma)$ vanishes identically.

Assume that $A \in \Psi^m_{\mathrm{b},h}(X)$ has compact support in a boundary coordinate patch $\mathcal{U} \subset X$, so that $(hD_x)A$ is a well defined operator. Applying $\widehat{N}$, it follows that $[hxD_x, A] \in xh\Psi^m_{\mathrm{b},h}(X)$ and $[x, A] \in xh\Psi^{m-1}_{\mathrm{b},h}(X)$. Therefore,

$$(hD_x)A = x^{-1}[hxD_x, A] + x^{-1}Ax(hD_x).$$

This can be rephrased as in the following lemma:

Lemma 3.3. *Given $A \in \Psi^m_{\mathrm{b},h}(X)$ with compact support in $\mathcal{U}$, there exist A', $A'' \in \Psi^m_{\mathrm{b},h}(X)$ with compact support in $\mathcal{U}$ such that*

$$(hD_x)A - A'(hD_x) = hA'', \tag{3.8}$$

where $A' = x^{-1}Ax$ and $A'' = x^{-1}[hxD_x, A]$.

Lemma 3.3 allows us to give a reasonable definition of differential operators with b-pseudodifferential coefficients:

Definition 3.4. Let $\mathrm{Diff}^k_h \Psi^m_{\mathrm{b},h}(X)$ denote the vector space of locally finite sums of the form $\sum P_j A_j$, where $P_j \in \mathrm{Diff}^k_h(X)$ and $A_j \in \Psi^m_{\mathrm{b},h}(X)$.

Using Lemma 3.3, it can shown that any $\sum P_j A_j \in \mathrm{Diff}^k_h \Psi^m_{\mathrm{b},h}(X)$ can also be written in the form $\sum A'_j P'_j$, where $A'_j \in \Psi^m_{\mathrm{b},h}(X)$ and $P'_j \in \mathrm{Diff}^k_h(X)$.

One can moreover show that the differential-b-pseudodifferential operators form a graded algebra in the following sense.

Lemma 3.5 (cf. [Vas3, Lemma 2.5])**.** *If $B_1 \in \mathrm{Diff}^{k_1}_h \Psi^{m_1}_{\mathrm{b},h}(X)$ and $B_2 \in \mathrm{Diff}^{k_2}_h \Psi^{m_2}_{\mathrm{b},h}(X)$, then the composition satisfies*

$$B_1 B_2 \in \mathrm{Diff}^{k_1+k_2}_h \Psi^{m_1+m_2}_{\mathrm{b},h}(X).$$

Furthermore,

$$[B_1, B_2] \in h\,\mathrm{Diff}^{k_1+k_2}_h \Psi^{m_1+m_2-1}_{\mathrm{b},h}(X).$$

We also have the following fundamental commutation result:

Lemma 3.6 (cf. [Vas3, Lemma 2.8])**.** *If $A \in \Psi^m_{\mathrm{b},h}(X)$ has compact support in a boundary coordinate patch $\mathcal{U}$, then there exist $A_1 \in \Psi^m_{\mathrm{b},h}(X)$ and $A_0 \in \Psi^{m-1}_{\mathrm{b},h}(X)$ satisfying*

$$i[hD_x, A] = hA_1 + hA_0(hD_x). \tag{3.9}$$

Here $\sigma_{\mathrm{b},h}(A_0) = \partial_\sigma a$ and $\sigma_{\mathrm{b},h}(A_1) = \partial_x a$.

Proof. The identity (3.9) follows from (3.8), since $A' - A = x^{-1}[A, x] \in h\Psi^{m-1}_{\mathrm{b},h}(X)$. The computation of the principal symbol follows by continuity from T^*X° as in [Vas3, Lemma 2.8] $\square$

For the next result we fix a Riemannian metric on X with respect to which all adjoints are taken. In particular, $(hD_x)^* = hD_x + h\,\mathrm{Diff}_h^0(X)$.

Lemma 3.7. *Let $A \in \Psi_{\mathrm{b},h}^m(X)$ have compact support in $\mathcal{U}$, and suppose that $a = \sigma_{\mathrm{b},h}(A)$ is real valued. Then there exist*

$$B_0 \in \Psi_{\mathrm{b},h}^{m-1}(X), \quad B_1 \in \Psi_{\mathrm{b},h}^m(X)$$

with $\sigma_{\mathrm{b},h}(B_0) = 2\partial_\sigma a$ and $\sigma_{\mathrm{b},h}(B_1) = 2\partial_x a$, such that

$$(i/h)[(hD_x)^* hD_x, A] = (hD_x)^* B_0 (hD_x) + (hD_x)^* B_1 + hR,$$

where $R \in \mathrm{Diff}_h^1 \Psi_{\mathrm{b},h}^{m-1}(X)$.

Proof. First, compute

$$\begin{aligned} i[(hD_x)^* hD_x, A] &= i(hD_x)^*[hD_x, A] - i[hD_x, A^*]^*(hD_x) \\ &= h(hD_x)^*(A_0 + A_0^*)(hD_x) + h\left((hD_x)^* A_1 + A_1^*(hD_x)\right), \end{aligned}$$

modulo $h\,\mathrm{Diff}_h^1 \Psi_{\mathrm{b},h}^{m-1}(X)$, where according to Lemma 3.6,

$$\sigma_{\mathrm{b},h}(A_0) = \partial_\sigma a, \quad \sigma_{\mathrm{b},h}(A_1) = \partial_x a.$$

Here we used that $A = A^* + h\Psi_{\mathrm{b},h}^{m-1}(X)$. In particular, $\sigma_{\mathrm{b},h}(A_0 + A_0^*) = 2\partial_\sigma a$. We then write

$$A_1^*(hD_x) = (hD_x)^* A_1 + h\,\mathrm{Diff}_h^1 \Psi_{\mathrm{b},h}(X)$$

according to Lemma 3.6. Therefore,

$$(i/h)[(hD_x^*) hD_x, A] = (hD_x)^* B_0 (hD_x) + (hD_x)^* B_1 + h\,\mathrm{Diff}_h^1 \Psi_{\mathrm{b},h}(X),$$

with $B_0 = A_0 + A_0^*$ and $B_1 = A_1$. $\square$

3.4. **Wavefront set and ellipticity.** In this section X continues to denote a smooth manifold with boundary. There is an operator wavefront set for elements of $\Psi_{\mathrm{bc},h}(X)$, which is naturally a subset of the fiber-radial compactification $\overline{{}^{\mathrm{b}}T^*}X$. As usual, $\mathrm{WF}_{\mathrm{b},h}(A)$ can be defined locally as the essential support of the total symbol a of $A \in \Psi_{\mathrm{bc},h}^m(X)$. Here the notion of essential support takes into account the conormal behavior of a: $q_0 \notin \mathrm{esssupp}(a)$ if there is a neighborhood of q_0 in $\overline{{}^{\mathrm{b}}T^*}X$ where a lies in $h^\infty S_{\mathrm{bc}}^{-\infty}({}^{\mathrm{b}}T^*X)$. If $a \in S_{\mathrm{b}}^m({}^{\mathrm{b}}T^*X)$, this automatically implies that a is locally in $h^\infty S_{\mathrm{b}}^{-\infty}({}^{\mathrm{b}}T^*X)$ near q. The operator wavefront set satisfies the usual relations

$$\begin{aligned} \mathrm{WF}_{\mathrm{b},h}(AB) &\subset \mathrm{WF}_{\mathrm{b},h}(A) \cap \mathrm{WF}_{\mathrm{b},h}(B), \\ \mathrm{WF}_{\mathrm{b},h}(A+B) &\subset \mathrm{WF}_{\mathrm{b},h}(A) \cup \mathrm{WF}_{\mathrm{b},h}(B). \end{aligned} \tag{3.10}$$

We write $\Psi_{\mathrm{bc},h}^{\mathrm{comp}}(X)$ for the subalgebra of operators whose wavefront sets are a compact subset of ${}^{\mathrm{b}}T^*X \subset \overline{{}^{\mathrm{b}}T^*}X$, and similarly for $\Psi_{\mathrm{b},h}^{\mathrm{comp}}(X)$.

Ellipticity is also defined as usual. For instance, fix a norm $|\cdot|$ on the fibers on ${}^{\mathrm{b}}T^*X$, and then set $\langle\zeta\rangle = (1+|\zeta|^2)^{1/2}$. We say that $A \in \Psi^m_{\mathrm{bc},h}(X)$ is elliptic at $q_0 \in \overline{{}^{\mathrm{b}}T^*}X$ if for some $h_0 > 0$

$$\langle\zeta\rangle^{-m}\,|\sigma_{\mathrm{b},h}(A)(z,\zeta)| > 0$$

for $h \in (0,h_0)$ in a neighborhood of $q_0 = (z_0,\zeta_0)$. The set of elliptic points is denoted $\mathrm{ell}_{\mathrm{b}}(A)$. The standard symbolic procedure for elliptic symbols allows one to construct microlocal elliptic parametrices: if $A \in \Psi^s_{\mathrm{bc},h}(X)$ and $B \in \Psi^m_{\mathrm{bc},h}(X)$ satisfy $\mathrm{WF}_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(B)$, then there is $Q \in \Psi^{s-m}_{\mathrm{bc},h}(X)$ such that

$$A - QB \in h^\infty\Psi^{-\infty}_{\mathrm{bc},h}(X), \quad A - BQ \in h^\infty\Psi^{-\infty}_{\mathrm{bc},h}(X). \tag{3.11}$$

Of course if $A, B \in \Psi_{\mathrm{b},h}(X)$, then both Q and the residual terms in (3.11) can be chosen in $\Psi_{\mathrm{b},h}(X)$.

A simple adaptation of [Vas3, Lemmas 3.2, 3.4] shows that each $A \in \Psi^0_{\mathrm{b},h}(X)$ defines a uniformly bounded map

$$A : \overline{H}^1_h(X) \to \overline{H}^1_h(X), \tag{3.12}$$

where $\overline{H}^1_h(X)$ is the space of extendible distributions in the sense of [Hör1, Appendix B.2]. The same is true if $\overline{H}^1_h(X)$ is replaced by $\dot{H}^1_h(X)$, the space of distributions supported on X, again in the sense of [Hör1, Appendix B.2]. By duality, A is uniformly bounded on $\overline{H}^{-1}_h(X)$ and $\dot{H}^{-1}_h(X)$ as well.

Lemma 3.8. *Each $A \in \Psi^1_{\mathrm{bc},h}(X)$ is uniformly bounded $A : \overline{H}^1_h(X) \to L^2(X)$.*

Proof. By a microlocal partition of unity we can assume that $\mathrm{WF}_{\mathrm{b},h}(A)$ is contained in the elliptic set of some vector field B (we can take $B = hW$ for some $W \in \mathcal{V}_{\mathrm{b}}(X)$). Thus $A = QB + R$ for a parametrix $Q \in \Psi^0_{\mathrm{bc},h}(X)$, where $R \in h^\infty\Psi^{-\infty}_{\mathrm{bc},h}(X)$. Hence

$$\|Au\|_{L^2} \leq C\|Bu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2} \leq C\|u\|_{\overline{H}^1_h}$$

since $B \in \mathrm{Diff}^1_h(X)$. $\square$

It will also be convenient to have a wavefront set for operators

$$A \in \mathrm{Diff}^k_h\,\Psi^m_{\mathrm{b},h}(X) + \Psi^l_{\mathrm{bc},h}(X).$$

For this, we define

$$\mathrm{WF}^k_{\mathrm{b},h}(A)^\complement = \bigcup\{\mathrm{ell}_{\mathrm{b}}(B) : B \in \Psi^0_{\mathrm{b},h}(X) \text{ and } BA \in h^\infty\,\mathrm{Diff}^k_h\,\Psi^{-\infty}_{\mathrm{b},h}(X) + h^\infty\Psi^{-\infty}_{\mathrm{bc},h}(X)\}.$$

If $A \in \Psi^m_{\mathrm{bc},h}(X)$, then $\mathrm{WF}^k_{\mathrm{b},h}(A) = \mathrm{WF}_{\mathrm{b},h}(A)$ for all $k \in \mathbb{N}$. Consider a concrete representation

$$A = \sum P_j A_j \in \mathrm{Diff}^k_h\,\Psi^m_{\mathrm{b},h}(X)$$

where $P_j \in \mathrm{Diff}^k_h(X)$. In that case, if $\mathrm{WF}_{\mathrm{b},h}(A_j) \subset U$ for some U, then $\mathrm{WF}^k_{\mathrm{b},h}(A) \subset U$ as well. In fact, the only reason we choose to introduce $\mathrm{WF}^k_{\mathrm{b},h}(A)$ is to bound certain quadratic forms. For this, we use the following observation: if $F \in \Psi_{\mathrm{b},h}(X)$ satisfies $\mathrm{WF}_{\mathrm{b},h}(F) \cap \mathrm{WF}^k_{\mathrm{b},h}(A) = \emptyset$ with A as above, then $FA \in h^\infty \operatorname{Diff}^k_h \Psi^{-\infty}_{\mathrm{b},h}(X)$.

Lemma 3.9. *If $A \in \operatorname{Diff}^2_h \Psi^0_{\mathrm{b},h}(X)$ and $G \in \Psi^0_{\mathrm{b},h}(X)$ satisfy $\mathrm{WF}^2_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(G)$, then*

$$|\langle Au, u\rangle| \leq C\|Gu\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h} \tag{3.13}$$

for each $u \in H^1_h(X)$, where the left hand side of (3.13) is the pairing of $Au \in H^{-1}_h(X)$ with $u \in H^1_h(X)$.

Proof. Choose $B \in \Psi^0_{\mathrm{b},h}(X)$ such that

$$\mathrm{WF}_{\mathrm{b},h}(B) \subset \mathrm{ell}_{\mathrm{b}}(G), \quad \mathrm{WF}_{\mathrm{b},h}(1-B) \cap \mathrm{WF}^2_{\mathrm{b},h}(A) = \emptyset.$$

Therefore $A = BA + h^\infty \operatorname{Diff}^2_h \Psi^{-\infty}_{\mathrm{b},h}(X)$. We can then choose a decomposition

$$BA = \sum_{i,j} BQ_jQ'_jA_{ij} + h^\infty \operatorname{Diff}^2_h \Psi^{-\infty}_{\mathrm{b},h}(X),$$

where $Q_i, Q'_j \in \operatorname{Diff}^1_h(X)$, and $A_{ij} \in \Psi^0_{\mathrm{b},h}(X)$ satisfies $\mathrm{WF}_{\mathrm{b},h}(A_{ij}) \subset \mathrm{ell}_{\mathrm{b}}(G)$. Therefore

$$|\langle Au, u\rangle| \leq \sum_{ij} |\langle Q'_jA_{ij}u, Q^*_jB^*u\rangle| + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h} \leq C\|Gu\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}$$

as desired. □

3.5. **b-Calculus relative to an interior hypersurface.** In this section we depart from the setting of manifolds with boundary, and instead consider a boundaryless manifold X with a distinguished hypersurface $Y \subset X$. For simplicity of exposition, we will work under the geometric assumption that Y is oriented, and that Y divides X into two manifolds with boundaries,

$$X = X_+ \cup X_-,$$

each of which satisfies $Y = \partial X_\pm$; the orientation is chosen so that X_+ is the positive side. In fact, all of our uses of this calculus will be local near a single point in Y, so neither the hypothesis of orientation nor that of bounding two components plays any role here: both are always true locally.

The space $\Psi^m_{\mathrm{b},h}(X,Y)$ of b-pseudodifferential operators (or $\Psi^m_{\mathrm{bc},h}(X,Y)$, with conormal coefficients) relative to Y is defined in analogy with boundary case discussed in Section 3.2. For instance, to define residual operators $h^\infty\Psi^{-\infty}_{\mathrm{bc},h}(X,Y)$, the stretched product X^2_{b} is replaced by the blow-up $[X^2; Y^2]$. The condition of vanishing to infinite order at the side faces is then replaced by requiring the kernel to be supported on the lift of $X^2_+ \cup X^2_-$.

In the case of smooth coefficients, we must impose an additional condition to ensure that the residual operators preserve $H_h^1(X)$. If $R \in h^\infty \Psi_{\mathrm{b},h}^{-\infty}(X,Y)$, then by restriction R defines two operators $R_\pm \in h^\infty \Psi_{\mathrm{b},h}^{-\infty}(X_\pm)$, and the action of R on $\mathcal{C}^\infty(X)$ is given by

$$R = e_+ R_+ r_+ + e_- R_- r_-,$$

where $r_\pm : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X_\pm)$ are the restriction maps, and $e_\pm$ is extension by zero from $X_\pm$ to X. A priori R does not preserve $\mathcal{C}^\infty(X)$. On the other hand, if we further require that the normal operators $\widehat{N}(R_\pm)(0)$ agree along Y, then R maps $\mathcal{C}^\infty(X)$ into piecewise continuous functions with smooth restrictions to $X_\pm$; this implies that R is uniformly bounded on $H_h^1(X)$ and $H_h^{-1}(X)$ by duality (cf. the discussion preceding [DHUV, Lemma 4.1]). *We thus always assume this matching condition for residual operators with smooth coefficients* (observe that this is meaningless for operators with conormal coefficients).

The symbol classes $S^m_{\mathrm{bc}}({}^{\mathrm{b}}T^*(X,Y))$ and $S^m_{\mathrm{b}}({}^{\mathrm{b}}T^*(X,Y))$ are defined in the obvious way, replacing the usual b-cotangent bundle by the relative space ${}^{\mathrm{b}}T^*(X,Y)$ discussed in Section 3.1. The quantization procedure (3.5) does not need modification, and hence Definition 3.2 goes through verbatim. In particular, if $a \in S^m_{\mathrm{b}}({}^{\mathrm{b}}T^*(X,Y))$ is a smooth b-symbol, then $\mathrm{Op}_{\mathrm{b},h}(a)$ automatically has matching normal operators.

Properties of $\Psi^m_{\mathrm{bc},h}(X,Y)$ are largely analogous to those in the boundary case. If X is compact then each $\Psi^0_{\mathrm{b},h}(X,Y)$ is uniformly bounded on $H^s_h(X)$ for $s \in \{-1,0,1\}$, cf. [DHUV, Lemma 4.1]. In the case of conormal coefficients, we still have uniform boundedness on $L^2(X)$.

Similarly, we can define $\mathrm{Diff}^k_h \Psi_{\mathrm{b},h}(X,Y)$ to consist of locally finite sums $\sum P_j A_j$, where $P_j \in \mathrm{Diff}^k_h(X)$ and $A_j \in \Psi_{\mathrm{b},h}(X,Y)$.

Finally, we define the wavefront set of a family $u = u(h)$ which is h-tempered in $H^s_h(X)$. Here, we will only consider the cases $s \in \{-1,0,1\}$. We say that $q_0 \notin \mathrm{WF}^{s,r}_{\mathrm{b},h}(u)$ if there exists $A \in \Psi^0_{\mathrm{b},h}(X)$ which is elliptic at q_0 and

$$\|Au\|_{H^s_h} \le Ch^r.$$

When $s = 0$ it suffices to test within the larger class of operators $A \in \Psi^0_{\mathrm{bc},h}(X)$, and we also abbreviate $\mathrm{WF}^r_{\mathrm{b},h}(u) = \mathrm{WF}^{0,r}_{\mathrm{b},h}(u)$. The action of b-pseudodifferential operators is then semiclassically pseudolocal in the sense that

$$\mathrm{WF}^{s,r}_{\mathrm{b},h}(Au) \subset \mathrm{WF}^{s,r}_{\mathrm{b},h}(u) \cap \mathrm{WF}_{\mathrm{b},h}(A).$$

In fact, the following result shows that for our purposes, the distinction between $\mathrm{WF}^{1,r}_{\mathrm{b},h}(u)$ and $\mathrm{WF}^r_{\mathrm{b},h}$ is irrelevant; the operator P is as in Section 1.1.

Lemma 3.10. *If u is h-tempered in $H^1_h(X)$, then*

$$\mathrm{WF}^{1,r}_{\mathrm{b},h}(u) = \mathrm{WF}^r_{\mathrm{b},h}(u) \cup \mathrm{WF}^{-1,r}_{\mathrm{b},h}(Pu).$$

Proof. The inclusion $\mathrm{WF}^r_{\mathrm{b},h}(u) \cup \mathrm{WF}^{-1,r}_{\mathrm{b},h}(Pu) \subset \mathrm{WF}^{1,r}_{\mathrm{b},h}(u)$ is obvious. The converse inclusion follows directly from Lemma 5.4, proved in Section 5.1 below. □

4. Bicharacteristics

4.1. **The characteristic set.** We return to the setting of Section 1.1: (X, g) is a smooth n dimensional Riemannian manifold with a distinguished hypersurface $Y \subset X$, and

$$P = h^2\Delta_g + V$$

where $V \in I^{[-1-\alpha]}(Y)$ for some $\alpha > 0$. In particular, we can consider multiplication by V as a b-pseudodifferential operator

$$V \in \Psi^0_{\mathrm{bc},h}(X, Y).$$

Since Y is fixed, for ease of notation we write ${}^{\mathrm{b}}T^*X$ instead of the more precise ${}^{\mathrm{b}}T^*(X, Y)$.

Given a point $y_0 \in Y$, we can find a coordinate patch $\mathcal{U} \ni y_0$ equipped with geodesic normal coordinates (x, y) with respect to g. In particular, $\mathcal{U} \cap Y = \{x = 0\}$. In these coordinates the metric is given by

$$g = dx^2 + k(x, y, dy),$$

where $x \mapsto k(x, \cdot)$ is family of metrics on Y depending smoothly on the parameter x. Therefore

$$P = (hD_x)^*(hD_x) + h^2\Delta_k + V,$$

where $(hD_x)^*$ is the adjoint of hD_x with respect to the metric density. If (x, y, ξ, η) are the corresponding canonical coordinates on T^*X, then the principal symbol is given by

$$p = \xi^2 + k^{ij}\eta_i\eta_j + V.$$

We also set

$$\tilde{P} = h^2\Delta_k + V \tag{4.1}$$

with principal symbol

$$\tilde{p} = k^{ij}\eta_i\eta_j + V.$$

Denote the characteristic set of P by $\Sigma = \{p = 0\} \subset T^*X$. The compressed characteristic set is then defined by

$$\dot{\Sigma} = \pi(\Sigma) \subset {}^{\mathrm{b}}\dot{T}^*X,$$

where $\pi : T^*X \to {}^{\mathrm{b}}\dot{T}^*X$ is the usual map. We equip $\dot{\Sigma}$ with the subspace topology inherited as a subset of ${}^{\mathrm{b}}T^*X$ (in particular, $\dot{\Sigma}$ is locally compact and metrizable). Note that Σ is compact in the fiber variables: if $K \subset X$ is compact, then so is $\Sigma \cap T^*_K X$. In particular, the restriction of π to Σ is proper.

We decompose the fiber-radial compactification $\overline{{}^{\mathrm{b}}\dot{T}^*}X$ into the elliptic, hyperbolic, and glancing regions, denoted by $\mathcal{E}, \mathcal{H}, \mathcal{G}$, respectively:

$$\begin{aligned} \mathcal{E} &= \{q \in \overline{{}^{\mathrm{b}}\dot{T}^*}X : \pi^{-1}(q) \cap \Sigma = \emptyset\}, \\ \mathcal{G} &= \{q \in {}^{\mathrm{b}}\dot{T}X : |\pi^{-1}(q) \cap \Sigma| = 1\}, \\ \mathcal{H} &= \{q \in {}^{\mathrm{b}}\dot{T}X : |\pi^{-1}(q) \cap \Sigma| \geq 2\}. \end{aligned} \tag{4.2}$$

Here $|\cdot|$ refers to the cardinality of a set. Since the restriction of π to $T^*(X \setminus Y)$ is $1-1$, it is clear that $\mathcal{H} \subset {}^{\mathrm{b}}T^*_Y X \cap \dot{\Sigma}$. Furthermore, if $T^*(X \setminus Y)$ is identified with its image under π, any point $q \in T^*(X \setminus Y)$ is either in $\mathcal{E}$ or $\mathcal{G}$, depending on whether $q \notin \Sigma$ or $q \in \Sigma$, respectively. Over a normal coordinate patch $\mathcal{U}$, the glancing region is given by

$$\mathcal{G} \cap {}^{\mathrm{b}}T^*_{\mathcal{U}}X = \{x = 0,\ \tilde{p} = 0\} \subset T^*Y \subset {}^{\mathrm{b}}T^*_Y X.$$

Likewise $\mathcal{H} \cap {}^{\mathrm{b}}T^*_{\mathcal{U}}X$ consists of those points $q \in T^*Y \subset {}^{\mathrm{b}}T^*_Y X$ for which $\tilde{p}(q) < 0$.

4.2. **Hamilton flow.** Formally, the Hamilton vector field of p on T^*X in normal coordinates is given by

$$\mathsf{H}_p = 2\xi\partial_x + 2k^{ij}\eta_j\partial_{y_i} - \left(\left(\partial_x k^{ij}\right)\eta_i\eta_j + \partial_x V\right)\partial_\xi - \left(\left(\partial_{y_i} k^{jk}\right)\eta_j\eta_k + \partial_{y_i} V\right)\partial_{\eta_i},$$

where Einstein summation is implied. This is a smooth vector field away from $T^*_Y X$, but in general only possesses $\mathcal{C}^{\alpha-1}_*$ coefficients due to the ∂_ξ component. Of course if $\alpha > 2$, then H_p has $\mathcal{C}^1$ (hence Lipschitz continuous) components, where the existence and uniqueness of solutions to Hamilton's equations are classical. Under the assumption that $\alpha > 0$, we define integral curves in the following sense:

Definition 4.1. If $I \subset \mathbb{R}$ is an interval, we say that an absolutely continuous map $\gamma : I \to T^*X$ is an integral curve of H_p if

$$\frac{d}{ds}\gamma(s) = \mathsf{H}_p(\gamma(s)) \tag{4.3}$$

for almost every $s \in I$. Such a curve is called a bicharacteristic.

Implicit in this definition is that $\mathsf{H}_p \circ \gamma$ itself has measurable, locally integrable components. For general $\alpha > 0$, there is no reason to expect existence, let alone uniqueness, of integral curves through an arbitrary point $q_0 \in T^*_Y X$.

On the other hand, near a point $q_0 = (0, y_0, \xi_0, \eta_0)$ with $\xi_0 \neq 0$, we can convert (4.3) into an equation to which the Carathéodory existence and uniqueness theorem applies. More generally, consider a vector field

$$F = \sum F_j \partial_{z_j}$$

on an open set $D \subset \mathbb{R}^m_z$ with arbitrary real coefficients. Generalizing Definition 4.1, we say that an absolutely continuous map $\gamma : I \to D$ is an integral curve of F if

$$\frac{d}{ds}\gamma(s) = F(\gamma(s)) \tag{4.4}$$

for almost every $s \in I$. The following lemma is a variation of [DHUV, Lemma 3.1]; when applied to $F = \mathsf{H}_p$, it allows us to treat the whole range of parameters $\alpha > 0$, whereas the given reference would only be valid for $\alpha > 1$.

Lemma 4.2. *Let $z = (z_1, z') \in \mathbb{R} \times \mathbb{R}^{m-1}$, with a corresponding decomposition $F = (F_1, F') : \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^{m-1}$. Assume that*

$$D = J_{z_1} \times O_{z'},$$

where $J \subset \mathbb{R}$ is an interval and $O \subset \mathbb{R}^{m-1}$. Suppose that F_1 is continuous and nonvanishing, and F' satisfies the following properties on D.

(1) F' is measurable in z_1 for all z', and continuous in z' for almost every z_1.

(2) There exists $m \in L^1(J; \mathbb{R}_+)$ such that $|F'(z)| \leq m(z_1)$.

(3) There exists $k \in L^1(J; \mathbb{R}_+)$ such that $|F'(z_1, x') - F'(z_1, y')| \leq k(z_1)|x' - y'|$.

Given $z_0 \in D$, there exists $\varepsilon > 0$ and a unique integral curve $\gamma : [-\varepsilon, \varepsilon] \to D$ such that $\gamma(0) = z_0$.

Furthermore, suppose that F' is continuous. If $\delta > 0$ is sufficiently small and $|z - z_0| \leq \delta$, then there is a unique integral curve

$$\gamma^{(z)} : [-\varepsilon, \varepsilon] \to D$$

satisfying $\gamma^{(z)}(0) = z$, and $\gamma^{(z)} \to \gamma^{(z_0)}$ uniformly on $[-\varepsilon, \varepsilon]$ as $z \to z_0$.

Proof. To avoid notational confusion, we reserve

$$\pi_1 : \mathbb{R} \times \mathbb{R}^{m-1} \to \mathbb{R}, \quad \pi' : \mathbb{R} \times \mathbb{R}^{m-1} \to \mathbb{R}^{m-1}$$

for projections onto the first and second factors, respectively. Suppose that γ is an integral curve of F. Since F_1 is continuous and nonvanishing, the map $s \mapsto (\pi_1 \circ \gamma)(s)$ has an absolutely continuous (and in fact $\mathcal{C}^1$) inverse $S = S(t)$. Define the time dependent vector field $G = (G_1, G')$ by

$$G_1(t, s, z') = 1/F_1(t, z'), \quad G'(t, s, z) = F'(t, z')/F_1(t, z').$$

Then the curve $\Gamma(t) = (S(t), (\pi' \circ \gamma)(S(t)))$ satisfies the equation

$$\frac{d}{dt}\Gamma(t) = G(t, \Gamma(t)). \tag{4.5}$$

This process can be reversed as well, in the sense that from an absolutely continuous solution $\Gamma(t)$ of (4.5) we can recover a solution $\gamma(s)$ of (4.4) by setting

$$\gamma(s) = (T(s), (\pi' \circ \Gamma)(T(s))), \tag{4.6}$$

where $T = T(s)$ is the inverse of $t \mapsto (\pi_1 \circ \Gamma)(t)$.

The equation (4.5) is well-posed in the sense of Carathéodory [CL, Theorems 1.1]. Thus, given $(z_1, z') \in D$, there exists $\varepsilon_0 > 0$ and a unique integral curve

$$\Gamma : [z_1 - \varepsilon_0, z_1 + \varepsilon_0] \to D$$

such that $\Gamma(z_0) = (0, z')$. Passing to a curve γ as in (4.6), we obtain a unique integral curve of F satisfying $\gamma(0) = (z_1, z')$ on a suitable interval $[-\varepsilon, \varepsilon]$.

If F' is continuous in its arguments, then solutions to (4.4) (which are unique by the argument above) depend continuously on the initial data [CL, Theorem 4.2], which implies the second point. □

Lemma 4.2 applies directly to the equation (4.3) in a neighborhood of the hyperbolic region.

Lemma 4.3. *Let $\alpha > 0$. Given $\varpi_0 \in \pi^{-1}(\mathcal{H})$, there exists $\varepsilon > 0$ and a unique integral curve $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ of H_p such that $\gamma(0) = \varpi_0$. Furthermore, if $\alpha > 1$, then the flow*

$$(s, \varpi) \mapsto \exp(sH_p)(\varpi)$$

exists and is continuous in a neighborhood of $(0, \varpi_0)$

Proof. Apply Lemma 4.2 to $F = \mathsf{H}_p$ with the splitting of variables $z_1 = x$ and $z' = (y, \xi, \eta)$. Since $F_1 = 2\xi$, it is continuous and nonvanishing in a small neighborhood of $\tilde{q}_0$. The hypotheses on the remaining components of F follows from Lemma 2.2 and (2.6). It remains to shows that $\gamma((-\varepsilon, \varepsilon)) \subset \Sigma$. If $x \neq 0$, then $\mathsf{H}_p p = 0$. On the other hand,

$$x(\gamma(s)) \neq 0 \text{ for } s \in (-\varepsilon, \varepsilon) \setminus 0,$$

since, by our assumption that $\varpi_0 \in \pi^{-1}(\mathcal{H})$, $F_1 = \mathsf{H}_p x \neq 0$. Thus $p \circ \gamma$ is locally constant on $(-\varepsilon, \varepsilon) \setminus 0$, which completes the proof since $p \circ \gamma$ is continuous and $p(\gamma(0)) = 0$.

Now suppose that $\alpha > 1$, in which case the properties of the flow in (s, q) follow from the second part of Lemma 4.2. □

Remark 4.4. For $\alpha < 2$, uniqueness of bicharacteristics can certainly fail in the glancing region, notwithstanding the special structure of Hamilton's equations. Consider for instance the symbol

$$p = \left(\xi^2 + \eta^2\right) - 1 - 4|x|^{3/2}$$

on $T^*\mathbb{R}^2$. The Hamilton vector field is

$$\mathsf{H}_p = 2\xi\partial_x + 6(\mathrm{sgn} x)|x|^{1/2}\partial_\xi + 2\eta\partial_y.$$

Clearly $(x = 0, \xi = 0, y = 2s, \eta = 1)$ is a null bicharacteristic. But on the other hand, so is $(x = s_+^4, \xi = 2s_+^3, y = 2s, \eta = 1)$. This example exhibits the possibility of bicharacteristics sticking to the interface Y for arbitrarily long times before detaching (cf. [HW] for further related examples of non-uniqueness of geodesics).

4.3. **Generalized broken bicharacteristics.** We now define the generalized broken bicharacteristic flow as initially introduced by Melrose–Sjöstrand [MS1]; cf. [Leb], [Vas3].

Definition 4.5. A function f on T^*X is π-invariant if $f(\varpi_1) = f(\varpi_2)$ whenever $\pi(\varpi_1) = \pi(\varpi_2)$.

Any π-invariant function f induces a function on ${}^{\mathrm{b}}\dot{T}^*X$, denoted by f_π. A rich class of π-invariant functions are those of the form π^*F, where F is a function on ${}^{\mathrm{b}}\dot{T}^*X$. In that case $F = (\pi^*F)_\pi$. If f is π-invariant, then in local coordinates (x, y, ξ, η) on T^*X,

$$\xi \mapsto f(0, y, \xi, \eta)$$

is constant for every fixed (y, η).

Lemma 4.6. *Let $\alpha > 0$. If $f \in \mathcal{C}^1(T^*X)$ is π-invariant, then $\mathsf{H}_p f$ admits a continuous extension to T^*X.*

Proof. The only obstruction to proving the lemma is the term $-(\partial_x V)\partial_\xi f$. On the other hand, since f is π-invariant, $\xi \mapsto f(0, y, \xi, \eta)$ is constant. Now $\partial_\xi f$ exists and vanishes along $T^*_Y X$, and hence $\partial_\xi f \in x\mathcal{C}^0(T^*X)$. Therefore $(\partial_x V)\partial_\xi f = (x\partial_x V)F$, where $F \in \mathcal{C}^0(T^*X)$, and this latter term vanishes along $T^*_Y X$ by Lemma 2.3. □

We now recall the definition of generalized broken bicharacteristics as given in [Vas2].

Definition 4.7. If $I \subset \mathbb{R}$ is an interval, we say that a continuous map $\gamma : I \to \dot{\Sigma}$ is a generalized broken bicharacteristic (GBB) if for each $s_0 \in I$ and $f \in \mathcal{C}^\infty(T^*X)$ which is π-invariant,

$$\liminf_{s\to s_0} \frac{f_\pi(\gamma(s)) - f_\pi(\gamma(s_0))}{s - s_0} \geq \inf\{(\mathsf{H}_p f)(\varpi) : \pi(\varpi) = \gamma(s_0),\ \varpi \in \Sigma\} \tag{4.7}$$

If s_0 is an endpoint of I, the left hand side of (4.7) is meant in the one-sided sense.

Note that in the case at hand, the infimum on the right hand side is in fact a minimum over at most two values.

This is of course the same as saying that both lower Dini derivatives $D_\pm(f_\pi \circ \gamma)(s_0)$ are no smaller than the right hand side of (4.7). Definition 4.7 makes it clear that

GBBs can be concatenated: if $\gamma : (s_0, s_1] \to \dot{\Sigma}$ and $\gamma' : [s_1, s_2) \to \dot{\Sigma}$ are two GBBs with $\gamma(s_1) = \gamma'(s_1)$, then we can define a GBB on (s_0, s_2) that restricts to γ on $(s_0, s_1]$ and γ' on $[s_1, s_2)$. This concise definition can be recast more concretely, as in work of Lebeau [Leb]:

Lemma 4.8. *If $I \subset \mathbb{R}$ and $\gamma : I \to \dot{\Sigma}$ is a continuous map, then the following are equivalent.*

(1) γ is a GBB *in the sense of Definition 4.7.*

(2) The following two conditions are satisfied for each $s_0 \in I$.

(a) If $q_0 = \gamma(s_0) \in \mathcal{G}$, then for each $f \in \mathcal{C}^\infty(T^*X)$ which is π-invariant,

$$\frac{d}{ds}(f_\pi \circ \gamma)(s_0) = (\mathsf{H}_p f)(\varpi_0), \tag{4.8}$$

where $\varpi_0 \in \Sigma$ is the unique point for which $\pi(\varpi_0) = q_0$.

(b) If $q_0 = \gamma(s_0) \in \mathcal{H}$, then there exists $\varepsilon > 0$ such that $0 < |s - s_0| < \varepsilon$ implies that $x(\gamma(s)) \neq 0$.

(3) For each $s_0 \in I$ there exist unique $\varpi_\pm \in \Sigma$ such that $\pi(\varpi_\pm) = \gamma(s_0)$ and for all π-invariant f,

$$\frac{d}{ds}(f_\pi \circ \gamma)_\pm(s_0) = (\mathsf{H}_p f)(\varpi_\pm). \tag{4.9}$$

Proof. (1) $\implies$ (2): Let $\gamma : I \to \dot{\Sigma}$ be a GBB and $s_0 \in I$. First assume that $q_0 = \gamma(s_0) \in \mathcal{G}$, in which case $\pi^{-1}(\{q_0\})$ consists of a single point ϖ_0. Applying (4.7) to f and $-f$ shows that (4.8) holds. If $q_0 \in \mathcal{H}$ instead, apply (4.7) to the π-invariant function

$$f = x\xi = \pi^*\sigma.$$

Then $\mathsf{H}_p f = 2\xi^2$ along $\pi^{-1}(\mathcal{H})$, so the infimum on the right hand side of (4.7) is positive for $q_0 \in \mathcal{H}$. On the other hand $f_\pi(\gamma(s_0)) = 0$, so $\sigma(\gamma(s)) \neq 0$ for small but nonzero values of $|s - s_0|$. Since γ takes values in $\dot{\Sigma}$, this implies that $x(\gamma(s)) \neq 0$ as well.

(2) $\implies$ (3): By definition this implication is clear for $\gamma(s_0) \in \mathcal{G}$, so we may assume that $\gamma(s_0) \in \mathcal{H}$, and that $s_0 = 0$. By hypothesis,

$$x(\gamma(s)) \neq 0 \text{ and } \gamma(s) \in \mathcal{G} \text{ for } s \in [-\varepsilon, \varepsilon] \setminus 0,$$

thus we can view $\gamma : (0, \varepsilon] \to \Sigma$. In particular, $\xi(\gamma(s)) = \pm(\tilde{p}(\gamma(s)))^{1/2}$ for one choice of sign, and since $\tilde{p}$ is π-invariant, the limit $\xi_+ = \lim_{s\to 0^+} \xi(\gamma(s))$ exists. We then set

$$\varpi_+ = (0, y(\gamma(0)), \xi_+, \eta(\gamma(0))).$$

Similarly, we can construct ϖ_-, and it is easy to check that (4.9) holds. The choices of $\xi_\pm$ are unique, since they are recovered by applying (4.9) to the π-invariant function x.

(3) $\implies$ (1): The condition (4.9) shows that the left hand side of (4.7) is equal to the minimum of $(\mathsf{H}_p f)(\varpi_\pm)$, which is clearly bigger than or equal to the infimum on the right hand side of (4.7). $\square$

Suppose that $q_0 \in \mathcal{H}$. In view of Lemma 4.8, we can construct a backward GBB on some $(-\varepsilon, 0]$ by solving (4.3) with a choice of initial data in $\pi^{-1}(\{q_0\})$ and then projecting to $\dot{\Sigma}$ by π; the same construction works in the forward direction. Conversely, any GBB through a point $q_0 \in \mathcal{H}$ is locally obtained by concatenating two solutions of (4.3) projected to $\dot{\Sigma}$.

If f is π-invariant, then $f_\pi \circ \gamma$ is Lipschitz on I. This follows from the fact that $f_\pi \circ \gamma$ has uniformly bounded one sided derivatives at each $s \in I$ by Lemma 4.8. Therefore

$$|(f_\pi \circ \gamma)(s_1) - (f_\pi \circ \gamma)(s_2)| \leq \sup\{|\mathsf{H}_p f(\varpi)| : \pi(\varpi) \in \gamma(I)\} \cdot |s_1 - s_2|,$$

so $f_\pi \circ \gamma$ is in fact Lipschitz on I with a constant independent of γ provided we assume that γ takes values in a fixed compact set K (cf. [Vas3, Corollary 5.3] and [Leb, Corollary 2]).

Furthermore, suppose that $\mathcal{U}$ is an adapted coordinate patch, so that ${}^{\mathrm{b}}T^*_{\mathcal{U}}X$ is equipped with the Euclidean distance induced by the coordinates (x, y, σ, η). If γ is any GBB with values in a fixed compact set $K \subset {}^{\mathrm{b}}T^*_{\mathcal{U}}X$, we conclude that

$$|\gamma(s_1) - \gamma(s_2)| \leq L|s_1 - s_2| \tag{4.10}$$

for a constant $L > 0$ depending only on K. Using these observations, one deduces some important topological information about the set of all GBBs. For a proof of the following proposition, the reader is referred to [Vas1, Proposition 5.4 and Corollary 5.6].

Proposition 4.9. *Given a compact set $K \subset \dot{\Sigma}$ and a compact interval $[a, b] \subset \mathbb{R}$, let*

$$\mathcal{R} = \{\gamma : [a, b] \to K : \gamma \text{ is a GBB}\}.$$

If $\mathcal{R} \neq \emptyset$, then $\mathcal{R}$ is compact with respect to the topology of uniform convergence. Furthermore, if $\gamma : (a, b) \to \dot{\Sigma}$ is a GBB*, then γ extends to a* GBB *on $[a, b]$.*

For a closely related result, see Lemma 5.18.

Lemma 4.10. *Let $U \subset \dot{\Sigma}$ be open and precompact, and $K \subset U$ be compact. There exists $\varepsilon_0 > 0$ such that if γ is any* GBB *defined on $[-\varepsilon, \varepsilon]$ with $\varepsilon \in (0, \varepsilon_0)$ and $\gamma(0) \in K$, then $\gamma([\varepsilon, \varepsilon]) \subset U$.*

Proof. It suffices to prove the result with $[0,\varepsilon]$ and $[-\varepsilon,0]$ replacing $[-\varepsilon,\varepsilon]$. We argue by contradiction. Fix $U' \supset K$ open with closure in U; we may thus assume that $d(U',\partial U) > c_0$ for some c_0. If the result does not hold, then we may choose a positive decreasing sequence $s_n \to 0$ and GBBs $\gamma_n : [0,s_n] \to \overline{U}$ such that $\gamma_n(s_n) \in \partial U$ and $\gamma_n(s_{n+1}) \in U'$. In particular,

$$d(\gamma_n(s_n),\gamma_n(s_{n+1})) > c_0$$

uniformly in n. Let q, q' denote subsequential limits of $\gamma_n(s_n), \gamma_n(s_{n+1})$, respectively; it follows that $q \neq q'$. On the other hand, if $f \in \mathcal{C}^\infty$ is π-invariant, then

$$|f_\pi(\gamma_n(s_n)) - f_\pi(\gamma_n(s_{n+1}))| \leq L(|s_n| + |s_{n+1}|)$$

where L is independent of n. Since functions of the form f_π separate points, this implies that $q = q'$, which is a contradiction. $\square$

We close this section with a brief description of the phenomenology allowed by the results above. Fix $\alpha > 1$, so that solutions to Hamilton's equations exist. A bicharacteristic curve arriving transversely at Y (hence at a point in $\mathcal{H}$) can be continued in just one way across the interface as a bicharacteristic curve. By contrast, the continuous trajectory in X obtained by flipping the sign of the normal momentum at the moment of impact is also the image of a GBB; these two curves are the only possible continuations of the incident bicharacteristic as a GBB, with the latter being "diffractive" in the heuristic terminology of the introduction. A bicharacteristic arriving tangent to Y, hence in $\mathcal{G}$, may, if $\alpha < 2$, stick to Y thereafter (and possibly re-release in a tangent direction at some later point). If $\alpha > 2$, uniqueness of bicharacteristics rules out this sticking at a point of simple tangency: the bicharacteristic brushes past Y and continues on its way. By contrast, the sticking behavior is always possible for a GBB.

5. Propagation of singularities along GBBs

Throughout this section we assume that $\alpha > 0$. We continue to write ${}^{\mathrm{b}}T^*X$ instead of ${}^{\mathrm{b}}T^*(X,Y)$, and also abbreviate $\Psi^m_{\mathrm{b},h} = \Psi^m_{\mathrm{b},h}(X,Y)$. To simplify various statements, assume that X is compact; as usual this is inessential.

5.1. **The elliptic region.** We will begin by studying the elliptic region. The main result here is the following:

Proposition 5.1. *If $A, G \in \Psi^0_{\mathrm{b},h}$ satisfy $\mathrm{WF}_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(G)$ and $\mathrm{WF}_{\mathrm{b},h}(A) \cap \dot{\Sigma} = \emptyset$, then*

$$\|Au\|_{H^1_h} \leq C\|GPu\|_{H^{-1}_h} + \mathcal{O}(h^\infty)\|u\|_{H^1_h}$$

for each $u \in H^1_h(X)$.

An immediate consequence of Proposition 5.1 is microlocal b-elliptic regularity, in the semiclassical sense.

Proposition 5.2. *If u is h-tempered in $H^1_h(X)$, then $\mathrm{WF}^{1,r}_{\mathrm{b},h}(u) \subset \mathrm{WF}^{-1,r}_{\mathrm{b},h}(Pu) \cup \dot{\Sigma}$ for each $r \in \mathbb{R} \cup \{+\infty\}$.*

Since this is just ordinary elliptic regularity away from Y, we will henceforth assume that all pseudodifferential operators have compact support in a normal coordinate chart $\mathcal{U}$. We begin by giving a simple microlocal estimate for the Dirichlet form associated with the operator P.

Lemma 5.3. *If $A, G \in \Psi^0_{\mathrm{b},h}$ satisfy $\mathrm{WF}_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(G)$, then*

$$\int_X h^2 |dAu|_g^2 + V|Au|^2 \, dg \leq C\varepsilon^{-1} \|GPu\|^2_{H_h^{-1}} + \varepsilon \|Au\|^2_{H^1_h} + Ch\|Gu\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}$$

for each $u \in H^1_h(X)$ and $\varepsilon > 0$.

Proof. By Green's formula, if $v \in H^1_h(X)$, then

$$\int_X h^2|dv|_g^2 + V|v|^2 \, dg = \langle Pv, v \rangle ,$$

where the right hand side is the pairing of $H^{-1}_h(X)$ with $H^1_h(X)$ induced by the volume density. Applying this to $v = Au \in H^1_h(X)$, it remains to estimate

$$\langle PAu, Au \rangle = \langle APu, Au \rangle + \langle [P, A]u, Au \rangle . \tag{5.1}$$

The first term on the right hand side of (5.1) is simply bounded by Cauchy–Schwarz,

$$| \langle APu, Au \rangle | \leq (1/4)\varepsilon^{-1}\|APu\|^2_{H_h^{-1}} + \varepsilon \|Au\|^2_{H^1_h}. \tag{5.2}$$

Since $\mathrm{WF}_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(G)$, we can use microlocal ellipticity to estimate $\|APu\|_{H_h^{-1}}$ by $\|GPu\|_{H_h^{-1}} + \mathcal{O}(h^\infty)\|Pu\|_{H_h^{-1}}$ on the right hand side of (5.2); we may of course further estimate $\|Pu\|_{H_h^{-1}} \leq C\|u\|_{H^1_h}$. As for the commutator, $\mathrm{WF}^2_{\mathrm{b},h}([P, A]) \subset \mathrm{WF}_{\mathrm{b},h}(A)$, and therefore by Lemma 3.9,

$$| \langle [P, A]u, Au \rangle | \leq Ch\|Gu\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}.$$

This completes the proof. ∎

Before proving Proposition 5.1 we record a corollary of Lemma 5.3 that will be important when studying the hyperbolic region. Since $V \in L^\infty(X)$, by choosing $\varepsilon > 0$ sufficiently small in Lemma 5.3 we can estimate

$$\|Au\|_{H^1_h} \leq C\|GPu\|_{H_h^{-1}} + Ch\|Gu\|_{H^1_h} + C_0\|Au\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{H^1_h}, \tag{5.3}$$

where crucially $C_0 > 0$ is independent of A. The remainder can also be improved, at the cost of losing control of C_0:

Lemma 5.4. *If $A, G \in \Psi^0_{\mathrm{b},h}$ satisfy $\mathrm{WF}_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(G)$, then*

$$\|Au\|_{H^1_h} \leq C\|GPu\|_{H^{-1}_h} + C\|Gu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{H^1_h}.$$

for each $u \in H^1_h(X)$.

Proof. The proof follows by inductively showing that for each $k \in \mathbb{N}$ and $u \in H^1_h(X)$, and for every A, G satisfying the hypotheses of the lemma,

$$\|Au\|_{H^1_h} \leq C\|GPu\|_{H^{-1}_h} + C\|Gu\|_{L^2} + Ch^k\|Gu\|_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}. \tag{5.4}$$

Now (5.4) holds for $k = 1$ by using (5.3) and then estimating

$$\|Au\|_{L^2} \leq C\|Gu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}. \tag{5.5}$$

In the inductive step, assume that (5.4) holds for $k = s$. Apply (5.4), replacing G with A' satisfying $\mathrm{WF}_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(A')$ and $\mathrm{WF}_h(A') \subset \mathrm{ell}_{\mathrm{b}}(G)$ to obtain

$$\|Au\|_{H^1_h} \leq C\|A'Pu\|_{H^{-1}_h} + C\|A'u\|_{L^2} + Ch^s\|A'u\|_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}. \tag{5.6}$$

Likewise, replacing A with A' in our inductive assumption gives

$$\|A'u\|_{H^1_h} \leq C\|GPu\|_{H^{-1}_h} + C\|Gu\|_{L^2} + Ch^s\|Gu\|_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}. \tag{5.7}$$

Then substitute (5.7) into the $\|A'u\|_{H^1_h}$ term on the right hand side of (5.6); the remaining A' terms on the right are estimated by the corresponding terms with G by elliptic regularity as in (5.5) (recall that the b-calculus is bounded on $H^{\pm 1}_h$ as well as L^2); this completes the inductive step. □

Note that the complement of $\dot{\Sigma}$ within $\overline{{}^{\mathrm{b}}T^*}X$ is the union of $\overline{{}^{\mathrm{b}}T^*X} \setminus \overline{{}^{\mathrm{b}}\dot{T}^*}X$ with $\mathcal{E}$. We begin by studying regularity on the former of these sets.

Lemma 5.5. *If $A \in \Psi^0_{\mathrm{b},h}$ has compact support in $\{|x| < \delta/\sqrt{2}\}$, where $\delta > 0$ satisfies*

$$|V| < \tfrac{1}{2}\delta^{-2}\sigma^2 \tag{5.8}$$

in a neighborhood of $\mathrm{WF}_h(A)$, and $G \in \Psi^0_{\mathrm{b},h}$ satisfies $\mathrm{ell}_{\mathrm{b}}(A) \subset \mathrm{WF}_{\mathrm{b},h}(G)$, then

$$\|Au\|_{H^1_h} \leq C\|GPu\|_{H^{-1}_h} + \mathcal{O}(h^\infty)\|u\|_{H^1_h}$$

for each $u \in H^1_h(X)$.

Proof. Since A is assumed to have compact support in $\{|x| < \delta/\sqrt{2}\}$,

$$\int_X \delta^{-2}|(xhD_x)Au|^2 + V|Au|^2\, dg \leq \int_X \tfrac{1}{2}h^2|dAu|^2 + V|Au|^2\, dg. \tag{5.9}$$

In view of (5.8), we can choose $B, F \in \Psi^1_{\mathrm{bc},h}(X, Y)$, where $\mathrm{WF}_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(B)$, such that

$$\mathrm{WF}_{\mathrm{b},h}((\delta^{-2}(hxD_x)^*(hxD_x) + V) - (B^*B + hF)) \cap \mathrm{WF}_{\mathrm{b},h}(A) = \emptyset.$$

Now integrate by parts in x to write the left hand side of (5.9) as

$$\int_X \delta^{-2}|hxD_x Au|^2 + V|Au|^2\, dg = \|BAu\|^2_{L^2} + h\,\langle FAu, Au\rangle + \mathcal{O}(h^\infty)\|u\|^2_{L^2}.$$

In particular, this implies that

$$\int_X \tfrac{1}{2}h^2|dAu|^2\, dg + \|BAu\|^2_{L^2} \leq \int_X h^2|dAu|^2 + V|Au|^2\, dg \\ + h\|FAu\|_{L^2}\|Au\|_{L^2} + \mathcal{O}(h^\infty)\|u\|^2_{L^2}.$$

Since B is elliptic on $\mathrm{WF}_{\mathrm{b},h}(A)$, the left hand side of the inequality above controls $\|Au\|^2_{H^1_h}$, whereas by Lemma 5.3 the right hand side is controlled by

$$C\varepsilon\|GPu\|^2_{H^{-1}_h} + (C\varepsilon^{-1} + Ch)\|Au\|^2_{H^1_h} + Ch\|Gu\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}.$$

Here we used Lemma 3.8 to bound the operator norm of $F \in \Psi^1_{\mathrm{bc},h}(X,Y)$. Thus for $\varepsilon > 0$ sufficiently small we can absorb the second term on the right hand side into the left hand side.

This establishes the result but with an extra term $Ch\|Gu\|^2_{H^1_h}$ on the right hand side. We now eliminate this term iteratively, just as in the proof of Lemma 5.4. □

Lemma 5.5 will also prove useful later in Section 7.4. The next step is to consider $A \in \Psi^0_{\mathrm{b},h}$ with wavefront set in a neighborhood of $q_0 \in \mathcal{E}$.

Lemma 5.6. *Let $q_0 \in \mathcal{E}$. There exists $A \in \Psi^0_{\mathrm{b},h}$ with $q_0 \in \mathrm{ell}_{\mathrm{b}}(A)$, such that if $G \in \Psi^0_{\mathrm{b},h}$ satisfies $\mathrm{WF}_{\mathrm{b},h}(A) \subset \mathrm{ell}_h(G)$, then*

$$\|Au\|_{H^1_h} \leq C\|GPu\|_{H^{-1}_h} + \mathcal{O}(h^\infty)\|u\|_{H^1_h}$$

for each $u \in H^1_h(X)$.

Proof. If $\mathrm{WF}_{\mathrm{b},h}(A)$ is a sufficiently small neighborhood of q_0, then there exists $c_0 > 0$ such that

$$(1 - c_0)k^{ij}\eta_i\eta_j + V > 0 \tag{5.10}$$

near $\mathrm{WF}_{\mathrm{b},h}(A)$. As in the proof of Lemma 5.5, we can choose $B, F \in \Psi^1_{\mathrm{bc},h}(X,Y)$, where $\mathrm{WF}_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(B)$, such that

$$\mathrm{WF}_{\mathrm{b},h}((1-c_0)k^{ij}(hD_{y^i})(hD_{y^j}) + V) - (B^*B + hF)) \cap \mathrm{WF}_{\mathrm{b},h}(A) = \emptyset.$$

Integrating by parts in y, it follows that

$$\int_X |(hD_x)Au|^2 + c_0k^{ij}(hD_{y^i}Au)(\overline{hD_{y^j}Au}) + |BAu|^2\, dg \\ \leq C\int_X h^2|dAu|^2 + V|Au|^2\, dg + h\,\langle FAu, Au\rangle + \mathcal{O}(h^\infty)\|u\|^2_{L^2}.$$

which completes the proof as above, since the left hand side controls a multiple of $\|Au\|^2_{H^1_h}$. ☐

Proposition 5.1 follows by combining Lemmas 5.5, 5.6 with a microlocal partition of unity argument.

5.2. **The hyperbolic region.** Since $\mathcal{H}$ is a compact subset of ${}^{\mathrm{b}}T^*X$, it suffices to work with pseudodifferential operators that are both compactly supported in a normal coordinate patch $\mathcal{U}$ and compactly microlocalized. Let $q_0 \in \mathcal{H}$. If (x, y, σ, η) are local coordinates near q_0, then

$$q_0 = (0, y_0, 0, \eta_0),$$

where $\tilde{p}(q_0) < 0$.

Proposition 5.7. *Suppose that u is h-tempered in $H^1_h(X)$ and $q_0 \notin \mathrm{WF}^{-1,r+1}_{\mathrm{b},h}(Pu)$, where $r \in \mathbb{R} \cup \{+\infty\}$. If q_0 has a neighborhood $U \subset \dot{\Sigma}$ such that*

$$U \cap \mathrm{WF}^{1,r}_{\mathrm{b},h}(u) \cap \{\sigma < 0\} = \emptyset,$$

then $q_0 \notin \mathrm{WF}^{1,r}_{\mathrm{b},h}(u)$.

Combined with b-elliptic regularity, this proposition implies that if

$$q_0 \in \mathrm{WF}^{1,r}_{\mathrm{b},h}(u) \backslash \mathrm{WF}^{-1,r+1}_{\mathrm{b},h}(Pu),$$

then q_0 is a limit point of $\mathrm{WF}^{1,r}_{\mathrm{b},h}(u) \cap T^*(X \setminus Y)$. This in turns suffices to prove propagation of singularities; see Section 5.4. The proposition is a restatement of the following quantitative result.

Proposition 5.8. *If $G \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ is elliptic at q_0, then there exist $Q, Q_1 \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$, where*

$$\mathrm{WF}_{\mathrm{b},h}(Q) \subset \mathrm{ell}_{\mathrm{b}}(G) \text{ and } q_0 \in \mathrm{ell}_{\mathrm{b}}(Q),$$
$$\mathrm{WF}_{\mathrm{b},h}(Q_1) \subset \mathrm{ell}_{\mathrm{b}}(G) \cap \{\sigma < 0\},$$

such that

$$\|Qu\|_{H^1_h} \leq Ch^{-1}\|GPu\|_{H^{-1}_h} + C\|Q_1 u\|_{H^1_h} + \mathcal{O}(h^\infty)\|u\|_{H^1_h},$$

for each $u \in H^1_h(X)$.

Proposition 5.8 holds verbatim if we replace σ with $-\sigma$ (corresponding to propagation in the backwards direction). We prove Proposition 5.8 by a positive commutator argument, closely following [Vas3, Section 6]. Define the functions

$$\omega = |x|^2 + |y - y_0|^2 + |\eta - \eta_0|^2, \quad \phi = \sigma + \frac{1}{\beta^2\delta}\omega.$$

Here the parameters $\delta, \beta \in (0, \infty)$ will be chosen later; δ will be chosen small, while in this argument β will ultimately be taken to be large.

Observe that $|W\phi| \leq C(1+\beta^{-2}\delta^{-1})\omega^{1/2}$, where $W \in \{\partial_x, x\partial_\sigma, \partial_{y_i}, \partial_{\eta_i}\}$. In particular, if $f \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*X)$, and U is a neighborhood of q_0 with compact closure in ${}^{\mathrm{b}}T^*X$, then using Lemma 2.1 we find

$$|\partial_x\phi| + |H_f\phi| \leq C_0(1+\beta^{-2}\delta^{-1})\omega^{1/2} \tag{5.11}$$

on U, where $C_0 > 0$ does not depend on β, δ. Choose cutoff functions χ_0, χ_1 with the following properties:

- χ_0 is supported in $[0,\infty)$, with $\chi_0(s) = \exp(-1/s)$ for $s > 0$.
- χ_1 is supported in $[0,\infty)$, with $\chi_1(s) = 1$ for $s \geq 1$, and $\chi_1' \geq 0$.

Now set

$$a = \chi_0(2-\phi/\delta)\chi_1(2+\sigma/\delta). \tag{5.12}$$

For each fixed $\beta > 0$, the support of a is controlled by the parameter $\delta > 0$ as follows.

Lemma 5.9. *Given a neighborhood $U \subset {}^{\mathrm{b}}T^*X$ of $q_0 \in \mathcal{H}$ and $\beta > 0$, there exists $\delta_0 > 0$ such that* $\operatorname{supp} a \subset U$ *for each $\delta \in (0,\delta_0)$.*

Proof. Necessary conditions to lie in the support of a are $\phi \leq 2\delta$ and $-2\delta \leq \sigma$. From the definition of ϕ,

$$|\sigma| \leq 2\delta, \quad 0 \leq \omega \leq \beta^2\delta(2\delta-\sigma) \leq 4\beta^2\delta^2$$

on $\operatorname{supp} a$, i.e.,

$$\operatorname{supp} a \subset \{|\sigma| \leq 2\delta,\ \omega^{1/2} \leq 2\beta\delta\}. \tag{5.13}$$

Finally, observe that any neighborhood of U of q_0 contains a set of the form $\{|\sigma| \leq 2\delta,\ \omega^{1/2} \leq 2\beta\delta\}$ provided δ is sufficiently small. ☐

If $A \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ has principal symbol a, the goal is to obtain negativity of the commutator $(i/h)[P, A^*A]$. This cannot be done symbolically within the b-calculus, since P is merely an element of Diff^2_h (for more motivational material, see [Vas3, Section 6]). Using the expression for P and the notation of Lemma 3.7 and (4.1),

$$\begin{aligned}(i/h)[P, A^*A] &= B_0(hD_x)^*(hD_x) + B_1(hD_x) + (i/h)[\tilde{P}, A^*A] + h\,\mathrm{Diff}^2_h\,\Psi^{\mathrm{comp}}_{\mathrm{b},h}\\ &= B_0P - B_0\tilde{P} + B_1(hD_x) + (i/h)[\tilde{P}, A^*A] + h\,\mathrm{Diff}^2_h\,\Psi^{\mathrm{comp}}_{\mathrm{b},h},\end{aligned} \tag{5.14}$$

where $\sigma_{\mathrm{b},h}(B_0) = 2\partial_\sigma(a^2)$ and $\sigma_{\mathrm{b},h}(B_1) = 2\partial_x(a^2)$. The last term is a b-pseudodifferential operator (with conormal coefficients) with principal symbol $\mathsf{H}^{\mathrm{b}}_{\tilde{p}}a^2$.

The symbols of the operators in (5.14) can be further decomposed, depending on whether the various derivatives fall onto χ_0 or χ_1 when a^2 is differentiated. Those terms differentiating χ_1 give rise to an term error supported on $\{-2\delta \leq \sigma \leq -\delta,\ \omega^{1/2} \leq 2\beta\delta\}$, whereas derivatives of χ_0 will yield positivity. To this end, define

$$b = 2\delta^{-1/2}(\chi_0'\chi_0)^{1/2}\chi_1, \quad B = \mathrm{Op}_h(b). \tag{5.15}$$

Here we have suppressed the arguments of χ_0, χ_1 as in (5.12).

Next, fix a neighborhood U_0 of q_0 with compact closure in ${}^{\mathrm{b}}T^*X$ such that $\tilde{p} < 0$ near U_0. Thus we can choose $\tilde{B} \in \Psi^{\mathrm{comp}}_{\mathrm{bc},h}$ such that

$$U_0 \subset \mathrm{ell}_{\mathrm{b}}(\tilde{B}), \quad \mathrm{WF}_{\mathrm{b},h}(\tilde{B}^*\tilde{B} + \tilde{P}) \cap U_0 = \emptyset.$$

The operators A, B depend on δ, β, whereas $\tilde{B}$ does not. Finally, fix $\alpha_0 \in (0, \alpha)$ and let $\theta = \min(1, \alpha_0) \in (0, 1]$. According to Lemma 2.3 and (2.7),

$$|x\partial_x V| \leq C|x|^\theta$$

on $\mathcal{U}$. We then have the following decomposition of $[P, A^*A]$:

Lemma 5.10. *Given $\beta > 0$, there exists $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$,*

$$(i/h)[P, A^*A] = B_0 P - B^*(\tilde{B}^*\tilde{B} + R_0 + (hD_x)^* R_1)B + E + hR, \tag{5.16}$$

where A, B, B_0, C are as above, and remaining operators in (5.16) have the following properties:

- *$R_0 \in \Psi^{\mathrm{comp}}_{\mathrm{bc},h}$ and $R_1 \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ satisfy*

$$|\sigma_{\mathrm{b},h}(R_i)| \leq C_1((\delta\beta)^\theta + \beta^{-1}),$$

  where $C_1 > 0$ does not depend on β, δ.
- *$E, R \in \mathrm{Diff}^2_h \Psi^{\mathrm{comp}}_{\mathrm{b},h} + \Psi^{\mathrm{comp}}_{\mathrm{bc},h}$, and $\mathrm{WF}^2_{\mathrm{b},h}(E) \subset \{-2\delta \leq \sigma \leq -\delta,\ \omega^{1/2} \leq 2\beta\delta\}$.*

The b-wavefront sets of R_0, R_1, R are contained in $\{|\sigma| \leq 2\delta,\ \omega^{1/2} \leq 2\beta\delta\}$.

Proof. Throughout the proof, we will use the notation E, R to denote any operators satisfying the hypotheses of the lemma; these may change from line to line. Fix a cutoff $\psi \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*X; [0, 1])$ such that $\psi = 1$ near $\{|\sigma| \leq 2\delta,\ \omega^{1/2} \leq 2\beta\delta\}$ with support in $\{|\sigma| < 3\delta,\ \omega^{1/2} < 3\beta\delta\}$.

(1) As in Lemma 5.9, given $\beta > 0$ we can choose $\delta_0 > 0$ so that $\mathrm{WF}_{\mathrm{b},h}(B) \subset U_0$ for $\delta \in (0, \delta_0)$; without loss we can assume that $\delta\beta \leq 1$. On the other hand,

$$\begin{aligned}\sigma_{\mathrm{b},h}(B_0) &= -4\delta^{-1}(\chi_0'\chi_0)\chi_1^2 + 4\delta^{-1}\chi_0^2(\chi_1'\chi_1) \\ &= -b^2 + e.\end{aligned} \tag{5.17}$$

Since e is supported in $\{-2\delta \leq \sigma \leq -\delta,\ \omega^{1/2} \leq 2\beta\delta\}$, if we denote its quantization by E, then

$$-B_0\tilde{P} = -B^*\tilde{B}^*\tilde{B}B + E + hR.$$

Here the error R arises since we have arranged equality at the level of principal symbols.

(2) Next, consider the term $B_1(hD_x)$. Since $\sigma_{\mathrm{b},h}(B_1) = -(\partial_x\phi)\psi b^2$, we can write

$$B_1(hD_x) = B^* R_1 (hD_x) B + hR,$$

where according to (5.11) we can bound $|\sigma_{\mathrm{b},h}(R_1)| \leq C_0(1+\beta^{-2}\delta^{-1})\omega^{1/2}$ on U_0 for some $C_0 > 0$ independent of β, δ (recall that U_0 is chosen in the paragraph preceding the lemma). Moreover, $\omega^{1/2} \leq 3\beta\delta$ on its support, so

$$|\sigma_{\mathrm{b},h}(R_1)| \leq 3C_0(\delta\beta + \beta^{-1})$$

as desired.

(3) We split up $(i/h)[\tilde{P}, A^*A] = (i/h)[h^2\Delta_k, A^*A] + (i/h)[V, A^*A]$. Temporarily writing $f = k^{ij}\eta_i\eta_j$, the first term has principal symbol

$$\mathsf{H}^{\mathrm{b}}_f a^2 = -(\mathsf{H}^{\mathrm{b}}_f\phi)\psi b^2 + e,$$

where $\operatorname{supp} e \subset \{-2\delta < \sigma < -\delta,\ \omega^{1/2} \leq 2\beta\delta\}$. As above, we can write

$$(i/h)[h^2\Delta_k, A^*A] = B^*R_0'B + E + hR,$$

where according to (5.11) we can bound $|\sigma_{\mathrm{b},h}(R_0')| \leq 3C_0(\delta\beta + \beta^{-1})$.

(4) Finally, consider $(i/h)[V, A^*A]$ with principal symbol

$$\mathsf{H}^{\mathrm{b}}_V a^2 = -(\mathsf{H}^{\mathrm{b}}_V\phi)\psi b^2 + e.$$

Now $\mathsf{H}^{\mathrm{b}}_V = (x\partial_x V)\partial_\sigma + (\partial_{y_i}V)\partial_{\eta_i}$ (with Einstein summation), so when bounding $|\mathsf{H}^{\mathrm{b}}_V\phi|$ we certainly have

$$|(\partial_{y_i}V)\partial_{\eta_i}\phi| \leq C_0(1+\beta^{-2}\delta^{-1})\omega^{1/2}$$

by (5.11).

This does not hold when ϕ is differentiated in σ. Instead, we bound $|x\partial_x V| \leq C_0'|x|^\theta \leq C_0'\omega^{\theta/2}$. Thus we can write

$$(i/h)[V, A^*A] = B^*R_0''B + E + hR,$$

where $|\sigma_{\mathrm{b},h}(R_0'')| \leq 3C_0((\beta\delta)+\beta^{-1}) + 3^\theta C_0'(\beta\delta)^\theta$ by the support properties of ψ. Letting $R_0 = R_0' + R_0''$ completes the proof of the lemma. □

Given $u \in H^1(X)$, apply Lemma 5.10 to write

$$\begin{aligned}-(2/h)\operatorname{Im}\langle APu, Au\rangle &= (i/h)\langle [A^*A, P]u, u\rangle \\ &= \|\tilde{B}Bu\|^2_{L^2} + \langle R_0Bu, Bu\rangle + \langle R_1Bu, (hD_x)Bu\rangle \\ &\quad - \langle Eu, u\rangle + h\langle Ru, u\rangle - \langle B_0Pu, u\rangle,\end{aligned}$$

noting that $A, B, \tilde{B}$ preserve $H^1_h(X)$ and B_0 preserves $H^{-1}_h(X)$ (these operators all have smooth coefficients).

First, we use the ellipticity of $\tilde{B}$ on $\mathrm{WF}_{\mathrm{b},h}(B)$ and (5.3) to estimate

$$c_0\|Bu\|^2_{H^1_h} \leq \|\tilde{B}Bu\|^2_{L^2} + C\|GPu\|^2_{H^{-1}_h} + Ch\|Gu\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}, \tag{5.18}$$

where $c_0 > 0$ independent of β, δ so long as $\delta \in (0, \delta_0)$, and where G is elliptic on $\mathrm{WF}_{\mathrm{b},h}(B)$. We fix β once and for all using the following lemma:

Lemma 5.11. *Given $\varepsilon > 0$, there exists $\beta > 0$ and $\delta_1 \in (0, \delta_0)$ such that*

$$|\,\langle R_0 Bu, Bu\rangle\,| + |\,\langle R_1 Bu, (hD_x)Bu\rangle\,| \leq \varepsilon \|Bu\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}.$$

for each $\delta \in (0, \delta_1)$ and $u \in H^1_h(X)$.

Proof. We bound

$$\begin{aligned} \|R_i v\|_{L^2} &\leq 2 \sup |\sigma_{\mathrm{b},h}(R_i)| \|v\|_{L^2} + \mathcal{O}(h^\infty)\|v\|_{L^2} \\ &\leq 2C_1((\delta\beta)^\theta + \beta^{-1})\|v\|_{L^2} + \mathcal{O}(h^\infty)\|v\|_{L^2}, \end{aligned}$$

where $C_1 > 0$ does not depend on β, δ. It suffices to first fix $\beta > 0$ sufficiently large, and then take $\delta_1 \in (0, \delta_0)$ sufficiently small. Applying this to $v = Bu$, along with Cauchy–Schwarz, finishes the proof. ☐

Now suppose that $G \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ is elliptic on $\mathrm{WF}_{\mathrm{b},h}(B)$, and $Q_1 \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ is elliptic on $\mathrm{WF}_{\mathrm{b},h}(E)$ with $\mathrm{WF}_{\mathrm{b},h}(Q_1) \subset \mathrm{ell}_{\mathrm{b}}(G) \cap \{\sigma < 0\}$ as in the statement of Proposition 5.8. Apply Lemma 5.11 by taking $\varepsilon = c_0/2$ (with c_0 defined by (5.18)). Combined with (5.18),

$$\begin{aligned} (c_0/2)\|Bu\|^2_{H^1_h} \leq (2/h)|\langle APu, Au\rangle| + C\|GPu\|^2_{H^{-1}_h} + Ch\|Gu\|^2_{H^1_h} \\ + |\langle Eu, u\rangle| + h|\langle Ru, u\rangle| + |\langle B_0 Pu, u\rangle| + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h} \end{aligned}$$

for $\delta \in (0, \delta_1)$. Using Cauchy–Schwarz on the B_0 term and estimating the E term by Q_1 using microlocal elliptic regularity bounds the second line by

$$\begin{aligned} &|\langle Eu, u\rangle| + h|\langle Ru, u\rangle| + |\langle B_0 Pu, u\rangle| \\ &\qquad \leq Ch^{-1}\|GPu\|^2_{H^{-1}_h} + Ch\|Gu\|^2_{H^1_h} + C\|Q_1 u\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}. \end{aligned}$$

Since $\mathrm{WF}_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(G)$ as well, we can also estimate

$$(2/h)|\langle APu, Au\rangle| \leq C\varepsilon^{-1}h^{-2}\|GPu\|^2_{H^{-1}_h} + C\varepsilon\|Au\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}.$$

Hence overall we obtain

$$\begin{aligned} (c_0/2)\|Bu\|^2_{H^1_h} \leq C\varepsilon^{-1}h^{-2}\|GPu\|^2_{H^{-1}_h} + Ch\|Gu\|^2_{H^1_h} + C\|Q_1 u\|^2_{H^1_h} \\ + C\varepsilon\|Au\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}. \end{aligned}$$

By construction $\chi_0(s) = s^2\chi_0'(s)$ for $s > 0$, so

$$a = (2 - \phi/\delta)(\chi_0'\chi_0)^{1/2}\chi_1 = \frac{1}{2}\delta^{1/2}(2 - \phi/\delta)b.$$

Thus we can write $A = FB + hF'$ for some $F, F' \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$. Choosing $\varepsilon > 0$ sufficiently small gives the estimate

$$\|Bu\|_{H^1_h} \leq Ch^{-1}\|GPu\|_{H^{-1}_h} + C\|Q_1 u\|_{H^1_h} + Ch^{1/2}\|Gu\|_{H^1_h} + \mathcal{O}(h^\infty)\|u\|_{H^1_h}.$$

We now finish the proof of Proposition 5.8.

Proof of Proposition 5.8. Let G be as in the statement of the proposition. Since $\mathrm{ell}_{\mathrm{b}}(G)$ is open, choose $\delta_\star \in (0, \delta_1)$ such that

$$\{|\sigma| \leq 2\delta_\star,\, \omega^{1/2} \leq 2\beta\delta_\star\} \subset \mathrm{ell}_{\mathrm{b}}(G).$$

Recall that δ_1, β are fixed in Lemma 5.11. Then, choose $Q_1 \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ such that

$$\{-2\delta_\star \leq \sigma \leq -\delta_\star,\, \omega^{1/2} \leq 2\beta\delta_\star\} \subset \mathrm{ell}_{\mathrm{b}}(Q_1), \quad \mathrm{WF}_{\mathrm{b},h}(Q_1) \subset \mathrm{WF}_{\mathrm{b},h}(G).$$

Take a sequence of operators $B_k \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ corresponding to decreasing sequence of δ_k in $(\delta_\star/2, \delta_\star)$. Then B_k is elliptic on $\mathrm{WF}_{\mathrm{b},h}(B_{k+1})$, so

$$\|B_{k+1}u\|_{H^1_h} \leq Ch^{-1}\|B_kPu\|_{H^{-1}_h} + C\|Q_1u\|_{H^1_h} + Ch^{1/2}\|B_ku\|_{H^1_h} + \mathcal{O}(h^\infty)\|u\|_{H^1_h}$$

for each k. Fix $Q \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$, elliptic at q_0, such that each B_k is elliptic on $\mathrm{WF}_{\mathrm{b},h}(Q)$. By induction, we conclude that

$$\|Qu\|_{H^1_h} \leq Ch^{-1}\|GPu\|_{H^{-1}_h} + C\|Q_1u\|_{H^1_h} + Ch^{k/2}\|Gu\|_{H^1_h} + \mathcal{O}(h^\infty)\|u\|_{H^1_h}$$

for each $k \in \mathbb{N}$, which completes the proof. □

5.3. **The glancing region.** As before, we assume that all b-pseudodifferential operators are supported in a fixed normal coordinate patch $\mathcal{U}$, and are compactly microlocalized. Before proceeding to the commutator argument, we need a variant of Lemma 5.4.

Lemma 5.12. *Given $\delta > 0$, let $U_\delta = \{q \in {}^{\mathrm{b}}T^*_{\mathcal{U}}X : |\tilde{p}| < \delta\}$. If $A, G \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ satisfy $\mathrm{WF}_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(G) \cap U_\delta$, then*

$$\int_X |hD_xAu|^2\, dg \leq Ch^{-1}\|GPu\|^2_{H^{-1}_h} + Ch\|Gu\|^2_{H^1_h} + 2\delta\|Au\|^2_{L^2} + \mathcal{O}(h^\infty)\|u\|_{H^1_h}$$

for each $u \in H^1_h(X)$.

Proof. Write $|hdv|^2_g = |hD_xv|^2 + k^{ij}(hD_{y_i}v)(\overline{hD_{y_j}v})$; now let $v = Au$ and apply Lemma 5.3 with $\varepsilon = h$ to see that

$$\int_X |hD_xAu|^2\, dg \leq -\int_X (\tilde{P}Au)\overline{Au}\, dg + Ch^{-1}\|GPu\|^2_{H^{-1}_h} + Ch\|Gu\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}$$

after integrating by parts in y. Choose $F \in \Psi^{\mathrm{comp}}_{\mathrm{bc},h}$ such that

$$\mathrm{WF}_{\mathrm{b},h}(F + \tilde{P}) \cap \mathrm{WF}_{\mathrm{b},h}(A) = \emptyset, \quad \mathrm{WF}_{\mathrm{b},h}(F) \subset U_\delta$$

One can always choose F such that with $f = \sigma_{\mathrm{b},h}(F)$,

$$\sup|f| \leq \delta.$$

Therefore we can bound

$$\langle Fv, v\rangle \leq 2\sup|f|\|v\|^2_{L^2} + \mathcal{O}(h^\infty)\|v\|^2_{L^2} \leq 2\delta\|v\|_{L^2} + \mathcal{O}(h^\infty)\|v\|_{L^2}.$$

Applying this to $v = Au$ and using that $\mathrm{WF}_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(G)$, we find that

$$\int_X |hD_x Au|^2 \, dg \leq Ch^{-1}\|GPu\|_{L^2}^2 + 2\delta\|Au\|_{L^2}^2 + Ch\|Gu\|_{H_h^1}^2 + \mathcal{O}(h^\infty)\|u\|_{H_h^1}$$

for each $u \in H_h^1(X)$. □

Define $\tilde{p}_0 \in \mathcal{C}^\infty(T^*X)$ over the normal coordinate patch $\mathcal{U}$ by

$$\tilde{p}_0(x, y, \xi, \eta) = k^{ij}(0, y)\eta_i\eta_j + V(0, y).$$

Given $q_0 \in \mathcal{G} \cap T^*Y$, let ϖ_0 denote the unique point in Σ such that $\pi(\varpi_0) = q_0$. Recall that a GBB passing through q_0 at $s = s_0$ is characterized by the equality

$$\frac{d}{ds}(f_\pi \circ \gamma)(s_0) = (\mathsf{H}_p f)(\varpi_0)$$

for each $f \in \mathcal{C}^\infty(T^*X)$ which is π-invariant. On the other hand, since $\xi(\varpi_0) = 0$, it follows that

$$(\mathsf{H}_p f)(\varpi_0) = (\mathsf{H}_{\tilde{p}_0} f)(\varpi_0). \tag{5.19}$$

Via the local coordinates (x, y, σ, η), we can also view $\tilde{p}_0$ as a function on ${}^{\mathrm{b}}T^*_{\mathcal{U}}X$. With this identification, $\tilde{p}_0$ can be considered as a function on ${}^{\mathrm{b}}T^*X$, and the flow $\exp(s\mathsf{H}^{\mathrm{b}}_{\tilde{p}_0})$ on ${}^{\mathrm{b}}T^*X$ makes sense.

As in Section 5.2, choose $\alpha_0 \in (0, \alpha)$ and let $\theta = \min(1, \alpha_0) \in (0, 1]$. Denote by $|\cdot|$ the Euclidean distance on ${}^{\mathrm{b}}T^*_{\mathcal{U}}X$ in local coordinates, and write $\mathsf{B}(q_0, \varepsilon)$ for the corresponding ball of radius $\varepsilon > 0$.

Proposition 5.13. *Suppose that u is h-tempered in $H_h^1(X)$ and $q_0 \notin \mathrm{WF}_{\mathrm{b},h}^{-1,r+1}(Pu)$, where $r \in \mathbb{R} \cup \{+\infty\}$. Let $K \subset \mathcal{G} \cap T^*_{\mathcal{U}\cap Y}Y$ be compact. There exists $C_0, \delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$ and $q_0 \in K$, if*

$$\mathsf{B}(\exp(-\delta\mathsf{H}^{\mathrm{b}}_{\tilde{p}_0})(q_0), C_0\delta^{2/(2-\theta)}) \cap \mathrm{WF}_h^{1,r}(u) = \emptyset,$$

then $q_0 \notin \mathrm{WF}_{\mathrm{b},h}^{1,r}(u)$.

Following [Vas3, Section 7], define the set

$$\mathsf{D}(q_0, \varepsilon) = \{q \in {}^{\mathrm{b}}T^*X : |x(q) - x(q_0)| + |y(q) - y(q_0)| + |\eta(q) - \eta(q_0)| \leq \varepsilon\}.$$

In order to prove Proposition 5.13, it suffices to replace B with D, possibly modifying C_0. Indeed, $\mathrm{WF}_{\mathrm{b},h}^{1,r}(u) \subset \dot{\Sigma}$, and on the compressed characteristic set $|\sigma| \leq C_1|x|$, where $C_1 > 0$ is uniform over compact subsets of X. Proposition 5.13 is then just a restatement of the following result:

Proposition 5.14. *Let $K \subset \mathcal{G} \cap T^*_{\mathcal{U}\cap Y}Y$ be compact. There exist $C_0, \delta_0 > 0$ such that the following property holds for each $\delta \in (0, \delta_0)$ and $q_0 \in K$. If $G \in \Psi_{\mathrm{b},h}^{\mathrm{comp}}$ is elliptic* *at q_0, then there exist $Q, Q_1 \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$, where*

$$\mathrm{WF}_{\mathrm{b},h}(Q) \subset \mathrm{ell}_{\mathrm{b}}(G) \text{ and } q_0 \in \mathrm{ell}_{\mathrm{b}}(Q),$$
$$\mathrm{WF}_{\mathrm{b},h}(Q_1) \subset \mathrm{ell}_{\mathrm{b}}(G) \cap \mathsf{D}(\exp(-\delta \mathsf{H}^{\mathrm{b}}_{\tilde{p}_0})(q_0), C_0\delta^{2/(2-\theta)})$$

such that

$$\|Qu\|_{H^1_h} \leq Ch^{-1}\|GPu\|_{H^{-1}_h} + C\|Q_1u\|_{H^1_h} + \mathcal{O}(h^\infty)\|u\|_{H^1_h}$$

for each $u \in H^1_h(X)$.

Just as with the hyperbolic estimate, Proposition 5.8, we can also reverse the direction of propagation here. Thus the same result holds verbatim if we replace $\exp(-\delta\mathsf{H}^{\mathrm{b}}_{\tilde{p}_0})(q_0)$ with $\exp(\delta\mathsf{H}^{\mathrm{b}}_{\tilde{p}_0})(q_0)$.

The rest of this section will be a proof of Proposition 5.14. View $\tilde{p}_0$ as a function on T^*Y, and thus $\mathsf{H}_{\tilde{p}_0}$ as a vector field on T^*Y. We may assume that $d\tilde{p}_0(q_0) \neq 0$ here viewed as a covector on T^*Y, as otherwise the result to be proved is vacuous. Then there are $2n-2$ functions $(\rho_0, \rho_1, \dots, \rho_{2n-3})$ on T^*Y, whose differentials are linearly independent at $\tilde{q}_0$, such that

$$(\mathsf{H}_{\tilde{p}_0}\rho_0)(q_0) > 0, \quad (\mathsf{H}_{\tilde{p}_0}\rho_j)(q_0) = 0, \quad \rho_1 = \tilde{p}_0.$$

We also arrange that these functions all vanish at q_0. Since it slightly simplifies matters, we can in fact arrange that $H_{\tilde{p}_0} = \partial_{\rho_0}$ near q_0 and thus

$$\mathsf{H}_{\tilde{p}_0}\rho_0 = 1, \quad \mathsf{H}_{\tilde{p}_0}\rho_j = 0 \text{ for } j = 1, \dots, 2n-3$$

identically. We extend $(\rho_0, \dots, \rho_{2n-3})$ to functions on ${}^{\mathrm{b}}T^*X$ by requiring them to independent of (x, σ), so that $(x, \sigma, \rho_0, \dots, \rho_{2n-3})$ are valid local coordinates on ${}^{\mathrm{b}}T^*X$ near q_0. Now define

$$\omega_0 = \sum_{j=1}^{2n-3} \rho_j^2, \quad \omega = \omega_0 + x^2.$$

In order to construct a commutant, let χ_0, χ_1 be as in Section 5.2. Define

$$\phi = \rho_0 + \frac{1}{\beta^2\delta}\omega.$$

We then set $A = \mathrm{Op}_h(a)$, where

$$a = \chi_0(2 - \phi/\delta)\chi_1(1 + (\rho_0 + \delta)/(\beta\delta)).$$

The difference compared with Section 5.2 is in the argument of χ_1. Indeed, there will be an error term (the analogue of E in Lemma 5.10) with wavefront set contained in

$$\{-\delta\beta - \delta \leq \rho_0 \leq -\delta,\ \omega^{1/2} \leq 2\beta\delta\}.$$

If $C > 0$ is sufficiently large, then this is certainly contained in the set

$$\mathsf{D}(\exp(-\delta\mathsf{H}^{\mathrm{b}}_{\tilde{p}_0})(q_0), C\beta\delta)$$

and thus lies inside a set of the form $\mathsf{D}(\exp(-\delta H_{\tilde{p}_0})(q_0), C_0\delta^{2/(2-\theta)})$ if we choose $\beta = c\delta^{\theta/(2-\theta)}$ (Note that this time, $\beta \in (0,\infty)$ will be taken to be small, rather than large as in the hyperbolic propagation argument.)

We also need to consider the difference between $\mathsf{H}^{\mathrm{b}}_{\tilde{p}}$ and $\mathsf{H}^{\mathrm{b}}_{\tilde{p}_0}$ (now vector fields on ${}^{\mathrm{b}}T^*X$). Here,

$$|\mathsf{H}^{\mathrm{b}}_{\tilde{p}}\phi - \mathsf{H}^{\mathrm{b}}_{\tilde{p}_0}\phi| \leq M\left(1 + \beta^{-2}\delta^{-1}\omega^{1/2}\right)\omega^{\theta/2} \tag{5.20}$$

locally, where $M > 0$ does not depend on β, δ.

Remark 5.15. The construction of ω above is meant to localize along GBBs through q_0. Using the same local coordinates $(\rho_0, \ldots, \rho_{2n-3})$, we could also localize at nearby points $q \in \mathcal{G} \cap T^*Y$ by setting

$$\omega_0 = \sum_{j=1}^{2n-3} |\rho_j - \rho_j(q)|^2,$$

if q is sufficiently close to q_0. If ω and ϕ are defined in the obvious way, then the constant $M > 0$ in (5.20) can then be taken uniform for q near q_0. As will be clear from the proof below, this implies uniformity of the constants C_0, δ_0 in Proposition 5.13 in a neighborhood of q_0. Thus by compactness, we can simply assume that $K = \{q_0\}$.

Now let $b = 2\delta^{-1/2}(\chi_0'\chi_0)^{1/2}\chi_1$ and $B = \mathrm{Op}_h(B)$ as before, and write

$$(i/h)[P, A^*A] = B_0(hD_x)^*(hD_x) + B_1(hD_x) + (i/h)[\tilde{P}, A^*A] + h\,\mathrm{Diff}^2_h\,\Psi^{\mathrm{comp}}_{\mathrm{b},h},$$

where $\sigma_{\mathrm{b},h}(B_0) = 2\partial_\sigma(a^2)$ and $\sigma_{\mathrm{b},h}(B_1) = 2\partial_x(a^2)$. We then have the following analogue of Lemma 5.10.

Lemma 5.16. *There exists $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$ and $\beta \in (0,1)$,*

$$(i/h)[P, A^*A] = B^*(hD_xR_1 + R_0 - 1)B + E + hR, \tag{5.21}$$

where A, B are as above, and remaining operators in (5.16) have the following properties:

- *$R_0 \in \Psi_{\mathrm{bc},h}$ and $R_1 \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ satisfy*

$$|\sigma_{\mathrm{b},h}(R_0)| \leq C_1\delta^\theta\beta^{\theta-1}, \quad |\sigma_{\mathrm{b},h}(R_1)| \leq C_1\beta^{-1}$$

  where $C_1 > 0$ does not depend on β, δ.

- *$E, R \in \mathrm{Diff}^2_h\,\Psi_{\mathrm{b},h} + \Psi_{\mathrm{bc},h}$, and*

$$\mathrm{WF}^2_{\mathrm{b},h}(E) \subset \{-\delta - \delta\beta \leq \rho_0 \leq -\delta,\, \omega \leq 2\beta\delta\}$$

The wavefront sets of R_0, R_1, R are contained in $\{|\rho_0| \leq 2\delta,\, \omega \leq 2\beta\delta\}$

Proof. As in the proof of Lemma 5.10, we use the notation E, R to denote any operators satisfying the hypotheses of the lemma; these may change from line to line. Fix a cutoff $\psi \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*X; [0,1])$ such that $\psi = 1$ near $\{|\rho_0| \le 2\delta,\ \omega^{1/2} \le 2\beta\delta\}$ with support in $\{|\rho_0| < 3\delta,\ \omega^{1/2} < 3\beta\delta\}$.

(1) First, following the notation of (5.14), consider the term $B_0(hD_x)^*(hD_x)$. Since a is independent of σ, it follows that $\sigma_{\mathrm{b},h}(B_0) = 0$ and hence $B_0 \in h\Psi^{\mathrm{comp}}_{\mathrm{b},h}(X,Y)$. Thus $B_0(hD_x)^*(hD_x)$ is part of the error hR.

(2) Now consider $B_1(hD_x)$, where $\sigma_{\mathrm{b},h}(B_1) = -(\partial_x\phi)\psi b^2$. Since $\partial_x\phi = 2\beta^{-2}\delta^{-1}x$ and $|x| \le \omega^{1/2} \le 3\beta\delta$ on $\operatorname{supp}\psi$, we can write $B_1(hD_x) = B^*(hD_xR_1)B + hR$; here

$$|\sigma_{\mathrm{b},h}(R_1)| \le 2\beta^{-2}\delta^{-1}\omega^{1/2} \le 6\beta^{-1}.$$

(3) Now we have dealt with the analogs of the first two terms in (5.14), and we turn to the term $(i/h)[\tilde{P}, A^*A]$. If $\tilde{P}_0$ is an operator with principal symbol $\tilde{p}_0$, write $\tilde{P} = \tilde{P}_0 + (\tilde{P} - \tilde{P}_0)$. The principal symbol of $(i/h)[\tilde{P} - \tilde{P}_0, A^*A]$ is given by

$$\mathsf{H}^{\mathrm{b}}_{\tilde{p}-\tilde{p}_0}(a^2) = (\mathsf{H}^{\mathrm{b}}_{\tilde{p}-\tilde{p}_0}\phi)\psi b^2 + e.$$

In view of (5.20), we can write $(i/h)[\tilde{P} - \tilde{P}_0, A^*A] = B^*R_0B + E + hR$, where

$$|\sigma_{\mathrm{b},h}(R_0)| \le (3^\theta M)\delta^\theta\beta^\theta(1 + 3\beta^{-1}).$$

Thus R_0 is as advertised, since $\beta < 1$.

(4) Finally, $(i/h)[\tilde{P}_0, A^*A]$ has principal symbol

$$\mathsf{H}^{\mathrm{b}}_{\tilde{p}_0}(a^2) = -b^2 + e,$$

hence we can write $(i/h)[\tilde{P}_0, A^*A] = -B^*B + E + hR$ as desired. □

We proceed as in Section 5.2, using Lemma 5.16 to write

$$\begin{aligned} -(2/h)\operatorname{Im}\langle APu, Au\rangle &= (i/h)\langle [A^*A, P]u, u\rangle \\ &= \|Bu\|^2_{L^2} + \langle R_0Bu, Bu\rangle + \langle R_1Bu, (hD_x)Bu\rangle \\ &\quad - \langle Eu, u\rangle - h\langle Ru, u\rangle \end{aligned}$$

for $u \in H^1_h(X)$. Applying (5.3) we can bound

$$c_0\|Bu\|^2_{H^1_h} \le \|Bu\|^2_{L^2} + C\|GPu\|^2_{H^{-1}_h} + Ch^2\|Gu\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h}, \tag{5.22}$$

where $c_0 > 0$ independent of β, δ, and where G is elliptic on $\mathrm{WF}_{\mathrm{b},h}(B)$. We now choose β depending on δ:

Lemma 5.17. *Let $\varepsilon > 0$. There exists $c > 0$ such that if $\delta \in (0, \delta_0)$ and $\beta = c\delta^{\theta/(2-\theta)}$, then*

$$\begin{aligned} |\langle R_0Bu, Bu\rangle| + |\langle R_1Bu, (hD_x)Bu\rangle| &\le \varepsilon\|Bu\|^2_{H^1_h} \\ &\quad + Ch^{-1}\|GPu\|^2_{H^{-1}_h} + Ch\|Gu\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h} \end{aligned}$$

for each $\delta \in (0, \delta_1)$ and $u \in H^1_h(X)$. The constant $C = C(\delta)$ depends on δ through β.

Proof. First consider R_1, in which case

$$\langle R_1 v, (hD_x)v\rangle \leq 2C_1\beta^{-1}\|v\|_{L^2}\|hD_x v\|_{L^2} + Ch\|v\|^2_{H^1_h},$$

where C_1 does not depend on β, δ. Apply this to $v = Bu$ and use Lemma 5.12. Indeed,

$$\mathrm{WF}_{\mathrm{b},h}(B) \subset \{|\rho_0| \leq 2\delta,\, \omega^{1/2} \leq 2\beta\delta\},$$

and by our choice of ρ_0 we conclude that $|x| \leq 2\beta\delta$ and $|\tilde{p}_0| \leq 2\beta\delta$ on $\mathrm{WF}_{\mathrm{b},h}(B)$. Now

$$|\tilde{p}| \leq |\tilde{p}_0| + |\tilde{p} - \tilde{p}_0| \leq 2\beta\delta + C|x|^\theta \leq C_1'(\beta\delta)^\theta$$

on $\mathrm{WF}_{\mathrm{b},h}(B)$, where C_1' does not depend on β, δ. Thus, by Lemma 5.12 and Cauchy–Schwarz,

$$\begin{aligned}\langle R_1 Bu, (hD_x)Bu\rangle \leq (\varepsilon/2)\|Bu\|^2_{L^2} + C_1''\beta^{-2}(\beta\delta)^\theta\|Bu\|^2_{L^2} \\ + Ch^{-1}\|GPu\|^2_{H^{-1}_h} + Ch\|Gu\|^2_{H^1_h} + \mathcal{O}(h^\infty)\|u\|^2_{H^1_h},\end{aligned}$$

where again $C_1'' > 0$ is independent of β, δ; here we have taken the δ in the notation of Lemma 5.12 to be a multiple of $(\beta\delta)^\theta$. Bounding $|\langle R_0 Bu, Bu\rangle|$ is done exactly as in Lemma 5.11, yielding

$$|\langle R_0 Bu, Bu\rangle| \leq C_1'''\beta^{-1}(\beta\delta)^\theta\|Bu\|^2_{L^2} + \mathcal{O}(h^\infty)\|u\|^2_{L^2}.$$

It therefore suffices to choose $\beta = c\delta^{\theta/(2-\theta)}$ with $c > 0$ sufficiently large. □

The rest of the argument in Section 5.2 goes through verbatim. Thus if $G \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ is elliptic on $\mathrm{WF}_{\mathrm{b},h}(B)$, and $Q_1 \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ is elliptic on $\mathrm{WF}_{\mathrm{b},h}(E)$ with $\mathrm{WF}_{\mathrm{b},h}(Q_1) \subset \mathrm{ell}_{\mathrm{b}}(G)$, then

$$\|Bu\|_{H^1_h} \leq Ch^{-1}\|GPu\|_{H^{-1}_h} + C\|Q_1 u\|_{H^1_h} + Ch^{1/2}\|Gu\|_{H^1_h} + \mathcal{O}(h^\infty)\|u\|_{H^1_h}.$$

Performing the inductive step requires that the commutant be slightly modified at each step; however this does not cause any problems, and proceeds exactly as in [Vas3].

5.4. **Proof of Theorem 1.** We now prove Theorem 1, following [Vas3, Section 8] quite closely. Without assuming that $Pu = 0$, we prove the slightly stronger statement that

$$F = \mathrm{WF}^{1,r}_{\mathrm{b},h}(u) \setminus \mathrm{WF}^{-1,r+1}_{\mathrm{b},h}(Pu)$$

is the union of maximally extended GBB within $\dot{\Sigma} \setminus \mathrm{WF}^{-1,r+1}_{\mathrm{b},h}(Pu)$ for each $r \in \mathbb{R} \cup \{+\infty\}$. It suffices to prove that for each $q_0 \in F$ there exists $\varepsilon > 0$ and a GBB

$$\gamma : [-\varepsilon, 0] \to F$$

satisfying $\gamma(0) = q_0$. Indeed, given any $Z \subset \dot{\Sigma}$, let $\mathcal{P}_Z$ denote the set of GBBs defined on open intervals $(\alpha, 0]$ with values in Z, such that $\gamma(0) = q_0$. There is a natural partial order on $\mathcal{P}_Z$ such that each chain has an upper bound. Thus, provided $\mathcal{P}_Z \neq \emptyset$, Zorn's lemma guarantees the existence of maximally extended GBB in Z on an interval $(\alpha_{\max}, 0]$, where possibly $\alpha_{\max} = -\infty$. We apply this argument with the set $Z = F$, but arguing verbatim as in [Vas3, Section 8], a maximal GBB within F is also maximal within $\dot{\Sigma} \setminus \mathrm{WF}_{\mathrm{b},h}^{-1,r+1}(Pu)$. Replacing the backwards propagation estimates with their forward counterparts, we similarly deduce the existence of a maximal GBB on $[0, \beta_{\max})$.

By Proposition 5.1 we can assume that $q_0 \in \mathcal{H}$ or $q_0 \in \mathcal{G}$. In the latter case it suffices to assume $q_0 \in \mathcal{G} \cap T^*Y$, since the semiclassical Duistermaat–Hörmander theorem on propagation of singularities applies when $q_0 \in \mathcal{G} \cap T^*(X \setminus Y)$, see [DZ, Appendix E] for example

We begin with the proof when $q_0 \in \mathcal{H}$. Fix a normal coordinate patch $\mathcal{U}$ such that ${}^{\mathrm{b}}T^*_{\mathcal{U}}X$ contains q_0. Since the complement of $\mathrm{WF}_{\mathrm{b},h}^{-1,r+1}(Pu)$ is open, first choose a precompact neighborhood $U \subset \dot{\Sigma} \cap {}^{\mathrm{b}}T^*_{\mathcal{U}}X$ of q_0 such that $U \cap \mathrm{WF}_{\mathrm{b},h}^{-1,r+1}(Pu) = \emptyset$. From the local compactness of $\dot{\Sigma}$, by further shrinking U we assume that

$$\mathsf{H}_p(x\xi) > 0 \text{ on } \pi^{-1}(U), \tag{5.23}$$

since this holds along $\pi^{-1}(\{q_0\})$. Also fix an open subset $U' \subset U$ containing q_0 with closure in U. By Lemma 4.10 there exists ε_0 such that every GBB defined on $[-\varepsilon_0, 0]$ with $\gamma(0) \in U'$ satisfies $\gamma([-\varepsilon_0, 0]) \subset U$. In particular, σ is increasing on any such GBB by (5.23). By Proposition 5.7, there is a sequence of points

$$q_n \in F \cap \{\sigma < 0\} \cap U'$$

tending to q_0. Since $q_n \in \dot{\Sigma}$ and $\sigma(q_n) < 0$, it follows that $x(q_n) \neq 0$. By the Duistermaat–Hörmander theorem on propagation of singularities, there is a maximally extended GBB

$$\gamma_n : (-\varepsilon_n, 0] \to F \cap T^*(X \setminus Y)$$

such that $\gamma_n(0) = q_n$.

Arguing as in [Vas3], the claim is that $\varepsilon_n \geq \varepsilon_0$. Indeed, since $\gamma_n(0) \in U'$, it would otherwise be the case that $\gamma_n(s) \in U$ for all $s \in (-\varepsilon_n, 0]$. Now γ_n extends to $[-\varepsilon_n, 0]$ by Proposition 4.9, and σ is increasing along γ_n. Therefore $\sigma(\gamma_n(-\varepsilon_n)) < 0$, so $x(\gamma_n(-\varepsilon_n)) \neq 0$, which contradicts maximality of γ_n. Thus we have a sequence of GBBs

$$\gamma_n|_{[-\varepsilon_0,0]} : [-\varepsilon_0, 0] \to F \cap \overline{U}$$

with values in a compact set. According to Proposition 4.9, there is a subsequence converging uniformly to a GBB

$$\gamma : [-\varepsilon_0, 0] \to F \cap \overline{U},$$

thus completing the proof.

For the proof when $q_0 \in \mathcal{G} \cap T^*Y$, we begin with a variant of Proposition 4.9. Fix a normal coordinate patch $\mathcal{U}$.

Lemma 5.18. *Let $K \subset {}^{\mathrm{b}}T^*_{\mathcal{U}}X$ be compact, $Z \subset \dot{\Sigma}$ be closed, and $[a,b] \subset \mathbb{R}$ a compact interval. Fix constants $r, C_0 > 0$. For each n, consider a partition*

$$a = s_{n,0} < s_{n,1} < \ldots < s_{n,k_n} = b.$$

Set $q_{n,j} = \gamma_n(s_{n,j})$ and $\delta_{n,j} = |s_{n,j} - s_{n,j-1}|$ for $j = 1, \ldots, k_n$. Suppose that

$$\gamma_n : [a,b] \to K$$

is a sequence of continuous maps, where the restriction of γ_n to $[s_{n,j-1}, s_{n,j}]$ is either a GBB *with values in Z, or the following holds*

- *$q_{n,j} \in Z \cap \mathcal{G} \cap T^*Y$ and $q_{n,j-1} \in Z$, where*
$$q_{n,j-1} \in \mathsf{B}(\exp(-\delta \mathsf{H}^{\mathrm{b}}_{\tilde{p}_0})(q_{n,j}), C_0 \delta_{n,j}^{1+r}), \quad r > 0. \tag{5.24}$$
- *The restriction of γ_n to $[s_{n,j-1}, s_{n,j}]$ is a line segment (in local coordinates), and $\delta_{n,j} \leq 2^{-n}|b-a|$.*

Then there is a subsequence of γ_n converging uniformly to a GBB *$\gamma : [a,b] \to K \cap Z$.*

Proof. Since $K \subset {}^{\mathrm{b}}T^*_{\mathcal{U}}X$ is compact, we can choose $L > 0$ such that

$$|\gamma_n(s) - \gamma_n(s')| \leq L|s - s'| \tag{5.25}$$

for $s, s' \in [a,b]$, uniformly in n. To see this, it suffices to consider the case when s, s' lie in a single interval $[s_{n,j-1}, s_{n,j}]$. By (4.10), the result is clear if the restriction of γ_n to $[s_{n,j-1}, s_{n,j}]$ is a GBB. If the restriction is a line segment, then (5.25) holds with

$$L = \frac{|q_{n,j-1} - q_{n,j}|}{\delta_{n,j}} \leq C_0 + \sup\{|\mathsf{H}^{\mathrm{b}}_{\tilde{p}_0}(q)| : q \in K\},$$

which is bounded uniformly in n. By the Arzelà–Ascoli theorem, there is a subsequence of γ_n converging to a curve $\gamma : [a,b] \to K$, and since Z is closed, γ actually maps into $K \cap Z$ under our hypotheses. It remains to check that γ is a GBB.

First, suppose that $\gamma(s_0) \notin \mathcal{G} \cap T^*Y$. Since $\mathcal{G} \cap T^*Y$ is closed in ${}^{\mathrm{b}}\dot{T}^*X$, there is a neighborhood O of $\gamma(s_0)$ that is also disjoint from $\mathcal{G} \cap T^*Y$. Choose $\delta > 0$ such that $\gamma_n(s) \in O$ for $s \in (s_0 - 2\delta, s_0 + 2\delta)$ and $n \geq N_0$. By assumption, the restriction of γ_n to $[s_0 - \delta, s_0 + \delta]$ is a GBB, increasing N_0 if necessary, so by Proposition 4.9, the restriction of γ to $[s_0 - \delta, s_0 + \delta]$ is a GBB.

On the other hand, suppose that $\gamma(s_0) \in \mathcal{G} \cap T^*Y$. Let $\varpi_0 = \pi^{-1}(q_0)$, and suppose that $f \in \mathcal{C}^\infty(T^*X)$ is π-invariant. We must show that

$$D_\pm(f_\pi \circ \gamma)(s_0) \geq (\mathsf{H}_p f)(\varpi_0).$$

Furthermore, at glancing points it suffices to check this when f is one of the π-invariant functions $\{x, y, \eta\}$. This follows from the fact that

$$f(x,y,\xi,\eta) = f_0(y,\eta) + x f_1(x,y,\xi,\eta)$$

and $x(\varpi_0) = \xi(\varpi_0) = 0$. Let $c_0 = (\mathsf{H}_p f)(\varpi_0)$. We show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f_\pi(\gamma(s)) - f_\pi(\gamma(s_0)) \geq (c_0 - \varepsilon)(s - s_0)$$

for each $s \in (s_0, s_0 + \delta)$.

Since the map π is proper and ${}^{\mathrm{b}}T^*X$ is locally compact, from the continuity of $\mathsf{H}_p f$ there is a neighborhood $O \subset \dot{\Sigma}$ of $\gamma(s_0)$ such that

$$\inf\{(\mathsf{H}_p f)(\varpi) : \varpi \in \pi^{-1}(O)\} \geq (c_0 - \varepsilon/4).$$

By uniform convergence, we choose $\delta > 0$ such that $\gamma_n(s) \in O$ for $s \in (s_0, s_0 + 2\delta)$ and $n \geq N_0$.

Fix the interval $[\alpha, \beta] = [s_{n,j-1}, s_{n,j}]$ containing s_0, where we choose $s_0 = s_{n,j-1}$ if s_0 happens to be an endpoint. For $s \in (s_0, s_0 + \delta)$ consider the function

$$F_n = (f_\pi \circ \gamma_n)(s) - (c_0 - \varepsilon/2)s.$$

If the restriction of γ_n to $[\alpha, \beta]$ is a GBB, then $D_+ F_n(s) \geq 0$ on the intersection $[\alpha, \beta] \cap (s_0, s_0 + \delta)$ by our choice of O. Otherwise the restriction of γ_n to $[\alpha, \beta]$ is the line segment

$$\gamma_n(s) = q_{n,j-1} + (s - \alpha)\frac{q_{n,j} - q_{n,j-1}}{\delta_{n,j}}.$$

Since f is one of $\{x, y, \eta\}$, it is clear that $f_\pi \circ \gamma_n$ is actually differentiable on $[\alpha, \beta]$, and

$$D_+(f_\pi \circ \gamma_n)(s) = \frac{f_\pi(q_{n,j}) - f_\pi(q_{n,j-1})}{\delta_{n,j}}$$

is *constant* on $[\alpha, \beta]$. By (5.24),

$$|D_+(f_\pi \circ \gamma_n)(s) - (\mathsf{H}^{\mathrm{b}}_{\tilde{p}_0} f_\pi)(q_{n,j})| \leq C_0 |\beta - \alpha|^r \leq C_0 2^{-nr}. \tag{5.26}$$

uniformly in n. By further increasing N_0 so that $2^{-N_0}|\beta - \alpha| < \delta$, we can assume that $\beta \in (s_0, s_0 + 2\delta)$ for $n \geq N_0$. Thus $q_{n,j} \in O \cap \mathcal{G} \cap T^*Y$. Let $\varpi_{n,j}$ satisfy $\pi(\varpi_{n,j}) = q_{n,j}$.

We now collect several observations. First, since $q_{n,j}$ is a glancing point over Y, the equality (5.19) holds. Furthermore, since $\tilde{p}_0$ depends only on (y, η) and f is one of $\{x, y, \eta\}$, it follows that

$$(\mathsf{H}_p f)(\varpi_{n,j}) = (\mathsf{H}_{\tilde{p}_0} f)(\varpi_{n,j}) = (\mathsf{H}^{\mathrm{b}}_{\tilde{p}_0} f_\pi)(q_{n,j}). \tag{5.27}$$

Therefore, since $\varpi_{n,j} \in \pi^{-1}(O)$ for $n \geq N_0$, by combining (5.26) and (5.27) we obtain

$$\begin{aligned} D_+ F_n(s) &= D_+(f_\pi \circ \gamma_n)(s) - (c_0 - \varepsilon/2) \\ &\geq (\mathsf{H}_p f)(\varpi_{n,j}) - C_0 2^{-nr} - (c_0 - \varepsilon/2) \\ &\geq (c_0 - \varepsilon/4) - C_0 2^{-nr} - (c_0 - \varepsilon/2) \geq 0 \end{aligned}$$

on $[\alpha, \beta] \cap (s_0, s_0 + \delta)$ if N_0 is increased so that $C_0 2^{-nr} \leq \varepsilon/4$ for $n \geq N_0$.

Thus we know that $D_+F_n(s) \geq 0$ on $(s_0, s_0+\delta)$ for $n \geq N_0$. Since F_n has a nonnegative lower right Dini derivative, it is non-decreasing, and so

$$f_\pi(\gamma_n(s)) - f_\pi(\gamma_n(s_0)) \geq (c_0 - \varepsilon/2)(s - s_0)$$

for $s \in (s_0, s_0+\delta)$ and $N \geq N_0$. We obtain the desired inequality for each $s \in (s_0, s_0+\delta)$ by choosing $n \geq N$ sufficiently large (depending on s_0 and s) so that

$$|f_\pi(\gamma_n(s)) - f_\pi(\gamma(s))| + |f_\pi(\gamma(s_0)) - f_\pi(\gamma_n(s_0))| \leq (\varepsilon/2)(s - s_0).$$

A similar argument applies for $s \in (s_0 - \delta, s_0)$. □

The proof of Theorem 1 for $q_0 \in \mathcal{G} \cap T^*Y$ is then a relatively straightforward application of Lemma 5.18. We again fix a precompact neighborhood $U \subset \dot{\Sigma} \cap {}^{\mathrm{b}}T^*_{\mathcal{U}}X$ of q_0 such that $U \cap \mathrm{WF}^{-1,r+1}_{\mathrm{b},h}(Pu) = \emptyset$. Let L be as in the proof of Lemma 5.18 where we take $K = \overline{U}$, and let C_0 be as in the statement of Proposition 5.13. By Lemma 4.10 we can choose $\varepsilon_0 > 0$ such that

$$U' = \mathsf{B}(q_0, (C_0 + L)\varepsilon_0) \cap \dot{\Sigma} \subset U$$

and if $\gamma : [-\varepsilon, 0]$ is a GBB with $\varepsilon \in (0, \varepsilon_0)$ such that $\gamma(0) \in U'$, then $\gamma([-\varepsilon, 0]) \subset U$. Let $\delta_n = 2^{-n}\varepsilon_0$. We then define a family of approximate GBB inductively. First, set $s_0 = 0$, and suppose that a continuous curve γ has already been defined on $[s_j, 0]$ such that if $q_j = \gamma(s_j)$, then

$$\gamma([s_j, 0]) \subset \mathsf{B}(q_0, (C_0 + L)s_j|) \subset U.$$

We then extend γ to an interval $[s_{j+1}, s_j]$ as follows.

If $q_j = \gamma(s_j) \in F \cap \mathcal{G} \cap T^*Y \cap U'$, then by Proposition 5.13 we can choose $q_{j+1} \in \mathrm{WF}^{1,r}_{\mathrm{b},h}(u)$ such that

$$|q_{j+1} - \exp(-\delta_n \mathsf{H}^{\mathrm{b}}_{\tilde{p}_0})(q_j)| \leq C_0(\delta_n)^{\theta/(2-\theta)} \leq C_0\delta_n.$$

Let $s_{j+1} = \max(-\varepsilon_0, s_j - \delta_n)$. In particular since $q_j \in \overline{U}$, the line connecting q_j to q_{j+1} is contained in $\mathsf{B}(q_0, (C_0 + L)|s_{j+1}|) \subset U'$. This also shows that $q_{j+1} \in F$.

Otherwise, $q_j \in F \setminus (\mathcal{G} \cap T^*Y)$. We know that there is a maximally extended GBB

$$\gamma' : (-\varepsilon', 0] \to F \setminus (\mathcal{G} \cap T^*Y)$$

with $\gamma'(-\varepsilon') = q_j$. Let $s_{j+1} = \max(-\varepsilon_0, s_j - \varepsilon')$. If $s_{j+1} = -\varepsilon_0$, then we can extend γ_N from $[s_j, 0]$ to all of $[-\varepsilon_0, 0]$ by concatenating with γ'. The other possibility is that $s_{j+1} > -\varepsilon_0$. Then γ' extends up to $-\varepsilon'$, so we concatenate with γ' and set $q_{j+1} = \gamma'(-\varepsilon')$. Either way, the extension by γ' has its image in $\mathsf{B}(q_0, (C_0 + L)|s_{j+1}|)$. In the second case, $q_{j+1} \in F \cap \mathcal{G} \cap T^*Y$ by maximality, and thus we proceed as in the first step.

The sequence γ_N just constructed satisfies the hypotheses of Lemma 5.18 with $K = \overline{U}$ and $Z = F$, letting $r = \theta/(2-\theta)$. This completes the proof of Theorem 1 when $q_0 \in \mathcal{G} \cap T^*Y$.

6. Semiclassical paired Lagrangian distributions

In this section we collect the technical tools that we will need on semiclassical paired Lagrangian distributions. We largely follow the discussion in [DHUV] in the homogeneous case (see also the introduction for further references), but we have been forced to revisit some of the foundations of the subject as there is no existing treatment in the semiclassical setting.

6.1. **Nested conormal distributions.**

Our paired Lagrangian distributions are locally modeled on oscillatory integrals in $\mathbb{R}^m$ associated with the conormal bundles of two nested submanifolds of $\mathbb{R}^m$.

Definition 6.1. We say that an h-dependent function $a \in \mathcal{C}^\infty(\mathbb{R}^m_x \times \mathbb{R}^k_{\xi'} \times \mathbb{R}^n_{\xi''})$ is in the symbol class $S^{r,p}_h(\mathbb{R}^m_x; \mathbb{R}^k_{\xi'}; \mathbb{R}^n_{\xi''})$ if

$$|(hD_{\xi'})^\alpha D^\beta_{\xi''} D^\gamma_x a(x,\xi)| \leq C_{\alpha\beta\gamma N} \langle \xi'/h \rangle^{r-|\alpha|} \langle (\xi', \xi'') \rangle^p$$

for all multiindices α, β, γ and $N \in \mathbb{R}$. We say $a \in S^{r,\mathrm{comp}}_h \subset S^{r,-\infty}_h$ if $\operatorname{supp} a$ is contained in an h-independent compact set.

Remark 6.2. An instructive example is as follows. Let $\chi \in \mathcal{C}^\infty_c(\mathbb{R})$, equal to 1 near the origin; let $\psi \in \mathcal{C}^\infty(\mathbb{R})$ equal 0 on $(-\infty, 1)$ and 1 on $(2, \infty)$. On $\mathbb{R}^1 \times \mathbb{R}^1$ we set

$$a(x', x'', \xi', \xi'') = \psi(\xi'/h)\chi(\xi')\chi(\xi'') \tag{6.1}$$

This symbol lies in $S^{0,-\infty}_h(\mathbb{R}^2; \mathbb{R}^1; \mathbb{R}^1)$.

Remark 6.3. This class of symbols can be interpreted in terms of a certain semiclassical blow-up as follows. Our symbols will be functions on $\mathbb{R}^m \times \mathbb{R}^{k+n} \times (0,1)_h$ that lift to certain conormal functions on $\mathbb{R}^n \times \mathsf{S}$, where S is defined as the *blow-up*

$$\mathsf{S} = [\mathbb{R}^{k+n} \times [0,1)_h; \{\xi' = 0,\ h = 0\}].$$

The space S has two boundary hypersurfaces, ff and sf, corresponding to the lifts of $\{\xi' = 0,\ h = 0\}$ and its complement within $\{h = 0\}$, respectively. We also fix

$$\rho_{\mathrm{sf}} = \langle \xi'/h \rangle^{-1},\ \rho_{\mathrm{ff}} = h \langle \xi'/h \rangle .$$

These lift to S as smooth, globally defined boundary defining functions for sf, ff, and $h = \rho_{\mathrm{sf}}\rho_{\mathrm{ff}}$.

The lift of $\{h \geq \varepsilon|\xi'|\}$ intersects the interior of the front face. Valid coordinates here are (x, ξ'', Ξ, h), where $\Xi = \xi'/h$, and in this region h is a boundary defining function for ff. Furthermore, elements of $\mathcal{V}_{\mathrm{b}}(\mathsf{S})$ (vector fields tangent to all boundary faces) that are supported near the interior of ff are spanned over $\mathcal{C}^\infty(\mathsf{S})$ by $\{\partial_x, \partial_{\xi''}, \partial_\Xi, h\partial_h\}$. In order, these vector fields are the lifts of $\{\partial_x, \partial_{\xi''}, h\partial_{\xi'}, h\partial_h + \xi' \cdot \partial_{\xi'}\}$.

A different set of coordinates is needed in $\{|\xi'| \geq \varepsilon h\}$. Restricting in addition to the set where $|\xi'_k| \geq \varepsilon |\xi'_j|$, we can use projective coordinates $(x, \xi'', \theta, \varrho, \Omega)$, where

$$\theta = \xi'_k, \quad \varrho = h/\xi'_k, \quad \Omega_j = \xi'_j/\xi'_k$$

for $j \neq k$. In particular, θ, ϱ are boundary defining function for ff and sf, respectively. In this case, $\mathcal{V}_{\mathrm{b}}(\mathsf{S})$ is spanned over $\mathcal{C}^\infty(\mathsf{S})$ by $\{\partial_x, \partial_{\xi''}, \theta\partial_\theta, \varrho\partial_\varrho, \partial_\Omega\}$, which are the lifts of $\{\partial_x, \partial_{\xi''}, \xi' \cdot \partial_{\xi'}, h\partial_h, \xi'_k\partial_{\xi'_j}\}$, in order.

Without localizing to the different regions of S, it follows from the previous two paragraphs that $\mathcal{V}_{\mathrm{b}}(\mathsf{S})$ is spanned over $\mathcal{C}^\infty(\mathsf{S})$ by the lifts of

$$\{\partial_{\xi''}, \xi'_j\partial_{\xi'_i}, h\partial_h, h\partial_{\xi'}\},$$

where $i, j \in \{1, \ldots, k\}$. Thus if we ignore $h\partial_h$ derivatives, $S_h^{r,\mathrm{comp}}$ corresponds exactly to compactly supported $\rho_{\mathrm{sf}}^{-r} L^\infty(\mathsf{S})$ functions that remain in the same space under arbitrary applications of $\mathcal{V}_{\mathrm{b}}(\mathsf{S})$.

On $\mathbb{R}^m$, consider a splitting of coordinates $x = (x', x'', x''') \in \mathbb{R}^k \times \mathbb{R}^{m-d-k} \times \mathbb{R}^d$, and consider the submanifolds

$$S_1 = \{x'' = 0\}, \quad S_0 = \{x' = 0,\, x'' = 0\}.$$

Thus $S_0 \subset S_1 \subset \mathbb{R}^m$ are nested with codimensions $\operatorname{codim} S_1 = m-d-k$ and $\operatorname{codim} S_0 = m - d$. In particular, we have $d = \dim S_0$. Their conormal bundles are given by

$$\begin{aligned} N^*S_1 &= \{x'' = 0,\, \xi' = 0,\, \xi''' = 0\}, \\ N^*S_0 &= \{x' = 0,\, x'' = 0,\, \xi''' = 0\}, \end{aligned}$$

where $(x', x'', x''', \xi', \xi'', \xi''')$ are canonical coordinates on $T^*\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$. We view these as model Lagrangians, writing

$$\Lambda_0 = N^*S_0, \quad \Lambda_1 = N^*S_1.$$

We then consider oscillatory integrals whose amplitudes are elements of

$$S_h^{r,-\infty} = S_h^{r,-\infty}(\mathbb{R}^m_x; \mathbb{R}^k_{\xi'}; \mathbb{R}^{d-m-k}_{\xi''}).$$

Observe that elements of $S^{r,-\infty}$ depend on (x, ξ', ξ''), but not ξ'''. Given $a \in S_h^{r,-\infty}$, define the oscillatory integral

$$u(x) = (2\pi h)^{-3m/4-k/2+d/2} \int e^{\frac{i}{h}(\langle x',\xi'\rangle + \langle x'',\xi''\rangle)} a(x, \xi', \xi'')\, d\xi' d\xi''. \tag{6.2}$$

Since a is rapidly decaying in (ξ', ξ'') for each fixed $h > 0$, this certainly defines a smooth function on $\mathbb{R}^m$. We now write

$$\bar{x} = (x', x''), \quad \bar{\xi} = (\xi', \xi'')$$

and analyze mapping properties of the Gauss transform $a \mapsto e^{-ih\langle D_{\bar{x}}, D_{\bar{\xi}}\rangle} a$ on $S_h^{r,-\infty}$.

Lemma 6.4. *If $a \in S_h^{r,-\infty}$, then $e^{-ih\langle D_{\bar{x}}, D_{\bar{\xi}}\rangle} a \in S_h^{r,-\infty}$. Furthermore,*

$$e^{-ih\langle D_{\bar{x}}, D_{\bar{\xi}}\rangle} a(x,\xi) - \sum_{j=0}^{N} \left\langle -iD_{\bar{x}}, hD_{\bar{\xi}} \right\rangle^j a(x,\xi)/j! \in S_h^{r-N-1,-\infty}.$$

Proof. Since the dependence on x''' is smooth and parametric, it suffices to consider the case $d = 0$, so that $x = \bar{x}$ and $\xi = \bar{\xi}$. Set

$$y = (h^{1/2}x', x''), \quad \eta = (h^{-1/2}\xi', \xi'').$$

For a given $a \in S_h^{r,-\infty}$, define the rescaled amplitude $b(y,\eta) = a(x,\xi)$, which therefore satisfies

$$|D_y^\alpha D_\eta^\beta b(y,\eta)| \le C_{\alpha\beta N} h^{-(|\alpha'|+|\beta'|)/2} \langle \eta'/h^{1/2}\rangle^{r-|\beta'|} \langle (h^{1/2}\eta', \eta'')\rangle^{-N},$$

where we have written $\alpha = (\alpha', \alpha'')$ and $\beta = (\beta', \beta'')$. After a change of variables,

$$e^{-ih\langle D_y, D_\eta\rangle} b(y,\eta) = (2\pi)^{-m} \int e^{i\langle z,\zeta\rangle} b(y - h^{1/2}z, \eta - h^{1/2}\zeta)\, dz d\zeta. \tag{6.3}$$

Write the integral on right hand side of (6.3) as a sum $A + B$, where A is the result of inserting a cutoff $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^{2m}; [0,1])$ into the integrand which is identically one for $|(z,\zeta)| \le 1$ and vanishes for $|(z,\zeta)| \ge 2$. We then estimate

$$\begin{aligned} |D_y^\alpha D_\eta^\beta A(y,\eta)| &\le C \sup_{|(z,\zeta)|\le 2} |(D_y^\alpha D_\eta^\beta b)(y - h^{1/2}z, \eta - h^{1/2}\zeta)| \\ &\le C_{\alpha\beta N} h^{-(|\alpha'|+|\beta'|)/2} \langle \eta'/h^{1/2}\rangle^{r-|\beta'|} \langle (h^{1/2}\eta', \eta'')\rangle^{-N}, \end{aligned}$$

since $\left\langle (\eta' - h^{1/2}\zeta')/h^{1/2} \right\rangle \asymp C \left\langle \eta'/h^{1/2} \right\rangle$ when $|\zeta'|$ is bounded. To estimate B, we integrate by parts using the operator

$$L = |(z,\zeta)|^{-2}(zD_\zeta + \zeta D_z),$$

defined for $|(z,\zeta)| \ge 1$ and satisfying $L(i\langle z,\zeta\rangle) = 1$. By Peetre's inequality,

$$\begin{aligned} &|D_y^\alpha D_\eta^\beta (L^t)^k (1 - \chi(z,\zeta)) b(y - h^{1/2}z, \eta - h^{1/2}\zeta)| \\ &\qquad \le C_{\alpha\beta N} h^{-(|\alpha'|+|\beta'|)/2} \langle \eta'/h^{1/2}\rangle^{r-|\beta'|} \langle (h^{1/2}\eta', \eta'')\rangle^{-N} \langle (z,\zeta)\rangle^{N(k)} \end{aligned}$$

for $|(z,\zeta)| \ge 1$, where $N(k) \to -\infty$ as $k \to \infty$. Integrating by parts k times for sufficiently large k shows that B satisfies the same symbol estimates as A. This establishes the desired mapping properties of $e^{-ih\langle D_x, D_\xi\rangle}$, since

$$e^{-ih\langle D_x, D_\xi\rangle} a(x,\xi) = e^{-ih\langle D_y, D_\eta\rangle} b(y,\eta).$$

To obtain the expansion, simply Taylor expand

$$e^{-ih\langle D_x, D_\xi\rangle} = \sum_{j=0}^{N} \frac{1}{j!} \langle -iD_x, hD_\xi\rangle^j + \frac{(-i)^N}{N!} \int_0^1 (1-t)^N e^{-ith\left\langle D_x, D_\xi\right\rangle} \left\langle D_x, hD_\xi\right\rangle^{N+1} dt.$$

The remainder can be estimated by replacing h with th and repeating the argument above. □

Lemma 6.4 shows that we can always write u given by (6.2) in the form

$$u(x) = (2\pi h)^{-3m/4-k/2+d/2} \int e^{\frac{i}{h}(\langle x',\xi'\rangle + \langle x'',\xi''\rangle)}\, c(x''',\xi',\xi'')\, d\xi' d\xi'' \tag{6.4}$$

where the amplitude $c \in S_h^{r,-\infty}$ depends only on x''' in the base variables. Indeed, by the Fourier inversion formula,

$$c(x''',\xi,\xi'') = e^{-ih\langle D_{\bar{x}}, D_{\bar{\xi}}\rangle} a(x,\xi',\xi'')|_{x'=x''=0}, \tag{6.5}$$

which defines an element of $S_h^{r,-\infty}$ by Lemma 6.4.

Lemma 6.5. *Let $b \in S^0(\mathbb{R}^m;\mathbb{R}^m)$. If $c \in S_h^{r,-\infty}$ and u is given by (6.4), then there is $\tilde{c} \in S_h^{r,-\infty}$ such that*

$$b(x,hD)u = (2\pi h)^{-3m/4-k/2+d/2} \int e^{\frac{i}{h}(\langle x',\xi'\rangle + \langle x'',\xi''\rangle)}\, \tilde{c}(x''',\xi',\xi'')\, d\xi' d\xi'', \tag{6.6}$$

and moreover

$$\tilde{c}(x''',\xi',\xi'',h) = e^{ih(-\langle D_{y'''}, D_{\xi'''}\rangle - \langle D_{\bar{x}}, D_{\bar{\xi}}\rangle)} b(x,\xi) c(y''',\xi',\xi'')|_{y'''=x''',\, x'=x''=\xi'''=0}.$$

Proof. This follows from the Fourier inversion formula and Lemma 6.4. □

We now define our class of compactly microlocalized paired Lagrangians in the model case of nested conormal distributions.

Definition 6.6. We say that $u \in I_h^{l,\mathrm{comp}}(\mathbb{R}^m;\Lambda_0,\Lambda_1)$ if $\operatorname{supp} u$ and $\mathrm{WF}_h(u)$ are compact, and

$$u(x) = (2\pi h)^{-3m/4-k/2+d/2} \int e^{\frac{i}{h}(\langle x',\xi'\rangle + \langle x'',\xi''\rangle)} a(x,\xi',\xi'')\, d\xi' d\xi'' \tag{6.7}$$

with $a \in S_h^{l-k/2,-\infty}$.

Even when $l = -\infty$, elements of $I_h^{-\infty,\mathrm{comp}}(\mathbb{R}^m;\Lambda_0,\Lambda_1)$ are not residual in the sense of $\mathcal{O}(h^\infty)$ remainders:

Lemma 6.7. *The following properties are satisfied.*

(1) $I_h^{-\infty,\mathrm{comp}}(\mathbb{R}^m;\Lambda_0,\Lambda_1) = I_h^{\mathrm{comp}}(\mathbb{R}^m;\Lambda_1)$.

(2) If l is fixed, then $h^\infty I_h^{l,\mathrm{comp}}(\mathbb{R}^m;\Lambda_0,\Lambda_1) = h^\infty \mathcal{C}_c^\infty(\mathbb{R}^m)$.

Proof. (1) We can write $u \in I_h^{-\infty,\mathrm{comp}}(\mathbb{R}^m;\Lambda_0,\Lambda_1)$ in the form (6.7) with $a \in S_h^{-\infty,-\infty}$. This implies that

$$a(x,\xi,h) = b(x,\xi'/h,\xi'',h)$$

where $b(x, \Xi, h)$ is rapidly decaying in $\Xi \in \mathbb{R}^{m-d}$, uniformly in h. Making the change of variables $\eta = \xi'/h$ and performing the η integral in the definition of u,

$$u = (2\pi h)^{-3m/4+k/2+d/2} \int e^{\frac{i}{h}\langle x'', \xi''\rangle} \tilde{b}(x, \xi'', h)\, d\xi'',$$

where $\tilde{b} \in \mathcal{C}_c^\infty(\mathbb{R}^m; \mathcal{S}(\mathbb{R}^{d-m-k}))$ uniformly in h. It now suffices to compare the power

$$-3m/4 + k/2 + d/2 = -(m-d-k)/2 - m/4$$

to the usual Lagrangian order convention (which is $m - d - k$ phase variables in m dimensions) to see that $u \in I_h^{\mathrm{comp}}(\mathbb{R}^m; \Lambda_1)$. The converse inclusion is obvious.

(2) This is just the observation that $h^\infty S_h^{r,-\infty} = h^\infty S^{-\infty}(\mathbb{R}^m; \mathbb{R}^{m-d})$ for any r. □

If a were a semiclassical symbol, then the wavefront set of u given by (6.7) would be contained in Λ_0. However, in this case the weaker symbolic properties of $a \in S_h^{r,-\infty}$ can generate additional singularities. We define the essential support of a as usual,

$$(\operatorname{esssupp} a)^\complement = \{(x, \xi', \xi'') : a \in h^\infty S^{-\infty} \text{ near } (x, \xi', \xi'')\},$$

where $h^\infty S_h^{r,-\infty} = h^\infty S^{-\infty}(\mathbb{R}^m; \mathbb{R}^{m-d})$ for each r, as was already observed in the proof of Lemma 6.7.

Lemma 6.8. *If $u \in I_h^{l,\mathrm{comp}}(\mathbb{R}^m; \Lambda_0, \Lambda_1)$ is given by (6.7), then*

$$\mathrm{WF}_h(u) \subset \{(x, \xi) \in N^*S_1 \cup N^*S_0) : (x, \xi', \xi'') \in \operatorname{esssupp} a\}.$$

Proof. Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^m)$, and write ψu in the form (6.4), where the amplitude $\tilde{a}$ is given by

$$\tilde{a}(x''', \xi', \xi'') = e^{-ih\langle D_{\bar{x}}, D_{\bar{\xi}}\rangle} \psi(x) a(x, \xi', \xi'')|_{x'=x''=0}.$$

Thus we can write $\tilde{a} = a_1 + a_2$, where $a_1 = 0$ if $(0, 0, x''') \notin \operatorname{supp} \psi$, and $a_2 \in S_h^{-\infty,-\infty}$. This gives rise to a corresponding decomposition $u = u_1 + u_2$. By the first part of Lemma 6.7, $\mathrm{WF}_h(u_2) \subset \Lambda_1 = N^*S_1$. On the other hand,

$$\mathcal{F}_h(\psi u_1)(\eta) = (2\pi h)^{m/4-k/2-d/2} \int e^{\frac{i}{h}\langle x''', \eta'''\rangle} a_1(x''', \eta)\, dx''',$$

so if $\eta_0''' \neq 0$, integrating by parts using the operator $L = |\eta'''|^{-2}\eta''' \cdot (hD_{x'''})$ shows that $\mathcal{F}_h(\psi u_1)(\eta) = \mathcal{O}(h^\infty)$ in a neighborhood of η_0. Therefore we find that $\mathrm{WF}_h(u_1) \subset \{\eta''' = 0\}$ and lies only over an arbitrarily small neighborhood of $\{(0, 0, x''') : x''' \in \operatorname{supp} \psi\}$, hence is in a small neighborhood of N^*S_0. Thus have have shown that

$$\mathrm{WF}_h(u) \subset N^*S_0 \cup N^*S_1.$$

On the other hand, $\mathrm{WF}_h(u) \subset \{(x, \xi) : (x, \xi', \xi'') \in \operatorname{esssupp} a\}$ by the second part of Lemma 6.7. □

It follows from Lemma 6.5 that any properly supported $B \in \Psi_h^0(\mathbb{R}^m)$ preserves $I_h^{l,\mathrm{comp}}(\mathbb{R}^m;\Lambda_0,\Lambda_1)$. Moreover, if $B = b(x,hD)$ with b a total symbol for B, we can write Bu in the form (6.6), where

$$\operatorname{esssupp} \tilde{c} \subset \{(x,\xi',\xi'') \in \operatorname{esssupp} a : (x,\xi) \in \mathrm{WF}_h(B))\} \tag{6.8}$$

As a consequence, we can always write $u \in I_h^{l,\mathrm{comp}}(\mathbb{R}^m;\Lambda_0,\Lambda_1)$ in the form (6.7), where $a \in S_h^{l-k/2,\mathrm{comp}}$ has compact support, modulo an $h^\infty \mathcal{C}_c^\infty(\mathbb{R}^m)$ remainder. We can also make Lemma 6.8 more precise by microlocalizing individually to each of the two Lagrangians Λ_0, Λ_1 carrying possible wavefront set.

Lemma 6.9. *The following hold for $u \in I_h^{l,\mathrm{comp}}(\mathbb{R}^m;\Lambda_0,\Lambda_1)$ and $B \in \Psi_h^0(\mathbb{R}^n)$ of proper support.*

(1) If $\mathrm{WF}_h(B) \cap \Lambda_1 = \emptyset$, then $Bu \in h^{-l} I_h^{\mathrm{comp}}(\mathbb{R}^m;\Lambda_0)$.

(2) If $\mathrm{WF}_h(B) \cap \Lambda_0 = \emptyset$, then $Bu \in I_h^{\mathrm{comp}}(\mathbb{R}^m;\Lambda_1)$.

Proof. Since $\mathrm{WF}_h(u) \subset \Lambda_0 \cup \Lambda_1$, we can assume that B is microlocalized near $\Lambda_0 \setminus \Lambda_1$ in the first case, and $\Lambda_1 \setminus \Lambda_0$ in the second case.

(1) By (6.8), we may assume that $|\xi'| \geq \varepsilon$ on $\operatorname{supp} a$. It follows that the symbol of Bu is an honest semiclassical symbol, and hence Bu is Lagrangian with respect to Λ_0. It remains to check the overall power of h.

(2) Again by (6.8), we may assume that $|x'| \geq \varepsilon$ on $\operatorname{supp} a$, at which point we proceed as in Lemma 6.8. □

6.2. **Change of variables.** In this section we show that $I_h^{l,\mathrm{comp}}(\mathbb{R}^m;\Lambda_0,\Lambda_1)$ is invariant under a diffeomorphism

$$\kappa : \mathbb{R}^m \to \mathbb{R}^m$$

preserving S_1 and S_0, and define a principal symbol. To simplify matters, we work with half-densities. Thus, given

$$u_\kappa = (2\pi h)^{-3m/4-k/2+d/2} \int e^{\frac{i}{h}(\langle x',\xi'\rangle + \langle x'',\xi''\rangle)} a_\kappa(x''',\xi',\xi'')\, d\xi' d\xi'', \tag{6.9}$$

with $a_\kappa \in S_h^{l-k/2,-\infty}$, we transform u_κ according to $u = |\det \kappa'|^{1/2}(\kappa^* u_\kappa)$. We write $\kappa = (\kappa_1,\kappa_2,\kappa_3)$ relative to the splitting $x = (x',x'',x''')$, and denote the Jacobian by

$$\kappa' = \begin{pmatrix} \kappa'_{11} & \kappa'_{12} & \kappa'_{13} \\ \kappa'_{21} & \kappa'_{22} & \kappa'_{23} \\ \kappa'_{31} & \kappa'_{32} & \kappa'_{33} \end{pmatrix}.$$

Since κ preserves S_1 and S_0, we see that κ'_{11} and κ'_{22} are nonsingular at points $(0,0,x''')$, and that $\kappa'_{21}, \kappa'_{13}, \kappa'_{23}$ vanish at such points. Let us write

$$\kappa(x) = (\psi_{11}(x)x' + \psi_{12}(x)x'', \psi_{22}(x)x'', \kappa_3(x)),$$

By Lemmas 6.8, 6.9 and a partition of unity, we can assume without loss of generality that ψ_{11} and ψ_{22} are nonsingular throughout the support of u, since the invariance properties of u away from S_0 are well known. Arguing precisely as in [Hör1, Theorem 18.2.9], we obtain the following.

Lemma 6.10. *If κ and $u_\kappa \in I^{l,\mathrm{comp}}(\mathbb{R}^m; \Lambda_0, \Lambda_1)$ are as above, then $u = |\det \kappa'|^{1/2}(\kappa^* u_\kappa)$ is of the form (6.7) with an amplitude $a \in S_h^{l-k/2,-\infty}$.*

Indeed, a is given by the expression

$$a(x,\xi',\xi'') = a_\kappa(\kappa_3(x), {}^t\psi_{11}(x)^{-1}\xi', {}^t\psi_{22}(x)^{-1}(\xi'' - {}^t(\psi_{11}(x)^{-1}\psi_{12}(x))\xi'))$$
$$\times |\det \kappa'(x)|^{1/2} |\det \psi_{11}(x)|^{-1} |\det \psi_{22}(x)|^{-1},$$

and it is easy to see that $a \in S_h^{l-k/2,-\infty}$. Of course the (x', x'') dependence can be eliminated as in (6.5).

To define the principal symbol along Λ_1, consider u_κ of the form

$$u_\kappa = (2\pi h)^{-3m/4-k/2+d/2} \int e^{\frac{i}{h}(\langle x',\xi'\rangle + \langle x'',\xi''\rangle)} a_\kappa(x', x''', \xi', \xi'')\, d\xi' d\xi''.$$

Compared with (6.9), we are not assuming that a_κ in necessarily independent of x'. We associate to $u_\kappa |dx|^{1/2}$ the half-density

$$b_\kappa(x', x''', \xi'')|dx'|^{1/2}|dx'''|^{1/2}|d\xi''|^{1/2}, \tag{6.10}$$

where

$$b_\kappa(x', x''', \xi'') = (2\pi h)^{-k} \int e^{\frac{i}{h}\langle x',\xi'\rangle} a_\kappa(x', x''', \xi', \xi'')\, d\xi'.$$

When $x' \neq 0$ (so away from $\Lambda_0 \cap \Lambda_1$), this is a representative of the principal symbol of $u_\kappa |dx|^{1/2}$ as a half-density valued element of $I_h^{\mathrm{comp}}(\mathbb{R}^m; \Lambda_1)$.

Definition 6.11. We say that $b(x', x''', \xi'') \in \mathcal{C}_c^\infty(\mathbb{R}^{m-d}_{x',x'''} \times \mathbb{R}^{m-d-k}_{\xi''})$ is in $S^{l,\mathrm{comp}}_{\Lambda_1}$ if there exists $a(x', x''', \xi', \xi'') \in S_h^{l-k/2,\mathrm{comp}}$ such that

$$b(x', x''', \xi'') = (2\pi h)^{-k} \int e^{\frac{i}{h}\langle x',\xi'\rangle} a(x', x''', \xi', \xi'')\, d\xi'$$

modulo $h^\infty \mathcal{C}_c^\infty(\mathbb{R}^{m-d} \times \mathbb{R}^{m-d-k})$.

That this class of symbols degenerates at $x' = 0$, on $\Lambda_0 \cap \Lambda_1$, can be seen by differentiating in x': there is a term with a factor ξ'/h arising from differentiation of the phase. Away from $x' = 0$ we may integrate by parts with respect to the operator $(h/x')D_{\xi'}$, which, falling on the amplitude a, produces a factor $\langle \xi'/h\rangle^{-1}$ that ensures the resulting amplitude enjoys the same estimates as a. This strategy fails at $x' = 0$, where the order of a is effectively raised to $l + 1$. The singularity only arises in $h \to 0$ asymptotics, though: b is smooth for every positive h since a is compactly supported in ξ.

Remark 6.12. If we apply this construction to the example a from Remark 6.2 (which does have compact support in the fiber variables, albeit not in the base), we obtain

$$b(x', \xi'') = \chi(\xi'')(2\pi h)^{-1} \int_0^\infty e^{ix'\xi'/h} \psi(\xi'/h)\chi(\xi')\, d\xi' \equiv b_0(x', \xi'') - b_1(x', \xi'')$$

where

$$b_0(x', \xi'') = \chi(\xi'')(2\pi h)^{-1} \int_0^\infty e^{ix'\xi'/h} \chi(\xi')\, d\xi',$$
$$b_1(x', \xi'') = \chi(\xi'')(2\pi h)^{-1} \int_0^\infty e^{ix'\xi'/h} (1-\psi)(\xi'/h)\, d\xi'$$

(dropping a factor of $\chi(\xi')$ in b_1 since it is moot for h small). Changing variables to $\eta = \xi'/h$ in the integral, we find that $b_1(x', \xi'') \in \mathcal{C}^\infty$ is independent of h. By contrast,

$$b_0(x', \xi'') = (2\pi)^{-1/2} h^{-1} \chi(\xi'')\mathcal{F}^{-1}(H\chi)(x'/h)$$

where H denotes the Heaviside function and $\mathcal{F}$ is the ordinary (non-semiclassical) Fourier transform. The distribution $\mathcal{F}^{-1}(H\chi)(y)$ is $\mathcal{C}^\infty$ and for large $|y|$ differs from $i(2\pi)^{-1/2} y^{-1}$ by a rapidly decreasing function, hence in the asymptotic regime where $x'/h \to \infty$, $b_0 \sim (i/2\pi)\chi(\xi'')/x'$. (Formally taking leading order terms in an expansion in $(x'/h)^{-1}$ is, in effect, our principal symbol construction.) For x'/h fixed, on the other hand, b_0 blows up like a multiple of h^{-1} as $h \to 0$.

Observe that $S^{l,\mathrm{comp}}_{\Lambda_1}$ is itself a degenerate version of the paired Lagrangian distributions we have been studying. Under κ, we see that b_κ is transformed to

$$b(x', x''', \xi'') = (2\pi h)^{-k} \int e^{\frac{i}{h}\langle x', \xi'\rangle} \left(e^{-ih\langle D_{x''}, D_{\xi''}\rangle} \tilde{b}(x', x'', x''', \xi', \xi'')|_{x''=0} \right) d\xi', \tag{6.11}$$

where we have defined

$$\tilde{b}(x', x'', x''', \xi', \xi'') = e^{\frac{i}{h}\langle \psi_{11}(x)^{-1}\psi_{12}(x)x'', \xi'\rangle} a_\kappa(\kappa_1(x), \kappa_3(x), {}^t\psi_{11}^{-1}(x)\xi', {}^t\psi_{22}^{-1}(x)\xi'')$$
$$\times |\det \kappa'(x)|^{1/2} |\det \psi_{22}(x)|^{-1} |\det \psi_{11}(x)|^{-1}.$$

The phase factor $e^{\frac{i}{h}\langle \psi_{11}(x)^{-1}\psi_{12}(x)x'', \xi'\rangle}$ is harmless when differentiating $e^{-ih\langle D_{x''}, D_{\xi''}\rangle}\tilde{b}$ in (x', x''', ξ', ξ''), since the result is evaluated at $x'' = 0$, and in particular

$$e^{-ih\langle D_{x''}, D_{\xi''}\rangle}\tilde{b} \in S_h^{l-k/2,\mathrm{comp}}.$$

On the other hand, the phase factor does appear in higher order expansion. Importantly,

$$e^{-ih\langle D_{x''}, D_{\xi''}\rangle}\tilde{b}(x', x'', x''', \xi', \xi'')|_{x''=0} = \tilde{b}(x', 0, x''', \xi', \xi'') + hS_h^{l-k/2+1,\mathrm{comp}}.$$

This shows that the equivalence class of $b_\kappa(x', x''', \xi'')|dx'|^{1/2}|dx'''|^{1/2}|d\xi''|^{1/2}$ is well defined in $S^{l,\mathrm{comp}}_{\Lambda_1}/hS^{l+1,\mathrm{comp}}_{\Lambda_1}$, since the pullback of b_κ as a half-density is precisely

$$(2\pi h)^{-k} \int e^{\frac{i}{h}\langle x', \xi'\rangle} \tilde{b}(x', 0, x''', \xi', \xi'')\, d\xi'.$$

We have proved the following:

Proposition 6.13. *The principal symbol*

$$\sigma_h^{\Lambda_1}(u_\kappa |dx|^{1/2}) = b_\kappa(x', x''', \xi'')|dx'|^{1/2}|dx'''|^{1/2}|d\xi''|^{1/2} \in S^{l,\mathrm{comp}}_{\Lambda_1}/hS^{l+1,\mathrm{comp}}_{\Lambda_1}$$

is well defined.

6.3. **Pseudodifferential operators with singular symbols.** In this section we discuss a calculus of pseudodifferential operators with singular symbols. Let X be an n-dimensional manifold, and $Y \subset X$ a codimension k submanifold. Consider an operator A with Schwartz kernel

$$K_A \in I_h^{l,\mathrm{comp}}(X \times X; N^*((X \times Y) \cap \mathrm{diag}), N^*\mathrm{diag}).$$

Since $\operatorname{supp} K_A$ and $\mathrm{WF}_h(K_A)$ are compact by assumption, it follows that

$$K_A : \mathcal{C}^{-\infty}(X) \to \mathcal{C}^\infty_c(X),$$

and K_A is h-tempered.

By the coordinate invariance discussed in Section 6.2, it suffices to construct this calculus on $X = \mathbb{R}^n$, where $Y = \{x' = 0\}$ for an appropriate splitting of coordinates

$$x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}.$$

If (x, y, ξ, η) are the corresponding coordinates on $T^*(\mathbb{R}^n \times \mathbb{R}^n)$, we work with the Lagrangian pair

$$\Lambda_1 = \{x = y,\, \eta = -\xi\}, \quad \Lambda_0 = \{x' = y' = 0,\, x'' = y'',\, \eta'' = -\xi''\}. \tag{6.12}$$

Working modulo $h^\infty \mathcal{C}^\infty_c(\mathbb{R}^{2n})$, we can write $K_A \in I_h^{l,\mathrm{comp}}(\mathbb{R}^{2n}; \Lambda_0, \Lambda_1)$ as a left quantization

$$K_A = (2\pi h)^{-n-k} \int e^{\frac{i}{h}(\langle x-y,\xi\rangle + \langle x',\eta'\rangle)} a(x'', \eta', \xi)\, d\eta' d\xi, \tag{6.13}$$

where $a \in S_h^{l-k/2,\mathrm{comp}}(\mathbb{R}^{n-k}; \mathbb{R}^k; \mathbb{R}^{n-k})$. This parametrization arises by using coordinates (z', z'', x', x'') on $\mathbb{R}^{2n}$, where $z = x - y$; thus

$$\Lambda_1 = N^*\{z = 0\}, \quad \Lambda_0 = N^*\{x' = 0, z = 0\}.$$

Alternatively, we can use coordinates (z', z'', y', y''), so that K_A can also be written as a right quantization

$$K_A = (2\pi h)^{-n-k} \int e^{\frac{i}{h}(\langle x-y,\xi\rangle + \langle y',\eta'\rangle)} a(y'', \eta', \xi)\, d\eta' d\xi, \tag{6.14}$$

with $a \in S_h^{l-k/2,\mathrm{comp}}(\mathbb{R}^{n-k}; \mathbb{R}^k; \mathbb{R}^{n-k})$. The principal symbol $\sigma_h^{\Lambda_1}(A)$ of A along $\Lambda_1 = N^*\mathrm{diag}$, which we define simply to be $\sigma_h^{\Lambda_1}(K_A)$, is

$$(2\pi h)^{-k} \int e^{\frac{i}{h}\langle x',\eta'\rangle} a(x'', \eta', \xi)\, d\eta'$$

in $S^{l,\mathrm{comp}}_{\Lambda_1}/hS^{l+1,\mathrm{comp}}_{\Lambda_1}$. As usual we use the canonical symplectic density on $N^*\mathrm{diag}$ to identify functions with half-densities.

Next we consider composition of two operators whose Schwartz kernels are of the form (6.13). The proof we give is closely based on [DHUV, Proposition 5.8]. Because of certain logarithmic terms, in some cases there appear arbitrarily small losses in the order of the composition. Since these losses are acceptable, we will not explicate when they can be avoided; for a more precise account, see [DHUV, Proposition 5.8]. We remind the reader that, the homogeneous paired Lagrangians considered in [DHUV], these semiclassical operators have smooth Schwartz kernels for all $h > 0$, so that the composition of two singular pseudodifferential operators is always a well-defined operator family. The constraints on orders only arise in interpreting the result as another element in the calculus.

Proposition 6.14. *Given $K_A \in I_h^{l,\mathrm{comp}}(\mathbb{R}^{2n};\Lambda_0,\Lambda_1)$ and $K_B \in I_h^{l',\mathrm{comp}}(\mathbb{R}^{2n};\Lambda_0,\Lambda_1)$ let*
$$L > \max(l, l', l + l' + k/2).$$
If $l + l' < 0$, then $K_{AB} \in I_h^{L,\mathrm{comp}}(\mathbb{R}^{2n};\Lambda_0,\Lambda_1)$. Furthermore, if $\delta \in (0,1]$ is such that $l + l' < -\delta$, then
$$\sigma_h^{\Lambda_1}(AB) = \sigma_h^{\Lambda_1}(A)\cdot\sigma_h^{\Lambda_1}(B)$$
in $S^{L,\mathrm{comp}}_{\Lambda_1}/h^\delta S^{L+\delta,\mathrm{comp}}_{\Lambda_1}$.

Proof. As remarked above, the proof is essentially the same as in [DHUV, Proposition 5.8]. We write A in the form (6.13) with amplitude $a(x'',\eta',\xi)$, and B in the form (6.14) with amplitude $b(y'',\mu',\xi)$. Now
$$\mathcal{F}_h(Bu)(\xi) = (2\pi h)^{-k}\int e^{\frac{i}{h}(-\langle y,\xi\rangle + \langle y',\mu'\rangle)} b(y'',\mu',\xi)u(y)\,dyd\mu',$$
and hence
$$K_{AB} = (2\pi h)^{-n-2k}\int e^{\frac{i}{h}(\langle x-y,\xi\rangle + \langle y',\mu'\rangle + \langle x',\eta'\rangle)} a(x'',\eta',\xi)b(y'',\mu',\xi)\,d\eta' d\mu' d\xi.$$
Following [DHUV, Proposition 5.8], we make the change of variables
$$\nu' = \eta' + \mu', \quad \zeta' = \xi' - \mu', \quad \zeta'' = \xi'',$$
leaving μ' unchanged (observe that ξ'' is being renamed for later convenience, but is otherwise unchanged). Rewriting the phase in terms of these new variables as $\langle x-y,\zeta\rangle + \langle x',\nu'\rangle$, it follows that
$$K_{AB} = (2\pi h)^{-n-k}\int e^{\frac{i}{h}(\langle x''-y'',\zeta''\rangle + \langle x'-y',\zeta'\rangle + \langle x',\nu'\rangle)} c(x'',\nu',\mu',\xi'')\,d\nu' d\mu' d\xi'',$$
where
$$c(x'',y'',\nu',\zeta',\zeta'') = (2\pi h)^{-k}\int a(x'',\nu'-\mu',\mu'+\zeta',\zeta'')b(y'',\mu',\mu'+\zeta',\zeta'')\,d\mu'.$$

It remains to verify that $c \in S_h^{L,\mathrm{comp}}(\mathbb{R}^{2(n-k)}; \mathbb{R}^k; \mathbb{R}^n)$. Observe that the integral defining c is over a compact set, since b has compact support in μ', and indeed c is itself compactly supported. We begin by giving sup-norm bounds on c, observing that

$$|c(x'', y'', \nu', \zeta', \zeta'')| \leq Ch^{-k} \int \langle(\nu' - \mu')/h\rangle^{l-k/2} \langle\mu'/h\rangle^{l'-k/2} \, d\mu'.$$

First, suppose that $|\nu'| \geq h$, in which case $\langle\nu'/h\rangle$ can everywhere be replaced by $|\nu'/h|$. We then consider the integral over four regions.

(1) $2|\mu'| \leq |\nu'|$. Here $\langle(\nu' - \mu')/h\rangle$ is comparable to $\langle\nu'/h\rangle$, so the integral over this region is bounded by

$$Ch^{-k} \langle\nu'/h\rangle^{l-k/2} \int_{|\mu'| \leq (1/2)|\nu'|} \langle\mu'/h\rangle^{l'-k/2} \, d\mu' \leq C|\nu'/h|^{l-k/2}(1 + |\nu'/h|^{l'+k/2+\varepsilon})$$

for any $\varepsilon > 0$.

(2) $2|\nu' - \mu'| \leq |\nu'|$. Here $\langle\mu'/h\rangle$ is comparable to $\langle\nu'/h\rangle$, so as above the integral over this region is bounded by

$$C|\nu'/h|^{l-k/2}(1 + |\nu'/h|^{l'+k/2+\varepsilon})$$

for any $\varepsilon > 0$.

(3) $2|\nu'| \leq |\mu'|$. Here $\langle(\nu' - \mu')/h\rangle$ is comparable to $\langle\mu'/h\rangle$, so the integral is bounded by

$$Ch^{-k} \int_{|\mu'| \geq 2|\nu'|} \langle\mu'/h\rangle^{l+l'-k} \, d\mu' \leq C|\nu'/h|^{l+l'},$$

since $l + l' < 0$.

(4) $(1/2)|\nu'| \leq |\mu'| \leq 2|\nu'|$ and $|\nu'| \leq 2|\nu' - \mu'|$. Here $\langle(\nu' - \mu')/h\rangle$ is comparable to $\langle\mu'/h\rangle$, so the integral is bounded by

$$Ch^{-k} \int_{(1/2)|\nu'| \leq |\mu'| \leq 2|\nu'|} \langle\mu'/h\rangle^{l+l'-k} \, d\mu' \leq C|\nu'/h|^{l+l'},$$

since $l + l' < 0$.

Thus, when $|\nu'| \geq h$, we conclude that

$$|c(x'', y'', \nu', \zeta', \zeta'')| \leq C(|\nu'/h|^{l+l'+\varepsilon} + |\nu'/h|^{l-k/2+\varepsilon} + |\nu'/h|^{l'-k/2+\varepsilon})$$

for any $\varepsilon > 0$. On the other hand, if $|\nu'| \leq h$, then

$$|c(x'', y'', \nu', \zeta', \zeta'')| \leq Ch^{-k} \int \langle\mu'/h\rangle^{l+l'-k} \, d\mu' \leq C.$$

provided that $l + l' < 0$. Bounds on the derivatives are established in precisely the same way.

It remains to prove the statement about the principal symbols. Note that the product $\sigma_h^{\Lambda_1}(A) \cdot \sigma_h^{\Lambda_1}(B)$ is the product

$$(\mathcal{F}_h')^{-1} a(x'', x', \zeta', \zeta'') \cdot (\mathcal{F}_h')^{-1} b(y'', y', \zeta', \zeta'')|_{x=y},$$

where the first inverse Fourier transform takes $\eta' \mapsto x'$, and the second takes $\mu' \mapsto y'$. Thus $\sigma_h^{\Lambda_1}(A) \cdot \sigma_h^{\Lambda_1}(B)$ at $(x'', \nu', \zeta', \zeta'')$ is the inverse Fourier transform of a convolution,

$$(2\pi h)^{-k} \int e^{\frac{i}{h}\langle x', \nu' \rangle} \int a(x'', \nu' - \mu', \zeta', \zeta'') b(x'', \mu', \zeta', \zeta'')\, d\mu' d\nu'.$$

On the other hand, $\sigma_h^{\Lambda_1}(AB)$ is given by

$$(2\pi h)^{-k} \int e^{\frac{i}{h}\langle x', \nu' \rangle} \int a(x'', \nu' - \mu', \mu' + \zeta', \zeta'') b(x'', \mu', \mu' + \zeta', \zeta'')\, d\mu'\, d\nu'$$

The only difference between these expressions is that ζ' in the first is replaced by $\mu' + \zeta'$ in the second. Taylor expanding a at ζ' in the second expression, we can write

$$\begin{aligned} a(x'', \nu' - \mu', \mu' + \zeta', \zeta'') &= a(x'', \nu' - \mu', \zeta', \zeta'') \\ &\quad + h \cdot \int_0^1 \langle \mu'/h, \partial_{\zeta'} a(x'', \nu' - \mu', \zeta' + t\mu', \zeta'') \rangle \, dt. \end{aligned}$$

The integral on the right hand is estimated by

$$Ch^\delta \langle \mu'/h \rangle^\delta \langle (\nu' - \mu')/h \rangle^{l-k/2},$$

for any $\delta \in [0,1]$, with similar bounds for its derivatives since μ' is bounded. A similar expansion also holds for $b(x'', \mu', \mu' + \zeta', \zeta'')$ in terms of $b(x'', \mu', \mu', \zeta'')$ modulo a remainder bounded by

$$Ch^\delta \langle \mu'/h \rangle^{l' + \delta - k/2},$$

along with derivative bounds. In particular,

$$a(x'', \nu' - \mu', \mu' + \zeta', \zeta'') b(x'', \mu', \mu' + \zeta', \zeta'') = a(x'', \nu' - \mu', \zeta', \zeta'') b(x'', \mu', \zeta', \zeta'')$$

modulo a remainder bounded by

$$Ch^\delta \langle (\nu' - \mu')/h \rangle^{l-k/2} \langle \mu'/h \rangle^{l' + \delta - k/2}$$

for any $\delta \in [0,1]$, along with derivative bounds. As above, if $l + l' < -\delta$, then the resulting integral in (μ', ν') yields an element of $h^\delta S_{\Lambda_1}^{L+\delta}$, since

$$L + \delta > \max(l, l' + \delta, l + l' + \delta - k/2).$$

Arguing similarly for the derivatives completes the proof. $\square$

We also need uniform L^2 mapping properties of operators with singular symbol. It suffices to consider the local situation $K_A \in I_h^{l,\mathrm{comp}}(\mathbb{R}^{2n}; \Lambda_0, \Lambda_1)$.

Lemma 6.15. *Let Λ_0, Λ_1 be given by (6.12). If $K_A \in I_h^{l,\mathrm{comp}}(\mathbb{R}^{2n}; \Lambda_0, \Lambda_1)$ and $s \geq 0$ is such that $l - s < -k/2$, then*

$$\|A\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq Ch^{-s}.$$

In particular, if $l < -k/2$, then A is uniformly bounded.

Proof. Using the left quantization (6.13), we may assume that K_A is parametrized by

$$K_A = (2\pi h)^{-n-k} \int e^{\frac{i}{h}(\langle x',\eta'+\xi'\rangle - \langle y',\xi'\rangle + \langle x''-y'',\xi''\rangle)} a(x'',\eta',\xi',\xi'')\, d\xi' d\xi'' d\eta', \tag{6.15}$$

where $a \in S_h^{l-k/2,\mathrm{comp}}(\mathbb{R}^{n-k}; \mathbb{R}^k; \mathbb{R}^{n-k})$. We bound this operator on $L^2(\mathbb{R}^n)$ by viewing it as a pseudodifferential operator on $\mathbb{R}^{n-k}_{x''}$ with values in uniformly bounded operators on $L^2(\mathbb{R}^k_{x'})$. Thus we write

$$K_A = (2\pi h)^{-n+k} \int e^{\frac{i}{h}\langle x''-y'',\xi''\rangle} \mathcal{A}(x'',\xi'')\, d\xi'',$$

where for each (x'',ξ'') the operator $\mathcal{A}(x'',\xi'')$ has kernel

$$K_{\mathcal{A}}(x',y';x'',\xi'') = (2\pi h)^{-2k} \int e^{\frac{i}{h}(\langle x',\zeta'\rangle - \langle y',\xi'\rangle)} a(x'',\zeta'-\xi',\xi',\xi'')\, d\zeta' d\xi'.$$

We now show that

$$\mathcal{A}(x'',\xi'') \in S^0(\mathbb{R}^{n-k}; \mathbb{R}^{n-k}; \mathcal{L}(L^2(\mathbb{R}^k))).$$

Because $\mathcal{A}(x'',\xi'')$ has compact support in (x'',ξ''), to prove the lemma it suffices to show

$$\|D^\alpha_{x''} D^\beta_{\xi''} \mathcal{A}(x'',\xi'')\|_{L^2(\mathbb{R}^k)\to L^2(\mathbb{R}^k)} \leq Ch^{-s}$$

for all multiindices α, β. On the other hand,

$$\|D^\alpha_{x''} D^\beta_{\xi''} \mathcal{A}(x'',\xi'')\|_{L^2(\mathbb{R}^k)\to L^2(\mathbb{R}^k)} = \|\mathcal{F}'_h (D^\alpha_{x''} D^\beta_{\xi''} \mathcal{A}(x'',\xi''))(\mathcal{F}')^{-1}_h\|_{L^2(\mathbb{R}^k)\to L^2(\mathbb{R}^k)},$$

since $h^{-k/2}\mathcal{F}'_h$ is unitary ($\mathcal{F}'_h$ denotes semiclassical Fourier transform only in the primed variables). The conjugated operator on the right, which we denote by $\widehat{\mathcal{A}}(x'',\xi'')$, has kernel

$$(\zeta',\xi') \mapsto (2\pi h)^{-k} D^\alpha_{x''} D^\beta_{\xi''} a(x'',\zeta'-\xi',\xi',\xi'').$$

Since $a(x'',\eta',\xi',\xi'')$ has compact support in η', for any $s \geq 0$,

$$(2\pi h)^{-k} |a(x'',\zeta'-\xi',\xi',\xi'')| \leq Ch^{-k} h^{-s} \langle (\zeta'-\xi')/h \rangle^{l-k/2-s}.$$

By Schur's lemma, it follows that $h^s \widehat{\mathcal{A}}(x'',\xi'')$ is uniformly bounded on $L^2(\mathbb{R}^k)$ provided that $l - k/2 - s < -k$, completing the proof. ∎

Remark 6.16. Lemma 6.15 is equally valid if K_A is given by the oscillatory integral (6.15) where the amplitude has compact support in η' uniformly with respect to the other variables, but is not of compact support in (x'',ξ',ξ''), provided that bounds of the form

$$|D^\alpha_{x''} D^\beta_{\xi''} a(x'',\eta',\xi',\xi'')| \leq C_{\alpha\beta} \langle \eta'/h \rangle^{l-k/2} \langle \xi'' \rangle^{-|\beta|}$$

are valid.

6.4. **Homogeneous paired Lagrangian distributions.** We also need another class of paired Lagrangian distributions, which have wavefront set at fiber-infinity. Again, it will suffice to consider conormal bundles of nested submanifolds. Let (x,ξ',ξ'') be coordinates on $\mathbb{R}^m\times\mathbb{R}^k\times\mathbb{R}^n$.

Definition 6.17. We say that an h-dependent function $a=a(x,\xi',\xi'';h)\in\mathcal{C}^\infty(\mathbb{R}^m_x\times\mathbb{R}^k_{\xi'}\times\mathbb{R}^n_{\xi''})$ is in $S^{q,r}(\mathbb{R}^m_x;\mathbb{R}^n_{\xi''};\mathbb{R}^k_{\xi'})$ if it satisfies the product-type estimates

$$|D^\alpha_{\xi'}D^\beta_{\xi''}D^\gamma_x a(x,\xi',\xi'')|\le C_{\alpha\beta\gamma}\left\langle(\xi',\xi'')\right\rangle^{q-|\beta|}\left\langle\xi'\right\rangle^{r-|\alpha|}$$

for all multiindices α,β,γ.

We use the same notation as in Section 6.1, so that $S_0\subset S_1\subset\mathbb{R}^m$, as well as $\Lambda_0=N^*S_0$ and $\Lambda_1=N^*S_1$. We consider oscillatory integrals of the form

$$(2\pi h)^{d-m}\int e^{\frac{i}{h}(\langle x',\xi'\rangle+\langle x'',\xi''\rangle)}a(x,\xi',\xi'')\,d\xi'd\xi'', \tag{6.16}$$

where $a\in S^{q,r}=S^{q,r}(\mathbb{R}^m_x;\mathbb{R}^{m-k-d}_{\xi''};\mathbb{R}^k_{\xi'})$.

Definition 6.18. We say that $u\in I^{p,l}_{\triangleleft,c}(\mathbb{R}^m;\Lambda_0,\Lambda_1)$ if $\operatorname{supp}u$ is compact, and u is of the form (6.2) for some $a\in S^{q,r}$, where $q=p-m/4+k/2+d/2$ and $r=l-k/2$.

This is a direct semiclassical adaptation of the paired Lagrangian distributions studied in [DHUV], and for this reason we take various facts for granted that were explicitly demonstrated for the related space $I^{l,\mathrm{comp}}_h(\mathbb{R}^m;\Lambda_0,\Lambda_1)$. We will need the following:

- Any u of the form (6.16) can be written in terms of an amplitude $\tilde a(x''',\xi',\xi'')$ depending only on x''' in the base variables.
- The space $I^{p,l}_{\triangleleft,c}(\mathbb{R}^m;\Lambda_0,\Lambda_1)$ is invariant under diffeomorphisms of $\mathbb{R}^m$ preserving S_1 and S_0, which allows for the definition of $I^{p,l}_{\triangleleft}$ on a general manifold.

However, we will not need to develop any symbol calculus for this class of distributions.

In this context, $I^{p,l}_{\triangleleft}(\mathbb{R}^m;\Lambda_0,\Lambda_1)$ arises when multiplying $u\in I^{\mathrm{comp}}_h(X;N^*Y)$ by $v\in I^{[\mu]}(Z)$, where Y,Z are two transverse submanifolds of a manifold X. It suffices to consider the model case; thus we take $X=\mathbb{R}^m$ with coordinates

$$(x',x'',x''')\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}\times\mathbb{R}^{m-d_1-d_2},$$

and then set

$$Y=\{x'=0\},\quad Z=\{x''=0\}.$$

Thus Y and Z have codimension d_1 and d_2 in $\mathbb{R}^m$, respectively, while $Y\cap Z$ has codimension d_1+d_2.

Lemma 6.19. *If* $u \in I_h^{\mathrm{comp}}(\mathbb{R}^m; N^*Y)$ *and* $v \in I_c^{[\mu]}(Z)$, *then*

$$\begin{aligned} uv \in\ & I_h^{\mu+d_2/2,\mathrm{comp}}(\mathbb{R}^m; N^*(Y\cap Z), N^*Y) \\ & + h^{-\mu-m/4+d_1/2} I_{\triangleleft}^{\mu-m/4+d_2/2,-\infty}(\mathbb{R}^m; N^*(Y\cap Z), N^*Z) \end{aligned}$$

Proof. Since $\operatorname{codim} Y = d_1$, modulo a $\mathcal{C}_c^\infty(\mathbb{R}^m)$ remainder we can write

$$u = (2\pi h)^{-m/4-d_1/2} \int e^{\frac{i}{h}\langle x',\xi'\rangle} a(x,\xi')\, d\xi',$$

where $a(x,\xi') \in \mathcal{C}_c^\infty(\mathbb{R}_x^m \times \mathbb{R}_{\xi'}^{d_1})$. On the other hand, modulo $\mathcal{C}_c^\infty(\mathbb{R}^m)$, we can find a Kohn–Nirenberg symbol $b(x,\eta'') \in S^\mu(\mathbb{R}_x^m; \mathbb{R}_{\xi''}^{d_2})$ such that

$$v = (2\pi h)^{-d_2} \int e^{\frac{i}{h}\langle x'',\xi''\rangle} b(x,\xi''/h)\, d\xi''. \tag{6.17}$$

Here we made the usual semiclassical change of variables $\eta'' = \xi''/h$. The product uv is given by

$$uv = (2\pi h)^{-m/4-d_1/2-d_2} \int e^{\frac{i}{h}(\langle x',\xi'\rangle + \langle x'',\xi''\rangle)} a(x,\xi') b(x,\xi''/h)\, d\xi' d\xi''. \tag{6.18}$$

Now insert a smooth cutoff function $\chi(\xi'')$ such that $\chi = 1$ near $\xi'' = 0$. Thus we may split $uv = w_0 + w_1$ as a sum of two oscillatory integrals where w_0 has amplitude χab, and w_1 has amplitude $(1-\chi)ab$. For the term w_0, let

$$c_0(x,\xi',\xi'') = \chi(\xi'') a(x,\xi') b(x,\xi''/h).$$

Thus $c_0 \in S_h^{\mu,\mathrm{comp}}$, and $-m/4 - d_1/2 - d_2 = -3m/4 - d_2/2 + \dim(Y\cap Z)/2$ since $\dim(Y\cap Z) = m - d_1 - d_2$. In particular,

$$\begin{aligned} w_0 &= (2\pi h)^{-m/4-d_1/2-d_2} \int e^{\frac{i}{h}(\langle x',\xi'\rangle + \langle x'',\xi''\rangle)} c_0(x,\xi',\xi'')\, d\xi' d\xi'' \\ &\in I_h^{\mu+d_2/2,\mathrm{comp}}(\mathbb{R}^m; N^*(Y\cap Z), N^*Y). \end{aligned}$$

For the second term w_1, observe that $|\xi''| \geq C_0$ on $\operatorname{supp}(1-\chi(\xi''))$ for some $C_0 > 0$. Let

$$c_1(x,\xi',\xi'') = h^\mu (1-\chi(\xi'')) a(x,\xi') b(x,\xi''/h).$$

Since c_1 is in fact compactly supported in ξ', we certainly have the symbol bounds

$$|D_{\xi'}^\alpha D_{\xi''}^\beta D_x^\gamma c_1(x,\xi',\xi'')| \leq C_{\alpha\beta\gamma N} \langle \xi' \rangle^{-N} \langle (\xi',\xi'') \rangle^{\mu-|\beta|}.$$

This shows that

$$\begin{aligned} w_1 &= h^{-\mu}(2\pi h)^{-m/4+d_1/2}(2\pi h)^{-d_1-d_2} \int e^{\frac{i}{h}(\langle x',\xi'\rangle + \langle x'',\xi''\rangle)} c_1(x,\xi',\xi'')\, d\xi' d\xi'' \\ &\in h^{-\mu-m/4+d_1/2} I_{\triangleleft}^{\mu-m/4+d_2/2,-\infty}(\mathbb{R}^m; N^*(Y\cap Z), N^*Z) \end{aligned}$$

as desired. □

Remark 6.20. If we assume that $(x,\xi'')\mapsto b(x,\xi''/h)$ in (6.18) has compact support in (x,ξ''), then we are left with only a w_0 term in the proof above, i.e., an element of $I_h^{\mu+d_2/2,\mathrm{comp}}(\mathbb{R}^m;N^*(Y\cap Z),N^*Y)$.

Remark 6.21. In the notation of the proof of Lemma 6.19, let $O\subset T^*\mathbb{R}^m$ be an open neighborhood of $\mathrm{WF}_h(u)\cap N^*Y$. Then $w_0\in I_h^{\mu+d_2/2,\mathrm{comp}}(\mathbb{R}^m;N^*(Y\cap Z),N^*Y)$ can always be chosen so that $\mathrm{WF}_h(w_0)\subset O$. Indeed, by Lemma 6.8 and semiclassical wavefront set calculus,

$$\begin{aligned}\mathrm{WF}_h(w_0)\subset{}&(N^*Y\cap\mathrm{WF}_h(u))\\&\cup\{(0,0,x''',\xi',\xi'',0):(0,0,x''',\xi',0,0)\in\mathrm{WF}_h(u),\ \xi''\in\operatorname{supp}\chi\}.\end{aligned}$$

Now $\mathrm{WF}_h(w_0)$ is closed, and $\mathrm{WF}_h(w_0)\cap N^*Y\subset O$; since O is open, the result follows by taking χ with sufficiently small support. Observe that this can be thought of as decomposing $v=v_0+v_1$ itself into a sum, where we insert $\chi(\xi'')$ and $1-\chi(\xi'')$ into (6.17).

We now return to the setting of Section 6.3: let X be an n-dimensional manifold, and $Y\subset X$ a codimension k submanifold. We then consider operators with Schwartz kernels

$$K_A\in I_\triangleleft^{p,l}(X\times X;N^*((X\times Y)\cap\mathrm{diag}),N^*(X\times Y)).$$

We also need to consider the case when $X\times Y$ is replaced with $Y\times X$. Although K_A is not compactly microlocalized, it nevertheless defines an h-tempered family of operators $A:\mathcal{C}^\infty(X)\to\mathcal{C}^{-\infty}(X)$.

As in Section 6.3, it suffices work on $X=\mathbb{R}^n$ with coordinates $x=(x',x'')\in\mathbb{R}^k\times\mathbb{R}^{n-k}$, where $Y=\{x'=0\}$. If (x,y,ξ,η) are the corresponding coordinates on $T^*(\mathbb{R}^n\times\mathbb{R}^n)$, let

$$\Lambda_R=N^*\{y'=0\},\quad \Lambda_L=N^*\{x'=0\}.$$

We work with the Lagrangian pair

$$\Lambda_1=\Lambda_R\text{ or }\Lambda_L,\quad \Lambda_0=\{x'=y'=0,\ x''=y'',\ \xi''=-\eta''\}.\tag{6.19}$$

For instance, if $\Lambda_1=\Lambda_R$, then we can parametrize $K_A\in I_\triangleleft^{p,l}(\mathbb{R}^{2n};\Lambda_0,\Lambda_1)$ by

$$K_A=(2\pi h)^{-n-k}\int e^{\frac{i}{h}(\langle x',\xi'\rangle-\langle y',\zeta'\rangle+\langle x''-y'',\xi''\rangle)}a(x'',\zeta',\xi',\xi'')\,d\xi'd\xi''d\zeta',\tag{6.20}$$

where now $a(y'',\zeta',\xi',\xi'')$ satisfies the symbol bounds

$$|D_{\zeta'}^\alpha D_\xi^\beta D_{y''}^\gamma a(y'',\zeta',\xi',\xi'')|\le C_{\alpha\beta\gamma}\left<\xi\right>^{l-n/2-|\beta|}\left<(\zeta',\xi)\right>^{p+(n-k)/2-|\alpha|}.$$

We need uniform mapping properties of A, which can be deduced as in [DHUV, Proposition 5.14].

Lemma 6.22. *Let Λ_0, Λ_1 be defined by (6.19). Let $K_A \in I^{p,l}_{\triangleleft}(\mathbb{R}^{2n}; \Lambda_0, \Lambda_1)$ be of the form (6.20). If $p + l < -k/2$ and $p < -n/2$, then*

$$\|A\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le Ch^{-k}.$$

Proof. For concreteness, assume that $\Lambda_1 = \Lambda_R$; the same proof can be repeated for Λ_L. We argue as in Lemma 6.15, viewing A as a pseudodifferential operator on $\mathbb{R}^{n-k}$ with an operator-valued symbol $\mathcal{A}(x'', \xi'')$ given by

$$K_{\mathcal{A}}(x', y'; y'', \xi'') = (2\pi h)^{-2k} \int e^{\frac{i}{h}(\langle x', \zeta'\rangle - \langle y', \xi'\rangle)} a(x'', \zeta', \xi', \xi'')\, d\zeta' d\xi'.$$

Conjugating by the Fourier transform as in Lemma 6.15, the problem is reduced to showing that the operator with Schwartz kernel

$$(\zeta', \xi') \mapsto \langle \xi'' \rangle^{|\beta|} D^\alpha_{y''} D^\beta_{\xi''} a(x'', \zeta', \xi', \xi'') \tag{6.21}$$

has uniformly bounded operator norm on $L^2(\mathbb{R}^k)$ (we multiplied by a factor of $(2\pi h)^k$). As in [DHUV, Proposition 5.14], it suffices to show that the Hilbert–Schmidt norm of this operator is uniformly bounded.

Write $a = a_1 + a_2$, where $\langle \xi \rangle \le \langle \zeta' \rangle$ on $\operatorname{supp} a_1$, and $\langle \zeta' \rangle \le 2 \langle \xi \rangle$ on $\operatorname{supp} a_2$. For a_1,

$$\langle \xi'' \rangle^{|\beta|} |D^\alpha_{y''} D^\beta_{\xi''} a_1(y'', \zeta', \xi', \xi'')| \le C \langle \xi \rangle^{l-n/2} \langle \zeta' \rangle^{p+(n-k)/2}$$

by the support assumption on a_1, and the proof proceeds just as in [DHUV, Proposition 5.14]. For a_2 the proof is even simpler, since then

$$\langle \xi' \rangle^{k/2+\delta} \langle \zeta' \rangle^{k/2+\delta} \langle \xi'' \rangle^{|\beta|} |D^\alpha_{y''} D^\beta_{\xi''} a_1(y'', \zeta', \xi', \xi'')| \le C \langle (\zeta', \xi) \rangle^{p+l+k/2+2\delta}.$$

If $\delta > 0$ is sufficiently small, then the right hand side is uniformly bounded, and this implies that the kernel (6.21) is uniformly square-integrable in (ζ', ξ'). □

We continue studying operators with kernels in $I^{p,l}_{\triangleleft}(\mathbb{R}^{2n}; \Lambda_0, \Lambda_1)$, but the results that follow are no longer coordinate invariant.

Lemma 6.23. *Let Λ_0, Λ_1 be as in (6.19). Let $K_A \in I^{p,l}_{\triangleleft}(\mathbb{R}^{2n}; \Lambda_0, \Lambda_1)$ be of the form (6.20). If $l < -n/2$ and $p < -n/2 - k/2$, then K_A is continuous, and*

$$|K_A(x', x'', y', y'')| \le Ch^{-n-k} \langle x'/h \rangle^{-N} \langle y'/h \rangle^{-N} \langle (x'' - y'')/h \rangle^{-N}$$

for each $N \ge 0$.

Proof. Again, assume that $\Lambda_1 = \Lambda_R$. As in Lemma 6.22, decompose $a = a_1 + a_2$, where $\langle \xi \rangle \le \langle \zeta' \rangle$ on $\operatorname{supp} a_1$, and $\langle \zeta' \rangle \le 2 \langle \xi \rangle$ on $\operatorname{supp} a_2$. The hypotheses imply that $a \in L^1(\mathbb{R}^{n+k})$, so K_A is continuous and $|K_A(x', x', y', y'')| \le Ch^{-n-k}$. Furthermore, integration by parts shows that

$$|x'/h|^{N_1} |y'/h|^{N_2} |(x'' - y'')/h|^{N_3} |K_A(x', x', y', y'')| \le Ch^{-n-k}$$

for every $N_1, N_2, N_3 \ge 0$. □

We now proceed with some $L^\infty(\mathbb{R}^k; L^2(\mathbb{R}^{n-k}))$ bounds which improve the loss in h that occurs in Lemma 6.22; these bounds will be essential to obtaining optimal estimates for the size of the reflected wave in our propagation argument later on.

We write $u(x')$ for the function $x'' \mapsto u(x', x'')$ on $\mathbb{R}^{n-k}$.

Lemma 6.24. *Let $K_A \in I^{p,l}_{\triangleleft}(\mathbb{R}^{2n}; \Lambda_0, \Lambda_1)$ be of the form (6.20). If $l < -n/2$ and $p < -n/2 - k/2$, then*

$$|\langle Au, v\rangle| \leq C\|u\|_{L^\infty(\mathbb{R}^k;L^2(\mathbb{R}^{n-k}))}\|v\|_{L^\infty(\mathbb{R}^k;L^2(\mathbb{R}^{n-k}))}.$$

Proof. Write the $L^2(\mathbb{R}^n)$ pairing,

$$\langle Au, v\rangle = \int \left(\int K_A(x', x'', y', y'')u(y', y'')v(x', x'')\, dx''dy'' \right) dx'dy'.$$

By Lemma 6.23 and Schur's lemma,

$$\begin{aligned} |\langle Au, v\rangle| &\leq Ch^{-2k} \int \langle x'/h\rangle^{-N} \langle y'/h\rangle^{-N} \|u(y')\|_{L^2(\mathbb{R}^{n-k})}\|v(x')\|_{L^2(\mathbb{R}^{n-k})}\, dx'dy' \\ &\leq C\|u\|_{L^\infty(\mathbb{R}^k;L^2(\mathbb{R}^{n-k}))}\|v\|_{L^\infty(\mathbb{R}^k;L^2(\mathbb{R}^{n-k}))}, \end{aligned}$$

which completes the proof. □

We also need an $L^\infty(\mathbb{R}^k; L^2(\mathbb{R}^{n-k})) \to L^1(\mathbb{R}^k; L^2(\mathbb{R}^{n-k}))$ boundedness result which similarly improves upon the loss in h in Lemma 6.22.

Lemma 6.25. *Let $K_A \in I^{p,l}_{\triangleleft}(\mathbb{R}^{2n}; \Lambda_0, \Lambda_1)$ be of the form (6.20). If $l < -n/2$ and $p < -n/2 - k/2$, then*

$$\|A\|_{L^\infty(\mathbb{R}^k;L^2(\mathbb{R}^{n-k}))\to L^1(\mathbb{R}^k;L^2(\mathbb{R}^{n-k}))} \leq C.$$

Proof. By Cauchy–Schwarz,

$$\begin{aligned} &\int \|Au(x')\|_{L^2(\mathbb{R}^{n-k})}\, dx' \\ &\qquad \leq \int \left(\int |K_A(x,y)||K_A(x,z)||u(y',y'')|^2 dy'dy''dz'dz''dx'' \right)^{1/2} dx', \end{aligned}$$

and by Lemma 6.23, changing variables to replace x'', z'' by $(x''-z'')/h$, $(x''-y'')/h$, and beginning with the y'' integral, we find that this is bounded by a constant times $\|u\|_{L^\infty(\mathbb{R}^k;L^2(\mathbb{R}^{n-k}))}$ □

Finally, we will need to consider composition of $A \in I^{p,-\infty}_{\triangleleft}(\mathbb{R}^{2n}; \Lambda_0, \Lambda_1)$ with a family of pseudodifferential operators on $\mathbb{R}^{n-k}$ depending parametrically on $\mathbb{R}^k$ (cf. the discussion of "tangential" operators in Section 2.2). Thus we consider an operator $Q \in \mathcal{C}^\infty(\mathbb{R}^k; \Psi^0_h(\mathbb{R}^{n-k}))$ with Schwartz kernel

$$K_Q(x', x'', y', y'') = (2\pi h)^{-n+k}\delta(x'-y') \int e^{\frac{i}{h}\langle x''-y'', \eta''\rangle} q(y', y'', \eta'')\, d\eta'', \tag{6.22}$$

where $q \in S^0(\mathbb{R}^n; \mathbb{R}^{n-k})$. Supposing that $\Lambda_1 = \Lambda_R$, compose with A given (6.20):

$$
\begin{aligned}
&K_{AQ}(x', x'', y', y'') \\
&\quad = (2\pi h)^{-n-k} \int e^{\frac{i}{h}(\langle x', \xi' \rangle - \langle y', \zeta' \rangle + \langle x'' - y'', \xi'' \rangle)} a(x'', \zeta', \xi', \xi'') q(y', y'', \xi'')\, d\xi d\zeta' dz'' d\eta'.
\end{aligned}
$$

Clearly the resulting operator is in $I_{\triangleleft}^{p,-\infty}(\mathbb{R}^{2n}; \Lambda_0, \Lambda_1)$. The same argument works if $\Lambda_1 = \Lambda_L$.

Lemma 6.26. *Let Q, A be given by (6.22) and (6.20), respectively. Suppose that*

$$(x'', \zeta', \xi', \xi'') \in \operatorname{esssupp}(a) \Longrightarrow (x', x'', \xi'') \notin \mathrm{WF}_h(Q) \text{ for each } x' \in \mathbb{R}^k.$$

If $p < -n/2$, then

$$\|AQ\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} = \mathcal{O}(h^\infty), \quad \|QA\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} = \mathcal{O}(h^\infty).$$

Proof. As in the proof of Lemma 6.22, we can view A as being a $\mathcal{L}(L^2(\mathbb{R}^k))$-valued operator, provided $p < -n/2$. Similarly, we can view Q as an operator on $\mathbb{R}^{n-k}$ with a $\mathcal{L}(L^2(\mathbb{R}^k))$-valued symbol; in this case the symbol of Q just acts on $L^2(\mathbb{R}^k)$ as a multiplication operator. The assumed relation between $\operatorname{esssupp}(a)$ and $\mathrm{WF}_h(Q)$ guarantees that essential supports of their operator valued-symbols do not intersect, hence

$$\|AQ\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} = \mathcal{O}(h^\infty)$$

by the calculus of operator-valued pseudodifferential operators. Either directly or by taking adjoints, QA is similarly negligible. $\square$

7. Diffractive improvements

We now return to our operator $P = -h^2\Delta_g + V$ on X and prove Theorems 2, 3. Recall that we establish these theorems only when $\alpha > 1$.

7.1. **Decomposing the potential.** We need to consider properties of the potential appearing in our operator $P = -h^2\Delta_g + V$ more carefully. All the material in this section applies to arbitrary codimension. Thus we let (X, g) be an n dimensional Riemannian manifold and $Y \subset X$ a codimension k submanifold. We work in a coordinate patch $\mathcal{U}$, identified with a subset of $\mathbb{R}^n$, with coordinates (x', x''), where $Y \cap \mathcal{U} = \{x' = 0\}$. We will frequently take advantage of this coordinate decomposition to write functions on $\mathcal{U}$ as functions in x' with values in some function space in x'', in order to obtain mixed-norm bounds. Assume that

$$V \in I^{[\mu]}(\mathbb{R}^n; N^*\{x' = 0\})$$

has compact support in $\mathcal{U}$. Thus we can write

$$V(x) = (2\pi h)^{-k} \int e^{\frac{i}{h}\langle x',\eta'\rangle} v(x,\eta'/h) d\eta'$$

for some $v(x,\eta') \in S^{\mu}(\mathbb{R}^n;\mathbb{R}^k)$ with compact support in the x variables. As in the remark following Lemma 6.19, we decompose $V = V_0 + V_1$, where

$$V_0(x) = (2\pi h)^{-k} \int e^{\frac{i}{h}\langle x',\eta'\rangle} \chi(\eta'/\tau) v(x,\eta'/h)\, d\eta', \tag{7.1}$$

and $V_1 = V - V_0$. Here $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^k;[0,1])$ is identically one near $\eta' = 0$, and $\tau > 0$ is a parameter which will be chosen small, so as to limit $\mathrm{WF}_h(V_0)$ to a neighborhood of the zero-section in the conormal bundle to $\{x' = 0\}$.

We remark for later use that provided $\mu < -k$, we have a trivial L^∞ estimate with decay in h,

$$\|V_1\|_{L^\infty} = \mathcal{O}(h^{-k-\mu}). \tag{7.2}$$

We also have a useful mixed-norm bound which will be used occasionally in place of Lemma 6.23 to directly bound certain multiplication operators (the proof is completely analogous to that of Lemma 6.23):

Lemma 7.1. *If $\mu < -k$, then $\|V_1\|_{L^1(\mathbb{R}^k;L^\infty(\mathbb{R}^{n-k}))} = \mathcal{O}(h^{-\mu})$.*

Proof. Recall that

$$V_1(x) = (2\pi h)^{-k} \int e^{\frac{i}{h}\langle x',\eta'\rangle} (1-\chi(\eta'/\tau)) v(x,\eta'/h)\, d\eta'.$$

Owing to the support properties of χ we have the symbol estimate

$$|D^\beta_{\eta'}(1-\chi(\eta'/\tau))v(x'',\eta'/h)| \le C_\beta h^{-\mu}\langle \eta'\rangle^{\mu-|\beta|}$$

for all multiindices β, where C_β depends on τ as well. Repeated integration by parts shows that

$$|V_1(x)| \le C_N h^{-\mu-k}\langle x'/h\rangle^{-N}$$

which implies the desired estimate by integration and change of variables. $\square$

Fix $A \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n)$ with compact support in $\mathcal{U}$, which will later play the role of the commutant in a positive commutator argument. Write $\Lambda_0 = N^*(\{x'=0\}\cap \mathrm{diag})$ and $\Lambda_1 = N^*\mathrm{diag}$. According to the proof of Lemma 6.19 (see Remark 6.20),

$$K_{V_0 A},\, K_{AV_0} \in I_h^{\mu+k/2,\mathrm{comp}}(\mathbb{R}^{2n},\Lambda_0,\Lambda_1).$$

The kernel of A has wavefront set a compact subset of $(O\times O')\cap N^*\mathrm{diag}$, where O is open in T^*X (with the usual notation $O' = \{(x,-\xi) : (x,\xi)\in O\}$.) As noted in Remark 6.21, by taking $\tau > 0$ sufficiently small in (7.1), we can arrange that the kernels satisfy

$$\mathrm{WF}_h(K_{V_0A}) \cup \mathrm{WF}_h(K_{AV_0}) \subset O \times O'. \tag{7.3}$$

This is therefore true of the commutator $[A, V_0]$ as well. We also need to compute the principal symbol of $[A, V_0]$ along $N^*\mathrm{diag}$. A priori,

$$K_{[A,V_0]} \in I_h^{\mu+k/2,\mathrm{comp}}(\mathbb{R}^{2n}, \Lambda_0, \Lambda_1),$$

but of course the principal symbol of $[A, V_0]$ along $N^*\mathrm{diag}$ vanishes, so in fact

$$K_{[A,V_0]} \in hI_h^{\mu+k/2+1,\mathrm{comp}}(\mathbb{R}^{2n}, \Lambda_0, \Lambda_1).$$

To compute the principal symbol of $[A, V_0]$, it is easiest to use the change of variables formulas from Section 6.2.

Lemma 7.2. *With $a = \sigma_h(A)$, the principal symbol of $(i/h)[A, V_0]$ along Λ_1 is $\mathsf{H}_a V_0$ in $S^{\mu+k/2+1}_{\Lambda_1}/hS^{\mu+k/2+2}_{\Lambda_1}$*

Proof. Set $b(y'', \eta') = e^{-ih\langle D_{y'}, D_{\eta'}\rangle} v(y, \eta'/h)\chi(\eta')|_{y'=0}$, so the kernel of AV_0 is

$$K_{AV_0}(x, y) = (2\pi h)^{-n-k} \int e^{\frac{i}{h}(\langle y', \eta'\rangle + \langle x-y, \xi\rangle)} a(x, \xi) b(y'', \eta')\, d\eta' d\xi,$$

where without loss we can assume that a is the total left symbol of A. To put this in the framework of Section 6.2, set $z = x - y$, so that in terms of coordinates (y', z, x''),

$$K_{AV_0}(x, y) = (2\pi h)^{-n-k} \int e^{\frac{i}{h}(\langle y', \eta'\rangle + \langle z, \xi\rangle)} a(y' + z', x'', \xi) b(x'' - z'', \eta')\, d\eta' d\xi.$$

It remains to express this in terms of coordinates (x', z, x''), namely we pull back by the map $(x', z, x'') \mapsto (x' - z', z, x'')$. By (6.11), the symbol of this pullback is

$$\begin{aligned} &e^{-ih\langle D_z, D_\xi\rangle} \big(e^{-\frac{i}{h}\langle z', \eta'\rangle} a(x', x'', \xi) b(x'' - z'', \eta')\big)|_{z=0} \\ &\qquad = a(x', x'', \xi) b(x'', \eta') + \langle \eta', \partial_{\xi'} a(x', x'', \xi'')\rangle b(x'', \eta') \\ &\qquad\qquad - ih\langle \partial_{\xi''} a(x', x'', \xi''), \partial_{x''} b(x'', \eta')\rangle + hS^{\mu+k/2+2,\mathrm{comp}}_{\Lambda_1}. \end{aligned}$$

In the same (x', z, x'') coordinates, the total symbol of $V_0 A$ along Λ_1 is

$$a(x', x'', \xi) b(x'', \eta').$$

Subtracting this second expression from the first, we obtain the desired result (after integration by parts in η'). $\square$

Remark 7.3. If $Q \in \Psi_h^0(\mathbb{R}^n)$, then the kernel of $[Q, V_0]$ is not strictly part of the paired Lagrangian calculus developed in the previous sections; we will need to consider such an operator in Lemma 7.6 below. We therefore record two facts that remain true for $[Q, V_0]$.

First, let $q(x, \xi)$ be the total left symbol of Q. Arguing as in Section 6.2, it follows that $K_{[Q,V_0]}$ can be written in the form (6.15), with amplitude

$$e^{-ih\langle D_{z''}, D_{\xi''}\rangle} \left(q(x, \xi' + \eta', \xi'') b(x'' - z'', \eta') - q(x, \xi', \xi'') b(x'', \eta')\right)|_{z''=0}.$$

Taylor expanding $q(x, \xi' + \eta', \xi'')$ about (x, ξ', ξ'') and integrating by parts in η' shows that the kernel of $h^{-1}[Q, W_1]$ can be written in the form (6.15), with an amplitude $a(x'', \eta', \xi', \xi'')$ that is compactly supported in η' and satisfies

$$|D^\alpha_{x''} D^\beta_{\xi''} a(x'', \eta', \xi', \xi'')| \leq C_{\alpha\beta} \langle \eta'/h \rangle^{\mu+1} \langle \xi'' \rangle^{-|\beta|}.$$

According to Remark 6.16, if $\mu < -k$ then this implies that for some $\gamma \in (0, 1/2]$,

$$\|[Q, V_1]\|_{L^2 \to L^2} = \mathcal{O}(h^{2\gamma}).$$

Secondly, let O be an open neighborhood of $\mathrm{WF}_h(Q)$ in $\overline{T^*}X$. Taking $\tau > 0$ sufficiently small in (7.1), we can still arrange that

$$\mathrm{WF}_h(K_{[Q,V_0]}) \subset O \times O',$$

as in (7.3). The point here is that this is true even when Q does not have compact microsupport.

As for the residual term V_1, we have

$$\begin{aligned} K_{AV_1} &\in h^{-\mu} I^{\mu - n/2 + k/2, -\infty}_{\triangleleft, c}(\mathbb{R}^{2n}, \Lambda_0, N^*\{y' = 0\}), \\ K_{V_1 A} &\in h^{-\mu} I^{\mu - n/2 + k/2, -\infty}_{\triangleleft, c}(\mathbb{R}^{2n}, \Lambda_0, N^*\{x' = 0\}). \end{aligned} \tag{7.4}$$

Observe that there is no gain in the commutator $[A, V_1]$ in terms of powers of h (or order of singularity) over AV_1 or $V_1 A$: we will simply estimate the summands in the commutator separately.

Lemma 7.4. *Let $A \in \Psi^{\mathrm{comp}}_h(\mathbb{R}^n)$. If $\mu < -k/2$, and $T \in \mathcal{C}^\infty_c(\mathbb{R}^k; \Psi^{\mathrm{comp}}_h(\mathbb{R}^{n-k}))$ satisfies*

$$(x, \xi) \in \mathrm{WF}_h(A) \Longrightarrow (x, \xi'') \in \mathrm{ell}_h(T),$$

then

$$\|AV_1 u\|_{L^2} + \|V_1 A u\|_{L^2} \leq C\|Tu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}.$$

Proof. Choose $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $A\psi = \psi A = A$, and let $t \in \mathcal{C}^\infty_c(\mathbb{R}^{n-k})$ be such that $t(\xi'') = 1$ on $\{\xi'' : (x, \xi) \in \mathrm{WF}_h(A)\}$. It suffices to prove the lemma with

$$T = \psi \mathrm{Op}_h(t) \psi.$$

To do this, we apply Lemma 6.26 with $Q = 1 - T_0$, with T_0 satisfying the same properties as T but microsupported in the elliptic set of T. This allows us to replace u with $T_0 u$ modulo $\mathcal{O}(h^\infty)\|u\|_{L^2}$ errors. We apply Lemma 6.22 to bound $V_1 A T_0 u$, while the $AV_1 T_0 u$ term is bounded similarly, following commutation of V_1 with T_0; by tangential smoothness of V, this yields an error term in the calculus $\mathcal{C}^\infty_c(\mathbb{R}^k; \Psi^{\mathrm{comp}}_h(\mathbb{R}^{n-k}))$ which can be estimated by $\|Tu\|_{L^2}$ where T is elliptic on $\mathrm{WF}_h(T_0)$. □

We will also need a slightly more refined decomposition of V_0 itself. With χ as in (7.1), write $V_0 = W_0 + W_1$, where

$$W_0 = (2\pi h)^{-k} \int e^{\frac{i}{h}\langle x', \eta'\rangle} \chi(\tilde\tau\eta/h) v(x, \eta'/h)\, d\eta'$$
$$= (2\pi)^{-k} \int e^{i\langle x', \eta'\rangle} \chi(\tilde\tau\eta) v(x, \eta')\, d\eta', \tag{7.5}$$

and $\tilde\tau > 0$ is a parameter. The point of this decomposition is that for $\mu + |\alpha| < -k$,

$$D^\alpha_{x',x''} W_1 \to 0 \text{ uniformly as } \tilde\tau \to 0, \tag{7.6}$$

whereas W_0 is smooth and independent of h. Also observe that the paired Lagrangian properties of AV_0 and V_0A described above also apply to AW_1 and W_1A.

7.2. **Elliptic estimates.** We prove an elliptic estimate for $P = -h^2\Delta_g + V$ involving ordinary semiclassical wavefront set. Although everything in this section applies to arbitrary codimension, for simplicity we restrict to codimension one; thus we assume that $V \in I^{[-1-\alpha]}(Y)$, where $\alpha > 0$.

Since we are ultimately interested in L^2 based wavefront set, the estimates we give are quite crude in terms of Sobolev regularity.

Proposition 7.5. *Let $\alpha > 0$ and $s \le \alpha + r$, where $s, r \in \mathbb{R} \cup \{+\infty\}$. Suppose that u is h-tempered in $H^1_h(X)$. If $\mathrm{WF}^r_h(u) = \emptyset$, then*

$$\mathrm{WF}^{1,s}_h(u) \subset \Sigma \cup \mathrm{WF}^{-1,s}_h(Pu).$$

Recall that the notation $\mathrm{WF}^{k,s}_h(u)$ for ordinary semiclassical wavefront set relative to $H^k_h(X)$ was introduced in Definition 2.6.

Proposition 7.5 follows from the quantitative estimate in Lemma 7.6 below; since stronger results are true away from $\overline{T^*_Y}X$, for the proof we assume that all operators have compact support in a coordinate patch $\mathcal{U}$ about Y.

We now obtain a semiclassical elliptic estimate. In contrast to Proposition 5.1, this estimate concerns ordinary, rather than b-pseudodifferential operators. Since the operators in question do not respect the interface Y, the resulting estimate has an α-dependent loss on the right side.

Lemma 7.6. *If $A, G \in \Psi^0_h$ satisfy $\mathrm{WF}_h(A) \subset \mathrm{ell}_h(G) \cap \mathrm{ell}_h(P)$, then*

$$\|Au\|_{H^1_h} \le C\|GPu\|_{H^{-1}_h} + Ch^\alpha \|u\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{H^1_h}$$

for each $u \in H^1_h(X)$.

Proof. The proof makes use of the decomposition $V = W_0 + W_1 + V_1$ described in Section 7.1. Let $P_{W_0} = -h^2\Delta_g + W_0$ and let p_{W_0} denote its principal symbol. Note that

$$\langle \zeta \rangle^{-2} p \neq 0 \text{ near } \mathrm{WF}_h(A),$$

where we have written $\zeta = (\xi, \eta)$. If $\tilde{\tau}_0 > 0$ is sufficiently small (where $\tilde{\tau}$ is the parameter appearing in (7.5)), then there is $c_0 > 0$ such that

$$\langle \zeta \rangle^{-2} |p_{W_0}| > c_0 \text{ near } \mathrm{WF}_h(A) \text{ for all } \tilde{\tau} \in (0, \tilde{\tau}_0). \tag{7.7}$$

Let $Z \in \Psi_h^{-2}$ be everywhere elliptic with principal symbol $\langle \zeta \rangle^{-2}$, and then set

$$q = \langle \zeta \rangle^2 \frac{\sigma_h(A)}{\sigma_h(P_{W_0})} \in S^0(T^*X). \tag{7.8}$$

If $Q \in \Psi^0$ has principal symbol q, then (7.7) and (7.8) show that we can take $\mathrm{WF}_h(Q) \subset \mathrm{WF}_h(A)$, and that

$$\|Qu\|_{L^2} \leq C_0 \|Au\|_{L^2} + Ch\|Gu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}$$

where $C_0 > 0$ is uniform in $\tilde{\tau} \in (0, \tilde{\tau}_0)$. Furthermore, we can write

$$A = ZQP_{W_0} + hF, \quad F \in \Psi_h^{-1},$$

where we may assume that $\mathrm{WF}_h(F) \subset \mathrm{ell}_h(G)$. Now estimate

$$\begin{aligned} \|Au\|_{H_h^1} &\leq \|ZQPW_0 u\|_{H_h^1} + Ch\|Fu\|_{H_h^1} \\ &\leq C\|Q(P - W_1 - V_1)u\|_{H_h^{-1}} + Ch\|Gu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}. \end{aligned}$$

Given $\varepsilon > 0$, choose $\tilde{\tau}$ sufficiently small so that $\|W_1\|_{L^\infty} \leq \varepsilon$. This yields

$$\begin{aligned} \|QW_1 u\|_{L^2} &\leq \|W_1 Qu\|_{L^2} + \|[Q, W_1]u\|_{L^2} \\ &\leq \varepsilon \|Qu\|_{L^2} + \|[Q, W_1]u\|_{L^2} \\ &\leq C_0 \varepsilon \|Au\|_{L^2} + \|[Q, W_1]u\|_{L^2} + Ch\|Gu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2} \end{aligned}$$

We need to bound the L^2 norm of $[Q, W_1]u$. If $Q \in \Psi_h^{\mathrm{comp}}$, then by Remark 6.20,

$$K_{[Q,W_1]} \in hI_h^{-\alpha+1/2,\mathrm{comp}}(X, N^*((X \times Y) \cap \mathrm{diag}), N^*\mathrm{diag})$$

and Lemma 6.15 would apply. However, since we are merely assuming that $Q \in \Psi_h^0$, the kernel of $[Q, W_1]$ is not strictly part of the paired Lagrangian calculus developed here.

On the other hand, the proof of Lemma 6.15 still applies in this setting, as explained in Remark 7.3. In the notation of latter remark, let O be an open neighborhood of $\mathrm{WF}_h(G)$ in $\overline{T^*}X$ such that $O \subset \mathrm{WF}_h(G)$. If $\tau > 0$ is sufficiently small so that $\mathrm{WF}_h(K_{[Q,W_1]}) \subset O \times O'$, then we can bound

$$\|[Q, W_1]u\|_{L^2} \leq Ch^{2\gamma}\|Gu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}.$$

for some $\gamma \in (0, 1/2]$.

In order to bound the final term $\|QV_1u\|_{L^2}$ simply use the estimate $\|V_1\|_{L^\infty} = \mathcal{O}(h^\alpha)$ by (7.2); hence

$$\|QV_1u\|_{L^2} \leq Ch^\alpha \|u\|_{L^2}.$$

By taking ε sufficiently small,

$$\|Au\|_{H^1_h} \leq \|QPu\|_{H^{-1}_h} + Ch^\alpha\|u\|_{L^2} + h^{\min(1/2,\gamma)}\|Gu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}.$$

Finally, recall that G is elliptic on $\mathrm{WF}_h(Q)$, hence QPu can be replaced with GPu. The proof is then finished by an iterative argument, increasing the semiclassical regularity by $\min(1/2,\gamma)$ at each step. □

Remark 7.7. The remainder term $h^\alpha\|u\|_{L^2}$ in Lemma 7.6 is not microlocalized. On the other hand, suppose that $A \in \Psi_h^{\mathrm{comp}}$ in Lemma 7.6. If $T \in \mathcal{C}^\infty_c(\mathbb{R}; \Psi_h^{\mathrm{comp}}(\mathbb{R}^{n-1}))$ satisfies

$$(x,y,\xi,\eta) \in \mathrm{WF}_h(A) \Longrightarrow (x,y,\eta) \in \mathrm{ell}_h(T),$$

then we can replace this term by $h^\alpha\|Tu\|_{L^2}$. This follows since by Lemma 6.26 (cf. Lemma 7.4) we can replace QV_1u in the proof with QV_1Tu modulo a $\mathcal{O}(h^\infty)_{L^2\to L^2}$ remainder.

7.3. **Ordinary and b-wavefront sets.** We now present two results relating ordinary and b-wavefront sets. The first allows us to replace microlocalization by b-pseudodifferential operators at $T^*Y \subset {}^{\mathrm{b}}T^*X$ with tangential operators.

Lemma 7.8. *Let $\alpha > 0$ and $s \in \mathbb{R} \cup \{+\infty\}$. Suppose that u is h-tempered in $H^1_h(\mathbb{R}^n)$. If $\mathrm{WF}^{-1,s}_{\mathrm{b},h}(Pu) = \emptyset$ and $q_0 = (0,y_0,0,\eta_0) \notin \mathrm{WF}^{1,s}_{\mathrm{b},h}(u)$, then there exists $T \in \mathcal{C}^\infty_c(\mathbb{R}; \Psi_h^{\mathrm{comp}}(\mathbb{R}^{n-1}))$ with $(0,y_0,\eta_0) \in \mathrm{ell}_h(T)$ such that*

$$\|Tu\|_{H^1_h} \leq Ch^s.$$

Proof. Let $T \in \mathcal{C}^\infty_c((-\delta,\delta); \Psi_h^{\mathrm{comp}}(\mathbb{R}^{n-1}))$ satisfy $(0,y_0,\eta_0) \in \mathrm{ell}_{\mathrm{b}}(T)$ and

$$\mathrm{WF}_h(T) \subset \{|x| + |y-y_0| + |\eta-\eta_0| < \delta\}$$

Define $f(x,y,\sigma,\eta) \in S^0_{\mathrm{b}}({}^{\mathrm{b}}T^*\mathbb{R}^n)$ by

$$f(x,y,\sigma,\eta) = \chi(\sigma^2/((C_0\delta)^2\langle\eta\rangle^2)),$$

where $\chi = \chi(s) \in \mathcal{C}^\infty_c(\mathbb{R})$ is one for $|s| \leq 1$ and vanishes when $|s| \geq 2$. The parameter $C_0 > 0$ will be chosen later. Let

$$F = \mathrm{Op}_{\mathrm{b},h}(f) \in \Psi^0_{\mathrm{b},h},$$

be properly supported. Since $|\sigma| \leq C\langle\eta\rangle$ on $\mathrm{supp}\, f$, it follows that $TF \in \Psi^0_{\mathrm{b},h}$, where

$$q_0 \in \mathrm{ell}_{\mathrm{b}}(TF), \quad \mathrm{WF}_h(TF) \subset \{|x| + |y-y_0| + |\sigma| + |\eta-\eta_0| < C_1\delta\}$$

for some $C_1 > 0$. Taking $\delta > 0$ sufficiently small implies that $\|TFu\|_{H^1_h} \leq Ch^s$.

On the other hand, we can write $T = T\varphi$, where $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ has $\operatorname{supp}\varphi \subset \{|x| < \delta\}$. Therefore,

$$\|T(1-F)u\|_{H_h^1} = \|T\varphi(1-F)u\|_{H_h^1} \leq C\|\varphi(1-F)u\|_{H_h^1},$$

as T is uniformly bounded on $H_h^1(\mathbb{R}^n)$. Now observe that $\varphi(1-F) \in \Psi_{\mathrm{b},h}^0$ has compact support in $\{|x| < \delta\}$, and

$$|\sigma| \geq C_0\delta \text{ on } \mathrm{WF}_{\mathrm{b},h}(\varphi(1-F)).$$

By taking $C_0 > 0$ sufficiently large, there exists $\delta_0 > 0$ such that Lemma 5.5 applies to $\varphi(1-F)$ for $\delta \in (0, \delta_0)$. In particular,

$$\|T(1-F)u\|_{H_h^1} \leq C\|Pu\|_{H_h^{-1}} + \mathcal{O}(h^\infty)\|u\|_{H_h^1},$$

which completes the proof. □

Note that the proof (or alternatively the Closed Graph Theorem) in fact yields the quantitative statement

$$\|Tu\|_{H_h^1} \leq Ch^s\|G_{\mathrm{b}}u\|_{H_h^1} + C\|Pu\|_{H_h^{-1}} + \mathcal{O}(h^\infty)\|u\|_{H_h^1}, \tag{7.9}$$

where $G_{\mathrm{b}} \in \Psi_{\mathrm{b},h}^0$ is elliptic near q_0.

Next, we show by a similar argument that at glancing points (or rather their preimages in T_Y^*X), microlocalization by ordinary semiclassical pseudodifferential operators can be replaced with microlocalization by tangential operators.

Lemma 7.9. *Let $\alpha > 0$ and $s \in \mathbb{R} \cup \{+\infty\}$. Suppose that u is h-tempered in $H_h^1(\mathbb{R}^n)$. Let*

$$\varpi_0 = (0, y_0, 0, \eta_0) \in \pi^{-1}(\mathcal{G}) \cap T_Y^*X$$

If $\mathrm{WF}_h^{-1,s}(Pu) = \emptyset$ and $\varpi_0 \notin \mathrm{WF}_h^s(u)$, then there exists $T \in \mathcal{C}^\infty(\mathbb{R}_x; \Psi_h^{\mathrm{comp}}(\mathbb{R}_y^{n-1}))$ with $(0, y_0, \eta_0) \in \mathrm{ell}_h(T)$ such that such that

$$\|Tu\|_{H_h^1} \leq Ch^s.$$

Proof. The proof is similar to that of Lemma 7.8. Let $T \in \mathcal{C}_c^\infty((-\delta, \delta); \Psi_h^{\mathrm{comp}}(\mathbb{R}^{n-1}))$ with total left symbol $t = t(x, y, \eta)$ satisfy $(0, y_0, \eta_0) \in \mathrm{ell}_{\mathrm{b}}(T)$ and

$$\mathrm{WF}_h(T) \subset \{|x| + |y - y_0| + |\eta - \eta_0| < \delta\}.$$

Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}\varphi \subset \{|x| + |y - y_0| < \delta\}$ and $T = \varphi T$. Because ϖ_0 is a glancing point, we know that $\tilde{p}(x_0, y_0, \eta_0) = 0$. Thus for $\delta > 0$ sufficiently small,

$$|\tilde{p}| \leq C'\delta^\theta \text{ on } \operatorname{supp} t.$$

where $\theta \in (0, 1]$ is a Hölder exponent for V (recall that $\alpha > 0$). Define $f(x, y, \xi, \eta) \in S^0(T^*\mathbb{R}^n)$ by

$$f(x, y, \xi, \eta) = \chi(\xi^2/(C_0\delta^{\theta/2}\langle\eta\rangle)^2),$$

where $\chi = \chi(s) \in \mathcal{C}_c^\infty(\mathbb{R})$ is one for $|s| \leq 1$ and vanishes when $|s| \geq 2$. Let

$$F = \mathrm{Op}_h(f) \in \Psi_h^0$$

As in Lemma 7.8, since $|\xi| \leq C\langle\eta\rangle$ on $\operatorname{supp} f$, it follows that $FT \in \Psi_h^{\mathrm{comp}}$, such that

$$\varpi_0 \in \mathrm{ell}_h(FT), \quad \mathrm{WF}_h(FT) \subset \{|x| + |y - y_0| + |\eta - \eta_0| < C_1\delta,\ |\xi| < C_1\delta^{\theta/2}\}.$$

Taking $\delta > 0$ sufficiently small implies that $\|FTu\|_{H_h^1} \leq Ch^s$.

Next, choose $T' \in \mathcal{C}_c^\infty((-\delta,\delta); \Psi_h^{\mathrm{comp}}(\mathbb{R}^{n-1}))$ with same properties as T, replacing δ with $(1+\varepsilon)\delta$ for $\varepsilon > 0$ arbitrarily small. We may choose T' so that

$$T = T'T + \mathcal{O}(h^\infty)_{H_h^1 \to H_h^1}.$$

Let t' be a total symbol for T'. Decompose the function $1 - f = f_1 + f_2$, where

$$C_0\delta^{\theta/2} \leq |\xi| \leq 2C_1\langle\eta\rangle \text{ on } \operatorname{supp} f_1, \quad |\xi| \geq C_1\langle\eta\rangle \text{ on } \operatorname{supp} f_2.$$

Writing, $F_i = \mathrm{Op}(F_i)$, we have that $F_1T' \in \Psi_h^{\mathrm{comp}}$ with principal symbol f_1t'. Now $|\xi| > C_0\delta^{\theta/2}$ on $\operatorname{supp}(f_1)$, so if $C_0 > 0$ is sufficiently large, then

$$p(x,y,\xi,\eta) = \xi^2 + k^{ij}(x,y)\eta_i\eta_j + V(x,y) > c_0\delta^\theta$$

on $\operatorname{supp}(f_1t')$, where $c_0 > 0$ does not depend on δ. Thus $\mathrm{WF}_h(F_1T') \subset \mathrm{ell}_h(P)$, so applying Lemma 7.6 to the function Tu,

$$\begin{aligned}\|F_1Tu\|_{H_h^1} &\leq \|(F_1T')Tu\|_{H_h^1} + \mathcal{O}(h^\infty)\|u\|_{H_h^1} \\ &\leq C\|PTu\|_{H_h^{-1}} + Ch^\alpha\|Tu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{H_h^1}.\end{aligned}$$

On the other hand, for the term F_2, if we take $C_1 > 0$ sufficiently large, then $p \geq c\langle\zeta\rangle^2$ on $\operatorname{supp}(\varphi f_2)$. Thus $\mathrm{WF}_h(F_2\varphi) \subset \mathrm{ell}_h(P)$, so again

$$\|F_2Tu\|_{H_h^1} = \|(F_2\varphi)Tu\|_{H_h^1} \leq C\|PTu\|_{H_h^{-1}} + Ch^\alpha\|Tu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{H_h^1}$$

In order to handle either of the terms involving F_1 or F_2, it therefore suffices to bound $\|PTu\|_{H_h^{-1}}$. This is done by writing $PTu = TPu + [P,T]u$ and bounding $\|TPu\|_{H_h^{-1}} \leq C\|Pu\|_{H_h^{-1}}$. As for the commutator,

$$[P,T] = h(hD_x)T_1 + hT_0 + [V,T],$$

where $T_i \in \mathcal{C}^\infty(\mathbb{R}; \Psi_h^{\mathrm{comp}}(\mathbb{R}^{n-1}))$. Here we can view $[V,T] \in \mathcal{C}^0(\mathbb{R}; h\Psi_h^{\mathrm{comp}}(\mathbb{R}^{n-1}))$. Since T' is elliptic on $\mathrm{WF}_h(T)$,

$$\|[P,T]u\|_{H_h^{-1}} \leq Ch\|T'u\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}.$$

Altogether, we have

$$\|Tu\|_{H_h^1} \leq C\|Pu\|_{H_h^{-1}} + C\|Au\|_{H_h^1} + Ch^{\min(1,\alpha)}\|T'u\|_{H_h^1} + \mathcal{O}(h^\infty)\|u\|_{H_h^1},$$

Since the wavefront set of T' is larger than that of T by an arbitrarily small amount the proof is finished by induction, improving the semiclassical regularity by $h^{\min(1,\alpha)}$ at each step. $\square$

Lemmas 7.8 and 7.9 can be combined using the following observation: if $A_{\mathrm{b}} \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ and $T \in \mathcal{C}^\infty(\mathbb{R}; \Psi^{\mathrm{comp}}_h(\mathbb{R}^{n-1}))$ are such that $(x,y,\eta) \in \mathrm{WF}_h(T)$ implies $(x,y,\sigma,\eta) \notin \mathrm{WF}_{\mathrm{b},h}(A_{\mathrm{b}})$ for any $\sigma \in \mathbb{R}$, then

$$\|A_{\mathrm{b}} u\|_{L^2} \leq \|Tu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}. \tag{7.10}$$

The proof is similar to that of Lemma 6.26.

Lemma 7.10. *Let $\alpha > 0$ and $r \in \mathbb{R} \cup \{+\infty\}$. Let $\varpi_0 \in \pi^{-1}(\mathcal{G})$ and $q_0 = \pi(\varpi_0)$. Suppose that u is h-tempered in $H^1_h(X)$ and $Pu \in L^2(X)$. If*

$$\mathrm{WF}^r_h(Pu) = \emptyset, \quad \varpi_0 \notin \mathrm{WF}^r_h(u),$$

then $q_0 \notin \mathrm{WF}^{1,r}_{\mathrm{b},h}(u)$.

Proof. This follows by combining Lemma 7.9 with (7.10) and Lemma 5.4. ☐

7.4. **Improvement at hyperbolic points.** We are now ready to prove Theorem 2. Fix $\varpi_0 \in \pi^{-1}(\mathcal{H})$, and write

$$\varpi_0 = (0, y_0, \xi_0, \eta_0)$$

with respect to a fixed normal coordinate patch $\mathcal{U}$, where $\xi_0 > 0$ for concreteness.

Proposition 7.11. *Let $\alpha > 1$ and $s \leq r + \alpha$, where $s, r \in \mathbb{R} \cup \{+\infty\}$. Suppose that u is h-tempered in $H^1_h(X)$ with $Pu \in L^2(X)$, such that*

$$\pi(\varpi_0) \notin \mathrm{WF}^{1,r}_{\mathrm{b},h}(u), \quad \mathrm{WF}^{s+1}_h(Pu) = \emptyset.$$

If there is a neighborhood $U \subset T^*X$ of ϖ_0 such that $U \cap \mathrm{WF}^s_h(u) \cap \{x < 0\} = \emptyset$, then $\varpi_0 \notin \mathrm{WF}^s_h(u)$.

As usual, Proposition 7.11 follows from a quantitative estimate via a positive commutator argument.

Proposition 7.12. *If $G \in \Psi^{\mathrm{comp}}_h$ is elliptic at ϖ_0 and $Q_{\mathrm{b}} \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ is elliptic at $\pi(\varpi_0)$, then there exist $Q, Q_1 \in \Psi^{\mathrm{comp}}_h$, where*

$$\mathrm{WF}_h(Q) \subset \mathrm{ell}_h(G) \text{ and } \varpi_0 \in \mathrm{ell}_h(Q),$$
$$\mathrm{WF}_h(Q_1) \subset \mathrm{ell}_h(G) \cap \{x < 0\},$$

such that

$$\|Qu\|_{L^2} \leq Ch^{-1}\|Pu\|_{L^2} + C\|Q_1 u\|_{L^2} + Ch^\alpha \|Q_{\mathrm{b}} u\|_{H^1_h} + \mathcal{O}(h^\infty)\|u\|_{L^2}, \tag{7.11}$$

for each $u \in H^1_h(X)$ with $Pu \in L^2(X)$.

Note that G can be used to control the sizes of $\mathrm{WF}_h(Q)$ and $\mathrm{WF}_h(Q_1)$, but the term involving Pu is not microlocalized. The term involving $Q_{\mathrm{b}}u$ is microlocalized, but only in the sense of b-wavefront set; by Theorem 1, it can be controlled by the singularities along backwards GBBs from $\pi(\varpi_0)$.

Remark 7.13. By a regularization argument it suffices to prove Proposition 7.12 (and also Proposition 7.17 in the next section) for $u \in \mathcal{C}^\infty(X)$. Indeed, given $u \in H_h^1(X)$ with $Pu \in L^2(X)$ we can choose $u_j \in \mathcal{C}^\infty(X)$ such that

$$u_j \to u \text{ in } H_h^1(X), \quad h^2\Delta_g u_j \to h^2 \Delta_g u \text{ in } L^2(X)$$

(see [DZ, Lemma E.47] for instance). This of course implies $Pu_j \to Pu$ in $L^2(X)$ as well.

One key to the proof of Proposition 7.12 is the use of the microlocal energy estimates discussed in Section 2.3. Suppose that $u \in \mathcal{C}^\infty(X)$ is supported in a normal coordinate patch near $Y \subset X$. If $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ are the corresponding normal coordinates, we can apply Lemma 2.7 to the operator $L = P$ with $x_1 = x$ and $x' = y$. Indeed,

$$P = (hD_x)^*(hD_x) - h^2\Delta_k + V,$$

and $V \in \mathcal{C}^1(X)$ since $\alpha > 1$. The hypotheses on $R = -\tilde{P} = h^2\Delta_k - V$ are satisfied in a sufficiently small neighborhood $(-\varepsilon, \varepsilon) \times U$ of a point $\tilde{q}_0 \in \pi^{-1}(\mathcal{H})$.

We use an approach quite close to that of Proposition 5.8. Define the functions

$$\omega = |\xi - \xi_0|^2 + |y - y_0|^2 + |\eta - \eta_0|^2, \quad \phi = x + \frac{1}{\beta^2\delta}\omega.$$

We use the same cutoffs χ_0, χ_1 as in Proposition 5.8. We also fix a cutoff $\psi \in \mathcal{C}^\infty(T^*X; [0,1])$ such that $\psi = 1$ near $\{|x| \leq 2\delta,\, \omega^{1/2} \leq 2\beta\delta\}$ with support in $\{|x| < 3\delta,\, \omega^{1/2} < 3\beta\delta\}$. Now set

$$a = \chi_0(2 - \phi/\delta)\chi_1(2 + x/\delta).$$

The support properties of a can be read off from the analogue of Lemma 5.9; in particular,

$$\operatorname{supp} a \subset \{|x| \leq 2\delta,\, \omega^{1/2} \leq 2\beta\delta\},$$

hence $\psi = 1$ on the support of a. Recall that we are assuming $V \in I^{[-1-\alpha]}(Y)$ with $\alpha > 1$. We will use a decomposition $V = V_0 + V_1$ as in Section 7.1 *which may depend on δ, but not β.*

To proceed with the positive commutator argument, write

$$-(2/h)\operatorname{Im}\langle APu - AV_1u, Au\rangle = (i/h)\left\langle [A^*A, -h^2\Delta_g + V_0]u, u\right\rangle.$$

The right hand side is treated symbolically within the paired Lagrangian calculus. For convenience, set

$$f = \sigma_h(-h^2\Delta_g).$$

Let $P_{V_0} = -h^2\Delta_g + V_0$ and $p_{V_0} = f + V_0$. For simplicity, write $z = (x, y)$ and $\zeta = (\xi, \eta)$. The point of the next lemma is that it holds uniformly with respect to the decomposition $V = V_0 + V_1$, i.e., with respect to the choice of τ in (7.1).

Lemma 7.14. *Let* $f = \sigma_h(-h^2\Delta_g) \in \mathcal{C}^\infty(T^*X)$*. There exists* $\beta, \delta_0, c_0, \tau_0 > 0$ *such that for each* $\delta \in (0, \delta_0)$ *and* $\tau \in (0, \tau_0)$ *in* (7.1),

$$\mathsf{H}_f \phi \geq 2c_0, \quad |\partial_{z_i} V_0 \cdot \partial_{\zeta_i} \phi| \leq \mathsf{H}_f \phi/(4n) \text{ for } i = 1, \dots, n.$$

on $\operatorname{supp}\psi$.

Proof. For any $g \in \mathcal{C}^\infty(T^*X)$,

$$|\mathsf{H}_g \omega| \leq C_0 \omega^{1/2}$$

uniformly on any fixed neighborhood U of ϖ_0. This is therefore true for the smooth part $f = \sigma_h(-h^2\Delta_g)$ of p_W. As for the potential, the crucial point here is that if U is a fixed neighborhood, then for $i = 1, \dots, n$,

$$|\partial_{z_i} V_0 \cdot \partial_{\zeta_i} \omega| \leq C_1 \omega^{1/2}$$

on U for a constant $C_1 > 0$ that is independent of the choice of the parameter $\tau > 0$ in (7.1); this is obvious from the oscillatory integral representation of V_0.

On the other hand,

$$\mathsf{H}_f x = 2\xi.$$

If we fix a sufficiently small neighborhood U of ϖ_0, it follows that $\mathsf{H}_f x \geq 3c_0$ on U for some $c_0 > 0$. Fix $\beta > 3(C_0 + 2nC_1)/c_0$, and suppose that $\delta_0 > 0$ is such that $\operatorname{supp}\psi \subset U$ for $\delta \in (0, \delta_0)$. Then,

$$\mathsf{H}_f \phi \geq 3c_0 - C_0 \beta^{-2} \delta^{-1} \omega^{1/2} \geq 3c_0 - 3C_0 \beta^{-1} \geq 2c_0 \tag{7.12}$$

on $\operatorname{supp}\psi$, and in addition

$$|\partial_{z_i} V_0 \cdot \partial_{\zeta_i} \phi| \leq 3C_1 \beta^{-1} \leq c_0/(2n) \leq \mathsf{H}_f \phi/(4n) \tag{7.13}$$

on $\operatorname{supp}\psi$. □

We now examine properties of the commutator as a whole; note that $\beta > 0$ and $\delta_0 > 0$ have been fixed by Lemma 7.14, and we are now taking $\delta \in (0, \delta_0)$. First, consider the smooth part $f = \sigma_h(-h^2\Delta_g)$ of p_{V_0}. Define

$$b = (2\delta)^{-1/2} (\mathsf{H}_f \phi)^{1/2} (\chi_0 \chi_0')^{1/2} \chi_1,$$

which is well-defined and smooth in light of (7.12). We then compute

$$\begin{aligned} \mathsf{H}_f(a^2) &= -2\delta^{-1}(\mathsf{H}_f \phi)(\chi_0 \chi_0') \chi_1^2 + 2\delta^{-1}(\mathsf{H}_f x) \chi_0^2 (\chi_1 \chi_1') \\ &= -b^2 + e, \end{aligned}$$

noting that $\operatorname{supp} e \subset \{-2\delta \leq x \leq -\delta\} \cap \operatorname{supp} b$. Fix compactly supported operators B and E in Ψ_h^{comp} with principal symbols b and e, respectively.

Next, fix compactly supported operators $R_1, \dots, R_n \in \Psi_h^{\mathrm{comp}}(X)$ with principal symbols

$$r_i = (\mathsf{H}_f \phi)^{-1} (\partial_{\zeta_i} \phi) \psi.$$

In particular, $\psi \mathsf{H}_{V_0}\phi = (\mathsf{H}_f\phi)\sum \partial_{z_i}V_0 \cdot r_i$, and $\sum |\partial_{z_i}V_0 \cdot r_i| \le 1/4$ by our choice of β. Moreover,

$$\mathsf{H}_{V_0}(a^2) = -2\delta^{-1}(\mathsf{H}_{V_0}\phi)(\chi_0\chi_0')\chi_1^2 = -b^2\left(\frac{\mathsf{H}_{V_0}\phi}{\mathsf{H}_f\phi}\right) = -b^2\sum \partial_{z_i}V_0 \cdot r_i,$$

since $\psi = 1$ on $\operatorname{supp} b$. Note that $\mathsf{H}_{V_0}(a^2) \ge -(1/4)b^2$, but we do not use this directly within the symbol calculus.

Instead, for a given $\delta \in (0, \delta_0)$, fix an open set $O \subset \operatorname{supp}\psi$ containing $\mathrm{WF}_h(B)$. Since $\mathrm{WF}_h(A) \subset \mathrm{WF}_h(B)$ we can choose V_0 such that

$$\mathrm{WF}_h(K_{[A^*A,V_0]}) \subset O \times O'.$$

By further shrinking τ in (7.1), we can arrange that the kernels of $B^*(\partial_{z_i}V_0)R_iB$ also have wavefront set contained in $O \times O'$, since the operators B, B^*, R_i are independent of V_0. By Proposition 6.14,

$$(i/h)[P_{V_0}, A^*A] + B^*B + B^*\sum(\partial_{z_i}V_0)R_iB + E \\ \in I_h^{-\alpha+(1/2)+\varepsilon_0,\mathrm{comp}}(X, N^*((X \times Y) \cap \mathrm{diag}), N^*\mathrm{diag})$$

for any $\varepsilon_0 > 0$. If this operator is denoted by F, then by construction the principal symbol of F along $N^*\mathrm{diag}$ vanishes, and hence

$$F \in hI_h^{-\alpha+(3/2)+\varepsilon_0,\mathrm{comp}}(X, N^*((X \times Y) \cap \mathrm{diag}), N^*\mathrm{diag}).$$

The key here is that since all of the operators above have kernels with wavefront set in $O \times O'$, so does F.

Now we consider the identity

$$\langle (i/h)[P_{V_0}, A^*A]u, u\rangle = \|Bu\|_{L^2}^2 + \sum \langle (\partial_{z_i}V_0)R_iBu, Bu\rangle + \langle Eu, u\rangle + \langle Fu, u\rangle\,. \quad (7.14)$$

The second, third, and fourth terms on the right hand side of (7.14) are bounded in absolute value as follows. For the second term, we use the bound

$$\|R_iu\|_{L^2} \le 2\sup|\sigma_h(R_i)|\|u\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2},$$

and the fact that $2\sum \sup|\partial_{z_i}V_0||r_i| \le 1/2$ by construction. Therefore

$$\sum |\langle (\partial_{z_i}V_0)R_iBu, Bu\rangle| \le (1/2)\|Bu\|_{L^2}^2 + \mathcal{O}(h^\infty)\|u\|_{L^2}.$$

To bound the third term, choose $Q_1 \in \Psi_h^{\mathrm{comp}}$ as in the statement of the proposition such that $\mathrm{WF}_h(E) \subset \mathrm{ell}_h(Q_1)$ and estimate

$$|\langle Eu, u\rangle| \le C\|Q_1u\|_{L^2}^2 + \mathcal{O}(h^\infty)\|u\|_{L^2}^2.$$

For the fourth term, we apply Lemma 6.15: since $\alpha > 1$, fix $\gamma > 0$ such that

$$-\alpha + 2\gamma + \varepsilon_0 < -1,$$

where recall $\varepsilon_0 > 0$ is arbitrarily small. Taking $s = 1 - 2\gamma$,

$$-\alpha + (3/2) + \varepsilon_0 - s = -\alpha + (1/2) + \varepsilon_0 + 2\gamma < -1/2.$$

Therefore, by Lemma 6.15,

$$\|F\|_{L^2 \to L^2} \leq Ch^{2\gamma}. \tag{7.15}$$

Let $G \in \Psi_h^{\mathrm{comp}}(X)$ be elliptic on $\mathrm{WF}_h(B)$; since O was an arbitrary neighborhood of $\mathrm{WF}_h(B)$, we can assume that $O \subset \mathrm{ell}_h(G)$ as well. Thus we can bound

$$|\langle Fu, u\rangle| \leq Ch^{2\gamma}\|Gu\|^2 + \mathcal{O}(h^\infty)\|u\|_{L^2}^2.$$

Combining (7.14) and (7.4), we obtain the useful bound

$$\begin{aligned}\|Bu\|_{L^2}^2 \leq Ch^{-1}\|APu\|_{L^2}\|Au\|_{L^2} + Ch^{-1}|\langle AV_1u, Au\rangle| \\ + Ch^{2\gamma}\|Gu\|_{L^2}^2 + \|Q_1u\|_{L^2}^2 + \mathcal{O}(h^\infty)\|u\|_{L^2}^2.\end{aligned}$$

Note that the various terms involving $\|Au\|_{L^2}$ can be bounded in terms of $\|Bu\|_{L^2}$. This is done as at the end of Section 5.2, yielding

$$\|Au\|_{L^2} \leq C\|Bu\|_{L^2} + Ch\|Gu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}. \tag{7.16}$$

It remains to bound the term $h^{-1}|\langle AV_1u, Au\rangle|$. Using Lemma 6.26 (cf. Lemma 7.4), we can choose a tangential operator $T \in \mathcal{C}_c^\infty(\mathbb{R}; \Psi_h^{\mathrm{comp}}(\mathbb{R}^{n-1}))$ with

$$\mathrm{WF}_h(T) \subset \{|x| < 3\delta,\, |y - y_0|^2 + |\eta - \eta_0|^2 \leq 9\beta^2\delta^2\},$$

such that

$$\|AV_1u\|_{L^2} = \|AV_1Tu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}. \tag{7.17}$$

The same lemma shows that

$$\|[V_1, A^*A]u\|_{L^2} = \|[V_1, A^*A]Tu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}. \tag{7.18}$$

The next step is to apply Lemmas 6.24, 6.25, 2.7.

Lemma 7.15. *For each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|\langle AV_1u, Au\rangle| \leq \varepsilon\|Bu\|_{L^2}^2 + C_\varepsilon\big(h^{-1}\|Pu\|_{L^2}^2 + \|Q_1u\|_{L^2}^2 + h^{2\gamma}\|Gu\|_{L^2}^2 + h^{2\alpha}\|Tu\|_{H_h^1}^2\big),$$

where $T \in \mathcal{C}_c^\infty(\mathbb{R}; \Psi_h^{\mathrm{comp}}(\mathbb{R}^{n-1}))$ is as above.

Proof. Recall from (7.4) that

$$AV_1 \in h^{-1-\alpha} I_{\triangleleft}^{-\alpha-n/2-1/2,-\infty}(X \times X; N^*((X \times Y) \cap \mathrm{diag}), N^*(X \times Y)).$$

Arguing as in the preceding paragraph,

$$|\langle AV_1u, Au\rangle| = |\langle V_1Tu, TA^*Au\rangle| + \mathcal{O}(h^\infty)\|u\|_{L^2}^2.$$

Instead of using Lemma 6.24, we may easily bound a pairing of the form $\langle V_1w, v\rangle$ by Lemma 7.1. This yields

$$|\langle V_1Tu, TA^*Au\rangle| \leq Ch^{\alpha+1}\|Tu\|_{L^\infty((-3\delta,3\delta);L^2(\mathbb{R}^{n-1}))}\|TA^*Au\|_{L^\infty((-3\delta,3\delta);L^2(\mathbb{R}^{n-1}))}.$$

Here we used that A has compact support in $\{|x| < 3\delta\}$. If δ is sufficiently small, then Lemma 2.7 is applicable. By Cauchy–Schwarz,

$$h^{-1}|\,\langle AV_1u, Au\rangle\,| \leq C_\varepsilon h^{2\alpha}\|Tu\|^2_{L^\infty((-3\delta,3\delta);L^2(\mathbb{R}^{n-1}))} + \varepsilon\|TA^*Au\|^2_{L^\infty((-3\delta,3\delta);L^2(\mathbb{R}^{n-1}))} + \mathcal{O}(h^\infty)\|u\|^2_{L^2}$$

for each $\varepsilon > 0$. Let $T_1 \in \mathcal{C}^\infty(\mathbb{R}_x; \Psi_h^{\mathrm{comp}}(\mathbb{R}_y^{n-1}))$ be elliptic on $\mathrm{WF}_h(T)$. Applying Lemma 2.7, we deduce that

$$\|Tu\|_{L^\infty((-\varepsilon,\varepsilon);L^2(\mathbb{R}^{n-1}))} \leq Ch^{-1}\|Pu\|_{L^2} + C\|T_1u\|_{H_h^1} + \mathcal{O}(h^\infty)\|u\|_{H_h^1}.$$

As for the next term, we again apply Lemma 2.7, but this time writing

$$\|TA^*Au\|_{L^\infty((-3\delta,3\delta);L^2(\mathbb{R}^{n-1}))} \leq Ch^{-1}\int_{-3\delta}^{3\delta}\|PA^*Au(s)\|_{L^2(\mathbb{R}^{n-1})}\,ds + C\|A^*Au\|_{H_h^1} + \mathcal{O}(h^\infty)\|u\|_{H_h^1}. \quad (7.19)$$

Since $A^* \in \Psi_h^{\mathrm{comp}}(X)$, the second term on the right hand side of (7.19) is estimated by $\|Au\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}$. The first term on the right hand side of (7.19) is bounded by a constant times

$$h^{-1}\int_{-3\delta}^{3\delta}\|A^*APu(s)\|_{L^2(\mathbb{R}^{n-1})} + \|[P, A^*A]u(s)\|_{L^2(\mathbb{R}^{n-1})}\,ds \leq Ch^{-1}(\|APu\|_{L^2} + \|[P_{V_0}, A^*A]u\|_{L^2}) + h^{-1}\int_{-3\delta}^{3\delta}\|[V_1, A^*A]u(s)\|_{L^2(\mathbb{R}^{n-1})}\,ds.$$

Now recall that $(i/h)[P_{V_0}, A^*A] = -B^*B + B^*\sum(\partial_{x_i}V_0)R_iB + E + F$, and hence

$$\begin{aligned} h^{-1}\|[P_{V_0}, A^*A]u\|_{L^2} &\leq \|B^*Bu\|_{L^2} + \sum\|B^*(\partial_{z_i}V_0)Bu\|_{L^2} + \|Eu\|_{L^2} + \|Fu\|_{L^2} \\ &\leq C(\|Bu\|_{L^2} + \|Q_1u\|_{L^2} + h^\gamma\|Gu\|_{L^2}). \end{aligned}$$

The final step is to replace $[V_1, A^*A]u$ by $[V_1, A^*A]Tu$ modulo a $\mathcal{O}(h^\infty)\|u\|_{L^2}$ error as in (7.18), and then apply Lemmas 6.25, 2.7:

$$\begin{aligned} h^{-1}\int_{-3\delta}^{3\delta}\|[V_1, A^*A]Tu(s)\|_{L^2(\mathbb{R}^{n-1})}\,ds &\leq Ch^\alpha\|Tu\|_{L^\infty((-3\delta,3\delta);L^2(\mathbb{R}^{n-1}))} \\ &\leq Ch^\alpha\big(h^{-1}\|Pu\|_{L^2} + \|T_1u\|_{H_h^1}\big). \end{aligned}$$

A final application of (7.16) finishes the proof with T_1 instead of T; this is of course not a restriction, since $\mathrm{WF}_h(T)$ can be shrunk at will. $\square$

Altogether, we have established the following:

Lemma 7.16. *There exists $\beta, \delta_0, \gamma > 0$ and such that the following holds for each $\delta \in (0, \delta_0)$. Let $G \in \Psi_h^{\mathrm{comp}}$ be elliptic on $\mathrm{WF}_h(B)$ and Q_1 be elliptic on $\mathrm{WF}_h(E)$. With $T \in \mathcal{C}^\infty(\mathbb{R}_x; \Psi_h^{\mathrm{comp}}(\mathbb{R}_y^{n-1}))$ as above,*

$$\|Bu\|_{L^2} \le Ch^{-1}\|Pu\|_{L^2} + Ch^\gamma \|Gu\|_{L^2} + C\|Q_1 u\|_{L^2} + Ch^\alpha \|Tu\|_{H_h^1}$$

for every $u \in H_h^1(X)$ with $Pu \in L^2(X)$.

We now make a further argument to eliminate the Gu term on the right hand side of our estimates. The semiclassical regularity is improved inductively by h^γ at each step. Each time, we reduce $\delta > 0$ by an arbitrarily small amount; notice that the decomposition $V = V_0 + V_1$ changes with every step as well by shrinking $\tau > 0$ in (7.1).

This nearly proves Proposition 7.12, except that we have a term $\|Tu\|_{H_h^1}$ on the right hand side involving a tangential operator; this is easily remedied by an application of 7.8, which allows us to estimate $\|Tu\|_{H_h^1}$ by $\|Q_{\mathrm{b}} u\|_{H_h^1}$ modulo acceptable terms.

Finally, we will prove Theorem 2 using Proposition 7.11.

Proof of Theorem 2. Let u be h-tempered in $H_h^1(X)$ with $Pu \in L^2(X)$, and assume that

$$\mathrm{WF}_h^{s+1}(Pu) = \emptyset.$$

In the notation of Theorem 2, let $\varpi_\pm = (0, y_0, \pm\xi_0, \eta_0)$, where without loss we assume $\xi_0 > 0$. Note that both $\gamma_\pm((-\varepsilon, 0))$ are disjoint from $T_Y^* X$ for sufficiently small. To prove the theorem, assume that there is a sequence of points $\varepsilon_n > 0$ tending to zero such that $\gamma_-(-\varepsilon_n) \notin \mathrm{WF}_h^r(u)$. We must then show that $\gamma_+([-\varepsilon_0, 0])$ is contained in $\mathrm{WF}_h^s(u)$ for some $\varepsilon_0 > 0$.

First let $s \in [r, r + \alpha]$. We can assume that $\pi(\varpi_\pm) \notin \mathrm{WF}_{\mathrm{b},h}^{1,r}(u)$, since otherwise by Theorem 1,

$$\gamma_+((-\varepsilon, 0)) \subset \mathrm{WF}_h^r(u) \subset \mathrm{WF}_h^s(u),$$

for some $\varepsilon > 0$, thus completing the proof. By Proposition 7.11, there is a sequence

$$\varpi_j \in \mathrm{WF}_h^s(u) \cap \{x < 0\}$$

tending to ϖ_+. By Lemma 4.2, if j is sufficiently large, then there exists $\varepsilon_0 > 0$ such that the backwards bicharacteristics γ_j from ϖ_j exists for $t \in [-\varepsilon_0, 0]$. Moreover, again by Lemma 4.2, $\gamma_j \to \gamma_+$ uniformly on $[-\varepsilon_0, 0]$. By Hörmander's theorem on propagation of singularities, $\gamma_j([-\varepsilon_0, 0])$ is contained within $\mathrm{WF}_h^s(u)$. Since $\mathrm{WF}_h^s(u)$ is closed, letting $j \to \infty$ shows that $\gamma_+([-\varepsilon_0, 0]) \subset \mathrm{WF}_h^s(u)$ as well.

If $s < r$, then apply the same argument but with $r' = s$ instead of r. $\square$

7.5. **Improvement at glancing points.** We begin proving Theorem 3 by establishing a local result similar to [DHUV, Proposition 7.4]. The difference is that the threshold condition is $s \le r + \alpha - 1$ rather than $s \le r + (\alpha - 1)/2$, and crucially we are able to microlocalize the background regularity more finely.

Given a normal coordinate patch $\mathcal{U}$, let $\mathsf{B}(\varpi, \varepsilon)$ denote the Euclidean ball about $\varpi \in T^*_{\mathcal{U}} X$ of radius $\varepsilon > 0$ induced by local coordinates (x, y, ξ, η). Also choose $\alpha_0 < \alpha$ and set

$$\theta = \min(1, \alpha_0 - 1) \in (0, 1].$$

Thus θ is a Hölder exponent for H_p. The following proposition applies equally well at glancing and hyperbolic points (but of course at hyperbolic points the threshold is weaker than the one established in Section 7.4).

Proposition 7.17. *Let $\alpha > 1$ and $s \le r + \alpha - 1$, where $s, r \in \mathbb{R}$. Suppose that u is h-tempered in $H^1_h(X)$ and $Pu \in L^2(X)$ and $\mathrm{WF}^{s+1}_h(Pu) = \emptyset$. Let*

$$K \subset \Sigma \cap T^*_{Y \cap \mathcal{U}} X$$

be compact. There exist $C_0, C_1, \delta_0 > 0$ such that for each $\varpi_0 \in K$ and $\delta \in (0, \delta_0)$, if

$$\begin{aligned} &\mathsf{B}(\exp(-\delta \mathsf{H}_p)(\varpi_0), C_0 \delta^{1+\theta}) \cap \mathrm{WF}^s_h(u) = \emptyset, \quad \mathrm{WF}^{s+1}_h(Pu) = \emptyset, \\ &\{|x| + |y - y_0| + |\sigma| + |\eta - \eta_0| < C_1 \delta\} \cap \mathrm{WF}^{1,r}_{\mathrm{b},h}(u) = \emptyset, \end{aligned} \tag{7.20}$$

then $\varpi_0 \notin \mathrm{WF}^s_h(u)$.

Proof. According to Remark 7.13, we can assume that $u \in \mathcal{C}^\infty(X)$. It will suffice to consider the case $K = \{\varpi_0\}$ (cf. Remark 5.15 and the discussion in [DHUV, Section 7]). We may also assume that $dp(\varpi_0) \neq 0$, otherwise there is nothing to prove.

Choose local coordinates $(\rho_0, \ldots, \rho_{2n-1})$ vanishing at ϖ_0 such that

$$(\mathsf{H}_p \rho_0)(\varpi_0) > 0, \quad (\mathsf{H}_p \rho_i)(\varpi_0) = 0 \text{ for } i = 1, \ldots, 2n-1.$$

We use the same decomposition of $V = W_0 + W_1 + V_1$ as in Section 7.1. As usual, set

$$\omega = \sum_{i=1}^{2n-1} \rho_i^2, \quad \phi = \rho_0 + \frac{1}{\beta^2 \delta} \omega.$$

Also, fix a cutoff $\psi \in \mathcal{C}^\infty(T^*X; [0,1])$ such that $\psi = 1$ near $\{|\rho_0| \le 2\delta,\, \omega^{1/2} \le 2\beta\delta\}$ with support in $\{|\rho_0| < 3\delta,\, \omega^{1/2} < 3\beta\delta\}$.

Fix a neighborhood U of ϖ_0 on which $\mathsf{H}_p \rho_0 > 4c_0$ for some $c_0 > 0$. On the other hand, using the Hölder regularity of $\mathsf{H}_p \in \mathcal{C}^{0,\theta}$,

$$|\mathsf{H}_p \omega| \le M \omega^{1/2} (\omega^{\theta/2} + |\rho_0|^\theta)$$

on U. Therefore $\mathsf{H}_p\phi \geq 4c_0 - 3M\beta^{-1}((3\beta\delta)^\theta + (3\delta)^\theta)$ on U. If we choose $\beta = c\delta^\theta$, with $c > 0$ sufficiently large, then we can arrange that

$$\mathsf{H}_p\phi > 3c_0$$

on $\operatorname{supp}\psi$. Given $\delta > 0$ (and setting $\beta = c\delta^\theta$ as above) we can choose $\tilde{\tau} > 0$ depending on δ such that

$$\mathsf{H}_{p_{W_0}}\phi > 2c_0 \text{ on } \operatorname{supp}\psi.$$

Further shrinking $\tilde{\tau}$ if necessary (again depending on δ) and using (7.6), we can also arrange that

$$|\partial_{z_i}W_1 \cdot \partial_{\zeta_i}\phi| \leq c_0/(2n) \leq \mathsf{H}_{p_{W_0}}\phi/(4n)$$

on $\operatorname{supp}\psi$.

Let $A = \mathrm{Op}_h(a)$, where $a = \chi_0(2 - \phi/\delta)\chi_1(1 + (\rho_0 + \delta)/(\beta\delta))$. Write

$$-(2/h)\operatorname{Im}\langle A(P - V_1)u, Au\rangle = (i/h)\langle [A^*A, P_{W_0} + W_1]u, u\rangle .$$

Now the term $(i/h)[P_{W_0}, A^*A] \in \Psi_h^{\mathrm{comp}}$ has principal symbol $\mathsf{H}_{p_{W_0}}a^2$. This we write as

$$\mathsf{H}_{p_{W_0}}a^2 = -b^2 + e,$$

where as usual, $b = (2\delta)^{1-2}(\mathsf{H}_{p_{W_0}}\phi)^{1/2}(\chi_0\chi_0')^{1/2}\chi_1$. On the other hand, $\operatorname{supp} e$ is contained in the set

$$\{-\delta - \delta\beta \leq \rho_0 \leq -\delta,\ \omega^{1/2} \leq 2\beta\delta\};$$

note that with the choice $\beta = c\delta^\theta$ this is contained in $\mathsf{B}(\exp(-\delta\mathsf{H}_p)(\varpi_0), C_0\delta^{1+\theta})$ for all δ sufficiently small.

Next, consider the term $(i/h)[W_1, A^*A]$. First, if $\delta > 0$ is given we can arrange the decomposition $V = W_0 + W_1 + V_1$ so that

$$\mathrm{WF}_h(K_{[W_1,A^*A]}) \subset O \times O',$$

where O is an arbitrary neighborhood of $\mathrm{WF}_h(A)$.

Exactly as in Section 7.4, fix compactly supported operators $R_1, \dots, R_n \in \Psi_h^{\mathrm{comp}}$ with principal symbols

$$r_i = (\mathsf{H}_{p_{W_0}}\phi)^{-1}(\partial_{\zeta_i}\omega)\psi.$$

In particular, $\psi\mathsf{H}_{W_1}\omega = (\mathsf{H}_f\phi)\sum\partial_{z_i}W_1 \cdot r_i$, and $\sum|\partial_{z_i}W_1 \cdot r_i| \leq 1/4$. Moreover,

$$\mathsf{H}_{W_1}(a^2) = -2\delta^{-1}(\mathsf{H}_{W_1}\phi)(\chi_0\chi_0')\chi_1^2 = -b^2\left(\frac{\mathsf{H}_{W_1}\phi}{\mathsf{H}_{p_{W_0}}\phi}\right) = -b^2\sum\partial_{z_i}W_1 \cdot r_i,$$

since $\psi = 1$ on $\operatorname{supp} b$. On the other hand, as compared to Section 7.4 there is an additional contribution to the commutator: fix compactly supported operators $L_1, \dots, L_n \in \Psi_h^{\mathrm{comp}}$ with principal symbols

$$\ell_i = (\beta\delta)^{-1}(\partial_{\zeta_i}\rho_0)\chi_0\chi_1'.$$

We can take

$$\mathrm{WF}_h(L_i) \subset \{-\delta - \delta\beta \le \rho_0 \le -\delta,\, \omega^{1/2} \le 2\beta\delta\}.$$

as well. By further refining the choice of $V = W_0 + W_1 + V_1$, we can arrange that the kernels of $B^*(\partial_{z_i} W_1) R_i B$ and $(\partial_{z_i} W_1) L_i$ also have wavefront set contained in $O \times O'$. By Proposition 6.14,

$$\begin{aligned}(i/h)[P_{W_0} + W_1, A^*A] + B^*B + B^* \sum (\partial_{z_i} W_1) R_i B + \sum (\partial_{z_i} W_1) L_i + E \\ \in I_h^{-\alpha+(1/2)+\varepsilon_0,\mathrm{comp}}(X, N^*((X \times Y) \cap \mathrm{diag}), N^*\mathrm{diag})\end{aligned}$$

for any $\varepsilon_0 > 0$, noting the additional terms involving L_i as compared to the corresponding expression in Section 7.4. If this operator is denoted by F, then by construction the principal symbol of F along $N^*\mathrm{diag}$ vanishes, and hence

$$F \in hI_h^{-\alpha+(3/2)+\varepsilon_0,\mathrm{comp}}(X, N^*((X \times Y) \cap \mathrm{diag}), N^*\mathrm{diag}).$$

Since all of the operators above have kernels with wavefront set in $O \times O'$, so does F. Now we consider the identity

$$\begin{aligned}\langle (i/h)[P_{W_0} + W_1, A^*A]u, u\rangle = \|Bu\|^2_{L^2} \\ + \sum \langle (\partial_{z_i} W_1) R_i Bu, Bu\rangle + \sum \langle (\partial_{z_i} W_1) L_i u, u\rangle + \langle Eu, u\rangle + \langle Fu, u\rangle. \quad (7.21)\end{aligned}$$

The second, third, and fourth terms on the right hand side of (7.14) are bounded in absolute value as in Section 7.4: for the second term, we use the bound

$$\|R_i u\|_{L^2} \le 2 \sup |\sigma_h(R_i)| \|u\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2},$$

and the fact that $2 \sum \sup |\partial_{z_i} W_1||r_i| \le 1/2$ by construction. Therefore

$$\sum |\langle (\partial_{z_i} W_1) R_i Bu, Bu\rangle| \le (1/2)\|Bu\|^2_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}$$

To bound the third and fourth terms, choose $Q_1 \in \Psi_h^{\mathrm{comp}}$ such that $\mathrm{WF}_h(E) \subset \mathrm{ell}_h(Q_1)$ and estimate

$$\sum |\langle (\partial_{z_i} W_1) L_i u, u\rangle| + |\langle Eu, u\rangle| \le C\|Q_1 u\|^2_{L^2} + \mathcal{O}(h^\infty)\|u\|^2_{L^2}.$$

For the fifth term, by Lemma 6.15,

$$\|F\|_{L^2 \to L^2} \le Ch^{2\gamma}$$

with the same exponent γ as in (7.15). Let $G \in \Psi_h^{\mathrm{comp}}(X)$ be elliptic on $\mathrm{WF}_h(B)$; since O was an arbitrary neighborhood of $\mathrm{WF}_h(B)$, we can assume that $O \subset \mathrm{ell}_h(G)$ as well. Thus we can bound

$$|\langle Fu, u\rangle| \le Ch^{2\gamma}\|Gu\|^2 + \mathcal{O}(h^\infty)\|u\|^2_{L^2}.$$

We therefore conclude that

$$\begin{aligned}\|Bu\|_{L^2}^2 \leq Ch^{-1}\|APu\|_{L^2}\|Au\|_{L^2} + Ch^{-1}|\,\langle AV_1 u, Au\rangle\,| \\ + Ch^{2\gamma}\|Gu\|_{L^2}^2 + \|Q_1 u\|_{L^2}^2 + \mathcal{O}(h^\infty)\|u\|_{L^2}^2.\end{aligned}$$

Here G is elliptic on $\mathrm{WF}_h(B)$, and

$$\mathrm{WF}_h(Q_1) \subset \mathrm{WF}_h(G) \cap \{-2\delta - 2\delta\beta \leq \rho_0 \leq -\delta/2,\, \omega^{1/2} \leq 3\beta\delta\}.$$

Note that the various terms involving $\|Au\|_{L^2}$ can be bounded in terms of $\|Bu\|_{L^2}$,

$$\|Au\|_{L^2} \leq C\|Bu\|_{L^2} + Ch\|Gu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}.$$

It remains to bound the term $h^{-1}|\,\langle AV_1 u, Au\rangle\,|$. As compared to Section 7.4, we are no longer able to use the energy estimates, which leads to a loss of h^{-1} in the threshold condition.

Just as in (7.17), if $C_1 > 0$ is sufficiently large we can choose a tangential psedudodifferential operator T with

$$\mathrm{WF}_h(T) \subset \{|x| + |y - y_0| + |\eta - \eta_0| < C_1\delta\}$$

such that

$$h^{-1}|\langle AV_1 u, Au\rangle| \leq h^{-1}\|V_1 ATu\| + \mathcal{O}(h^\infty)\|u\|_{L^2}^2.$$

Then (7.2) yields

$$h^{-1}|\langle AV_1 u, Au\rangle| \leq \varepsilon\|Au\|_{L^2}^2 + C_\varepsilon h^{2\alpha-2}\|Tu\|_{L^2}^2 + \mathcal{O}(h^\infty)\|u\|_{L^2}^2.$$

On the other hand, by (7.9), we can choose $Q_\mathrm{b} \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ so that

$$\|Tu\|_{L^2}^2 \leq C\|Pu\|_{L^2}^2 + \|Q_\mathrm{b} u\|_{L^2}^2,$$

where $\mathrm{WF}_h(Q_\mathrm{b}) \subset \{|x| + |y - y_0| + |\sigma| + |\eta - \eta_0|\} < C_1\delta\}$, increasing C_1 if necessary. An inductive argument completes the proof (the commutant must be modified slightly at each step, as pointed out at the end of Section 5.3). $\square$

Proof of Theorem 3. Let $\varpi_0 \in \pi^{-1}(\mathcal{G}) \cap T^*_Y X$, and suppose that no bicharacteristic segment of the form $\gamma(-\varepsilon, 0)$, where $\gamma(0) = \varpi_0$, is contained in $\mathrm{WF}^r_h(u)$ for any $\varepsilon > 0$; we wish to show that $\varpi_0 \notin \mathrm{WF}^r_h(u)$. Let s be such that $\varpi_0 \notin \mathrm{WF}^s_h(u)$; this always exists by our tempered assumption. According to Lemma 7.10, this also implies that $q_0 = \pi(\varpi_0) \notin \mathrm{WF}^{1,s}_{\mathrm{b},h}(u)$. We now show that

$$\varpi_0 \notin \mathrm{WF}^{s_0}_h(u) \text{ for } s_0 = \min(r, s + \alpha - 1).$$

Observe that $s + \alpha - 1 > s$ since $\alpha > 1$. Since $\varpi_0 \notin \mathrm{WF}^s_h(u)$, let U be a neighborhood of ϖ_0 of the form $U = \mathsf{B}(\varpi_0, \varepsilon_0)$, where $\varepsilon_0 > 0$ is chosen so that $U \cap \mathrm{WF}^s_h(u) = \emptyset$. By further shrinking ε_0, we can also assume that $U_\mathrm{b} \cap \mathrm{WF}^{1,s}_{\mathrm{b},h}(u) = \emptyset$, where

$$U_\mathrm{b} = \{|x| + |y - y_0| + |\sigma| + |\eta - \eta_0| < \varepsilon_0\}.$$

By Lemma 7.8 and Remark 7.7, we can conclude that

$$\mathrm{WF}_h^{s_0}(u) \cap U \subset \Sigma.$$

We now argue as in [DHUV, Lemma 8.1]: using Proposition 7.17 and ordinary semiclassical propagation of singularities away from Y, we can therefore construct a backward bicharacteristic segment through ϖ_0 contained in $\mathrm{WF}_h^{s_0}(u)$; the proof is an even simpler analogue of Lemma 5.18. This yields a contradiction, and thus we may reach the desired regularity $s = r$ by iteration. □

Appendix A. Proof of Proposition 1.1

A.1. **Plane wave solutions.** We construct exact solutions of $(P-1)u = 0$ on $[0, x_0)$ of the form

$$u_\pm(x) = e^{\pm ix/h}(1 + b_\pm(x)),$$

subject to the conditions $b_\pm(0) = 0$ and $b'_\pm(0) = 0$. We then obtain $\mathcal{C}^2$ solutions to $(P-1)u = 0$ on $(-\infty, x_0)$ after extending $b_\pm$ by zero to $(-\infty, 0)$. Thus $u_\pm$ are precisely the continuations of the plane wave solutions $e^{\pm ix/h}$ from $(-\infty, 0)$ to $(-\infty, x_0)$.

Although the functions $b_\pm$ are globally defined on $[0, x_0)$, their region of asymptotic validity is small (in an h-dependent way). First consider the case $b = b_+$, so that b_+ satisfies the equation

$$h^2 b''(x) + 2ihb'(x) = (1 + b(x))V(x). \tag{A.1}$$

Viewing the right hand side as a correction, the unperturbed equation has linearly independent solutions 1 and $e^{-2ix/h}$. By variation of parameters, (A.1) is equivalent to the integral equation $b = Jb$, where

$$(Jb)(x) = \frac{1}{2ih}\int_0^x \left(1 - e^{2i(s-x)/h}\right) V(s)(1 + b(s))\, ds.$$

This equation can be solved by successive approximation. Thus we set $b_0 = 0$, inductively define $b_{n+1} = Jb_n$. Let

$$\sigma(x) = \frac{1}{h}\int_0^x |V(s)|\, ds = \frac{x^{\alpha+1}}{(\alpha+1)h}$$

on $[0, x_0)$. A simple inductive argument shows that

$$|b_{n+1}(x) - b_n(x)| \le \frac{\sigma(x)^{n+1}}{(n+1)!}, \quad |b'_{n+1}(x) - b'_n(x)| \le \frac{2\sigma(x)^{n+1}}{h(n+1)!}$$

for $n \ge 0$. Differentiating once more and using the formula for J, it follows that

$$b = \sum_{n=0}^{\infty} (b_{n+1} - b_n)$$

is a $\mathcal{C}^2([0,\infty))$ function solving (A.1) with $b(0)=0$ and $b'(0)=0$. Moreover, $b=b_1+\varepsilon$, where $b_1=J(0)$ and the remainder satisfies

$$|\varepsilon(x)| \le e^{\sigma(x)}-1-\sigma(x), \quad |h\varepsilon'(x)/2| \le e^{\sigma(x)}-1-\sigma(x)$$

on $[0,\infty)$. We now find the behavior of $b_1(x)$ as $x/h\to\infty$. We will frequently use the rescaled variable $y=x/h$, and by a slight abuse of notation write $b_1=b_1(y)$ when convenient.

Lemma A.1. *In terms of $y=x/h$, the function b_1 satisfies*

$$h^{-\alpha}b_1(y) = -2^{-\alpha-2}e^{i\alpha\pi/2}\,\Gamma(\alpha+1)\,e^{-2iy} + \frac{y^{\alpha+1}}{2i(\alpha+1)} + \frac{y^\alpha}{4} + \mathcal{O}(y^{\alpha-1})$$

as $y\to\infty$, where the right hand side does not depend on h.

Proof. Integrating by parts once,

$$\begin{aligned} h^{-\alpha}b_1(y) &= \frac{e^{-2iy}}{\alpha+1}\int_0^y e^{2is}s^{\alpha+1}\,ds \\ &= 2^{-\alpha-2}e^{i(\alpha+2)\pi/2}\left(\Gamma(\alpha+2)-\Gamma(\alpha+2,-2iy)\right), \end{aligned} \tag{A.2}$$

where $\Gamma(a,z)$ is the incomplete gamma function, defined as

$$\Gamma(a,z)\equiv\int_z^\infty t^{a-1}e^{-t}\,dt,$$

with the integral taken along any path not crossing the negative real axis. Since y is real, there is an asymptotic expansion

$$\Gamma(\alpha+2,-2iy)\sim(-2iy)^{\alpha+1}e^{2iy}\sum_{k=0}^\infty a_k(-2iy)^{-k}$$

as $y\to\infty$, where $a_0=1$ and $a_k=(\alpha+2-1)\cdots(\alpha+2-k)$ for $k>0$ (see [Olv, Chapter 3, §1.1]). Truncating after two terms,

$$\Gamma(\alpha+2,-2iy) = e^{2iy}\left(2^{\alpha+1}e^{-i(\alpha+1)\pi/2}y^{\alpha+1} + (\alpha+1)2^\alpha e^{-i\alpha\pi/2}y^\alpha + \mathcal{O}(y^{\alpha-1})\right).$$

Plugging this into (A.2) finishes the proof. □

For future use, define the quantity

$$\gamma_\pm(\alpha) = -2^{-\alpha-2}e^{\pm i\alpha\pi/2}\Gamma(\alpha+1).$$

Since V is real, we can define the complementary solution u_- simply by $u_-=\bar{u}_+$, so that $u_-=e^{-iy}(1+\bar{b}_1+\bar{\varepsilon})$.

A.2. **WKB solutions.** We would like to connect the solutions $u_\pm$ with a WKB-type solution which is valid for $x \in (0, \infty)$. To do this we will require precise remainder estimates that will permit matching solutions at an h-dependent family of points $x_0 \to 0$ that satisfies $x_0/h \to \infty$ so Lemma A.1 will apply. Let $f = 1 - V$, so $P - 1 = (hD_x)^2 - f$. Define the phase

$$\phi(x) = \int_0^x f^{1/2}(s)\, ds.$$

According to [Olv, Chapter 6, Theorem 2.2], there exists an exact solution to $Pu = 0$ on $(0, 1)$ of the form

$$v_+(x) = f(x)^{-1/4} e^{i\phi(x)/h}(1 + \delta(x)).$$

The remainder satisfies

$$|\delta(x)| \leq e^{h\tau(x)} - 1, \quad f(x)^{-1/2}|h\delta'(x)| \leq e^{h\tau(x)} - 1,$$

where

$$\tau(x) = \int_x^\infty \left| f(s)^{-1/4} \partial_s^2 \left(f(s)^{-1/4} \right) \right| ds.$$

In particular, $v_+(x) = f(x)^{-1/4} e^{i\phi(x)/h} + \mathcal{O}(h)$ uniformly on any compact subset of $(0, \infty)$. Observe that $f = 1$ and δ vanishes outside the support of V, since $\tau(x)$ vanishes. Thus $v_+ = c_0 e^{ix/h}$ for $x \gg 0$, where

$$c_0 = \int_0^{x_1} f^{1/2}(s)\, ds - x_1$$

for any fixed point $x_1 \gg 0$ outside the support of V.

There exist constants A, B such that $v_+ = Au_+ + Bu_-$. Setting $u = v_+ = Au_+ + Bu_-$, the solution u satisfies

$$u = \begin{cases} Ae^{ix/h} + Be^{-ix/h} & x < 0, \\ c_0 e^{ix/h}, & x \gg 0. \end{cases}$$

Therefore $R = B/A$ and $T = c_0/A$, where R, T are as in (1.3). The constants A, B are found by computing the semiclassical Wronskians

$$\mathcal{W}_h(u_\pm, v_+)(x) = hu_\pm(x) \cdot hv_+'(x) - hu_\pm'(x) \cdot v_+(x)$$

at an appropriate h-dependent point (the Wronskian is of course constant). Indeed, we have the identity

$$v_+ = \frac{\mathcal{W}(u_+, v_+)}{\mathcal{W}(u_+, u_-)} u_- - \frac{\mathcal{W}(u_-, v_+)}{\mathcal{W}(u_+, u_-)} u_+. \tag{A.3}$$

A.3. **Wronskians.** We continue to write $y = x/h$. Fix η satisfying

$$\frac{2+\alpha}{2(\alpha+1)} < \eta < 1.$$

and set $x_0 = h^\eta$. Then $y_0 = x_0/h \to \infty$ whereas

$$h^\alpha y_0^{\alpha+1} = x_0^{\alpha+1}/h = o(h^{\alpha/2}). \tag{A.4}$$

Since $x_0^{\alpha+1}/h \to 0$, we see that

$$e^{i\phi(x_0)/h} = e^{ix_0/h}e^{i(\phi(x_0)-x_0)/h} = e^{iy_0}\Big(1 + h^\alpha \frac{iy_0^{\alpha+1}}{2i(\alpha+1)} + \zeta(x_0)\Big),$$

where $\zeta(x_0) = \mathcal{O}\left(x_0^{2\alpha+2}/h^2\right) + \mathcal{O}\left(x^{2\alpha+1}/h\right)$. We then check that

$$x^{\alpha+2}/h = h^{(\alpha+2)\eta-1} < h^{((2+\alpha)^2/2(\alpha+1))-1} = h^{1+\alpha},$$

so in particular, $\zeta(x_0) = o(h^\alpha)$ and $h\zeta'(x_0) = o(h^\alpha)$. Since $\alpha \in (0,1)$, it follows that $f(x)^{-1/4} \sim 1 + x^\alpha/4$ as $x \to 0^+$, hence

$$\tau(x) \sim \alpha x^{\alpha-1}/4 \text{ as } x \to 0^+.$$

Therefore

$$f(x_0)^{-1/4} = 1 + x_0^\alpha/4 + o(h^\alpha), \quad h(f^{-1/4})'(x_0) = o(h^\alpha).$$

Finally, the errors in $u_\pm(x_0)$ and $v_+(x_0)$ are bounded by

$$|\varepsilon(x_0)| + |h\varepsilon(x_0)| = o\left(h^\alpha\right), \quad |\delta(x_0)| + |h\delta'(x_0)| = o(h^\alpha).$$

From this we conclude that

$$v_+(x_0) = e^{iy_0}\Big(1 + h^\alpha\Big(\frac{y_0^{\alpha+1}}{2i(\alpha+1)} + \frac{y_0^\alpha}{4}\Big) + o(h^\alpha)\Big),$$

$$v_+'(x_0) = ie^{iy_0}\Big(1 + h^\alpha\Big(\frac{y_0^{\alpha+1}}{2i(\alpha+1)} - \frac{y_0^\alpha}{4}\Big) + o(h^\alpha)\Big).$$

Similarly,

$$u_\pm(x_0) = e^{\pm iy_0}\Big(1 \pm h^\alpha\Big(\frac{y_0^{\alpha+1}}{2i(\alpha+1)} \pm \frac{y_0^\alpha}{4}\Big) + o(h^\alpha)\Big) + h^\alpha e^{\mp iy_0}\gamma_\pm(\alpha),$$

$$hu_\pm'(x_0) = \pm ie^{\pm iy_0}\Big(1 \pm h^\alpha\Big(\frac{y_0^{\alpha+1}}{2i(\alpha+1)} \mp \frac{y_0^\alpha}{4}\Big) + o(h^\alpha)\Big) \mp ih^\alpha e^{\mp iy_0}\gamma_\pm(\alpha).$$

Calculating the Wronskians by evaluating at x_0,

$$\mathcal{W}(u_+, v_+) = 2ih^\alpha\gamma_+(\alpha) + o(h^\alpha), \quad \mathcal{W}(u_-, v_+) = 2i + o(h^\alpha).$$

We also have $\mathcal{W}(u_+, u_-) = -2i$ by evaluating the Wronskian at $x = 0$. Using (A.3), we see that $v_+ = Au_+ + Bu_-$ with $A = 1 + o(1)$ and $B = -h^\alpha\gamma_+ + o(h^\alpha)$. Dividing through by A also shows that $u_+ + Ru_- = Tv_+$, where the reflection and transmission coefficients satisfy

$$R = 2^{-\alpha-2}e^{i\alpha\pi/2}\Gamma(\alpha+1)h^\alpha + o(h^\alpha), \quad T = c_0 + o(1),$$

thereby completing the proof.

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