ERRATUM TO *SEMICLASSICAL SECOND MICROLOCAL PROPAGATION OF REGULARITY AND INTEGRABLE SYSTEMS*

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Nalini Anantharaman has pointed out to the authors that the proof of Corollary 6.2 of [2] is incomplete, as the assertion of the penultimate sentence is wrong. We have been unable to complete the proof of the corollary as stated, but note that the incomplete proof does yield several interesting results, including a new proof of the somewhat mysterious results of [3]. It is our hope that adopting some of the second microlocal methods used by Anantharaman-Maciá [1] in the special setting of flat tori may lead to stronger results.

We emphasize that the gap in the proof of Corollary 6.2, which is an application of the main propagation results of the paper, does not affect any of the other results of [2]. In particular the key results, Theorem 4.1 and Theorem 5.1, are valid with unchanged proofs. Also, as we show below, Corollary 6.2 does hold in dimension two.

In order to state our general replacement for Corollary 6.2, we first introduce a definition. (We continue with the notation of [2].)

Definition 0.1. Let $p \in \mathcal{L}$, and $V \subset T_p\mathcal{L} \cong \mathbb{R}^n$ a vector subspace. Let $\overline{\exp_p(V)}$ denote the closure of the flowout starting at p along all constant vector fields in V.

Our result now describes how local Lagrangian regularity propagates, based on our results on propagation of the second microlocal wavefront set. The idea is that regularity on a set $U \subset \mathcal{L}$ spreads automatically to certain points $p \in \mathcal{L}$: By Theorem 5.1 of [2], for any point $q \in {}^2\mathrm{WF}(u)$ lying over a point p, ${}^2\mathrm{WF}(u)$ would automatically fill out the orbit closure under $\operatorname{span}\{\mathsf{H}_1, \mathsf{H}_2\}$ (with orbits emanating from q), hence if if all of these orbit closures intersect U, there cannot be any ${}^2\mathrm{WF}$ over p. Although $\mathsf{H}_1(q)$ regarded as a vector in (i.e. pushed forward to) $T_p\mathcal{L}$ is independent of q, in the same sense $\mathsf{H}_2(q)$ varies with q, requiring a hypothesis on general two-dimensional V below:

Proposition 0.2. *Assume that the hypotheses of Theorem 5.1 hold and that, additionally, $\mathcal{L}$ is isoenergetically nondegenerate. Let $U \subset \mathcal{L}$ and let*

$$\widetilde{U} = \Big\{p \in \mathcal{L} : \ \overline{\exp_p(V)} \cap U \neq \emptyset \text{ for every two-dimensional } V \subset \mathbb{R}^n \text{ containing } \sum \overline{\omega}_j \partial_{\theta_j}\Big\}.$$

If u is locally Lagrangian on $U \subset \mathcal{L}$ relative to L^2 then u is locally Lagrangian on $\widetilde{U}$ relative to $h^{-\epsilon}L^2$ for all $\epsilon > 0$.

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Proof. As stated in the proof of Corollary 6.2, ${}^2\mathrm{WF}^{\infty,-1/2}(u)$ is invariant under the flow along

$$\mathsf{H}_1 = \sum \overline{\omega}_j \partial_{\theta_j}$$

and

$$\mathsf{H}_2 = \sum \overline{\omega}_{ij} \hat{I}_i \partial_{\theta_j}$$

and hence under any linear combination of these vector fields. We also remark that ${}^2\mathrm{WF}^{\infty,-1/2}(u)$ is contained in the characteristic set

$$\Sigma_2 = \{\sum \overline{\omega}_j \hat{I}_j = 0\}.$$

We claim that $\mathsf{H}_1, \mathsf{H}_2$ are never linearly dependent on Σ_2 : if they were linearly dependent at $\hat{I} = \xi$, we would have by rescaling ξ appropriately:

$$\sum \overline{\omega}_{ij} \xi_i = \overline{\omega}_j$$

for each j, hence

$$\begin{pmatrix} \overline{\omega}_{11} & \dots & \overline{\omega}_{1n} & \overline{\omega}_1 \\ \vdots & \ddots & \vdots & \vdots \\ \overline{\omega}_{n1} & \dots & \overline{\omega}_{nn} & \overline{\omega}_n \\ \overline{\omega}_1 & \dots & \overline{\omega}_n & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum \xi_j \overline{\omega}_j \end{pmatrix}.$$

On Σ_2, this vector would vanish, contradicting isoenergetic nondegeneracy. Thus, $\mathrm{span}\{\mathsf{H}_1, \mathsf{H}_2\}$ is always two-dimensional.

If $p \in \widetilde{U}$, then for every value of ξ, we have $(p, \xi) \notin {}^2\mathrm{WF}^{\infty,-1/2} u$, since if (p, ξ) did lie in the wavefront set, then by closedness, all of $\overline{\exp_p(V)}$ would lie in the wavefront set with $V = \mathrm{span}\{\mathsf{H}_1, \mathsf{H}_2\}$, and this set intersects U by hypothesis. Thus, u enjoys Lagrangian regularity with respect to $h^{-1/2}L^2$ on $\widetilde{U}$. Since $u \in L^2$, an interpolation yields the desired regularity. □

Consequently, we do recover the result stated in [2] in the special case $n = 2$:

Corollary 0.3. *Let $n = 2$. If u is locally Lagrangian relative to L^2 near some point $p \in \mathcal{L}$, then u is locally Lagrangian on relative to $h^{-\epsilon}L^2$ on all of $\mathcal{L}$.*

We also recover the results of [3], showing that Lagrangian regularity on an annular neighborhood of a closed orbit "fills in" the interior:

Corollary 0.4. *Assume that all the $\overline{\omega}_j$ are rationally related, and let γ denote a closed orbit under H_1. Let N_δ denote a delta-neighborhood of γ with respect to the flat metric on $\mathcal{L}$. If u is locally Lagrangian relative to L^2 on all of $N_{2\delta} \backslash N_\delta$ then u is locally Lagrangian relative to $h^{-\epsilon}L^2$ on γ.*

Proof. We simply choose $U = N_{2\delta} \backslash N_\delta$ and remark that for $p \in \gamma$, any two-dimension orbit-closure through p passes through U. □

References

[1] Fabricio Maciá and Nalini Anantharaman, *Semiclassical measures for the schrödinger equation on the torus*, arXiv:1005.0296, 2010.

[2] András Vasy and Jared Wunsch, *Semiclassical second microlocal propagation of regularity and integrable systems*, J. Anal. Math. **108** (2009), 119–157. MR MR2544756

[3] Jared Wunsch, *Spreading of lagrangian regularity on rational invariant tori*, Comm. Math. Phys. **279** (2008), 487–496.

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