Geometric optics and its limitations

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Wave equation

How does a wave move? Mathematical description of waves, at least to first approximation, is the same in many different settings: The (scalar) wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = 0,$$

with

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

for u a function on $\mathbb{R}^{1+n}$.

This (approximately) describes sound waves, light waves (omitting polarization), water waves, etc.

A close relative is the *Schrödinger* wave equation

$$\frac{\hbar}{i}\frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2}\Delta\psi = 0,$$

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Question: What do solutions *look like*?

In one dimension, we can write explicit solutions:

$$u(t,x) = f_+(t-x) + f_-(t+x)$$

gives a solution for any functions $f_\pm$ (differentiable or not!).

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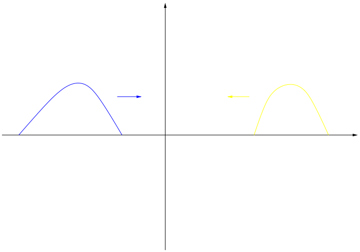

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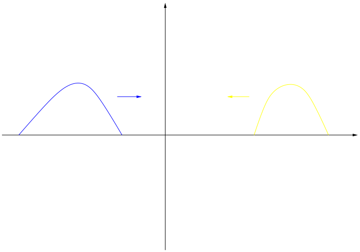

So waves in one dimensional space can propagate as localized objects, at unit speed: they move just like particles! This is the idea of "geometric optics:" we can understand a lot about wave propagation via particle motion.

Question: What about in higher dimension?

Answer: Not as simple as in 1D. "Wave packets" may propagate, but they *spread* as they do so.

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Wavefront set

Let $f(x)$ be a function (or even a distribution, i.e., generalized function) on $\mathbb{R}^k$.

Definition

$x_0 \in \mathbb{R}^k$ is in the *singular support* of f (sing-supp f) if there is no neighborhood of x_0 on which $f \in \mathcal{C}^\infty$.

The singular support is the set of points at which a function is *not smooth*, i.e. is *singular*.

A refinement, due to Hörmander (cf. Sato in analytic case) is the *wavefront set*, $\mathrm{WF} f$. This measures where f is singular, *and in what direction*:

$$\mathrm{WF} f \subset T^*\mathbb{R}^k,$$

with

$$\pi(\mathrm{WF} f) = \operatorname{sing-supp} f.$$

Can be explicitly characterized in terms of Fourier transform: use the fact that $g(x) \in L^2$ is smooth iff $\hat{g}(\xi)$ is rapidly decreasing, then think about isolating rapid decrease in different directions.

Studying WF of solutions to wave equation is studying high-frequency limit as need high-frequency Fourier modes to synthesize singularities.

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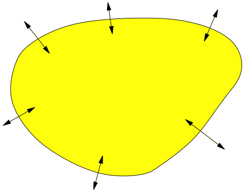
Wavefront set example



Let $f(x) = 1_\Omega$, the indicator function of an open set Ω with smooth boundary.

$$\text{sing-supp}\, f = \partial\Omega,$$

$$\mathrm{WF} f = N^*(\partial\Omega) = \{(x,\xi) : x \in \partial\Omega,\ \xi \perp T(\partial\Omega)\}$$

Propagation of singularities

Theorem

For a solution u to the wave equation on $\mathbb{R}^{1+n}$,

- $\mathrm{WF}\, u \subset \{\tau^2 = |\xi|^2\}$
- $(t_0, x_0, \tau_0, \xi_0) \in \mathrm{WF}\, u$ *if and only if*

$$(t_0 - s\tau_0, x_0 + s\xi_0, \tau_0, \xi_0) \in \mathrm{WF}\, u \text{ for all } s \in \mathbb{R}.$$

Thus,

- Spacetime singularities lie in the "light cone."
- Wavefront set propagates as time evolves by moving in straight lines at unit speed, in direction given by ξ_0, the "momentum" variable; hence moves as a particle would.

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Variable coefficients

What happens on a manifold? Let (X, g) be a Riemannian manifold. Replace Δ in the wave equation by

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i} g^{ij} \sqrt{g} \frac{\partial}{\partial x_j}$$

with $g = \det(g_{ij})$ ("Laplace-Beltrami operator") and consider the wave equation on $\mathbb{R} \times X$.

Can also think of wave propagation in $\mathbb{R}^n$ in an inhomogeneous, anisotropic medium (e.g. earth!).

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For a solution to the wave equation on $\mathbb{R} \times X$, $\mathrm{WF}\, u$ propagates along geodesics, lifted to $T^*(\mathbb{R} \times X)$, inside the light cone.

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Spectral geometry

A SHORT DIGRESSION INTO SPECTRAL GEOMETRY

(X, g) compact Riemannian manifold, Δ_g Laplacian. There is an orthonormal basis of $L^2(X)$ of eigenfunctions $\varphi_j(x)$ of Δ

$$\Delta\varphi_j = -\lambda_j^2\varphi_j.$$

where eigenvalues $-\lambda_j^2 \to -\infty$.

Can construct solutions to the wave equation by separation of variables

$$u(t, x) = e^{\pm i\lambda_j t}\varphi_j(x)$$

so λ_j's are the characteristic frequencies of vibration, or the overtone series.

Kac (1966): what can you determine about (X, g) from λ_j's?
("Can one hear the shape of a drum?")
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Wave trace

Consider the family of operators $U(t) = \cos t\sqrt{-\Delta}$
i.e., the operator acting on φ_j by

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$u(t,x) = U(t)f$ solves the wave equation with initial data

$$u(0,x) = f(x), \quad u_t(0,x) = 0,$$

i.e. $U(t)$ is the *solution operator.*

Consider

$$\operatorname{Tr} U(t) = \sum \cos(\lambda_j t).$$

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Singularities of the wave trace

Using the propagation of singularities theorem, it is not hard to prove "Poisson relation:"

Theorem (Chazarain, Duistermaat-Guillemin, '74)

$\text{sing-supp}\,\mathrm{Tr}\,U(t) \subset \{0\} \cup \big\{ \pm \textit{lengths of closed geodesics on } X \big\}.$

- More or less: you can "hear" lengths of closed geodesics.
- Much harder: there is a considerably deeper trace theorem of Duistermaat-Guillemin, giving the leading-order description of the singularities of the wave trace (cf. Selberg on quotients of symmetric spaces). Gives subtler dynamical information.
- Applications of these ideas include theorem of Zelditch: Isospectral convex analytic domains in $\mathbb{R}^2$ with an axis of symmetry are isometric. (This involves. . .)

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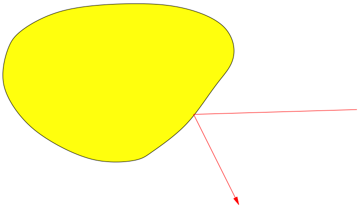
Boundaries

Say we impose Dirichlet boundary conditions on wave equation. What happens to wavefront set that reaches the boundary of an obstacle?

Case 1: transverse reflection. Reflects according to "angle of incidence equals angle of reflection," just like a billiard ball.

More precisely: energy and momentum tangent to obstacle are conserved; normal momentum jumps.



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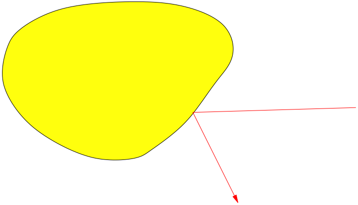

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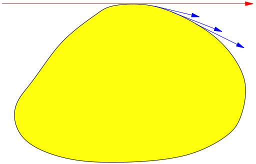
Case 2: tangency to a convex obstacle

The question here is as follows: can a ray carrying WF tangent to a convex obstacle stick to it and rerelease in the "shadow region," or does it simply pass on by?



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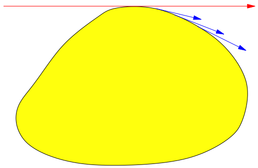

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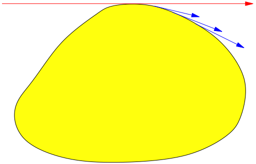

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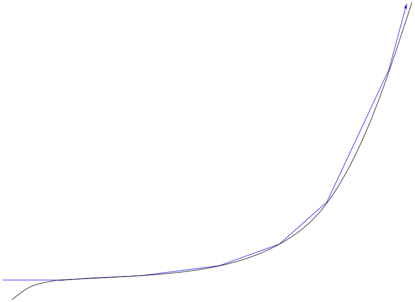
Glancing

Case 3: tangency, no convexity

In this case, if the ray has infinite order tangency with the boundary, even deciding what should constitute the continuation of a ray striking the boundary is difficult.

Example of Taylor (1976) of many possible continuations of a given ray that hits boundary with infinite-order tangency.



Theorem on propagation on appropriate "generalized geodesics" (Melrose-Sjöstrand, 1982).

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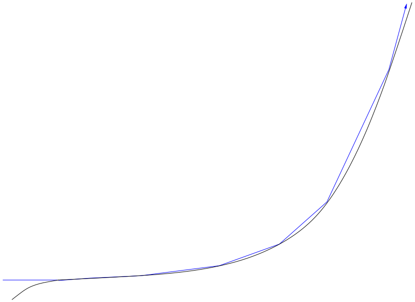

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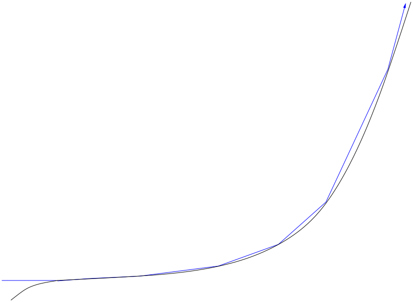

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The rules of geometric optics are no longer so simple (!) when we are not on a manifold with smooth boundary. Singularities may indeed go into the shadow region.

Diffraction studied (and named) first by Grimaldi, 1665—led to wave theory of light. Cf. Huygens, Newton; also Fresnel, Poisson, Arago, 1818.

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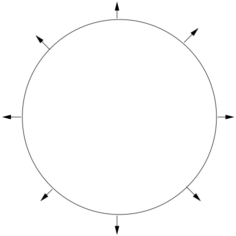
$$u(0,x) = 0, \quad u_t(0,x) = \delta(x).$$

Wavefront set is initially just $(0,\xi)$ for all ξ (δ is singular at the origin, in every direction).

On $\mathbb{R}^n$, singularities spread outward in an expanding sphere.



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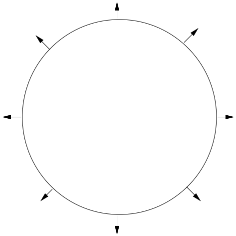
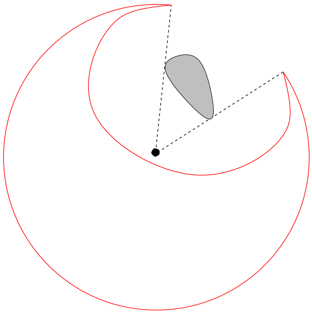
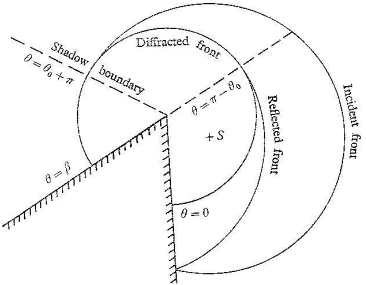


In an exterior domain, there is a reflected wave of singularities as well, but of course still nothing on the far side of the obstacle, by Melrose-Taylor.

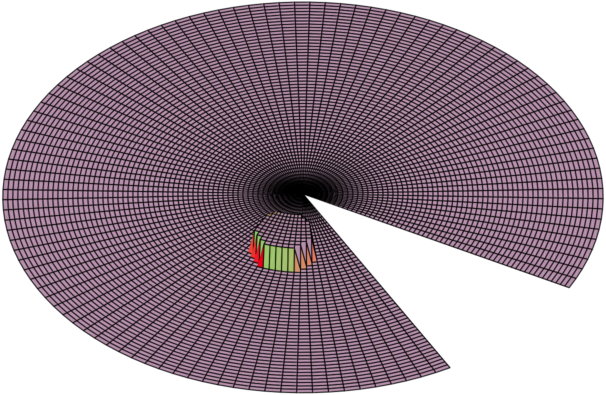


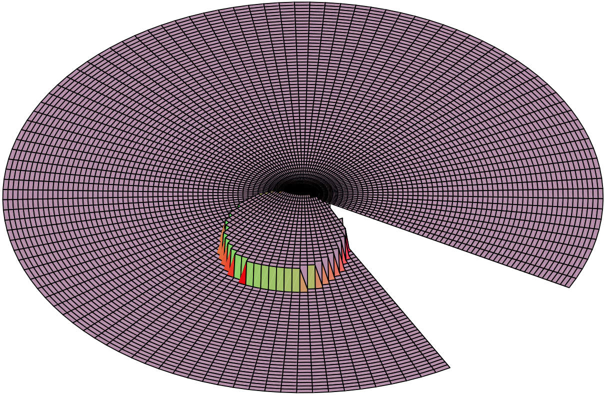
Sommerfeld

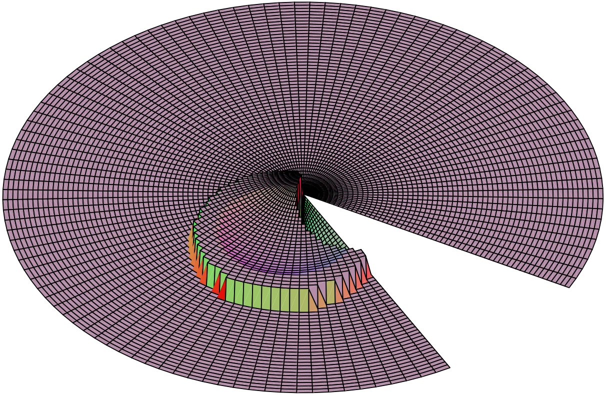
But Sommerfeld (1896) explicitly solved the following example, showing there are singularities in the shadow region: A spherical wave of singularities is emitted by the wedge tip at the time the wavefront strikes it:

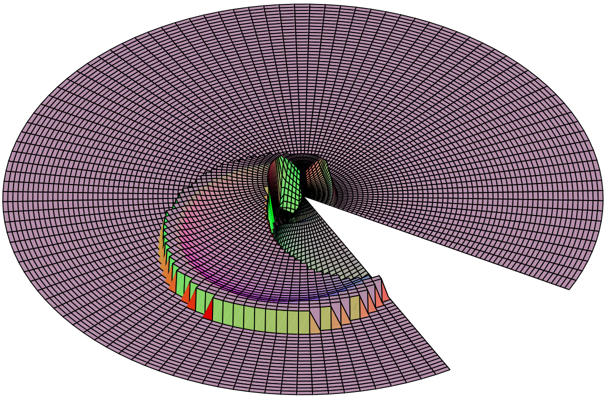


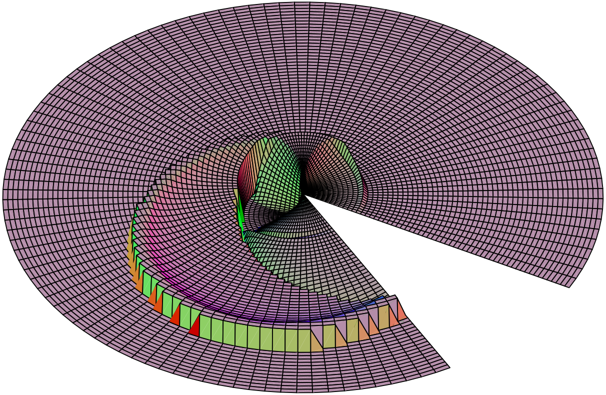
(Figure from F.G. Friedlander, *Sound Pulses*.)

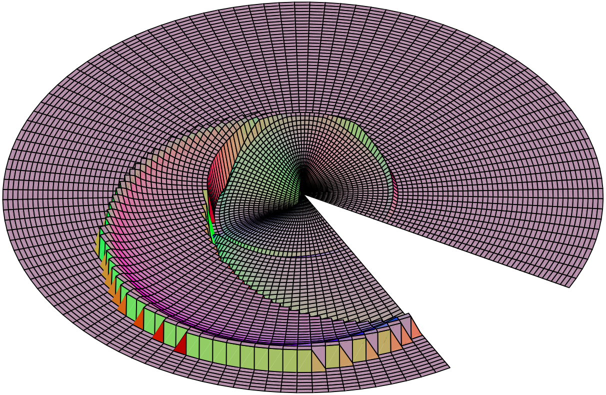


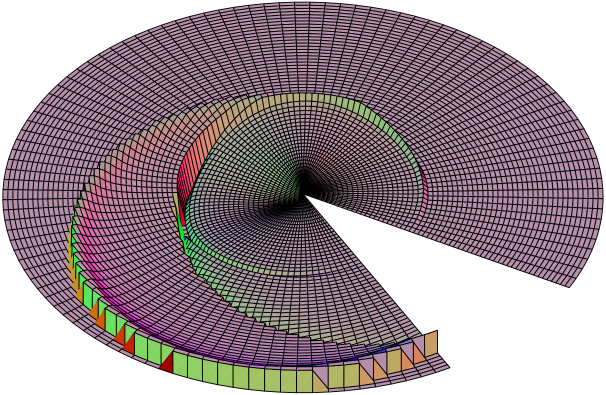


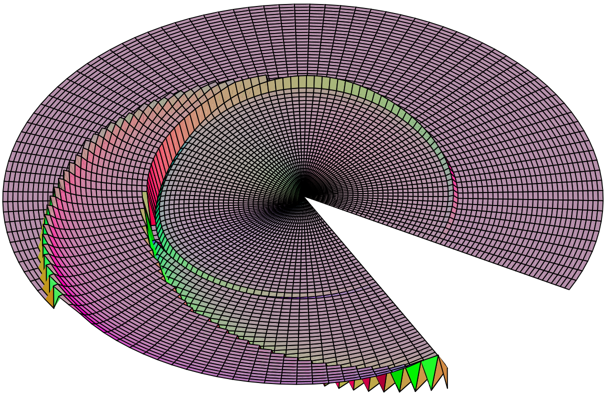












Other geometries

- A wedge is, of course, rather special. Sommerfeld was able to employ separation of variables, and deep knowledge of Bessel functions.
- More generally, we might consider something like a singularity striking an edge, a corner, or a cone point—some singular structure in our space. The laws of geometric optics have to be modified to allow diffraction by interaction with these structures:
- E.g. a singularity hitting a cone point creates a whole spherical wave of "vertex diffracted" singularities in every direction from the cone tip.

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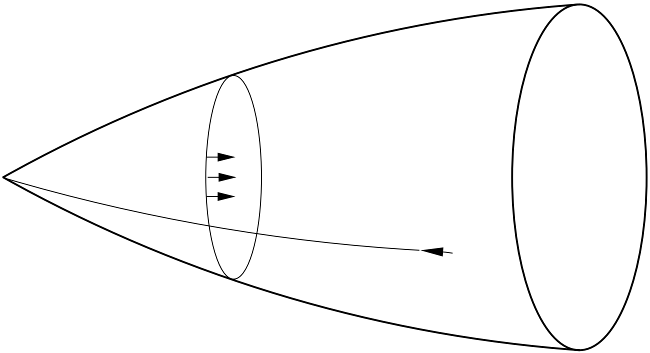
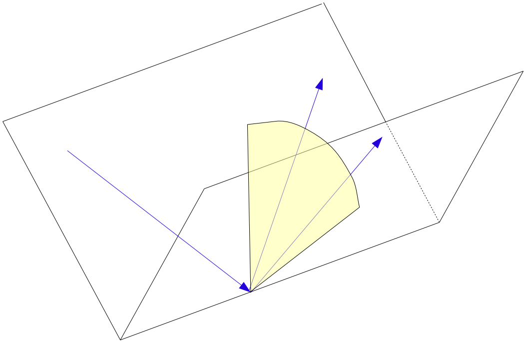
Diffracted front from a cone tip

(Keller: "it's not much of a law, but at least it's democratic.")



Edges

Geometry is a bit more interesting for edges or corners. E.g. a singularity hitting an edge between two walls can create a whole cone of outgoing singularities. Specular reflection holds to the degree that it makes sense, i.e. energy and tangential momentum are conserved:



A number of theorems substantiate this:

- Keller, extensive heuristics—"geometric theory of diffraction"
- Cheeger-Taylor, exact cones (1982)
- Lebeau, manifolds with corners (analytic setting) (1997)
- Melrose-W., manifolds with conic singularities (2004)
- Vasy, manifolds with corners (2008)
- Melrose-Vasy-W., manifolds with edge singularities (2008)

Sobolev regularity

Measure strength of singularities by what *Sobolev space* they lie in. Recall that for $k \in \mathbb{N}$,

$$H^k(X) = \{u(x) : u, Du, \ldots, D^k u \in L^2(X)\},$$

and this can be generalized to $k \in \mathbb{R}$, using interpolation and duality or Fourier analysis.

For $k > 0$, $H^{-k}(X)$ consists of (generalized) functions that can be as singular as "k derivatives of an L^2 function." ($H^0(X) = L^2(X)$.)

Regularity of fundamental solution

Regularity of the fundamental solution to the wave equation above: direct and reflected fronts are in H^{-0} while diffracted front is in $H^{1/2-0}$.
I.e., half a derivative of smoothing of diffractive wave relative to main one.

Question: Is the diffracted wave always smoother?

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Nonfocusing

It turns that the diffracted wavefront is not always weaker: if the incident wave is *focused* on (say) a cone point or corner, it is just as strong as the main singularity!

Nonfocusing hypothesis ensures otherwise; holds for fundamental solution.

Theorem

Let $U(t)$ be the fundamental solution to the wave equation. The diffracted front is smoother than the main front by $f/2$ derivatives where f is

- $n-1$ *if X is an n-manifold with conic singularities*
- $d-1$ *if X is a cone bundle ("edge manifold") with fiber dimension d*
- $c-1$ *if X is a manifold with corners, and the wave is diffracted by a corner of codimension c.*

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Loosely speaking, then, diffraction off a singular stratum of higher codimension produces more regular diffracted wave.

- Cheeger-Taylor, exact cones (1982)
- Gérard-Lebeau, planar angles in analytic setting (1993)
- Melrose-W., conic manifolds (2004)
- Melrose-Vasy-W., manifolds with edge singularities (2008)
- Melrose-Vasy-W., manifolds with corners, (?)

Poisson relation

On a conic manifold we have two kind of geodesics: Let diffractive geodesics enter and leave a boundary component on any pair of rays, while geometric geodesics enter and leave on geodesics that are limits of geodesics missing cone point.
Let

$$\mathrm{DIFF} = \{\pm \text{ lengths of closed diffractive geodesics}\} \cup \{0\}$$

and

$$\mathrm{GEOM} = \{\pm \text{ lengths of closed geometric geodesics}\} \cup \{0\}.$$

Theorem
$\mathrm{Tr} \cos t\sqrt{-\Delta} \in \mathcal{C}^{-n-0}(\mathbb{R}) \cap \mathcal{C}^{-1-0}(\mathbb{R}\backslash \mathrm{GEOM}) \cap \mathcal{C}^{\infty}(\mathbb{R}\backslash \mathrm{DIFF})$.

Flat surfaces with conic singularities

Hillairet has proved an actual *trace formula* on flat surfaces with conic singularities. Allows us to know, among other things, that these *possible* singularities will (usually) actually exist.

Consequence: spectral rigidity of triangles, since we can "hear" the length of the shortest altitude (originally proved by Durso). (Area and perimeter can be "heard" using $\operatorname{Tr} e^{t\Delta}$—cf. Kac.)

(Note that a triangle can be "doubled" across the edges to make a flat surface with three cone points.)

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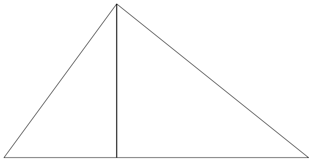

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Future directions

- More general singular spaces (e.g. to encompass complex projective varieties?). Use some kind of iterated cone/edge construction?
- Trace theorems.
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