THE DIFFRACTIVE WAVE TRACE ON MANIFOLDS WITH CONIC SINGULARITIES

G. AUSTIN FORD AND JARED WUNSCH

Abstract. Let (X, g) be a compact manifold with conic singularities. Taking Δ_g to be the Friedrichs extension of the Laplace-Beltrami operator, we examine the singularities of the trace of the half-wave group $e^{-it\sqrt{\Delta_g}}$ arising from strictly diffractive closed geodesics. Under a generic nonconjugacy assumption, we compute the principal amplitude of these singularities in terms of invariants associated to the geodesic and data from the cone point. This generalizes the classical theorem of Duistermaat–Guillemin on smooth manifolds and a theorem of Hillairet on flat surfaces with cone points.

0. Introduction

In this paper, we consider the trace of the half-wave group $\mathcal{U}(t) \stackrel{\text{def}}{=} e^{-it\sqrt{\Delta_g}}$ on a compact manifold with conic singularities (X, g). Our main result is a description of the singularities of this trace at the lengths of closed geodesics undergoing diffractive interaction with the cone points. Under the generic assumption that

the cone points of X are pairwise nonconjugate along the geodesic flow,

the resulting singularity at such a length $t = L$ has the oscillatory integral representation

$$\int_{\mathbb{R}_\xi} e^{-i(t-L)\cdot\xi}\, a(t, \xi)\, d\xi,$$

where the amplitude a is to leading order

$$a(t,\xi) \sim L \cdot (2\pi)^{\frac{kn}{2}}\, e^{\frac{ik\pi(n-3)}{4}} \cdot \chi(\xi)\, \xi^{-\frac{k(n-1)}{2}} \times \left[\prod_{j=1}^{k} i^{-m_{\gamma_j}} \cdot \boldsymbol{D}_{\alpha_j}(q_j, q_j') \cdot \operatorname{dist}_g^{\gamma_j}(Y_{\alpha_{j+1}}, Y_{\alpha_j})^{-\frac{n-1}{2}} \cdot \Theta^{-\frac{1}{2}}(Y_{\alpha_j} \to Y_{\alpha_{j+1}}) \right]$$

as $|\xi| \to \infty$ and the index j is cyclic in $\{1, \ldots, k\}$. Here, n is the dimension of X and k the number of diffractions along the geodesic, and χ is a smooth function supported in $[1, \infty)$ and equal to 1 on $[2, \infty)$. The product is over the diffractions undergone by the geodesic, with $\boldsymbol{D}_{\alpha_j}$ a quantity determined by the functional calculus of the Laplacian on the link of the j-th cone point Y_{α_j}, the factor $\Theta^{-\frac{1}{2}}(Y_{\alpha_j} \to Y_{\alpha_{j+1}})$ is (at least on a formal level) the determinant of the differential of the flow between the j-th and $(j+1)$-st cone points, and m_{γ_j} is the Morse index of the geodesic segment γ_j from the j-th to $(j+1)$-st cone points. All of these factors are described in more detail below.

To give this result some context, we recall the known results for the Laplace-Beltrami operator $\Delta_g = d_g^* \circ d$ on a *smooth* ($\mathcal{C}^\infty$) compact Riemannian manifold (X, g). In this setting, there is a countable orthonormal basis for $L^2(X)$ comprised of eigenfunctions φ_j of Δ_g with eigenvalues $\{\lambda_j^2\}_{j=0}^{\infty}$ of finite multiplicity and tending to infinity. The celebrated trace formula of Duistermaat and Guillemin [DG75], a generalization of the Poisson summation formula to this setting, establishes that the quantity

$$\sum_{j=0}^{\infty} e^{-it\lambda_j}$$

is a well-defined distribution on $\mathbb{R}_t$. Moreover, it satisfies the "Poisson relation": it is singular only on the *length spectrum* of (X, g),

$$\mathrm{LSp}(X, g) \overset{\text{def}}{=} \{0\} \cup \{\pm L \in \mathbb{R} : L \text{ is the length of a closed geodesic in } (X, g)\}.$$

(This was shown independently by Chazarain [Cha74]; see also [CdV73].) Subject to a nondegeneracy hypothesis, the singularity at the length $t = \pm L$ of a closed geodesic has a specific leading form encoding the geometry of that geodesic—the formula involves the linearized Poincaré map and the Morse index of the geodesic. The proofs of these statements center around the identification

$$\sum_{j=0}^{\infty} e^{-it\lambda_j} = \mathrm{Tr}\, \mathcal{U}(t)$$

where $\mathcal{U}(t) = e^{-it\sqrt{\Delta_g}}$ is (half of) the propagator for solutions to the wave equation on X; in particular, $\mathcal{U}(t)$ is a Fourier integral operator.

In this paper, we prove an analogue of the Duistermaat–Guillemin trace formula on compact *manifolds with conic singularities* (or "conic manifolds"), generalizing results of Hillairet [Hil05] from the case of flat surfaces with conic singularities. We again consider the trace $\mathrm{Tr}\, \mathcal{U}(t)$, a spectral invariant equal to $\sum_{j=0}^{\infty} e^{-it\lambda_j}$. The Poisson relation is complicated here by the fact that closed geodesics may have two different meanings on a manifold with conic singularities. On the one hand, we may regard a geodesic striking a cone point as being legitimately continued by any other geodesic emanating from a cone point. On the other hand, we may only take those geodesics which are (locally) uniformly approximable by families of geodesics that miss the cone point. It turns out that singularities of solutions to the wave equation can propagate along all geodesics in the former, broader interpretation, and this is the phenomenon of "diffraction." It was extensively (and explicitly) studied for cones admitting a product structure by Cheeger and Taylor [CT82a, CT82b]. We refer to geodesics of this broader type as *diffractive geodesics,* and call them *strictly diffractive* if they are *not* (locally) approximable by ordinary geodesics. We refer to the (locally) approximable geodesics as *geometric geodesics*. In [MW04], Melrose and the second author showed that singularities of solutions to the wave equation propagate along diffractive geodesics, although the singularities at strictly diffractive geodesics are generically weaker than the singularities at the geometric geodesics. In [Wun02] the second author used this fact to prove that the singularities of the wave trace on a conic manifold are a subset of the length spectrum $\mathrm{LSp}(X, g)$, consisting again of zero and the positive and negative lengths of closed geodesics. A new wrinkle in this case is the fact that singularities at closed strictly diffractive geodesics are weaker than the singularities at ordinary or geometric closed geodesics, reflecting the analogous phenomenon for the propagation of singularities.

Expanding upon these previous works, we describe explicitly in this paper the leading order behavior of the singularities at lengths of closed diffractive geodesics. We must assume that these geodesics are isolated and appropriately nondegenerate, essentially in the sense that no pair of cone points are mutually conjugate along the geodesic flow of the manifold. Note that these hypotheses are generically satisfied and moreover, the diffractive closed geodesics are generically the only closed geodesics apart from interior ones, where the contribution to the trace is already known [DG75].

To describe these leading order asymptotics, we need to briefly set up some of the framework. Let us suppose that γ is a diffractive geodesic undergoing diffraction with k cone points and repeating with period T (so T is the primitive length of γ). We let γ_j denote the segment of geodesic from the j-th to the $(j+1)$-st cone point, and we denote by m_{γ_j} the Morse index of each of these segments. As the link of the j-th cone point is a Riemannian manifold Y_j, we may consider the operator

$$\nu_j \overset{\text{def}}{=} \sqrt{\Delta_{Y_j} + \left(\frac{2-n}{2}\right)^2}$$

on the link as well as the operators in its functional calculus. The ordinary propagation of singularities implies that the kernel of the half-Klein-Gordon propagator $e^{-it\nu_j}$ is smooth away from points distance $|t|$ apart. In particular, if q_j and q_j' are the initial and terminal points on Y_j of the geodesic segments γ_j and γ_{j-1} respectively along a diffractive geodesic, then the Schwartz kernel of the half-Klein-Gordon propagator $e^{-i\pi\nu_j}$ is smooth near (q_j, q_j'). We write $\boldsymbol{D}_{\alpha_j}(q_j, q_j')$ for this value $\mathcal{K}\big[e^{-i\pi\nu_j}\big](q_j, q_j')$. Finally, for each segment γ_j we define an invariant $\Theta(Y_{\alpha_j} \to Y_{\alpha_{j+1}})$ in Section 1.3 below, letting j range cyclically over $\{1, \dots, k\}$. This invariant can be viewed in more than one way: it looks formally like the determinant of the differential of the flow from one cone point to the next, but owing to the singular of nature of this flow, we employ a definition in terms of (singular) Jacobi fields; alternatively, it measures the tangency of the intersection along γ_j of the geodesic "spheres" centered at the successive cone points, via an interpretation in terms of Wronskians of Jacobi fields vanishing at successive cone points (see Section 5 for the latter interpretation).

Main Theorem. *For t sufficiently close to L, the wave trace* $\operatorname{Tr}\mathcal{U}(t)$ *is a conormal distribution with respect to $t = L$ of the form*

$$\int_{\mathbb{R}_\xi} e^{-i(t-L)\cdot\xi}\, a(t,\xi)\, d\xi.$$

Its amplitude $a \in S^{-\frac{k(n-1)}{2}}(\mathbb{R}_t \times \mathbb{R}_\xi)$ has the leading order asymptotics

$$\begin{aligned} a(t,\xi) &\equiv L \cdot (2\pi)^{\frac{kn}{2}}\, e^{\frac{ik\pi(n-3)}{4}} \cdot \chi(\xi)\, \xi^{-\frac{k(n-1)}{2}} \\ &\times \left[\prod_{j=1}^{k} i^{-m_{\gamma_j}} \cdot \boldsymbol{D}_{\alpha_j}(q_j, q_j') \cdot \operatorname{dist}_g^{\gamma_j}(Y_{\alpha_{j+1}}, Y_{\alpha_j})^{-\frac{n-1}{2}} \cdot \Theta^{-\frac{1}{2}}(Y_{\alpha_j} \to Y_{\alpha_{j+1}})\right] \end{aligned}$$

modulo elements of $S^{-\frac{k(n-1)}{2}-\frac{1}{2}+\varepsilon}$ for any $\varepsilon > 0$, where $\chi(\xi) \in \mathcal{C}^\infty(\mathbb{R})$ is supported in $[1,\infty)$ and equal to 1 for $\xi > 2$.

The rest of the paper is organized as follows. In Section 1 we review the geometry of manifolds with conic singularities, in particular the geometry of geodesics and Jacobi fields. Section 2 contains the calculation of the principal amplitude of the diffractive part of the half-wave propagator near the cone point of a metric cone, and Section 3 generalizes this calculation to the wider class of conic manifolds. In Section 4, we use the previous work to calculate the principal amplitude of a multiply-diffracted wave on a manifold with (perhaps multiple) cone points, and the proofs of the required results make up Sections 5 and 6. Finally, using a microlocal partition of unity developed in Section 7, we compute the trace of the half-wave group along the diffractive closed geodesics in Section 8. At the end, we include Appendix A as a brief review of the theory of Lagrangian distributions and their amplitudes.

Notation. We use the following pieces of notation throughout this work.

- If $\{\mathcal{Y}^s : s \in \mathbb{R}\}$ is an $\mathbb{R}$-filtered collection of vector spaces with inclusions $\mathcal{Y}^s \subseteq \mathcal{Y}^t$ if $s < t$, then we write
$$\mathcal{Y}^{t-0} \stackrel{\text{def}}{=} \bigcup_{s<t} \mathcal{Y}^s.$$
Similarly, if the inclusions are of the form $\mathcal{Y}^s \supseteq \mathcal{Y}^t$ if $s < t$, then we write
$$\mathcal{Y}^{t+0} \stackrel{\text{def}}{=} \bigcap_{s>t} \mathcal{Y}^s.$$
- If $\mathrm{pr} : E \longrightarrow Z$ is a vector bundle over a manifold Z and $V \subseteq E$ is a subset of this bundle, then we write $V^\flat \stackrel{\text{def}}{=} \mathrm{pr}[V]$ for the projection of this subset to the base manifold.
- If Z is a smooth manifold, then we let $\dot{T}^*Z \stackrel{\text{def}}{=} T^*Z \backslash \mathbf{0}$ be its punctured cotangent bundle, where $\mathbf{0} \subseteq T^*Z$ is the zero section.
- We write $\mathcal{F}[u](\xi) \stackrel{\text{def}}{=} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, u(x)\, dx$ for the unitary normalization of the Fourier transform.

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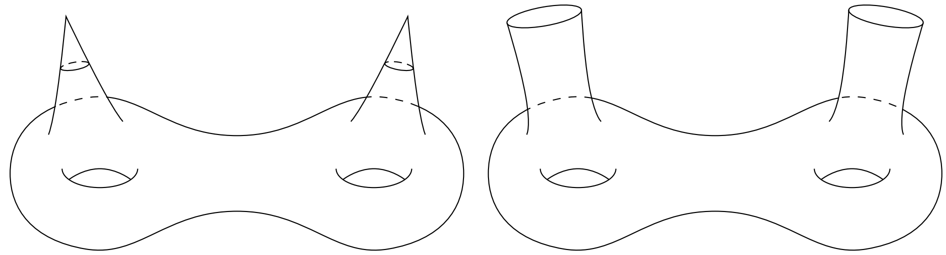
1. Conic geometry

In this section we recall the basic notions of the geometry of manifolds with conic singularities and the analysis of distributions on them. A conic manifold of dimension n is a Riemannian manifold with boundary (X, g) whose metric is nondegenerate over the interior X° but singular at the boundary ∂X. Near the boundary it is assumed to take the form
$$g = dx^2 + x^2\, h(x, dx; y, dy)$$
for some boundary defining function[1] x, where h is a smooth symmetric tensor of rank 2 restricting to be a metric on ∂X. Writing Y for the boundary ∂X, this restriction has the effect of reducing each of the components Y_α of Y to a "cone

[1] x is a defining function for the boundary if $\partial X = \{x = 0\}$ and $dx\big|_{\partial X} \neq 0$.



(A) The metric picture (B) The topological (or "blown-up") picture

FIGURE 1.1. A manifold with conic singularities

point." Thus, we refer to these components Y_α as the "cone points" of our manifold throughout the rest of this work.

In Section 1 of [MW04] the authors show the existence of a defining function x decomposing a collar neighborhood of the boundary into a product $[0,\varepsilon)_x \times Y$ such that in the associated local product coordinates (x, y) the metric decomposes as a product

$$g = dx^2 + x^2\, h(x, y, dy). \tag{1.1}$$

In particular, when written in these coordinates h is a smooth family in x of metrics on Y. Equivalently, the family of curves $[0,\varepsilon) \times \{y_0\}$ parametrized by $y_0 \in Y_\alpha$ are geodesics reaching the boundary component Y_α and foliating the associated collar neighborhood. These are moreover the *only* geodesics reaching Y_α. *Henceforth, we will always assume that the metric has been reduced to this normal form.* We write

$$h_0 \overset{\text{def}}{=} h\big|_{x=0} \quad \text{and} \quad h_\alpha \overset{\text{def}}{=} h\big|_{Y_\alpha}$$

for the induced metric on the boundary as a whole and an individual boundary component Y_α respectively. Additionally, we define $x_* \overset{\text{def}}{=} \sup x$ to be the supremum of the value of the designer boundary defining function, i.e., an upper bound on the distance from ∂X for which x is defined. We also write x_α for the restriction of x to the corresponding connected neighborhood of the cone point Y_α; it is thus a designer boundary defining function for this cone point.

1.1. **Operators and domains.** Let $(z_1, \ldots, z_n)$ be a local system of coordinates on the interior X°. We consider the Friedrichs extension of the Laplace operator

$$\Delta_g \overset{\text{def}}{=} \frac{1}{\sqrt{\det g(z)}} \sum_{j,k=1}^{n} D_{z_j}\, g^{jk}(z) \sqrt{\det g(z)}\, D_{z_k}$$

from the domain $\mathcal{C}_c^\infty(X^\circ)$. Here, $D_{z_j} = \frac{1}{i}\,\partial_{z_j}$ is the Fourier-normalization of the z_j-derivative. Using our preferred product coordinates (x, y) near the boundary, we compute

$$\Delta_g = D_x^2 - \frac{i\,[(n-1) + x\, e(x)]}{x}\, D_x + \frac{1}{x^2}\,\Delta_{h(x)}, \tag{1.2}$$

where $e(x)$ is the function

$$e(x) = \frac{1}{2}\frac{\partial \log\det h(x)}{\partial x} = \frac{1}{2}\,\mathrm{tr}\left[h^{-1}(x)\,\frac{\partial h(x)}{\partial x}\right].$$

This is the Laplace operator considered as an operator on scalars. In the sequel, however, we will primarily work with the version of the operator Δ_g acting on half-densities. To define this, we trivialize the half-density bundle using the convention that the metric half-density ω_g, which in the local z-coordinates is

$$\omega_g = [\det g(z)]^{\frac{1}{4}} \left|dz_1 \wedge \cdots \wedge dz_n\right|^{\frac{1}{2}},$$

is annihilated by Δ_g. In other words, the action of Δ_g on a general half-density $f(z)\,\omega_g$ is

$$\Delta_g\left[f(z)\,\omega_g\right] = \left[\frac{1}{\sqrt{\det g(z)}} \sum_{j,k=1}^{n} D_{z_j}\, g(z)^{jk} \sqrt{\det g(z)}\, D_{z_k} f(z)\right] \omega_g.$$

Note that near the boundary we have

$$\omega_g = x^{\frac{n-1}{2}} \left[\det h(x)\right]^{\frac{1}{4}} \left|dx \wedge dy_1 \wedge \cdots \wedge dy_{n-1}\right|^{\frac{1}{2}}, \tag{1.3}$$

with $h(x)$ the family of induced metrics on the boundary.

From Δ_g we construct its complex powers Δ_g^z via the functional calculus. We use the domains of the real powers,

$$\mathcal{D}_s \overset{\text{def}}{=} \mathrm{Dom}\Big[\Delta_g^{s/2} : L^2\Big(X; |\Omega|^{\frac{1}{2}}(X)\Big) \longrightarrow L^2\Big(X; |\Omega|^{\frac{1}{2}}(X)\Big)\Big],$$

as the principal regularity spaces in this work. Here, $L^2\Big(X; |\Omega|^{\frac{1}{2}}(X)\Big)$ are the L^2-half-densities on X, so each domain is a space of distributional half-densities.

An alternate characterization of these domains comes in terms of b-Sobolev spaces, which we now recall. First, set $\mathcal{V}_{\mathrm{b}}(X)$ to be the Lie algebra of all vector fields on X tangent to ∂X, and let $\mathrm{Diff}_{\mathrm{b}}^*(X)$ denote the filtered algebra of differential operators generated by these vector fields over $\mathcal{C}^\infty(X)$. For $m \in \mathbb{Z}_{\geqslant 0}$ we define the b-Sobolev space

$$H_{\mathrm{b}}^m(X) = \Big\{u \in L^2\Big(X; |\Omega|^{\frac{1}{2}}(X)\Big) : Au \in L^2\Big(X; |\Omega|^{\frac{1}{2}}(X)\Big) \text{ for all } A \in \mathrm{Diff}_{\mathrm{b}}^m(X)\Big\}.$$

More generally, we define the b-Sobolev spaces $H_{\mathrm{b}}^s(X)$ for all real orders s by either interpolation and duality or by substituting Melrose's b-calculus of pseudodifferential operators $\Psi_{\mathrm{b}}^s(X)$ for the differential b-operators—see [Mel93] for further details on the latter method.

Proposition 1.1 ([MW04][2])**.** *For $|s| < \frac{n}{2}$ there is an identification*

$$\mathcal{D}_s = x^{-s}\, H_{\mathrm{b}}^s(X).$$

It further follows from the analysis in [MW04] that for every $s \in \mathbb{R}$ we have

$$\sqrt{\Delta_g} : x^{-1} H_{\mathrm{b}}^s(X) \longrightarrow H_{\mathrm{b}}^{s-1}(X) \text{ if } n > 2,$$

while in the case $n = 2$ (which is nearly always a borderline case in such computations),

$$\sqrt{\Delta_g} : x^{-1+\varepsilon}\, H_{\mathrm{b}}^s(X) \longrightarrow H_{\mathrm{b}}^{s-1}(X) \text{ for all } \varepsilon > 0 \text{ (when } n = 2).$$

Moreover, it follows from the more detailed description of these complex powers Δ_g^z in [Loy03] that $\sqrt{\Delta_g}$ is *microlocal* over the interior manifold X°: its Schwartz

[2]We use a different convention for the density with respect to which L^2 is defined than was used in [MW04]. The b-weight $\frac{dx}{x}\, dy$ was used in that work rather than the metric weight ω_g^2.

kernel is that of a pseudodifferential operator over compact sets in $X^\circ \times X^\circ$. This is a fact which will be implicitly used in our analysis below.

From the Laplace-Beltrami operator Δ_g we construct the d'Alembertian (or wave operator) acting on the spacetime $\mathbb{R} \times X$,

$$\square_g \stackrel{\text{def}}{=} D_t^2 - \Delta_g,$$

and we consider this operator acting on half-densities on X lifted to the product spacetime. We define the *half-wave group* $\mathcal{U}(t)$ as

$$\mathcal{U}(t) \stackrel{\text{def}}{=} e^{-it\sqrt{\Delta_g}},$$

again considered as acting on half-densities. As usual, note that $\square_g[\mathcal{U}(t)\mu] = 0$ for all $\mu \in L^2\Big(X; |\Omega|^{\frac{1}{2}}(X)\Big)$ (or more general distributional half-densities μ). We remark that conjugating $\mathcal{U}(t)$ to get between scalars and half-densities has no effect on the overall trace of the group, and hence the introduction of half-densities is merely for computational convenience and clarity.

1.2. **Diffractive and geometric geodesics.** Two different notions of geodesic exist on a conic manifold X, one more restrictive and one less restrictive. We now recall these, following the exposition from [BW13].

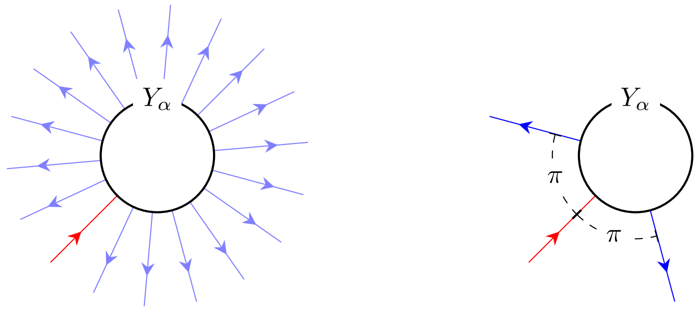
Definition 1.2. Suppose γ is a *broken geodesic*, i.e., a union of a finite number of closed, oriented geodesic segments $\gamma_1, \ldots, \gamma_N$ in X. Let γ_j be parametrized by the interval $[T_j, T_{j+1}]$.

(**D**) The curve γ is a *diffractive geodesic* in X if
   (i) all end points except possibly the initial point $\gamma_1(T_1)$ and the final point $\gamma_N(T_{N+1})$ of γ lie in the boundary $Y \stackrel{\text{def}}{=} \partial X$, and
   (ii) the intermediate terminal points $\gamma_j(T_{j+1})$ lie in the same boundary component as the initial points $\gamma_{j+1}(T_{j+1})$ for each $j = 1, \ldots, N-1$.

(**G**) The curve γ is a *(partially) geometric geodesic* if it is a diffractive geodesic such that for some $j = 1, \ldots, N-1$ the intermediate terminal point $\gamma_j(T_{j+1})$ and the initial point $\gamma_{j+1}(T_{j+1})$ are connected by a geodesic of length π in the boundary component Y_α in which they lie (with respect to the boundary metric $h_0 \stackrel{\text{def}}{=} h\big|_Y$). If this is true for all $j = 1, \ldots, N-1$, then we call γ a *strictly geometric geodesic*; if it is never true for $j = 1, \ldots, N-1$, then γ is a *strictly diffractive geodesic*.

As described in [MW04], the geometric geodesics are those that are locally realizable as limits of families of ordinary geodesics in the interior X°; see also [BW13, Section 2.1] for a discussion of the question of local versus global approximability.

We shall also use the geodesic flow at the level of the cotangent bundle, and it is helpful to see how this behaves uniformly up to ∂X (although in this paper, microlocal considerations will only arise over X°—a considerable simplification). To describe this flow, let ${}^{\mathrm{b}}T^*X$ denote the b-cotangent bundle of X, i.e., the dual of the bundle ${}^{\mathrm{b}}TX$ whose sections are $\mathcal{V}_{\mathrm{b}}(X)$, the smooth vector fields tangent to ∂X. We write ${}^{\mathrm{b}}S^*X$ for the corresponding sphere bundle. The b-cotangent bundle comes equipped with a canonical 1-form ${}^{\mathrm{b}}\alpha$, which in our product coordinates near



(A) Diffractive geodesics (B) Geometric geodesics

FIGURE 1.2. Diffractive and geometric geodesics

the boundary is

$$^{\mathrm{b}}\alpha = \xi\,\frac{dx}{x} + \eta\cdot dy,$$

and we write $^{\mathrm{b}}\sigma \overset{\text{def}}{=} d\big({}^{\mathrm{b}}\alpha\big)$ for the associated symplectic form. This defines the natural coordinate system $(x, \xi; y, \eta)$ on $^{\mathrm{b}}T^*X$ near Y. (We refer the reader to Chapter 2 of [Mel93] for a detailed explanation of "b-geometry," of which we only use the rudiments here.)

Let $\mathbf{H}_g$ be the Hamilton vector field (with respect to $^{\mathrm{b}}\sigma$) of the metric function $\frac{1}{2}\,g$, the symbol of $\frac{1}{2}\Delta_g$ on $^{\mathrm{b}}T^*X$; in product coordinates near Y, this is

$$\frac{g}{2} = \frac{1}{2}\,\frac{\xi^2 + h(x, y, \eta)}{x^2},$$

With this normalization, $\mathbf{H}_g$ is the geodesic spray in $^{\mathrm{b}}T^*X$ with velocity $\sqrt{g}$. It is convenient to rescale this vector field so that it is both tangent to the boundary of X and homogeneous of degree zero in the fibers. Near a boundary component Y_α, we have (see [MW04])

$$\mathbf{H}_g = x^{-2}\left\{\mathbf{H}_{Y_\alpha}(x) + \left[\xi^2 + h(x, y, \eta) + \frac{x}{2}\,\frac{\partial h}{\partial x}\right]\partial_\xi + \xi x\,\partial_x\right\}, \tag{1.4}$$

where $\mathbf{H}_{Y_\alpha}(x)$ is the geodesic spray in Y_α with respect to the family of metrics $h_\alpha(x, \cdot)$. Hence, our desired rescaling is

$$\mathbf{Z} \overset{\text{def}}{=} \frac{x}{\sqrt{g}}\,\mathbf{H}_g.$$

(Note that g here refers to the metric function on the b-cotangent bundle and not the determinant of the metric tensor.) By the homogeneity of $\mathbf{Z}$, if we radially compactify the fibers of the cotangent bundle and identify $^{\mathrm{b}}S^*X$ with the "sphere at infinity", then $\mathbf{Z}$ is tangent to $^{\mathrm{b}}S^*X$ and may be restricted to it. Henceforth, we let $\overline{\mathbf{Z}}$ denote this restriction of $\frac{x}{\sqrt{g}}\,\mathbf{H}_g$ to the *compact* manifold $^{\mathrm{b}}S^*X$. On the sphere bundle $^{\mathrm{b}}S^*X$, we replace the b-dual coordinates $(x, \xi; y, \eta)$ in a neighborhood of the boundary by the (redundant) coordinate system

$$\big(x, \bar{\xi}; y, \bar{\eta}\big) \overset{\text{def}}{=} \left(x, \frac{\xi}{\sqrt{\xi^2 + h(x, y; \eta)}}; y, \frac{\eta}{\sqrt{\xi^2 + h(x, y; \eta)}}\right).$$

Using these coordinates, it is easy to see that $\overline{\mathbf{Z}}$ vanishes only at the critical manifold $\{x = 0, \bar{\eta} = 0\}$ over ∂X, and thus the closures of maximally extended integral curves of this vector field can only begin and end over ∂X. Since $\overline{\mathbf{Z}}$ is tangent to the boundary, such integral curves either lie entirely over ∂X or lie over ∂X only at their limit points. Hence, the interior and boundary integral curves giving rise to our broken geodesics can meet only at their limit points in this critical submanifold $\{x = 0, \bar{\eta} = 0\} \subseteq {}^{\mathrm{b}}S^*X$.

We now introduce a way of measuring the lengths of the integral curves of $\overline{\mathbf{Z}}$. Suppose γ is such an integral curve over the interior X°. Let k be a Riemannian metric on ${}^{\mathrm{b}}S^*X^\circ$ such that $\overline{\mathbf{Z}}$ has unit length, i.e., $k(\overline{\mathbf{Z}}, \overline{\mathbf{Z}}) = 1$, and let

$$\lambda = x\, k(\cdot, \overline{\mathbf{Z}}) \in \Omega^1\left({}^{\mathrm{b}}S^*X\right). \tag{1.5}$$

Then

$$\int_\gamma \lambda = \int_\gamma x\, k\left(\frac{d\gamma}{ds}, \overline{\mathbf{Z}}\right) ds = \int_\gamma \frac{x}{\sqrt{g}}\, k\big(\mathbf{H}_g, \overline{\mathbf{Z}}\big)\, ds = \int_\gamma ds = \operatorname{length}(\gamma),$$

where s parametrizes γ as an integral curve of $\frac{1}{\sqrt{g}}\mathbf{H}_g$, the unit speed geodesic spray. Given this setup, we may define two symmetric relations between points in ${}^{\mathrm{b}}S^*X$: a "geometric" relation and a "diffractive" relation. These correspond to the two different possibilities for linking these points via geodesic flow through the boundary.

Definition 1.3. Let p and p' be points of the b-cosphere bundle ${}^{\mathrm{b}}S^*X$.

(a) We write $p \underset{\mathsf{D}}{\overset{t}{\sim}} p'$ if there exists a piecewise smooth but not necessarily continuous curve $\gamma : [0,1] \to {}^{\mathrm{b}}S^*X$ with $\gamma(0) = p$ and $\gamma(1) = p'$ and such that $[0,1]$ can be decomposed into a finite union of closed subintervals I_j, intersecting at their endpoints, where
   (i) on each I_j°, γ is a (reparametrized) positively oriented integral curve of $\overline{\mathbf{Z}}$ in ${}^{\mathrm{b}}S^*X^\circ$;
   (ii) the final point of γ on I_j and the initial point of γ on I_{j+1} lie over the same component Y_α of ∂X; and
   (iii) $\int_\gamma \lambda = t$ for $\lambda \in \Omega^1\left({}^{\mathrm{b}}S^*X\right)$ as in (1.5).

(b) We write $p \underset{\mathsf{G}}{\overset{t}{\sim}} p'$ if there exists a *continuous* and piecewise smooth curve $\gamma : [0,1] \to {}^{\mathrm{b}}S^*X$ with $\gamma(0) = p$ and $\gamma(1) = p'$ such that $[0,1]$ can be decomposed into a finite union of closed subintervals I_j, intersecting at their endpoints, where
   (i) on each I_j°, γ is a (reparametrized) positively oriented integral curve of $\overline{\mathbf{Z}}$ in ${}^{\mathrm{b}}S^*X$;
   (ii) on successive intervals I_j and I_{j+1}, interior and boundary curves alternate; and
   (iii) $\int_\gamma \lambda = t$.

We know from the preceding discusion that the integral curves of $\overline{\mathbf{Z}}$ over X° are lifts of geodesics in X°. It follows from the formula (1.4) for $\mathbf{H}_g$ near the boundary that the maximally extended integral curves of $\overline{\mathbf{Z}}$ in the restriction ${}^{\mathrm{b}}S^*_{\partial X}X$ of the b-cosphere bundle to the boundary are the lifts of geodesics of length π in ∂X (see [MW04] for details). Hence, we may conclude the following proposition.

Proposition 1.4. *Suppose that p and p' are points of ${}^{\mathrm{b}}S^*X$. Then*

(a) $p \overset{t}{\underset{\mathbf{G}}{\sim}} p'$ if and only if p and p' are connected by a (lifted) geometric geodesic of length t, and

(b) $p \overset{t}{\underset{\mathbf{D}}{\sim}} p'$ if and only if p and p' are connected by a (lifted) diffractive geodesic of length t.

An important feature of these equivalence relations, proved in [Wun02], is the following:

Proposition 1.5 ([Wun02, Prop. 4])**.** *The sets*

$$\left\{(p,p',t) : p \overset{t}{\underset{\mathbf{G}}{\sim}} p'\right\} \quad \textit{and} \quad \left\{(p,p',t) : p \overset{t}{\underset{\mathbf{D}}{\sim}} p'\right\}$$

are closed subsets of ${}^{\mathrm{b}}S^*X \times {}^{\mathrm{b}}S^*X \times \mathbb{R}_+$.

We remark that based on pure dimensional considerations, closed geodesics that involve geometric interactions should not generically exist, just as closed geodesics on a smooth manifold passing through a fixed, finite set of marked points are non-generic.[3] For this reason we focus our attention in this paper on the contributions to the wave trace of closed diffractive geodesics; they, together with closed geodesics in X°, should typically give the only singularities of the wave trace.

1.3. **Jacobi fields and Fermi normal coordinates.** In the course of our analysis, we will encounter two classes of Jacobi fields along the geodesics γ of X. The first of these, the *cone Jacobi fields*, are the solutions to the Jacobi equation with respect to the metric g which are sections of the *cone tangent bundle*

$$ {}^{\mathrm{cone}}TX \overset{\mathrm{def}}{=} x^{-1} \cdot {}^{\mathrm{b}}TX \tag{1.6}$$

along γ. The sections of this bundle, which we denote by $\mathcal{V}_{\mathrm{cone}}(X)$, are spanned over $\mathcal{C}^\infty(X)$ by the vector fields ∂_x and $x^{-1} \cdot \partial_{y_j}$ for $j = 1, \dots, n-1$ near Y and restrict to be smooth vector fields in the interior. While these vector fields ∂_x and $x^{-1}\,\partial_{y_j}$ are singular at the boundary, we note that they are the vector fields of unit length with respect to g and thus, from a Riemannian point of view, naturally extend the smooth vector fields from the interior. Indeed, the standard Riemannian objects such as the metric g, the volume half-density ω_g, and the curvature tensor Riem are smooth sections of bundles constructed from ${}^{\mathrm{cone}}TX$ and its dual bundle, ${}^{\mathrm{cone}}T^*X = x \cdot {}^{\mathrm{b}}T^*X$.

[3]A simple way to demonstrate that a residual set of conic metrics admits no closed geodesics with geometric interactions at cone points is as follows. Suppose we are given a conic manifold with no pair of cone points which are conjugate. (Genericity of this situation should also be easy to demonstrate by perturbing in X°.) Now consider all geodesics of length less than A both starting and ending at cone points; the endpoints of these at the boundary components then form a discrete (and hence finite) set S_A, by nonconjugacy. Now we scale the boundary metric h_0 while leaving the metric g unchanged except in a small neighborhood of the boundary: for β close to 1 we set

$$g_\beta = dx^2 + x^2\,[\psi(x)\,\beta + (1-\psi(x))]\,h(x)$$

where ψ is supported in $[0,\varepsilon)$ and equal to 1 in $[0,\frac{\varepsilon}{2}]$. Then for all but finitely many values of β no pair of points in S_A are exactly distance π apart, hence no concatenation of any two is a geometric geodesic. If ε is chosen small, we have not introduced any new geodesics of length less than A that connect cone points, implying that we have killed off all geometric geodesics of length less than A by this simple perturbation.

The second class of Jacobi fields we encounter are the *b-Jacobi fields*, introduced by Baskin and the second author in [BW13]. They are the sections of the b-tangent bundle ${}^{\mathrm{b}}TX$ along γ which satisfy the Jacobi equation with respect to g, i.e., they are the smooth Jacobi fields which are tangent to Y. In particular, if γ connects two different cone points, this definition entails tangency to both the starting and ending components of the boundary. We note that nonzero b-Jacobi fields are necessarily normal to a geodesic: in the boundary coordinates (x, y), the geodesics reaching the boundary are exactly the curves $\{y = y_0\}$. The smooth Jacobi fields along these geodesics are spanned by the coordinate vector fields ∂_x and ∂_{y_j}, and the constant linear combinations of these which are tangent to the boundary are precisely the span of the ∂_{y_j}'s.

The b-Jacobi fields with respect to a cone point should be regarded as the analogue on a conic manifold of the ordinary Jacobi fields on a smooth manifold that vanish at a given point—they are precisely the cone Jacobi fields on X which vanish simply *as a cone vector field* at the cone point. Since the Jacobi equation is second order, these cone vector fields are specified by their derivatives at this initial point, and thus the corresponding b-Jacobi fields are specified by the point in the boundary component Y_α from which they emanate, owing to the uniqueness of the geodesic striking that point in the boundary. This measures an "angle of approach" to the cone point when viewed metrically, i.e., in the blown-down picture shown in Figure 1.1a. As in [BW13], we use these Jacobi fields to define when the endpoints of geodesics emanating from a cone point are conjugate.

Definition 1.6 ([BW13, Section 2.4 and Appendix A])**.**

(a) We say that a point $p \in X^\circ$ is *conjugate* to a cone point Y_α along a geodesic γ if there exists a nonzero cone Jacobi field $\mathbf{J} \in \mathcal{V}_{\mathrm{cone}}(\gamma)$ along γ which vanishes at both p and Y_α. Equivalently, p is conjugate to Y_α along γ if there exists a nonzero b-Jacobi field ${}^{\mathrm{b}}\mathbf{J}$ along γ vanishing at p.
(b) We say two cone points Y_α and Y_β are *conjugate* along a geodesic γ if there exists a nonzero cone Jacobi field $\mathbf{J} \in \mathcal{V}_{\mathrm{cone}}(\gamma)$ along γ vanishing at both Y_α and Y_β. Equivalently, they are conjugate along γ if there exists a nonzero b-Jacobi field ${}^{\mathrm{b}}\mathbf{J}$ along γ.

Out of these Jacobi fields we build the corresponding classes of *Jacobi endomorphisms* (cf. [KV86]). These are the smooth sections J of the bundles

$$\mathrm{End}({}^{\mathrm{cone}}TX) \quad \text{and} \quad \mathrm{End}\big({}^{\mathrm{b}}TX\big)$$

along a geodesic $\gamma : [0, T]_t \longrightarrow X$ satisfying the analogous Jacobi equation

$$\ddot{\mathsf{J}}(t) + \mathsf{Riem}(t) \circ \mathsf{J}(t) = \mathbf{0},$$

where $\mathsf{Riem} \in \mathrm{End}({}^{\mathrm{cone}}TX)\big|_\gamma$ is the endomorphism defined by applying the Riemann curvature tensor to the tangent vector $\dot\gamma(t)$. (In other words, Jacobi endomorphisms are simply matrices of Jacobi fields.) We only use a few facts about these endomorphisms in our work. The first, which is a consequence of a simple calculation using the connection, is that the Wronskian

$$\mathcal{W}(\mathsf{X}, \mathsf{Y}) \overset{\mathrm{def}}{=} \dot{\mathsf{X}}^{\mathrm{t}}(t) \cdot \mathsf{Y}(t) - \mathsf{X}^{\mathrm{t}}(t) \cdot \dot{\mathsf{Y}}(t) \tag{1.7}$$

of any two solutions to this equation is constant, where $\mathsf{X}^{\mathrm{t}}(t)$ is the adjoint endomorphism determined by the metric. The second is the standard orthogonal decomposition into the tangential and normal blocks of the endomorphism coming from the Gauss lemma:

$$\mathbf{J}(t) = \mathbf{J}^{\parallel}(t) \oplus \mathbf{J}^{\perp}(t).$$

Here, $\mathbf{J}^{\parallel}(t) : T_{\gamma(t)}\gamma \longrightarrow T_{\gamma(t)}\gamma$ and $\mathbf{J}^{\perp}(t) : N_{\gamma(t)}\gamma \longrightarrow N_{\gamma(t)}\gamma$. We note that the constancy of the Wronskian for the whole Jacobi endomorphisms descends to constancy of the Wronskians of the tangential and normal endomorphisms as well.

We will often use *Fermi normal coordinates* (ν, ℓ) along the geodesics γ of X. These are defined by choosing a basepoint $z_0 = \gamma(t_0)$ and a g-orthonormal basis $(\partial_\nu, \partial_\ell)$ of ${}^{\mathrm{cone}}T_{z_0}X$ with $\partial_\ell = \dot{\gamma}(t_0)$ the tangent vector to the geodesic at the basepoint. Extending this basis to all of γ by parallel transport and exponentiating then yields the Fermi normal coordinates (ν, ℓ) on a small tube around the geodesic. The orthonormal basis $(\partial_\nu, \partial_\ell)$ at the basepoint generally will be the initial data for a set of Jacobi fields $(\mathbf{J}_1, \dots, \mathbf{J}_n)$ or, equivalently, a Jacobi endomorphism $\mathbf{J}$. Indeed, we will primarily deal with the Jacobi endomorphism $\mathbf{J}(t)$ along a geodesic $\gamma(t)$ whose initial data is

$$\mathbf{J}(0) = \mathbf{0} \quad \text{and} \quad \dot{\mathbf{J}}(0) = \mathrm{Id}\,.$$

Using Fermi normal coordinates along γ, we may show this endomorphism is computable by the derivative of the exponential map at $\gamma(0)$:

$$\mathbf{J}(t) \cdot (a \cdot \partial_\nu + b\,\partial_\ell) = D\exp_{\gamma(0)}(-)\Big|_{t\dot{\gamma}(0)} \cdot t\,(a \cdot \partial_\nu + b\,\partial_\ell)\,. \tag{1.8}$$

In particular, we have the identification

$$t^{-1} \cdot \mathbf{J}(t) \cdot (a \cdot \partial_\nu + b\,\partial_\ell) = D\exp_{\gamma(0)}(-)\Big|_{t\dot{\gamma}(0)} \cdot (a \cdot \partial_\nu + b\,\partial_\ell)\,. \tag{1.9}$$

Note that the map $\exp_p(v)$, defined on $(x, v) \in TX^\circ$, does *not* extend smoothly to the boundary, since the only geodesics that reach the boundary have tangents that are multiples of ∂_x (and the same issue arises even if we view the bicharacteristic flow in the b-cotangent bundle: bicharacteristics only limit to $x = 0$ from the interior at the radial points of the flow, which are at $\mathbb{R}dx/x + 0 \cdot dy$—see [MW04] for details). However, we can still define a flowout map from the boundary: employing a variant on the notation of [MVW08], we denote the "flowout map"

$$\mathcal{F} : [0, x_*) \times Y \to X$$
$$\mathcal{F} : (\ell, y) \to (x = \ell, y).$$

We view this as time-ℓ flow along a geodesic (described here in our product coordinates), however we can just as easily view it as the "identity map" that identifies a piece of a *model cone*

$$X_0 \equiv [0, x_*) \times Y$$

with a neighborhood of ∂X in X.

Now we may take the b-differential of $\mathcal{F}$ as in [Mel92] to obtain

$${}^{\mathrm{b}}\mathcal{F}_* : {}^{\mathrm{b}}TX_0 \to {}^{\mathrm{b}}TX.$$

Since multiplication by x^{-1} identifies b- and cone-tangent spaces, this gives a map (for which we recycle the same notation)

$${}^{\mathrm{cone}}\mathcal{F}_* : {}^{\mathrm{cone}}TX \to {}^{\mathrm{cone}}TX_0$$

Now equip X_0 with the "product metric"

$$g_0 = dx^2 + x^2\,h(0, y, dy).$$

As remarked above $\omega_g(x,y) = |\det h(y)|^{\frac{1}{4}}\, |dx \wedge x\, dy_1 \wedge \cdots \wedge x\, dy_{n-1}|^{\frac{1}{2}}$ is naturally a smooth section of the vector bundle $|\bigwedge^n[{}^{\text{cone}}T^*X]|^{\frac{1}{2}}$, the *cone half-densities* on X; likewise X_0 has its own metric half-density in its cone half-density bundle.

For any $y_0 \in Y$, we thus have an isometry, for which we employ the notation[4]

$$\left|\det_g D\mathcal{F}\big|_{(x_0,y_0)}\right|^{-\frac{1}{2}} : \left|\bigwedge\nolimits^n \left[{}^{\text{cone}}T^*_{(0,y_0)}X\right]\right|^{\frac{1}{2}} \xrightarrow{\cong} \left|\bigwedge\nolimits^n \left[{}^{\text{cone}}T^*_{(x_0,y_0)}X\right]\right|^{\frac{1}{2}}$$

which maps $\omega_g(0,y_0)(= \omega_{g_0}(0,y_0) = \omega_{g_0}(x_0,y_0))$ to $\omega_g(x_0,y_0)$. We now write

$$\begin{aligned} \Theta(Y_\alpha \to p_0) &\stackrel{\text{def}}{=} \left|\det_g D\mathcal{F}\big|_{(x_0,y_0)}\right| \\ \Theta(p_0 \to Y_\alpha) &\left|\det_g D\mathcal{F}^{-1}\big|_{(x_0,y_0)}\right|; \end{aligned} \tag{1.10}$$

note the slight abuse of notation, in which the particular choice of y_0 such that flowout from y_0 hits p_0 (which might be globally non-unique) is not specified. In this notation, then, we have obtained

$$\omega_{g_0}(p_0) = \Theta^{-1/2}(Y_\alpha \to p_0)\, \omega_g(p_0). \tag{1.11}$$

We now work somewhat more globally. To begin, note that (1.10) makes equally good sense for any $x_0 \in \mathbb{R}$, where we are interpreting $\mathcal{F}$ as the geodesic flow along the geodesic γ emanating from $(0,y_0) \in Y_\alpha$, a given component of Y; hence the definition extends to map

$$[0,\infty) \times Y \to X.$$

The fact that

$$\mathcal{F}_* : \partial_{y_j} \to \partial_{y_j}$$

in our special coordinate system near the cone point easily generalized to show that $\mathcal{F}_*$ maps the tangent space of Y_α to b-Jacobi fields. Thus, we may let $\mathsf{J}(t)$ denote the basis of b-Jacobi fields along a flowout geodesic γ with $\mathsf{J}(0)$ equal to an orthonormal basis of TY_α; recall, as discussed in [BW13] that this suffices to specify unique Jacobi fields, as the remaining solutions to the Jacobi equation are singular at Y_α. Then if $p_0 = \mathcal{F}(x_0,y_0)$, we simply have

$$\Theta(Y_\alpha \to p_0) = |\det_g \mathsf{J}(x_0)|.$$

Returning to our discussion of interior points above, we may reinterpret this definition as follows: In the special case where Y_α is a “fictional” cone point obtained by blowing up a point z_0 on a smooth manifold, these b-Jacobi fields correspond to Jacobi fields on the original manifold that vanish at z, and it is easily seen that we may reinterpret Θ as the determinant of the differential of the exponential map $\exp_z(-)$.

Now suppose that $(0,y_0) \in Y_\alpha$, a given boundary component, and that there exists x_0 such that

$$\mathcal{F}(x_0,y_0) \in Y_\beta,$$

a different boundary component. We may define[5]

$$\Theta(Y_\alpha \to Y_\beta) \stackrel{\text{def}}{=} |\det_g \mathsf{J}(x_0)| \tag{1.12}$$

[4]Here, $\det_g D\mathcal{F}\big|_{(x_0,y_0)}$ is the determinant of the matrix representing $\det_g D\mathcal{F}\big|_{(x_0,y_0)}$ in g-orthonormal bases of ${}^{\text{cone}}T_{(0,y_0)}X$ and ${}^{\text{cone}}T_{(x_0,y_0)}X$; note that we have now stopped using pullback and pushforward notation explicitly.

[5]As before, we omit from the notation the choice of which geodesic we are using to connect Y_α and Y_β although this certainly matters.

just as at interior points, where we simply note that while the Jacobi fields in J are generically *singular* at Y_β owing to the singularity of the Jacobi equation, they are precisely sections of ${}^{\mathrm{cone}}TX$ there, hence the metric is well-defined and the resulting quantity is finite.

The quantity $\Theta(Y_\alpha \to Y_\beta)$ is a measure of the convexity of the flowout manifold $\mathcal{F}(-, Y_\alpha)$—analogous to a geodesic sphere in the smooth manifold case—as it reaches Y_β. The manipulations involved in the proof of Theorems 4.2 and 4.3 will allow another interpretation, namely as the Hessian of the difference of shape operators of the flowouts of Y_α and Y_β respectively at a point p_0 along a geodesic γ connecting them, rescaled by certain half-density factors; this product arises via a Wronskian of the two sets of b-Jacobi fields, one set coming from each of the two cone points, and can be seen to be a measure of the degree of tangency of the two flowouts at p_0; it is intriguing that the result is independent of the choice of p_0 along γ, but we will not dwell on this issue here.

2. Single diffraction on a product cone

In this section we review the results of Cheeger and Taylor on the symbol of the diffracted wavefront on a product cone [CT82a, CT82b]. Note that in our work the space dimension is denoted n, while this dimension is denoted $m+1$ in these references.

Suppose Y is a connected, closed manifold of dimension $n-1$ (such as a single component Y_α of the boundary appearing in the previous discussion). As above, let X_0 denote the "product cone" over Y, the noncompact cylinder $[0,\infty)\times Y$ equipped with the scale-invariant metric

$$g_0 = dx^2 + x^2\, h_0(y, dy)$$

(we have slightly changed notation so that X_0 has an infinite end, but our considerations are all local in any case). Here, h_0 is a Riemannian metric on Y (e.g., $h_\alpha \overset{\text{def}}{=} h\big|_{Y_\alpha}$ from the previous section). Let Δ_0 denote the Laplace-Beltrami operator acting on half-densities on X_0, and let $\Box_0$ denote the d'Alembertian $D_t^2 - \Delta_0$ on the half-densities of the associated spacetime $\mathbb{R}\times X_0$. Following Cheeger and Taylor, we define a shifted square-root of the Laplacian

$$\nu \overset{\text{def}}{=} \sqrt{\Delta_{h_0} + \left(\frac{2-n}{2}\right)^2},$$

determined in the functional calculus of Δ_{h_0} on Y. For a function $f \in L^\infty(\mathbb{R})$, we let $\mathcal{K}[f(\nu)](y, y')$ (or sometimes simply $f(\nu)$) denote the Schwartz kernel of the corresponding element of the functional calculus.

Having set up the framework, we now state a mild reinterpretation of the results of Cheeger and Taylor calculating the asymptotics of the sine propagator on $\mathbb{R}\times X_0$,

$$\mathbf{W}_0(t) \overset{\text{def}}{=} \frac{\sin\big(t\sqrt{\Delta_0}\big)}{\sqrt{\Delta_0}}.$$

In what follows, we let $u \overset{\text{def}}{=} (x+x')-t$ denote the defining function for the diffracted wavefront, and we write $N^*\{x+x'=t\}$ for its conormal bundle.

Proposition 2.1. *Suppose $p = (x, y)$ and $p' = (x', y')$ are strictly diffractively related[6] points in X_0°, i.e.,*

$$p \underset{\mathbf{D}}{\overset{t}{\sim}} p' \quad \text{and} \quad p \underset{\mathbf{G}}{\overset{t}{\not\sim}} p'.$$

Then near $(t, p, p') \in \mathbb{R} \times X_0^\circ \times X_0^\circ$, the Schwartz kernel $\boldsymbol{E}_0$ of the sine propagator $\mathbf{W}_0(t)$ lies locally in the space of Lagrangian distributions[7]

$$I^{-\frac{5}{4}-\frac{n-1}{2}}\Big(\mathbb{R} \times X_0^\circ \times X_0^\circ, N^*\{x + x' = t\}; |\Omega|^{\frac{1}{2}}(X_0^\circ \times X_0^\circ)\Big).$$

In particular, $\boldsymbol{E}_0$ has an oscillatory integral representation

$$\boldsymbol{E}_0(t, x, y; x', y') = \int_{\mathbb{R}_\xi} e^{i(x+x'-t)\cdot\xi}\, e(t, x, y; x', y'; \xi)\, d\xi, \tag{2.1}$$

where the amplitude has the leading order behavior

$$e(t, x, y; x', y'; \xi) \equiv \mathcal{E}(x, y; x', y'; \xi) \left(\operatorname{mod} S^{-\frac{3}{2}}\Big(\mathbb{R} \times X_0^\circ \times X_0^\circ; |\Omega|^{\frac{1}{2}}(X_0^\circ \times X_0^\circ)\Big)\right)$$

for

$$\begin{aligned} \mathcal{E}(x, y; x', y'; \xi) \overset{\text{def}}{=}\, & \frac{(xx')^{-\frac{n-1}{2}}}{2\pi} \frac{\chi(\xi)}{2|\xi|} \\ & \times \left[H(\xi)\, \mathcal{K}[e^{-i\pi\nu}](y, y') + H(-\xi)\, \mathcal{K}[e^{i\pi\nu}](y, y')\right] \omega_{g_0}(x, y)\, \omega_{g_0}(x', y'). \end{aligned} \tag{2.2}$$

Here, $\rho \in \mathcal{C}^\infty(\mathbb{R}_\xi)$ is a smooth function satisfying $\rho \equiv 1$ for $|\xi| > 2$ and $\rho \equiv 0$ for $|\xi| < 1$, and H is the Heaviside function.

Remark 2.2. We call attention to a few aspects of this proposition.

(a) The sine propagator has order $-\frac{5}{4} - \frac{n-1}{2}$ as a Lagrangian distribution, showing the diffractive improvement of $\frac{n-1}{2}$ derivatives.
(b) The principal part of the amplitude, $\mathcal{E}(x, y'; x', y'; \xi)$, is only given here modulo symbolic half-densities of $\frac{1}{2}$-order lower rather than those a full order lower.
(c) The operators $e^{i\pi\nu}$ and $e^{-i\pi\nu}$ are half-Klein-Gordon propagators for time-π, and hence their Schwartz kernels have singular support only at distance π from the diagonal.

Proof of Proposition 2.1. Varying slightly upon Cheeger and Taylor, let δ be the defining function

$$\delta \overset{\text{def}}{=} \operatorname{sgn}\big(t^2 - (x + x')^2\big) \cdot \left| \frac{t^2 - (x + x')^2}{xx'} \right|^{\frac{1}{2}}.$$

From Theorems 5.1 and 5.3 of [CT82b], we know that uniformly away from the set $\{\operatorname{dist}_{h_0}(y, y') = \pi\}$ there is a complete asymptotic expansion of $\boldsymbol{E}_0$ of the form

$$\boldsymbol{E}_0(t, x, y; x', y') \sim \sum_{j=0}^{\infty} a_j(t, x, y; x', y')\, \delta^j + \sum_{k=0}^{\infty} b_k(t, x, y; x', y')\, \delta^{2k} \log|\delta|$$

[6]Note that it is only possible for a geodesic to undergo one diffraction on X_0.
[7]See Appendix A for a definition of Lagrangian distributions.

as $|\delta| \to 0$, where the leading terms are[8]

$$a_0 = \frac{1}{\pi(xx')^{\frac{n-1}{2}}} \Bigg[\log(2\sqrt{2}) \cos(\pi\nu) \\ + \int_{s=0}^{\pi} \frac{\cos(s\nu) - \cos(\pi\nu)}{2\cos\left(\frac{s}{2}\right)} \, ds - H(-\delta) \, \frac{\pi}{2} \sin(\pi\nu) \Bigg]$$

$$b_0 = -\frac{1}{\pi(xx')^{\frac{n-1}{2}}} \cos(\pi\nu).$$

(This may be thought of as a $\frac{1}{2}$-step quasipolyhomogeneous expansion of the kernel of $\mathbf{W}_0(t)$ in distributions associated to $\{\delta = 0\}$.) The existence of this expansion establishes the Lagrangian structure of $\boldsymbol{E}_0$ in this region, and moreover the leading order singularity of $\boldsymbol{E}_0$ at the diffractive front is

$$\frac{1}{\pi(xx')^{\frac{n-1}{2}}} \left[-\frac{\pi}{2} \sin(\pi\nu) H(-\delta) - \cos(\pi\nu) \log|\delta| \right]. \tag{2.3}$$

Note that the Schwartz kernels of the propagators $\cos(\pi\nu)$ and $\sin(\pi\nu)$ are in fact smooth since we are localized away from the submanifold $\{\mathrm{dist}_{h_0}(y, y') = \pi\}$.

We now convert this expansion in δ to one in our defining function $u \overset{\text{def}}{=} x + x' - t$ using the relation

$$\delta \sim \mathrm{sgn}(u) \left| \frac{2t}{xx'} \right|^{\frac{1}{2}} |u|^{\frac{1}{2}} \text{ as } |\delta| \to 0.$$

Thus (2.3) becomes

$$\frac{1}{\pi(xx')^{\frac{n-1}{2}}} \left[-\frac{\pi}{2} \sin(\pi\nu) H(-u) - \frac{1}{2} \cos(\pi\nu) \log|u| \right] \tag{2.4}$$

modulo a smooth function. Using the oscillatory integral representations

$$H(-u) = \int_{\mathbb{R}_\xi} e^{iu\xi} \, \frac{i}{2\pi} \frac{1}{\xi + i0} \, d\xi \quad \text{and} \quad \log|u| = \int_{\mathbb{R}_\xi} e^{iu\xi} \left[-\frac{1}{2|\xi|} - \gamma \, \delta(\xi) \right] d\xi$$

where γ is the Euler-Mascheroni constant, we may represent the leading order singularity of (2.4) as the oscillatory integral

$$\frac{1}{2\pi \, (xx')^{\frac{n-1}{2}}} \int_{\mathbb{R}_\xi} e^{iu\xi} \left[\frac{1}{2|\xi|} \cos(\pi\nu) + \gamma \, \delta(\xi) \, \cos(\pi\nu) - \frac{i}{2} \frac{1}{\xi + i0} \sin(\pi\nu) \right] d\xi.$$

The singularities at $\xi = 0$ in this expression are superfluous since we are only interested in the large-$|\xi|$ behavior of this function. Thus, this distribution is equivalent (up to introducing a smooth error) to

$$\frac{1}{2\pi \, (xx')^{\frac{n-1}{2}}} \int_{\mathbb{R}_\xi} e^{iu\xi} \, \rho(\xi) \left[\frac{1}{2|\xi|} \cos(\pi\nu) - \frac{i}{2\xi} \sin(\pi\nu) \right] d\xi.$$

[8] In fact the expansions given here differ slightly from those stated in Theorems 5.1 and 5.3 of [CT82b]. In equation (5.20) of [CT82b] the factor of $1/(4 - u^2)^{1/2}$ at the end of the first line should in fact read $1/(1 - u^2/4)^{1/2}$, leading to the replacement of the factor $\log \sqrt{2/\delta}$ in the statement of these theorems with $\log(2\sqrt{2}/\delta)$.

where $\rho \in \mathcal{C}^\infty(\mathbb{R}_\xi)$ is a smooth function satisfying $\rho \equiv 1$ for $|\xi| > 2$, say, and $\rho \equiv 0$ for $|\xi| < 1$. To make this expression more intuitive, we replace the sine and cosine Klein-Gordon propagators on the link with their half-wave counterparts, yielding

$$\frac{1}{2\pi\,(xx')^{\frac{n-1}{2}}}\int_{\mathbb{R}_\xi} e^{iu\xi}\,\rho(\xi)\,\frac{H(\xi)\,e^{-i\pi\nu}+H(-\xi)\,e^{i\pi\nu}}{2|\xi|}d\xi,$$

and therefore the principal amplitude of this distribution is

$$\frac{(xx')^{-\frac{n-1}{2}}}{2\pi}\cdot\rho(\xi)\,\frac{H(\xi)\cdot e^{-i\pi\nu}+H(-\xi)\,e^{i\pi\nu}}{2|\xi|}. \tag{2.5}$$

As we have calculated it, this is the principal amplitude of the propagator acting on scalars, and thus it is missing a right density factor, i.e., $\boldsymbol{E}_0\,\omega_g^2(x',y')$ is the (right-density) kernel of the operator on scalars. Letting ${}^{\frac{1}{2}}\boldsymbol{E}_0$ denote the kernel acting on half-densities, we calculate:

$${}^{\frac{1}{2}}\boldsymbol{E}_0 = \left[\boldsymbol{E}_0\,\omega_g^2(x',y')\right]\omega_g(x,y)\,\omega_g^{-1}(x',y') = \boldsymbol{E}_0\,\omega_g(x,y)\,\omega_g(x',y'),$$

yielding the expression (2.2). $\square$

We now use the Lagrangian structure of $\boldsymbol{E}_0$ near the diffractive front to conclude the analogous structure for the Schwartz kernel of the (forward) half-wave propagator $\mathcal{U}_0(t) \overset{\text{def}}{=} e^{-it\sqrt{\Delta_0}}$ on X_0.

Corollary 2.3. *Suppose $p = (x,y)$ and $p' = (x',y')$ are strictly diffractively related points in X_0° as above. Then near $(t,p,p') \in \mathbb{R}\times X_0^\circ \times X_0^\circ$, the Schwartz kernel $\boldsymbol{U}_0$ of the half-wave propagator lies locally in the Lagrangian distributions:*

$$\boldsymbol{U}_0 \in I^{-\frac{1}{4}-\frac{n-1}{2}}\left(\mathbb{R}\times X_0^\circ\times X_0^\circ, N^*\{t = x+x'\}; |\Omega|^{\frac{1}{2}}(X_0^\circ\times X_0^\circ)\right).$$

Using the phase function $\phi(t,x,x',\xi) = (x+x'-t)\cdot\xi$, its principal amplitude is

$$\mathcal{D}(x,y';x',y';\xi) \overset{\text{def}}{=} \frac{(xx')^{-\frac{n-1}{2}}}{2\pi i}\,\rho(\xi)\,\mathcal{K}[e^{-i\pi\nu}](y,y')\,\omega_{g_0}(x,y)\,\omega_{g_0}(x',y'), \tag{2.6}$$

modulo elements of $S^{-\frac{1}{2}}$. Here, $\rho \in \mathcal{C}^\infty(\mathbb{R}_\xi)$ is a smooth function satisfying $\rho \equiv 1$ for $\xi > 2$ and $\rho \equiv 0$ for $\xi < 1$.

Proof. We compute the kernel of the half-wave operator via Euler's formula:

$$e^{-it\sqrt{\Delta_0}} = \cos\left(t\sqrt{\Delta_0}\right) - i\sin\left(t\sqrt{\Delta_0}\right).$$

Differentiating the oscillatory integral expression for $\boldsymbol{E}_0$ in the ξ-variable produces an expression for the Schwartz kernel of the cosine propagator

$$\dot{\mathbf{W}}_0(t) \overset{\text{def}}{=} \cos\left(t\sqrt{\Delta_0}\right),$$

bringing a factor of $\frac{1}{i}\,\xi$ into the amplitude from differentiating the phase. Thus, the principal term in the amplitude of $\dot{\boldsymbol{E}}_0 \overset{\text{def}}{=} \mathcal{K}\left[\dot{\mathbf{W}}_0(t)\right]$ is

$$(xx')^{-\frac{n-1}{2}}\,\frac{\rho(\xi)}{4\pi i}\,\frac{\xi}{|\xi|}\left[H(\xi)\,e^{-i\pi\nu}+H(-\xi)\,e^{i\pi\nu}\right]\omega_g(x,y)\,\omega_g(x',y').$$

To determine the principal amplitude of the operator $\sin\Big(t\sqrt{\Delta_0}\Big)$, we recall[9] from [Loy03] that $\sqrt{\Delta_0}$ is a pseudodifferential operator over X_0° (it lies in the "big b-calculus" of Melrose [Mel93]), and hence applying it to the conormal distribution given in Proposition 2.1 has the effect of multiplying the amplitude by the value of the symbol of the pseudodifferential operator $\sqrt{\Delta_0}$ along the Lagrangian submanifold $N^*\{x+x'=t\}$, to wit, $|\xi|$. Thus, the leading order amplitude of $-i\sin\Big(t\sqrt{\Delta_0}\Big)$ away from $\{\mathrm{dist}_{h_0}(y,y')=\pi\}$ is

$$(xx')^{-\frac{n-1}{2}}\,\frac{\rho(\xi)}{4\pi i}\left[H(\xi)\,e^{-i\pi\nu}+H(-\xi)\,e^{i\pi\nu}\right]\omega_g(x,y)\,\omega_g(x',y').$$

Adding these two contributions produces (2.6). $\square$

3. Single diffraction on a non-product cone

In this section, we use the information from Corollary 2.6 on the structure of diffraction on the product cone to understand the analogous structure on our more general conic manifold X. The finite speed of propagation implies that we only need to understand a single diffraction, and thus we may work in a small collar neighborhood in X of a boundary component Y_α,

$$C_\alpha \overset{\text{def}}{=} \left[0,\frac{x_*}{2}\right)\times Y_\alpha,$$

with x_* as in Section 1. We write $C_\alpha^\circ \overset{\text{def}}{=} \left(0,\frac{x_*}{2}\right)\times Y_\alpha$ for the interior of this collar neighborhood.

We work with two different metrics on the collar neighborhood[10] C_α: the conic metric in designer form $g = dx^2 + x^2\,h(x,y,dy)$ and the associated product metric $g_0 = dx^2 + x^2\,h(0,y,dy)$ coming from the boundary metric $h(0)$. Associated to these metrics are their (nonnegative) Laplace-Beltrami operators, Δ_g and $\Delta_0 \overset{\text{def}}{=} \Delta_{g_0}$, and their wave operators, $\Box_g \overset{\text{def}}{=} D_t^2-\Delta_g$ and $\Box_0 \overset{\text{def}}{=} D_t^2-\Delta_0$. We now show that when half-density factors are taken into account, the D_x-terms in $\Box_g$ and $\Box_0$ agree. This is very helpful in proving the conormality of the diffractive part of the propagator as the remaining first-order D_y-terms act harmlessly on distributions associated to $N^*\{t=x+x'\}$, where we now consider this as a Lagrangian submanifold of $\dot{T}^*(\mathbb{R}\times C_\alpha^\circ\times C_\alpha^\circ)$.

Lemma 3.1. *As operators on half-densities, we have*

$$\Box_g-\Box_0\in x^{-1}\mathcal{C}^\infty\left(\left[0,\frac{x_*}{2}\right);\mathrm{Diff}^2(Y)\right).$$

Proof. By definition, $\Box_g$ and $\Box_0$ act trivially on two different trivializations of the half-density bundle: $\Box_g\omega_g=0$ and $\Box_0\omega_{g_0}=0$, where

$$\omega_g \overset{\text{def}}{=} x^{\frac{n-1}{2}}\left[\det h(x)\right]^{\frac14}|dx\wedge dy|^{\frac12}\quad\text{and}\quad \omega_{g_0}\overset{\text{def}}{=} x^{\frac{n-1}{2}}\left[\det h(0)\right]^{\frac14}|dx\wedge dy|^{\frac12}.$$

[9] Technically, the noncompactness of X_0 makes these results not directly applicable, but "closing up" the large end of the cone yields a compact manifold that is locally identical to X_0 near the cone points on which we may as well (by finite speed of propagation) study the propagators.

[10] Technically, we should consider these objects to live on the distinct manifolds X_0 and X, identified by the flowout map $\mathcal{F}$ defined above; however, we will abuse notation to the extent of leaving this identification tacit.

In order to compare the operators we write them both in terms of the new half-density

$$\varpi \overset{\text{def}}{=} x^{\frac{n-1}{2}} \, |dx \wedge dy|^{\frac{1}{2}} \, ,$$

defined (non-invariantly!) in a local coordinate patch in y. Then

$$\Box_g(f\,\varpi) = (Pf)\,\varpi \quad \text{and} \quad \Box_0(f\,\varpi) = (P_0 f)\,\varpi$$

with operators on the coefficients of ϖ locally given by

$$P \overset{\text{def}}{=} [\det h(x)]^{\frac{1}{4}} \; \Box_{g,\text{scal}} \; [\det h(x)]^{-\frac{1}{4}} \quad \text{and} \quad P_0 \overset{\text{def}}{=} [\det h(0)]^{\frac{1}{4}} \; \Box_{0,\text{scal}} \; [\det h(0)]^{-\frac{1}{4}} \, .$$

Here, we use the notation $\Box_{\bullet,\text{scal}}$ to indicate the usual scalar d'Alembertian operators

$$\Box_{g,\text{scal}} \overset{\text{def}}{=} D_t^2 - \left[\frac{1}{x^{n-1}} \; [\det h(x)]^{-\frac{1}{2}} \, D_x \, [\det h(x)]^{\frac{1}{2}} \; x^{n-1} \, D_x + \frac{1}{x^2} \, \Delta_{h(x),\text{scal}} \right]$$

and

$$\Box_{0,\text{scal}} \overset{\text{def}}{=} D_t^2 - \left[\frac{1}{x^{n-1}} \, D_x \, x^{n-1} \, D_x + \frac{1}{x^2} \, \Delta_{h(0),\text{scal}} \right]$$

(with scalar Laplacians on Y denoted in the analogous way). Thus,

$$\begin{aligned} P = D_t^2 - \Bigg[-\frac{i(n-1)}{x} \, D_x + [\det h(x)]^{-\frac{1}{4}} \, D_x \, [\det h(x)]^{\frac{1}{2}} \, D_x \, [\det h(x)]^{-\frac{1}{4}} \\ + \frac{1}{x^2} \, [\det h(x)]^{\frac{1}{4}} \; \Delta_{h(x),\text{scal}} \, [\det h(x)]^{-\frac{1}{4}} \Bigg] , \end{aligned}$$

while since $\partial_x[\det h(0)] = 0$,

$$P_0 = D_t^2 - \left[-\frac{i(n-1)}{x} \, D_x + D_x^2 + \frac{1}{x^2} \, [\det h(0)]^{\frac{1}{4}} \; \Delta_{h(0),\text{scal}} \, [\det h(0)]^{-\frac{1}{4}} \right] .$$

Thus,

$$\begin{aligned} -(P - P_0) = [\det h(x)]^{-\frac{1}{4}} \, D_x \, [\det h(x)]^{\frac{1}{2}} \, D_x \, [\det h(x)]^{-\frac{1}{4}} - D_x^2 \\ + \frac{1}{x^2} \, [\det h(x)]^{\frac{1}{4}} \; \Delta_{h(x),\text{scal}} \, [\det h(x)]^{-\frac{1}{4}} \\ - \frac{1}{x^2} \, [\det h(0)]^{\frac{1}{4}} \; \Delta_{h(0),\text{scal}} \, [\det h(0)]^{-\frac{1}{4}} \, . \end{aligned}$$

We note that the operator on $\mathcal{C}^\infty(\mathbb{R}_x)$ given by

$$[\det h(x)]^{-\frac{1}{4}} \, D_x \, [\det h(x)]^{\frac{1}{2}} \, D_x \, [\det h(x)]^{-\frac{1}{4}}$$

is formally self-adjoint on $L^2(\mathbb{R}_x, dx)$, has real coefficients when written in terms of ∂_x, and has principal symbol ξ^2. Thus, the subprincipal term—the coefficient of ∂_x—vanishes, and the above operator equals $D_x^2 +$ zeroth order term (with the zeroth order term smooth, to boot). Since we also have $h(x) = h(0) + \mathrm{O}(x)$, we now find that all the D_x^2- and D_x-terms cancel, and hence

$$P - P_0 \in x^{-1} \mathcal{C}^\infty \left(\left[0, \frac{x_*}{2} \right) ; \mathrm{Diff}^2(Y) \right) .$$

The lemma follows. $\Box$

We now state the principal result of this section.

Theorem 3.2. *Uniformly away from the set* $\left\{\operatorname{dist}_{h(0)}(y,y')=\pi\right\}\subseteq C_\alpha^\circ\times C_\alpha^\circ$, *the amplitudes*[11] *of the diffracted waves for the half-wave propagators* $\mathcal{U}(t)$ *and* $\mathcal{U}_0(t)$ *(as operators on half-densities) agree in the space*

$$\frac{S^0\Big(\mathbb{R}_t\times C_\alpha^\circ\times C_\alpha^\circ\times\mathbb{R}_\xi;|\Omega|^{\frac12}(C_\alpha^\circ\times C_\alpha^\circ)\Big)}{S^{-\gamma+0}\Big(\mathbb{R}_t\times C_\alpha^\circ\times C_\alpha^\circ\times\mathbb{R}_\xi;|\Omega|^{\frac12}(C_\alpha^\circ\times C_\alpha^\circ)\Big)},$$

where $\gamma=\frac12$ *unless* $h(x)-h(0)=\mathrm{O}(x^2)$, *in which case* $\gamma=1$.

Let us recall what is already known about each of the half-wave propagators $\mathcal{U}(t)$ and $\mathcal{U}_0(t)$. From Corollary 2.6, we know that near strictly diffractively related points in $\mathbb{R}_t\times C_\alpha^\circ\times C_\alpha^\circ$ the half-wave kernel $\boldsymbol{U}_0\overset{\text{def}}{=}\mathcal{K}[\mathcal{U}_0(t)]$ is locally an element of

$$\begin{aligned}I^{-\frac14-\frac{n-1}{2}}&\Big(\mathbb{R}\times C_\alpha^\circ\times C_\alpha^\circ,N^*\{t=x+x'\};|\Omega|^{\frac12}(C_\alpha^\circ\times C_\alpha^\circ)\Big)\\&=IB^{-\frac12}_{2,\infty}\Big(\mathbb{R}\times C_\alpha^\circ\times C_\alpha^\circ,N^*\{t=x+x'\};|\Omega|^{\frac12}(C_\alpha^\circ\times C_\alpha^\circ)\Big),\end{aligned}$$

and its principal amplitude is (2.6) when the phase function $\phi(t,x,x',\xi)=(x+x'-t)\cdot\xi$ is used in the oscillatory integral representation (see Appendix A for the definitions of the spaces employed here).[12] We also know from the main result of the work of Melrose and the second author [MW04] that in this same region $\boldsymbol{U}\overset{\text{def}}{=}\mathcal{K}[\mathcal{U}(t)]$ is locally in

$$IH^{-\frac12-0}\Big(\mathbb{R}\times C_\alpha^\circ\times C_\alpha^\circ,N^*\{t=x+x'\};|\Omega|^{\frac12}(C_\alpha^\circ\times C_\alpha^\circ)\Big),$$

the space of Lagrangian distributional half-densities associated to $N^*\{t=x+x'\}$ with iterated $H^{-\frac12-0}_{\mathrm{loc}}$-regularity, though that theorem gives no further information on its amplitude. Thus, to show the amplitudes are the same to leading order, we shall need to relate these two spaces of Lagrangian half-densities and, in particular, relate the amplitudes associated to each. This comparision comes in the form of the following lemma, where we test difference $\mathcal{U}(t)-\mathcal{U}_0(t)$ against data which is Lagrangian with respect $N^*\{x=x_0\}$, the conormal bundle to a transverse "slice" of the cone.

Lemma 3.3. *Suppose* y_0 *and* y_0' *are points of* Y_α *such that* $\operatorname{dist}_{h(0)}(y_0,y_0')\neq\pi$, *i.e., points in the boundary whose flowouts into the interior are strictly diffractively related. Let* $s\geqslant\frac12$, *and let* $f\in H^s\Big(C_\alpha^\circ;|\Omega|^{\frac12}(C_\alpha^\circ)\Big)$ *be a distributional half-density which is Lagrangian with respect to* $N^*\{x=x_0\}$ *and supported in a sufficiently small neighborhood of* (x_0,y_0'). *Then for times* $t\in I\overset{\text{def}}{=}\left(x_0,x_0+\frac{x_*}{2}\right)$, *we have*

$$[\mathcal{U}(t)-\mathcal{U}_0(t)]\,f\in H^{s+\gamma-0}\Big(I\times C_\alpha^\circ;|\Omega|^{\frac12}(C_\alpha^\circ)\Big)$$

locally near $\{x=t-x_0\}$, *where* γ *is as in Theorem 3.2.*

---

[11]Note that this requires we use the same phase function in both oscillatory integral representations.

[12]We may instead use the phase function $\psi(t,x,x',\eta)=(t-x-x')\cdot\eta$, which would simply correspond to changing the sign of the phase variable in the formula (2.6).

Remark 3.4. Note that we do not need to bother specifying whether conormality is with respect to $H^s\Big(C_\alpha^\circ; |\Omega|^{\frac{1}{2}}(C_\alpha^\circ)\Big)$ or some other Sobolev space with less regularity. Indeed, by interpolation we obtain conormality with respect to $H^{s-0}\Big(C_\alpha^\circ; |\Omega|^{\frac{1}{2}}(C_\alpha^\circ)\Big)$, and this will suffice to prove the lemma. We may therefore assume without loss of generality that f is an element of

$$IH^{\frac{1}{2}}\Big(C_\alpha^\circ, N^*\{x = x_0\}; |\Omega|^{\frac{1}{2}}(C_\alpha^\circ)\Big),$$

the minimal regularity required.

Proof of Lemma 3.3. Let y_0 and y_0' be as in the statement of the lemma. Taking into account the previous remark, let f be in $IH^{\frac{1}{2}}\Big(C_\alpha^\circ, N^*\{x = x_0\}; |\Omega|^{\frac{1}{2}}(C_\alpha^\circ)\Big)$ with support in a small neighborhood of (x_0, y_0'). We observe that f is then an element of the domain $\mathcal{D}_{\frac{1}{2}}$ by microlocality of the powers of $\sqrt{\Delta_g}$ away from the boundary. As it is sufficient to show $[\mathcal{U}(t) - \mathcal{U}_0(t)]\, f$ is locally an element of $H^{\frac{1}{2}+\gamma-0}\Big(I \times C_\alpha^\circ; |\Omega|^{\frac{1}{2}}(C_\alpha^\circ)\Big)$ away from the geometric rays emanating from the support of f, this is how we will proceed. We start with the general case $\gamma = \frac{1}{2}$.

Define $u(t) \stackrel{\text{def}}{=} \mathcal{U}_0(t) f$ to be the associated solution to $\Box_0 u(t) = 0$ on $\mathbb{R}_t \times C_\alpha^\circ$ with initial half-wave data f. By unitarity of $\mathcal{U}_0(t)$, the solution $u(t)$ is an element of $L^\infty\Big(\mathbb{R}_t; \mathcal{D}_{\frac{1}{2}}\Big)$, implying in particular that

$$u(t) \in L^2\Big(I \times C_\alpha^\circ; |\Omega|^{\frac{1}{2}}(C_\alpha^\circ)\Big). \tag{3.1}$$

It is moreover a Lagrangian distributional half-density with respect to this regularity and the Lagrangian $N^*\{x = t - x_0\}$. Writing $E \stackrel{\text{def}}{=} \Box_g - \Box_0$, we compute that

$$\Box_g u(t) = [\Box_0 + E]\, u(t) = E u(t).$$

We now compute via Duhamel's formula that

$$u(t) = \mathcal{U}(t) f + \int_0^t \mathcal{U}(t-s)\, E u(s)\, ds,$$

and in particular

$$[\mathcal{U}_0(t) - \mathcal{U}(t)]\, f = \int_0^t \mathcal{U}(t-s)\, E u(s)\, ds. \tag{3.2}$$

Suppose for the moment that the spatial dimension satisfies $n \geqslant 3$. We then have the *Morawetz estimate* for the wave equation on the product cone $((\mathbb{R}_+)_x \times Y_\alpha, g_0)$ (cf. [BFM13, Theorem 4.1 and Remark 4.2] or the proof of Theorem 2 of [BPSTZ03]):

$$u(t) \in xL^2\Big(\mathbb{R}_t \times (\mathbb{R}_+)_x \times Y_\alpha; |\Omega|^{\frac{1}{2}}((\mathbb{R}_+)_x \times Y_\alpha)\Big).$$

As we are only interested in the submanifold C_α° of $(\mathbb{R}_+)_x \times Y_\alpha$ and times $t \in I$, we also have

$$u(t) \in xL^2\Big(I \times C_\alpha^\circ; |\Omega|^{\frac{1}{2}}(C_\alpha^\circ)\Big)$$

since the volume measures arising from g and g_0 are comparable on C_α°. Interpolating this last estimate with the Lagrangian regularity with respect to the space in

(3.1), we find that the solution $u(t)$ is in fact Lagrangian with respect to the regularity space $x^{1-0}L^2\Big(I \times C_\alpha^\circ; |\Omega|^{\frac{1}{2}}(C_\alpha^\circ)\Big)$. Hence, the result of Lemma 3.1 implies

$$Eu(t) \in x^{-0}L^2\Big(I \times C_\alpha^\circ; |\Omega|^{\frac{1}{2}}(C_\alpha^\circ)\Big) \subseteq L^2(I; \mathcal{D}_{-0}),$$

where the inclusion follows from Proposition 1.1. Thus, the right-hand side of (3.2) is contained in $L^\infty(I; \mathcal{D}_{1-0})$ as it solves an inhomogeneous equation with inhomogeneity in $L^2(I; \mathcal{D}_{-0})$, implying that this term is more regular than $u(t)$ by $\frac{1}{2} - 0$ derivatives, as claimed.

Now, if the spatial dimension is $n = 2$, we must make a slight adjustment to the argument above in order to apply the Morawetz estimate. Let Π_0 be the projection onto the zero mode of Y_α (i.e., the constants):

$$[\Pi_0 f](x) = \frac{1}{\mathrm{vol}(Y_\alpha)} \int_{Y_\alpha} f(x,y)\, \omega_{h(0)}.$$

We decompose f into its projection onto the zero mode and the positive modes:

$$f = (f - \Pi_0 f) + \Pi_0 f.$$

The Morawetz estimate argument above applies verbatim to $f - \Pi_0 f$. For the final piece, $\Pi_0 f$, we note that $E\,(\mathcal{U}_0(t) \circ \Pi_0 f)$ is smooth on $I \times C_\alpha^\circ$ since E acts as a multiplication operator on the range of Π_0. Hence, $E\,(\mathcal{U}_0(t) \circ \Pi_0 f) \in L^\infty\Big(I; \mathcal{D}_{\frac{1}{2}}\Big)$, implying that

$$[\mathcal{U}(t) - \mathcal{U}_0(t)] \circ \Pi_0 f \in L^\infty\Big(I; \mathcal{D}_{\frac{3}{2}}\Big),$$

an even stronger estimate than is required. Altogether, this establishes the result for $n = 2$.

When $\gamma = 1$, i.e., when we have the stronger product cone structure given by $h(x) - h(0) = \mathrm{O}(x^2)$, then the operator E is simply a family of differential operators on Y with smooth coefficients: there is no x^{-1} singularity. This improves the Duhamel term above so that we gain a full derivative in solving the inhomogeneous equation for $\mathcal{U}(t)f - \mathcal{U}_0(t)f$. □

Finally, we return to the proof of Theorem 3.2.

Proof of Theorem 3.2. From the discussion following the statement of this theorem, we know that near strictly diffractively related points of $\mathbb{R} \times C_\alpha^\circ \times C_\alpha^\circ$ our two half-wave propagators $\boldsymbol{U}$ and $\boldsymbol{U}_0$ are each Lagrangian distributional half-densities of class $IH^{-\frac{1}{2}-0}$ associated to $N^*\{t = x + x'\}$. Therefore, over sufficiently small open subsets $I \times U \times U'$ of $\mathbb{R} \times C_\alpha^\circ \times C_\alpha^\circ$ they each have local oscillatory integral representations of the form

$$\int_{\mathbb{R}_\xi} e^{i(x+x'-t)\cdot\xi}\, a(t,x,y;x',y';\xi)\, d\xi, \tag{3.3}$$

where a is a symbolic half-density of class

$$S_c^{\frac{1}{2}+0}L^2\Big(I \times U \times U' \times \mathbb{R}_\xi; |\Omega|^{\frac{1}{2}}(U \times U')\Big).$$

Now, by Lemma 3.3 we also know that for all choices of initial data $f \in IH^{\frac{1}{2}}\Big(C_\alpha^\circ, N^*\{x = x_0\}; |\Omega|^{\frac{1}{2}}(C_\alpha^\circ)\Big)$ we have

$$[\mathcal{U}(t) - \mathcal{U}_0(t)]\, f \in H^{\frac{1}{2}+\gamma-0}\Big(\mathbb{R}_t \times C_\alpha^\circ; |\Omega|^{\frac{1}{2}}(C_\alpha^\circ)\Big). \tag{3.4}$$

These initial data have oscillatory integral representations

$$f(x,y) \equiv \int_{\mathbb{R}_\eta} e^{-i(x-x_0)\cdot\eta}\, b(y,\eta)\, d\eta \,(\mathrm{mod}\ \mathcal{C}^\infty)$$

with b in $S^{-\frac{1}{2}}L^2(V\times\mathbb{R}_\eta; |\Omega|^{\frac{1}{2}}(V))$ for some open $V \subseteq Y_\alpha$; this is valid locally in a neighborhood of $\{x_0\}\times V$ in C_α°. Taking a and a_0 to be the amplitudes in (3.3) of $\boldsymbol{U}$ and $\boldsymbol{U}_0$ respectively, we then compute

$$\begin{aligned} &([\mathcal{U}(t)-\mathcal{U}_0(t)]\, f)(x,y) \\ &= \int_{C_\alpha}\int_{\mathbb{R}_\xi} e^{i(x+x'-t)\cdot\xi}\,[a-a_0](t,x,y;x',y';\xi)\int_{\mathbb{R}_\eta} e^{-i(x'-x_0)\cdot\eta}\, b(y',\eta)\, d\eta d\xi dx' dy', \end{aligned}$$

again up to a smooth error. Applying stationary phase in the (x',η)-variables, this becomes

$$([\mathcal{U}(t)-\mathcal{U}_0(t)]\, f)(x,y) \equiv \int_{\mathbb{R}_\xi} e^{i(x+x_0-t)\cdot\xi}\, e(t,x,y;\xi)\, d\xi \,(\mathrm{mod}\ \mathcal{C}^\infty)\,,$$

where

$$e(t,x,y;\xi) \equiv C\int [a-a_0](t,x,y;x_0,y';\xi)\, b(y',\xi)\, dy' \left(\mathrm{mod}\ S_{\mathrm{c}}^{-\frac{3}{2}+0}L^2\right)$$

for a constant C. In particular, e is a priori an element of $S_{\mathrm{c}}^{-\frac{1}{2}+0}L^2(W\times\mathbb{R}_\xi; |\Omega|^{\frac{1}{2}}(W))$ for some open $W\subseteq \mathbb{R}_t\times C_\alpha^\circ$ by (A.3). However, by (3.4) and an interpolation argument we know that $[\mathcal{U}(t)-\mathcal{U}_0(t)]f$ is locally an element of

$$IH^{\frac{1}{2}+\gamma-0}\left(\mathbb{R}_t\times C_\alpha^\circ, N^*\{x=t-x_0\}; |\Omega|^{\frac{1}{2}}(C_\alpha^\circ)\right),$$

so e is actually an element of the lower-order symbol space $S_{\mathrm{c}}^{-\frac{1}{2}-\gamma+0}L^2(W\times\mathbb{R}_\xi; |\Omega|^{\frac{1}{2}}(W))$. Hence, Lemma A.2 implies that

$$a-a_0 \in S_{\mathrm{c}}^{-\gamma+0}(U\times\mathbb{R}_\xi; |\Omega|^{\frac{1}{2}}(U)),$$

and this proves the theorem. □

Now we define

$$\boldsymbol{D}_\alpha(y,y') \overset{\mathrm{def}}{=} \mathcal{K}\big[e^{-i\pi\nu_\alpha}\big](y,y'). \tag{3.5}$$

We employ the comparison of product and non-product metrics in (1.11) to express the principal amplitude of $\boldsymbol{U}$ given in (2.6) in terms of the ambient (nonproduct) metric half-density ω_g.

Theorem 3.5. *Let $p=(x,y)$ and $p'=(x',y')$ be strictly diffractively related points in C_α°. Then near $(t,p,p')\in\mathbb{R}\times C_\alpha^\circ\times C_\alpha^\circ$, the Schwartz kernel $\boldsymbol{U}$ of the half-wave propagator $\mathcal{U}(t)$ has an oscillatory integral representation*

$$\boldsymbol{U}(t,x,y;x',y') = \int_{\mathbb{R}_\xi} e^{i(x+x'-t)\cdot\xi}\, d(t,x,y;x',y';\xi)\, d\xi \tag{3.6}$$

whose amplitude $d\in S^0$ is

$$\frac{(xx')^{-\frac{n-1}{2}}}{2\pi i}\,\rho(\xi)\cdot\boldsymbol{D}_\alpha(y,y')\cdot\Theta^{-\frac{1}{2}}(Y_\alpha\to y)\,\Theta^{-\frac{1}{2}}(y'\to Y_\alpha)\,\omega_g(x,y)\,\omega_g(x',y') \tag{3.7}$$

modulo elements of $S^{-\frac{1}{2}+0}$. Here, $\rho\in\mathcal{C}^\infty(\mathbb{R}_\xi)$ is a smooth function satisfying $\rho\equiv 1$ for $\xi>2$ and $\rho\equiv 0$ for $\xi<1$.

4. The amplitude of a multiply-diffracted wave

We now return to the setting of a general conic manifold (X, g). Before calculating the trace, we calculate the amplitude of $\mathcal{U}(t)$ microlocally along a geodesic γ undergoing multiple (strictly) diffractive interactions with the cone points of X. Of central importance is a calculation of the amplitude along geodesics in the interior of X with a particularly convenient choice of phase function, which we may treat as a calculation in the smooth manifold setting. To our knowledge, no version of this calculation currently exists in the literature, although many similar analyses of the propagator have been made close to the diagonal (within the injectivity radius).

4.1. **The γ-microlocalization of the half-wave group.** To define what we mean by the amplitude of $\mathcal{U}(t)$ microlocally along a geodesic, let us fix a (broken) geodesic segment $\gamma : [0,T] \longrightarrow {}^{\mathrm{b}}S^*X$ whose endpoints $\gamma(0)$ and $\gamma(T)$ lie over the interior X° and which undergoes k strictly diffractive interactions with the cone points of X. Thus, γ is a piecewise smooth curve in the b-cosphere bundle with jump discontinuities at each of the boundary components $Y_{\alpha_1}, \ldots, Y_{\alpha_k}$ through which it passes (note boundary components may repeat in this sequence). Writing

$$\gamma^\flat \overset{\text{def}}{=} \mathrm{pr} \circ \gamma : [0,T] \longrightarrow X$$

for the projection of this geodesic to the base manifold, we label the endpoints as $p_0^\flat \overset{\text{def}}{=} \gamma^\flat(0)$ and $p_1^\flat \overset{\text{def}}{=} \gamma^\flat(T)$. By shortening the geodesic slightly, we may arrange that

$$p_0^\flat \text{ is not conjugate to } Y_{\alpha_1} \text{ and } p_1^\flat \text{ is not conjugate to } Y_{\alpha_k} \text{ along } \gamma^\flat, \tag{4.1}$$

the analogue of the nonconjugacy assumption (0) at this stage. We label the segments and endpoints of $\gamma^\flat$ as follows:

(i) $\gamma_0^\flat$ is the segment connecting $p_0^\flat$ to $q_1' \in Y_{\alpha_1}$;
(ii) $\gamma_j^\flat$ is the segment connecting $q_j \in Y_{\alpha_j}$ to $q_j' \in Y_{\alpha_{j+1}}$ for $j = 1, \ldots, k-1$; and
(iii) $\gamma_k^\flat$ is the segment connecting $q_k \in Y_{\alpha_k}$ to $p_1^\flat$.

We partition the domain of γ as

$$0 = T_0 < T_1 < \cdots < T_{2k} < T_{2k+1} = T$$

so that for $m = 1, \ldots, k$ the projections of $\gamma(T_{2m-1})$ and $\gamma(T_{2m})$ to the base each lie in the collar neighborhoods $C_{\alpha_m}^\circ$ of the boundary component Y_{α_m} and so that the m-th diffraction of γ occurs between $t = T_{2m-1}$ and $t = T_{2m}$. By perturbing such a partition slightly, we may also arrange that none of the points $\gamma^\flat(T_m)$ in X° are conjugate to one another or to the cone points along γ, refining the assumption (4.1). See Figure 4.1 for an illustration of such a partition.

We define $t_m \overset{\text{def}}{=} T_m - T_{m-1}$ to be the interim time between the designated points along γ, and we choose microlocalizers $A_m \in \Psi_{\mathrm{c}}^0(X^\circ)$ whose microsupports are contained in sufficiently small neighborhoods of the points $\gamma(t_m) \in S^*X^\circ$, e.g., within ε-balls with respect to the metric g restricted to the unit sphere bundle for an $\varepsilon > 0$ as small as we need.

Definition 4.1. Let $\gamma : [0,T] \longrightarrow {}^{\mathrm{b}}S^*X$ be a broken geodesic segment partitioned as above. We define the *γ-microlocalization* of the half-wave propagator to be

$$\mathcal{U}^\gamma(t) \overset{\text{def}}{=} A_{2k+1}\,\mathcal{U}(t - T_{2k})\,A_{2k}\,\mathcal{U}(t_{2k})\,A_{2k-1}\cdots A_1\,\mathcal{U}(t_1)\,A_0. \tag{4.2}$$

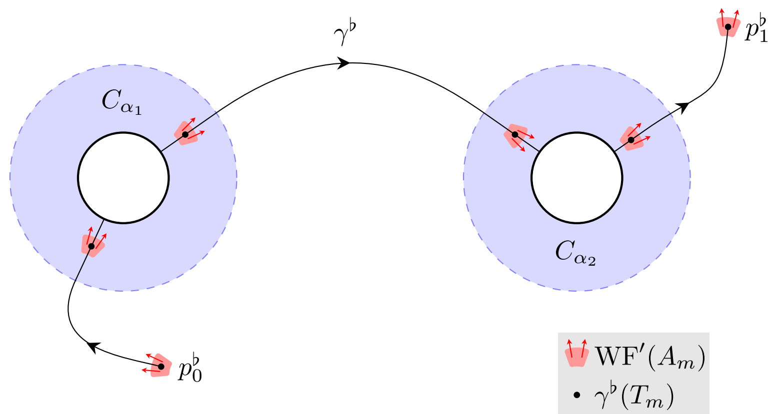


FIGURE 4.1. A projected geodesic segment $\gamma^\flat$ and its partitioning

The rest of our efforts in this section go towards calculating the principal amplitude of $\mathcal{U}^\gamma(t)$. We note that the factors in this operator are of two basic types. The first, the factors $A_{2m+2} \circ \mathcal{U}(t_{2m+2}) \circ A_{2m+1}$, microlocalize to within the collar neighborhoods $C_{\alpha_{m+1}}$ capturing the propagation through the cone point $Y_{\alpha_{m+1}}$. The computation of their amplitudes is an application of Theorem 3.5 above. The other factors, of the form $A_{2m+1} \circ \mathcal{U}(t_{2m+1}) \circ A_{2m}$, microlocalize the propagator to within the interior X° and thus capture the propagation along γ away from the cone points. We now calculate their amplitudes.

4.2. **The amplitude in the interior.** Since the factors $A_{2m+1} \circ \mathcal{U}(t_{2m+1}) \circ A_{2m}$ are only microlocally nontrivial in a compact subset of the interior of X, it is equivalent to consider the same framework in a closed, smooth manifold Z. Therefore, suppose $\gamma : [0, T] \longrightarrow \dot{T}^*Z$ is a geodesic segment of length T with endpoints

$$\gamma(0) = (z_0, \zeta_0) \quad \text{and} \quad \gamma(T) = (z_1, \zeta_1).$$

We make the analogous assumption to (4.1) in this case:

$$z_0 \text{ and } z_1 \text{ are not conjugate along } \gamma^\flat \text{ in } Z. \tag{4.3}$$

This implies the existence of fiber-homogeneous neighborhoods $U_0 \ni (z_0, \zeta_0)$ and $U_1 \ni (z_1, \zeta_1)$ in $\dot{T}^*Z$ between which the exponential map is a diffeomorphism.

Using these neighborhoods, we choose microlocalizers A_0 and A_1 in $\Psi^0(Z)$ such that

$$(z_0, \zeta_0) \in \mathrm{WF}'(A_0) \subseteq U_0 \quad \text{and} \quad (z_1, \zeta_1) \in \mathrm{WF}'(A_1) \subseteq U_1.$$

By choosing the microsupports of these operators sufficiently small, we may write A_0 as the right quantization of a compactly-supported symbol $a_0 \in S^0_{\mathrm{c}}(U_0)$, and we may write A_1 as the left quantization of $a_1 \in S^0_{\mathrm{c}}(U_1)$. Composing these, we form the associated γ-microlocalization of the half-wave group $\mathcal{U}^\gamma(t) = A_1\,\mathcal{U}(t)\,A_0$. From the calculus of Fourier integral operators [Hör71] and Hörmander's result on the structure of $\mathcal{U}(t)$ [Hör68], we conclude

$$\boldsymbol{U}^\gamma \overset{\text{def}}{=} \mathcal{K}[\mathcal{U}^\gamma(t)] \in I_{\mathrm{c}}^{-\frac{1}{4}}\Big(\mathbb{R} \times Z \times Z, G^t[A_0, A_1]'; |\Omega|^{\frac{1}{2}}(Z \times Z)\Big),$$

where

$$G^t[A_0, A_1] \stackrel{\text{def}}{=} \left\{ (t,\tau;z,\zeta;z',\zeta') : \begin{array}{l} \tau = |\zeta'|_g,\ (z',\zeta') \in \mathrm{WF}'(A_0), \\ \text{and } (z,\zeta) = G^t(z',\zeta') \in \mathrm{WF}'(A_1) \end{array} \right\}$$

is the graph of geodesic flow from $\mathrm{WF}'(A_0)$ to $\mathrm{WF}'(A_1)$ and $(\cdot)'$ denotes the fiber-twist $(t,\tau;z,\zeta;z',\zeta')' = (t,\tau;z,\zeta;z',-\zeta')$ making $G^t[A_0,A_1]'$ into a Lagrangian submanifold of $\dot{T}^*(\mathbb{R} \times Z \times Z)$.

For points (z,z') in $U_1^\flat \times U_0^\flat$, where $U_j^\flat$ is the projection of U_j to Z, we may find a variation $c^\flat : [0,t]_s \longrightarrow Z$ of $\gamma^\flat$ by a Jacobi field such that $c^\flat(0) = z'$ and $c^\flat(t) = z$. We then define

$$\operatorname{dist}_g^\gamma(z,z') \stackrel{\text{def}}{=} \operatorname{length}(c^\flat) \tag{4.4}$$

to be the function measuring the distance between z' and z "along γ". Our geometric assumptions imply that this distance function is smooth for $(z,z') \in U_1^\flat \times U_0^\flat$, and moreover we have

$$G^t[A_0,A_1]' \subseteq N^*\left\{\operatorname{dist}_g^\gamma(z,z') = t\right\}.$$

Theorem 4.2. *Assume the nonconjugacy condition* (4.3)*, and suppose $t > \varepsilon$ for some $0 < \varepsilon \ll 1$. Provided the microsupports of A_0 and A_1 are chosen sufficiently small, the Schwartz kernel of $\mathcal{U}^\gamma(t)$ on $\mathbb{R} \times Z \times Z$ has the representation*

$$\boldsymbol{U}^\gamma(t,z,z') = \int_{\mathbb{R}_\theta} e^{i\left[\operatorname{dist}_g^\gamma(z,z') - t\right]\theta}\, b(t,z,z',\theta)\, d\theta \tag{4.5}$$

whose amplitude $b \in S_{\mathrm{c}}^{\frac{n-1}{2}}\left(\mathbb{R} \times U_1^\flat \times U_0^\flat \times \mathbb{R}_\theta; |\Omega|^{\frac12}(U_1^\flat \times U_0^\flat)\right)$ is congruent to

$$\begin{aligned} &a_1\big(z, \partial_z \operatorname{dist}_g^\gamma(z,z') \cdot \theta\big)\, a_0\big(z', -\partial_{z'} \operatorname{dist}_g^\gamma(z,z') \cdot \theta\big) \\ &\qquad \times \frac{e^{-\frac{i\pi(n-1)}{4}}\, i^{-m_\gamma}}{(2\pi)^{\frac{n+1}{2}}} \cdot \frac{\chi(\theta)\, \theta^{\frac{n-1}{2}}}{\operatorname{dist}_g^\gamma(z,z')^{\frac{n-1}{2}}} \cdot \Theta^{-\frac12}(z' \to z) \cdot \omega_g(z)\, \omega_g(z') \end{aligned} \tag{4.6}$$

modulo elements of $S^{\frac{n-1}{2}-1}$. In the above,

- *$\chi \in \mathcal{C}^\infty(\mathbb{R}_\theta)$ satisfies $\chi \equiv 0$ for $\theta < 1$ and $\chi \equiv 1$ for $\theta > 2$;*
- *m_γ is the Morse index of γ; and*
- *$\Theta(z' \to z) \stackrel{\text{def}}{=} \left| \det_g \left[D \exp_{z'}(-)\right] \big|_{\exp_{z'}^{-1}(z)} \right|$ is the determinant of the matrix representing $\left[D\exp_{z'}(-)\right]\big|_{\exp_{z'}^{-1}(z)}$ in g-orthonormal bases of $T_{z'}Z$ and T_zZ.*

Note that the use of Θ here to denote the determinant of the derivative of the exponential map at a point in X° is consistent with the definitions we have made of the analogous quantity at cone points above.

We prove this theorem in Section 5.

4.3. **Assembling the pieces.** We assemble the calculations from Sections 3 and 4.2 to compute the principal amplitude of $\mathcal{U}^\gamma(t)$. In the following, we let $\Pi_\alpha^\gamma : X^\circ \longrightarrow Y_\alpha$ be the map taking a point in the interior of X to the point in Y_α to which it is linked by a geodesic near $\gamma^\flat$ (in the sense of small variations by cone Jacobi fields as above), and we set the geodesic segments $\tilde{\gamma}_0^\flat$ and $\tilde{\gamma}_k^\flat$ in X to be

$$\tilde{\gamma}_0^\flat \stackrel{\text{def}}{=} \gamma^\flat\big|_{[0,T_1]} \quad \text{and} \quad \tilde{\gamma}_k^\flat = \gamma^\flat\big|_{[T_{2k},T]}.$$

Theorem 4.3. *Let $\mathcal{U}^\gamma(t) = A_{2k+1}\,\mathcal{U}(t - T_{2k})\,A_{2k}\,\mathcal{U}(t_{2k})\,A_{2k-1}\cdots A_1\,\mathcal{U}(t_1)\,A_0$ be a γ-microlocalization of the half-wave propagator undergoing k diffractions through the cone points $Y_{\alpha_1},\ldots,Y_{\alpha_k}$, no pair of which are conjugate. Then the Schwartz kernel of $\mathcal{U}^\gamma(t)$ has an oscillatory integral representation*

$$\boldsymbol{U}^\gamma \equiv \int e^{i\phi}\, b(t,z,z',\xi)\,d\xi \,(\mathrm{mod}\ \mathcal{C}^\infty) \tag{4.7}$$

with phase function

$$\phi \stackrel{\text{def}}{=} \left[\operatorname{dist}_g^{\gamma_k}(z, Y_{\alpha_{k-1}}) + \sum_{j=2}^{k-1} \operatorname{dist}_g^{\gamma_{j-1}}(Y_{\alpha_j}, Y_{\alpha_{j-1}}) + \operatorname{dist}_g^{\gamma_0}(Y_{\alpha_1}, z') - t\right]\xi \tag{4.8}$$

and amplitude $b \in S_{\mathrm{c}}^{-\frac{k(n-1)}{2}}\Big(\mathbb{R}_t \times U_1^\flat \times U_0^\flat \times \mathbb{R}_\xi; |\Omega|^{\frac12}\big(U_1^\flat \times U_0^\flat\big)\Big)$ given by

$$\begin{aligned}
&\boldsymbol{a}(z,z',\xi)\cdot \frac{e^{\frac{i\pi(n-1)(k-1)}{4}}\,(2\pi)^{\frac{(n+1)(k-1)}{2}}}{(2\pi i)^k}\cdot \chi(\xi)\,\xi^{-\frac{(k-1)(n-1)}{2}}\cdot\left[\prod_{j=0}^{k} i^{-m_{\gamma_j}}\right]\\
&\quad\times \boldsymbol{D}_{\alpha_k}(\Pi^{\gamma_k}_{\alpha_k}(z), q_k')\cdot\left[\prod_{j=2}^{k-1}\boldsymbol{D}_{\alpha_j}(q_j,q_j')\right]\cdot \boldsymbol{D}_{\alpha_1}(q_1, \Pi^{\gamma_1}_{\alpha_1}(z'))\\
&\quad\times \operatorname{dist}_g^{\gamma_k}(z,Y_{\alpha_k})^{-\frac{n-1}{2}}\cdot\left[\prod_{j=1}^{k-1}\operatorname{dist}_g^{\gamma_j}(Y_{\alpha_{j+1}},Y_{\alpha_j})^{-\frac{n-1}{2}}\right]\cdot \operatorname{dist}_g^{\gamma_0}(Y_{\alpha_1},z')^{-\frac{n-1}{2}}\\
&\quad\times \Theta^{-\frac12}(z'\to Y_{\alpha_1})\cdot\left[\prod_{j=1}^{k-1}\Theta^{-\frac12}(Y_{\alpha_j}\to Y_{\alpha_{j+1}})\right]\cdot\Theta^{-\frac12}(Y_{\alpha_k}\to z)\cdot\omega_g(z)\,\omega_g(z')
\end{aligned} \tag{4.9}$$

modulo elements of $S^{-\frac{(k-1)(n-1)}{2}-\frac12+0}$. In the above,

- *the symbol $\boldsymbol{a} \in S^0_{\mathrm{c}}\Big(U_1^\flat \times U_0^\flat \times \mathbb{R}_\xi; |\Omega|^{\frac12}(U_1^\flat \times U_0^\flat)\Big)$ is the combined amplitude of the microlocalizers $A_j \in \Psi^0_{\mathrm{c}}(X^\circ)$,*

$$\begin{aligned}
\boldsymbol{a}(z,z',\xi) &\stackrel{\text{def}}{=} a_{2k+1}\Big(z, \partial_z \operatorname{dist}_g^{\tilde\gamma_k}(z,w_k)\cdot\xi\Big)\\
&\times\left[\prod_{j=1}^{k} a_{2j}\big(w_j, -\partial_{w_j}\operatorname{dist}_g(w_j,Y_{\alpha_j})\cdot\xi\big)\cdot a_{2j-1}\Big(w_j', \partial_{w_j'}\operatorname{dist}_g(Y_{\alpha_j},w_j')\cdot\xi\Big)\right]\\
&\qquad\times a_0\Big(z', -\partial_{z'}\operatorname{dist}_g^{\tilde\gamma_0}(w_1',z')\cdot\xi\Big)
\end{aligned} \tag{4.10}$$

evaluated at the critical points

$$w_1' \stackrel{\text{def}}{=} \exp_{z'}\left(\frac{T_1}{\operatorname{dist}_g^{\gamma_0}(Y_{\alpha_1}, z')} \exp_{z'}^{-1}(Y_{\alpha_1})\right),$$

$$w_j' \stackrel{\text{def}}{=} \gamma_{j-1}^\flat\left(T_{2j-1} - \sum_{\ell=1}^{j-2} \operatorname{dist}_g^{\gamma_\ell}(Y_{\alpha_{\ell+1}} Y_{\alpha_\ell}) - \operatorname{dist}_g^{\gamma_0}(Y_{\alpha_1}, z')\right), \ j = 2, \dots, k,$$

$$w_j \stackrel{\text{def}}{=} \gamma_j^\flat\left(\stackrel{\text{def}}{=} T_{2j} - \sum_{\ell=1}^{j-1} \operatorname{dist}_g^{\gamma_\ell}(Y_{\alpha_{\ell+1}}, Y_{\alpha_\ell}) - \operatorname{dist}_g^{\gamma_0}(Y_{\alpha_1}, z')\right), \ j = 1, \dots, k-1, \text{ and}$$

$$w_k \stackrel{\text{def}}{=} \exp_z\left(\frac{t - T_{2k}}{\operatorname{dist}_g^{\gamma_k}(z, Y_{\alpha_k})} \exp_z^{-1}(Y_{\alpha_k})\right);$$

- $\chi \in \mathcal{C}^\infty(\mathbb{R}_\xi)$ *satisfies* $\chi \equiv 0$ *for* $\xi < 1$ *and* $\chi \equiv 1$ *for* $\xi > 2$*;*
- m_{γ_j} *is the Morse index of* $\gamma_j^\flat$*;*
- $\boldsymbol{D}_\alpha(y, y')$ *is the Schwartz kernel of the half-Klein-Gordon propagator* $e^{-i\pi\nu_\alpha}$ *on* Y_α*; and*
- $\Theta(z' \to z) \stackrel{\text{def}}{=} \left|\det_g[D\exp_{z'}(-)]\big|_{\exp_{z'}^{-1}(z)}\right|$ *is the determinant of the matrix representing* $[D\exp_{z'}(-)]\big|_{\exp_{z'}^{-1}(z)}$ *in* g*-orthonormal bases of* ${}^{cone}T_{z'}X$ *and* ${}^{cone}T_zX$*.*

We prove this theorem in Section 6.

5. Proof of Theorem 4.2

We begin by fixing an element $(t, z, z') \in \mathbb{R}_+ \times U_1^\flat \times U_0^\flat$ at which we want to compute the representation (4.5) of $\mathcal{U}^\gamma(t)$. If (t, z, z') is not the image of a point $(t, \tau; z, \zeta; z', \zeta') \in G^t[A_0, A_1]$ under the projection map, then the amplitude of (4.5) is residual (i.e., in $S^{-\infty}$) there. Therefore, we restrict to those (t, z, z') which are in this projection. To each such point corresponds a geodesic $c : [0, t]_s \longrightarrow \dot{T}^*Z$ with $c(0) = (z', \zeta')$ and $c(t) = (z, \zeta)$ which we use to compute the amplitude. We partition the domain of c as

$$0 = t_0 < t_1 < \dots < t_{M-1} < t_M = t,$$

with these times chosen so that

(i) the interim times $s_m \stackrel{\text{def}}{=} t_m - t_{m-1}$ are each less than the injectivity radius $\operatorname{inj}(Z, g)$;
(ii) for $m = 1, \dots, M-1$, none of the points $c^\flat(t_m)$ is conjugate to $c^\flat(0)$;
(iii) and for $m = 1, \dots, M-1$ there is at most one point between $c^\flat(t_m)$ and $c^\flat(t_{m+1})$ along $c^\flat$ which is conjugate to $c^\flat(0)$.

Using this partition and the group property, we decompose the operator as

$$\mathcal{U}^\gamma(s) = A_1\,\mathcal{U}(s - t_{M-1})\,\mathcal{U}(s_{M-1}) \cdots \mathcal{U}(s_2)\,\mathcal{U}(s_1)\,A_0, \tag{5.1}$$

a microlocalized composite of short-time propagators analogous to (4.2).

We prove Theorem 4.2 by induction on M, the number of short-time propagators required in this decomposition of $\mathcal{U}^\gamma(t)$ corresponding to the point (t, z, z'). The base case is $M = 1$, corresponding to the γ-microlocalizations of a single short-time propagator. The core of this is the following.

Lemma 5.1. *Let* I_0 *and* W_0 *be the open sets*

$$I_0 \stackrel{\text{def}}{=} \{0 < t < \operatorname{inj}(Z)\} \subseteq \mathbb{R}_t \quad \text{and} \quad W_0 \stackrel{\text{def}}{=} \{\operatorname{dist}_g(z, z') < \operatorname{inj}(Z)\} \subseteq Z \times Z.$$

Then for $(t, z, z') \in I_0 \times W_0$, the Schwartz kernel of $\mathcal{U}(t)$ has the representation

$$\boldsymbol{U}(t, z, z') \equiv \int_{\mathbb{R}_\theta} e^{i\left[\operatorname{dist}_g(z,z') - t\right]\theta}\, b^\natural(t, z, z', \theta)\, d\theta \,(\operatorname{mod} \mathcal{C}^\infty) \tag{5.2}$$

whose amplitude $b^\natural \in S^{\frac{n-1}{2}}\left(I_0 \times W_0 \times \mathbb{R}_\theta; |\Omega|^{\frac{1}{2}}(W_0)\right)$ is to leading order

$$\frac{t\, e^{-\frac{i\pi(n-1)}{4}}}{\pi^{\frac{n+1}{2}}\left[\operatorname{dist}_g(z, z') + t\right]^{\frac{n+1}{2}}} \cdot \chi(\theta)\, \theta^{\frac{n-1}{2}} \cdot \Theta^{-\frac{1}{2}}(z \to z') \cdot \omega_g(z)\, \omega_g(z') \tag{5.3}$$

modulo elements of $S^{\frac{n-1}{2}-1}$. The quantities here are the same as in Theorem 4.2.

Proof. The oscillatory integral representation (5.2) is a modification of Bérard's Hadamard parametrix construction for the sine propagator $\mathbf{W}(t)$ on functions from [Bér77]. Rephrasing Bérard's calculation in terms of the antidifferentiated half-wave propagator $\underset{\cdot}{\boldsymbol{U}} \overset{\text{def}}{=} \mathcal{K}\left[\frac{e^{it\sqrt{\Delta}}}{\sqrt{\Delta}}\right]$, we have the *exact* equality

$$\underset{\cdot}{\boldsymbol{U}}(t, z, z') = \int_{\mathbb{R}_\eta} e^{i\left[\operatorname{dist}_g^2(z,z') - t^2\right]\eta} \cdot C_0\, e^{-\frac{i\pi(n-1)}{2}} \left\{\sum_{k=0}^{\infty} U_k(z, z')\, \eta_+^{\frac{n-3}{2}-k}\right\} \omega_g^2(z')\, d\eta \tag{5.4}$$

for $(t, z, z') \in \{|t| < \operatorname{inj}(Z)\} \times W_0$, where C_0 is a constant depending only on the dimension n and $U_0(z, z') = \Theta^{-\frac{1}{2}}(z \to z')$. (The lower-order U_k's are all explicit in terms of the geometry of Z; see (11) and (13) in [Bér77].)

We transform this into a representation for $\boldsymbol{U}$ by differentiating with respect to t, dividing by i, and replacing t by $-t$. This yields

$$\boldsymbol{U}(t, z, z') = \int_{\mathbb{R}_\eta} e^{i\left[\operatorname{dist}_g^2(z,z') - t^2\right]\eta} \cdot 2C_0\, t\, e^{-\frac{i\pi(n-1)}{4}} \left\{\sum_{k=0}^{\infty} U_k(z, z')\, \eta_+^{\frac{n-1}{2}-k}\right\} \omega_g^2(z')\, d\eta.$$

We now restrict to $t \in I_0$ so that $t > 0$. After introducing the new phase variable $\theta = [\operatorname{dist}_g(z, z') + t]\, \eta$, which we note is a positive multiple of the original phase variable η, this expression becomes

$$\begin{aligned}\boldsymbol{U}(t, z, z') = \int_{\mathbb{R}_\theta} & e^{i\left[\operatorname{dist}_g(z,z') - t\right]\theta} \cdot 2C_0\, t\, e^{-\frac{i\pi(n-1)}{4}} \\ & \times \left\{\sum_{k=0}^{\infty} U_k(z, z') \cdot \frac{\theta_+^{\frac{n-1}{2}}}{\left[\operatorname{dist}_g(z, z') + t\right]^{\frac{n+1}{2}}} \cdot \frac{\theta_+^{-k}}{\left[\operatorname{dist}_g(z, z') + t\right]^{-k}}\right\} \omega_g^2(z')\, d\theta.\end{aligned}$$

To convert this to the Schwartz kernel of the operator acting between half-densities, we multiply by the factor $\omega_g(z)\, \omega_g^{-1}(z')$. Modulo the calculation of the constant C_0, this yields the desired representation (5.2) once we insert the cutoff χ localizing in $\theta > 1$ (producing an overall smooth, and thus microlocally negligible, error).

To finish, we briefly indicate how to calculate the constant C_0. Starting with the classical expression for the antidifferentiated half-wave kernel on $\mathbb{R}^n$,

$$\underset{\cdot}{\boldsymbol{U}}(t, z, z') = \frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{n+1}{2}}} \lim_{\varepsilon \downarrow 0} \left[|z - z'|^2 - (t - i\varepsilon)^2\right]^{-\frac{n-1}{2}} |dz'|,$$

we obtain the oscillatory integral representation

$$\underset{\cdot}{\boldsymbol{U}}(t, z, z') = \frac{e^{-\frac{i\pi(n-1)}{4}}}{2 \cdot \pi^{\frac{n+1}{2}}} \int_{\theta=0}^{\infty} e^{i\left[|z-z'|^2 - (t-i0)^2\right]\theta}\, \theta^{\frac{n-3}{2}}\, |dz'|\, d\theta$$

using the distributional identity

$$\int_{\theta=0}^{\infty} e^{i(u+i0)\theta}\,\theta^{\alpha}\,d\theta = \Gamma(\alpha+1)\,e^{\frac{-i\pi(\alpha+1)}{4}}\,(u+i0)^{-(\alpha+1)}, \qquad u\in\mathbb{R}.$$

Comparing this with (5.4), we see that $C_0 = \frac{1}{2}\pi^{-\frac{n+1}{2}}$, concluding the proof. $\square$

Corollary 5.2. *Choose* $0<\varepsilon\ll 1$, *and suppose* $(t,z,z')\in(\varepsilon,\operatorname{inj}(Z))\times W_0$. *Then there is a representation*

$$\boldsymbol{U}(t,z,z') \equiv \int_{\mathbb{R}_\theta} e^{i\left[\operatorname{dist}_g(z,z')-t\right]\theta}\, b(z,z',\theta)\,d\theta\,(\operatorname{mod}\,\mathcal{C}^\infty)$$

whose amplitude $b\in S^{\frac{n-1}{2}}\left(W_0\times\mathbb{R}_\theta;|\Omega|^{\frac12}(W_0)\right)$ *is to leading order*

$$\frac{e^{-\frac{i\pi(n-1)}{4}}}{(2\pi)^{\frac{n+1}{2}}}\cdot\frac{\chi(\theta)\,\theta^{\frac{n-1}{2}}}{\operatorname{dist}_g(z,z')^{\frac{n-1}{2}}}\cdot\Theta^{-\frac12}(z\to z')\cdot\omega_g(z)\,\omega_g(z')\left(\operatorname{mod}\,S^{\frac{n-1}{2}-1}\right). \tag{5.5}$$

Proof. This follows from Lemma 5.1 by introducing $s=\operatorname{dist}_g(z,z')-t$ into (5.3) to replace the t-variable and applying Lemma 18.2.1 of [Hör07]. $\square$

To finish the base case we must introduce the microlocalizers A_0 and A_1 and compute the amplitude of $\mathcal{U}^\gamma(t)=A_1\,\mathcal{U}(t)\,A_0$. By the above, we may write $\boldsymbol{U}^\gamma$ as

$$\int e^{i\psi}\,a_1(z,\zeta)\,b(z',z'',\theta)\,a_0(z''',\zeta''')\,d\theta d\zeta d\zeta''' dz' dz''$$

with $\psi=(z-z')\cdot\zeta+[\operatorname{dist}_g(z',z'')-t]\,\theta+(z''-z''')\cdot\zeta'''$. Applying the method of stationary phase in the variables (ζ,ζ''',z',z'') then yields this case's version of the formula (4.6) for the principal part of the amplitude.

We now move on to the induction step in the proof. We assume that the representation (4.5)–(4.6) holds for all γ-microlocalizations of $\mathcal{U}(t)$ at all points (t,z,z') whose associated decomposition is into M short-time propagators. (In particular, we assume this for all microlocalizers A_0 and A_1.) We shall show this representation also holds at those points requiring a decomposition into $M+1$ short-time propagators. Thus, let $\mathcal{U}^\gamma(t)$ be a γ-microlocalization of $\mathcal{U}(t)$, and let (t,z,z') be a point in the kernel spacetime with associated decomposition

$$\mathcal{U}^\gamma(t)=A_1\,\mathcal{U}(t-t_M)\,\mathcal{U}(s_M)\,\cdots\mathcal{U}(s_1)\,A_0.$$

Recalling that $t_M=s_1+\cdots+s_M$, choose $A\in\Psi^0_{\mathrm{c}}(Z)$ to be a microlocalizer whose symbol a is identically 1 on the set

$$G^{t_M}\left[\mathrm{WF}'(A_0)\right]\cap G^{-(t-t_M)}\left[\mathrm{WF}'(A_1)\right]$$

and which is microsupported in a small neighborhood of this set. We may then write

$$\mathcal{U}^\gamma(t)\equiv[A_1\,\mathcal{U}(t-t_M)A]\circ[A\,\mathcal{U}(s_M)\cdots\mathcal{U}(s_1)\,A_0]\,(\operatorname{mod}\,\mathcal{C}^\infty)\,.$$

If we define γ_0 and γ_1 to be the geodesics

$$\gamma_0\overset{\text{def}}{=}\gamma\big|_{[0,t_M]}\qquad\text{and}\qquad\gamma_1\overset{\text{def}}{=}\gamma\big|_{[t_M,T]}$$

and shrink the microsupports of A_0 and A_1 if necessary, then we may arrange for $A\,\mathcal{U}(s_M)\cdots\mathcal{U}(s_1)A_0$ to be the decomposition of $\mathcal{U}^{\gamma_0}(t_M)$ associated to the point $(t_M,c^\flat(t_M),z')$ and for $A_1\,\mathcal{U}(t-t_M)\,A$ to be the decomposition of $\mathcal{U}^{\gamma_1}(t-t_M)$ associated to $(t-t_M,z,c^\flat(t_M))$. Since each consists of fewer than $M+1$ short-time propagators, both of these operators satisfy the induction hypothesis, and thus the expressions (4.5)-(4.6) hold for each. This implies that

$$\boldsymbol{U}^\gamma(t,z,z') \equiv \int_{Z_w} \int_{\mathbb{R}_\theta} \int_{\mathbb{R}_\eta} e^{i\Phi}\, b_1(z,w,\eta)\, b_0(w,z',\theta)\, d\eta d\theta \,(\operatorname{mod} \mathcal{C}^\infty)\,, \tag{5.6}$$

where the phase function is

$$\Phi = [\operatorname{dist}_g(z,w) - (t-t_M)]\,\eta + \left[\operatorname{dist}_g^{\gamma_0}(w,z') - t_M\right]\theta$$

and the amplitudes satisfy

$$\begin{aligned} b_0(w,z',\theta) &\equiv a_{\mathrm{L}}(w, \partial_w \operatorname{dist}_g(w,z')\cdot\eta)\, a_0(z', -\partial_{z'} \operatorname{dist}_g(w,z')\cdot\eta) \\ &\times \frac{e^{-\frac{i\pi(n-1)}{4}}}{(2\pi)^{\frac{n+1}{2}}} \cdot \frac{\chi(\eta)\,\eta^{\frac{n-1}{2}}}{\operatorname{dist}_g(w,z')^{\frac{n-1}{2}}} \cdot \Theta^{-\frac12}(w\to z')\cdot \omega_g(w)\,\omega_g(z') \left(\operatorname{mod} S^{\frac{n-1}{2}-1}\right) \end{aligned}$$

and

$$\begin{aligned} b_1(z,w,\theta) &\equiv a_1\big(z, \partial_z \operatorname{dist}_g^{\gamma_0}(z,w)\cdot\theta\big)\, a_{\mathrm{R}}\big(w, -\partial_w \operatorname{dist}_g^{\gamma_0}(z,w)\cdot\theta\big) \\ \times \frac{e^{-\frac{i\pi(n-1)}{4}}\, i^{-m_{\gamma_0}}}{(2\pi)^{\frac{n+1}{2}}} &\cdot \frac{\chi(\theta)\,\theta^{\frac{n-1}{2}}}{\operatorname{dist}_g^{\gamma_0}(z,w)^{\frac{n-1}{2}}} \cdot \Theta^{-\frac12}(z\to w)\cdot \omega_g(z)\,\omega_g(w) \left(\operatorname{mod} S^{\frac{n-1}{2}-1}\right). \end{aligned}$$

Here, a_{L} and a_{R} are the left and right symbols of A respectively, and we note there is a density factor in the w-variable so that the integration makes sense.

We now apply the method of stationary phase to the integral (5.6) in the (η, w)-variables. Φ is critical in these variables precisely when

$$\begin{aligned} \partial_\eta \Phi &= \operatorname{dist}_g(z,w) - (t-t_M) = 0 \\ \partial_w \Phi &= \eta\cdot\partial_w \operatorname{dist}_g(z,w) + \theta\cdot\partial_w \operatorname{dist}_g^{\gamma_0}(w,z') = 0; \end{aligned} \tag{5.7}$$

on the support of the amplitude, this is where

- $\operatorname{dist}_g(z,w) = t - t_M$;
- $\eta = \theta$; and
- $\partial_w \operatorname{dist}_g(z,w) = -\partial_w \operatorname{dist}_g^{\gamma_0}(w,z')$, implying that w lies on the geodesic $c^\flat$ between z' and z.

Therefore, the critical set consists of the single point $w_* \overset{\text{def}}{=} c^\flat(t_M)$. Hence, the expression (5.6) is equivalent to an oscillatory integral of the form

$$\int_{\mathbb{R}_\theta} e^{i\left[\operatorname{dist}^\gamma(z,z') - t\right]\theta}\, b(z,z',\theta)\, d\theta \tag{5.8}$$

modulo smooth kernels. The amplitude $b \in S^{\frac{n-1}{2}}$ satisfies

$$\begin{aligned} &b(z,z',\theta) \\ &\quad\equiv (2\pi)^{\frac{n+1}{2}}\, e^{\frac{i\pi \operatorname{sgn}(\mathbf{Q})}{4}}\, b_1(z,w,\eta)\, b_0(w,z',\theta)\, |\det(\mathbf{Q})|^{-\frac12}\Big|_* \left(\operatorname{mod} S^{\frac{n-1}{2}-1}\right), \end{aligned} \tag{5.9}$$

where $\mathbf{Q}$ is the matrix

$$\mathbf{Q} \overset{\text{def}}{=} \begin{bmatrix} 0 & \partial_w \operatorname{dist}_g(z,w) \\ \partial_w \operatorname{dist}_g(z,w) & \eta\cdot\nabla_w^2 \operatorname{dist}_g(z,w) + \theta\cdot\nabla_w^2 \operatorname{dist}_g^{\gamma_0}(w,z') \end{bmatrix} \tag{5.10}$$

and $(-)\big|_*$ denotes restriction to the critical set described above.

Observe that $|\det(\mathbf{Q})|^{-\frac{1}{2}}\Big|_* = \theta^{-\frac{n-1}{2}}\,|\det(\mathbf{Q}_0)|^{-\frac{1}{2}}$, where $\mathbf{Q}_0$ is a matrix no longer depending on phase variables:

$$\mathbf{Q}_0 \overset{\text{def}}{=} \begin{bmatrix} 0 & \partial_w \operatorname{dist}_g(z,w_*) \\ \partial_w \operatorname{dist}_g(z,w_*) & \nabla^2_w \operatorname{dist}_g(z,w_*) + \nabla^2_w \operatorname{dist}^{\gamma_0}_g(w_*,z') \end{bmatrix}. \tag{5.11}$$

Since $\theta > 0$ on the support of the amplitude, the signatures of $\mathbf{Q}\big|_*$ and $\mathbf{Q}_0$ also agree. Substituting in this together with the principal parts of b_0 and b_1 and noting that a is identically 1 on the critical set, (5.9) becomes

$$\begin{aligned} b(z,z',\theta) \equiv{}& a_1(z,\partial_z \operatorname{dist}^\gamma_g(z,z')\cdot\theta)\, a_0(z',-\partial'_z \operatorname{dist}^\gamma_g(z,z')\cdot\theta) \\ &\times \frac{e^{\frac{i\pi\,\mathrm{sgn}(\mathbf{Q}_0)}{4}}\, e^{-\frac{i\pi(n-1)}{2}}\, i^{-m_{\gamma_0}}}{(2\pi)^{\frac{n+1}{2}}} \cdot \chi(\theta)^2\,\theta^{\frac{n-1}{2}} \cdot \omega_g(z)\,\omega_g(z') \\ &\times \frac{\omega^2_g(w_*)\cdot\Theta^{-\frac{1}{2}}(z\to w_*)\,\Theta^{-\frac{1}{2}}(w_*\to z')}{|dw|\cdot \operatorname{dist}^{\gamma_0}_g(z,w_*)^{\frac{n-1}{2}}\cdot \operatorname{dist}_g(w_*,z')^{\frac{n-1}{2}}}\,|\det(\mathbf{Q}_0)|^{-\frac{1}{2}} \left(\bmod S^{\frac{n-1}{2}-1}\right). \end{aligned} \tag{5.12}$$

We compute the signature of $\mathbf{Q}_0$ via the following lemma.

Lemma 5.3. *Let $c^\flat : [0,T] \longrightarrow Z$ be a geodesic in Z with endpoints $z_1 \overset{\text{def}}{=} c^\flat(0)$ and $z_2 \overset{\text{def}}{=} c^\flat(T)$. For $0 < S < T$ with $T - S < \operatorname{inj}(Z)$, let*

$$c^\flat_1 \overset{\text{def}}{=} c^\flat\Big|_{[0,S]} \qquad \text{and} \qquad c^\flat_2 \overset{\text{def}}{=} c^\flat\Big|_{[S,T]}$$

be a decomposition of $c^\flat$ with $w_* \overset{\text{def}}{=} c^\flat(S)$ the common endpoint. Assume w_* is not conjugate to either z_1 or z_2 along $c^\flat$, and assume also that there is at most one conjugate point to z_1 along $c^\flat_2$. Writing m_c and m_{c_1} for the Morse indices[13] *of $c^\flat$ and $c^\flat_1$, respectively, we have*

$$m_c = m_{c_1} + \operatorname{ind}\Big(\operatorname{Hess}\big[\operatorname{dist}^{c_1}_g(z_1,w) + \operatorname{dist}^{c_2}_g(w,z_2)\big]\big|_{w=w_*}\Big), \tag{5.13}$$

where $\operatorname{ind}(-)$ is the index of a quadratic form, i.e., the sum of the dimensions of the eigenspaces associated to negative eigenvectors.

Proof. Define $m_* \overset{\text{def}}{=} m_c - m_{c_1}$, and set

$$i_* \overset{\text{def}}{=} \operatorname{ind}\Big(\operatorname{Hess}\big[\operatorname{dist}^{c_1}_g(z_1,w) + \operatorname{dist}^{c_2}_g(w,z_2)\big]\big|_{w=w_*}\Big).$$

We first claim that $m_* \leqslant i_*$. If $m_* = 0$, this is trivially true, so we assume $m_* \geqslant 1$, i.e., that there exists a conjugate point $z_{\mathrm{c}} = c^\flat_2(s_{\mathrm{c}})$ to z_1 along $c^\flat_2$. Therefore, we may choose independent normal Jacobi fields $(\mathbf{J}_1,\ldots,\mathbf{J}_{m_*})$ along $c^\flat$ such that $\mathbf{J}_j(0) = \mathbf{J}_j(s_{\mathrm{c}}) = \mathbf{0}$. We arrange that their values at w_* (i.e., at $s = S$) are g-orthonormal, and we extend these to a maximal collection $(\mathbf{J}_1,\ldots,\mathbf{J}_{n-1})$ whose values at w_* form an orthonormal basis of $N_{w_*}c^\flat$. These values then define a Fermi normal coordinate system (ν,ℓ) along $c^\flat$; we use these coordinates to calculate i_*.

For each $j = 1,\ldots,n-1$ let $\mathbf{V}_j(s)$ be the unique broken (normal) Jacobi field along $c^\flat$ satisfying

$$\mathbf{V}_j(0) = \mathbf{V}_j(T) = \mathbf{0} \qquad \text{and} \qquad \mathbf{V}_j(S) = \mathbf{J}_j(S).$$

[13]Note that the Morse index of $c^\flat_2$ vanishes since it is a distance-minimizing geodesic, i.e., $m_{c_2} = 0$.

We construct the variation of $c^\flat$ with respect to these broken Jacobi fields $\mathbf{V} = (\mathbf{V}_1, \ldots, \mathbf{V}_{n-1})$:

$$c^\flat_{\mathbf{V}}(s;\nu) \overset{\text{def}}{=} \exp_{c^\flat(s)}[\nu_1\,\mathbf{V}_1(s) + \cdots + \nu_{n-1}\,\mathbf{V}_{n-1}(s)]\,.$$

By definition, $c^\flat_{\mathbf{V}}(s;\nu)$ agrees to first order with the path realizing the broken distance between z_1 and z_2 with intermediate point $(\nu, \ell = 0)$ in the local manifold $\{\ell = 0\}$ transverse to $c^\flat$. Since the second variation formula involves only the first derivative of the variation, this implies that

$$\text{length}\Big(c^\flat_{\mathbf{V}}(\cdot;\nu)\Big) = \operatorname{dist}_g^{c_1}(z_1;\nu,0) + \operatorname{dist}_g^{c_2}(\nu,0;z_2) + O(\nu^3). \tag{5.14}$$

By putting the Hessian in (5.13) into our Fermi normal coordinate system, we see that

$$i_* = \operatorname{ind}\Big(\,\partial_\nu^2\left[\operatorname{dist}_g^{c_1}(z_1;\nu,0) + \operatorname{dist}_g^{c_2}(\nu,0;z_2)\right]\big|_{\nu=0}\Big)\,,$$

that is, the index in question is the same as the index of the Hessian in the ν-variables only.

We show the existence of a negative eigenvalue of the ν-Hessian of (5.14) by using a standard piece of Riemannian geometry. For any $\varepsilon > 0$ small, we may construct a smooth vector field $\mathbf{X}_1(s)$ along $c^\flat$ satisfying the following:

(i) $\mathbf{X}_1(s)$ agrees with $\mathbf{J}_1(s)$ for $0 \leqslant s < s_{\mathrm{c}} - \varepsilon$;
(ii) $\mathbf{X}_1(s) \equiv \mathbf{0}$ for $s_{\mathrm{c}} + \varepsilon < s \leqslant T$; and
(iii) the variation $c^\flat_{\mathbf{X}_1}(s;\nu_1)$ is shorter than $c^\flat$ for small $\nu_1 > 0$.

(For the details, see, e.g., [Jos11, Theorem 5.3.1].) By choosing $\varepsilon < s_{\mathrm{c}} - S$, we produce a variation of $c^\flat$ which agrees with the broken variation $c^\flat_{\mathbf{V}}(s;\nu_1,0,\ldots,0)$ obtained previously for $0 \leqslant s \leqslant S$. On the other hand, as long as ν_1 is small, we are guaranteed that $c^\flat_{\mathbf{V}}(s;\nu_1,0,\ldots,0)$ is shorter than the injectivity radius of Z for $S \leqslant s \leqslant T$ since $c^\flat_2$ has this property. This implies that it is distance minimizing for these values of s, which in turn implies

$$\text{length}\Big(c^\flat_{\mathbf{V}}(\cdot;\nu_1,0,\ldots,0)\Big) \leqslant \text{length}\Big(c^\flat_{\mathbf{X}_1}(\cdot;\nu_1)\Big) < \text{length}\Big(c^\flat\Big)\,.$$

Hence, the length of $c^\flat_{\mathbf{V}}(\cdot;\nu)$ is decreasing in the ν_1-direction at $\nu = 0$, and therefore the Hessian $\partial_\nu^2\left[\operatorname{dist}_g^{c_1}(z_1;\nu,0) + \operatorname{dist}_g^{c_2}(\nu,0;z_2)\right]\big|_{\nu=0}$ is negative in that direction. This implies the existence of a negative eigenvalue $\mu_1 < 0$ and an associated μ_1-eigenvector $\mathbf{v}_1$ of that Hessian, as desired.

To generate the remaining eigenvectors of the Hessian, we apply this argument inductively. If $(\mathbf{v}_1, \ldots, \mathbf{v}_\ell)$ are distinct eigenvectors of the Hessian associated to negative eigenvalues, then we may find independent Jacobi fields $\Big(\mathbf{J}^\sharp_1, \ldots, \mathbf{J}^\sharp_{n-\ell-1}\Big)$ in the span of our original collection $(\mathbf{J}_1, \ldots, \mathbf{J}_{n-1})$ such that $\mathbf{J}^\sharp_j(s_{\mathrm{c}}) = \mathbf{0}$ for at least $j = 1, \ldots, m_* - \ell$ and whose values at $s = S$ are an orthonormal basis of the g-orthocomplement of the span of $(\mathbf{v}_1, \ldots, \mathbf{v}_\ell)$ in $N_{w_*}c^\flat$. Thus, as long as $m_* - \ell > 0$, our previous argument shows there is another negative eigenvalue. Altogether, this shows that $m_* \leqslant i_*$, finishing the first claim.

We now show the remaining inequality, $i_* \leqslant m_*$. Suppose that $(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$ are the distinct eigenvectors of our Hessian, the first i_* of which are associated to negative eigenvalues μ_j. We may then use these vectors as the coordinate vector fields ∂_{ν_j} at w_*, extending along $c^\flat$ as before. For each j, we let $\mathbf{V}_j(s)$ be the broken Jacobi field along $c^\flat$ satisfying

$$\mathbf{V}_j(0) = \mathbf{V}_j(T) = \mathbf{0} \quad \text{and} \quad \mathbf{V}_j(S) = \mathbf{v}_j,$$

and we again construct the (approximate) joint broken variation $c^\flat_{\mathbf{V}}(s;\nu)$ of $c^\flat$ coming from these broken Jacobi fields. Thus

$$\partial^2_{\nu_j} \operatorname{length}\Big(c^\flat_{\mathbf{V}_j}(\cdot;\nu_j)\Big)\Big|_{\nu_j} = \mu_j,$$

which is negative when $j = 1, \dots, i_*$.

Finally, let $z_{\mathrm{c}(\kappa)} = c^\flat(s_{\mathrm{c}(\kappa)})$ be the points which are conjugate to z_1 along $c^\flat$ for times $0 < s_{\mathrm{c}(\kappa)} < S$, and let $\mathbf{M}_{\kappa,j}(s)$ be independent broken Jacobi fields along $c^\flat$ such that $\mathbf{M}_{\kappa,j}(0) = \mathbf{M}_{\kappa,j}(s_{\mathrm{c}(\kappa)}) = \mathbf{0}$ and $\mathbf{M}_{\kappa,j}(s) \equiv \mathbf{0}$ for $s_{\mathrm{c}(\kappa)} \leqslant s \leqslant T$. There are exactly m_{c_1} of these broken Jacobi fields. Since our vector fields $\mathbf{V}_j$ are nonzero at $s = S$, they do not lie in the span of the $\mathbf{M}_{\kappa,j}$'s. Moreover, since the index form is negative on $\mathbf{V}_j$ for $j = 1, \dots, m$, there must be a conjugate point $z_{\mathrm{c}} = c^\flat(s_{\mathrm{c}})$ for some $S < s_{\mathrm{c}} < T$. This conjugate point is unique by our assumptions on the decomposition of $[0,T]_s$, so it must have multiplicity m_*. This concludes the proof. $\square$

As the signature of a matrix is invariant under small pertubations, we may adjust the microsupports of A_0, A_1, and A to ensure that $m_{\gamma_0} = m_{c_0}$, where we recall that m_{γ_0} is the Morse index of the geodesic segment $\gamma_0^\flat$. Similarly, we may arrange that $m_\gamma = m_c$, and hence

$$e^{\frac{i\pi \operatorname{sgn}(\mathsf{Q}_0)}{4}} e^{-\frac{i\pi(n-1)}{2}} i^{-m_{\gamma_0}} = e^{-\frac{i\pi(n-1)}{4}} i^{-m_\gamma}. \tag{5.15}$$

The remainder of the proof of Theorem 4.2 deals with the third line of (5.12).

Lemma 5.4. *We have the following identifications:*

$$\frac{\omega_g^2(w_*) \cdot \Theta^{-\frac{1}{2}}(z \to w_*)\, \Theta^{-\frac{1}{2}}(w_* \to z')}{|dw| \cdot \operatorname{dist}_g^{\gamma_0}(z, w_*)^{\frac{n-1}{2}} \cdot \operatorname{dist}_g(w_*, z')^{\frac{n-1}{2}}} \cdot |\det(\mathsf{Q}_0)|^{-\frac{1}{2}} = \frac{\Theta^{-\frac{1}{2}}(z \to z')}{\operatorname{dist}^\gamma(z,z')^{\frac{n-1}{2}}} \tag{5.16}$$

and

$$\Theta^{-\frac{1}{2}}(z \to z') = \Theta^{-\frac{1}{2}}(z' \to z). \tag{5.17}$$

Proof. We shall express the quantities in (5.16) and (5.17) in terms of Jacobi endomorphisms along $TZ\big|_{c^\flat}$. To set these up, let (ν', ℓ') be a Fermi normal coordinate system along $c^\flat$ based at $z' = c^\flat(0)$, and let $\mathsf{J}(s)$ be the Jacobi endomorphism satisfying

$$\mathsf{J}(0) = \mathbf{0} \quad \text{and} \quad \dot{\mathsf{J}}(0) = \operatorname{Id}.$$

Now, let $\mathsf{L}(s)$ be the analogous Jacobi endomorphism arising from a Fermi normal coordinate system (ν_*, ℓ_*) based at $w_* = c^\flat(t_M)$, and let and $\mathsf{K}(s)$ be that coming from a Fermi normal coordinate system (ν, ℓ) based at $z = c^\flat(t)$. We may then identify $\Theta(z \to w_*)$ as

$$\Theta(z \to w_*) \stackrel{\text{def}}{=} \Big|\det{}_g[D\exp_z(-)]|_{-(t-t_M)\,\partial_\ell}\Big| = (t - t_M)^{-(n-1)} \Big|\det \mathsf{K}^\perp(t_M)\Big|$$

using (1.9) and the fact that $\mathbf{K}^{\parallel}(t_M)\cdot\partial_\ell = (t - t_M)\cdot\partial_\ell$. Similarly, $\Theta(w_* \to z') = t_M^{-(n-1)}\left|\det \mathbf{L}^{\perp}(0)\right|$, and from the Wronskian identity

$$\left|\det \mathbf{L}^{\perp}(0)\right| = \left|\det \mathcal{W}(\mathbf{J}^{\perp}, \mathbf{L}^{\perp})\big|_{s=0}\right| = \left|\det \mathcal{W}(\mathbf{L}^{\perp}, \mathbf{J}^{\perp})\big|_{s=t_M}\right| = \left|\det \mathbf{J}^{\perp}(t_M)\right|$$

we may rewrite this as $\Theta(w_* \to z') = t_M^{-(n-1)}\left|\det \mathbf{J}^{\perp}(t_M)\right|$. Hence, upon identifying the distance factors with values of $s \in [0,t]$, we have

$$\frac{\Theta^{-\frac{1}{2}}(z \to w_*)\,\Theta^{-\frac{1}{2}}(w_* \to z')}{\operatorname{dist}_g^{\gamma_0}(z,w_*)^{\frac{n-1}{2}}\cdot \operatorname{dist}_g(w_*,z')^{\frac{n-1}{2}}} = \left|\det\Big[\mathbf{J}^{\perp}(t_M)\cdot\mathbf{K}^{\perp}(t_M)\Big]\right|^{-\frac{1}{2}}. \tag{5.18}$$

We now turn to the factor $\frac{\omega_g^2(w_*)}{|dw|}\cdot\left|\det \mathbf{Q}_0\right|^{-\frac{1}{2}} = \left|\det\big[g^{-1}(w_*)\cdot\mathbf{Q}_0\big]\right|^{-\frac{1}{2}}$. Observe that the lower right block of (5.11) may be viewed invariantly[14] as

$$\nabla_w^2 d_w\left[\operatorname{dist}_g(z,w) + \operatorname{dist}_g^{\gamma_0}(w,z')\right]\big|_{N_{w_*}c^\flat\otimes N_{w_*}c^\flat},$$

the Hessian of $\operatorname{dist}_g(z,w) + \operatorname{dist}_g^{\gamma_0}(w,z')$ acting on normal vectors to our geodesic $c^\flat$ at the critical point w_*, and therefore

$$\left|\det\big[g^{-1}(w_*)\cdot\mathbf{Q}_0\big]\right| = \left|\det\bigg[\Big[\operatorname{Hess}[\operatorname{dist}_g(z,\cdot)]^\sharp + \operatorname{Hess}\big[\operatorname{dist}_g^{\gamma_0}(\cdot,z')\big]^\sharp\Big]\Big|_{N_{w_*}c^\flat}\bigg]\right|,$$

where the raising operator $(-)^\sharp$ converts these Hessians into endomorphisms of $N_{w_*}c^\flat$ via $\operatorname{Hess}(-)^\sharp = \nabla\operatorname{grad}(-)$. These gradients are

$$\operatorname{grad}_w(\operatorname{dist}_g(z,w)) = \partial_\ell \quad\text{and}\quad \operatorname{grad}_w\big(\operatorname{dist}_g^{\gamma_0}(w,z')\big) = \partial_{\ell'},$$

which are the radial vector fields for the geodesic spheres with centers z and z' respectively when restricted to $c^\flat$. Applying the connection then returns the shape operators for these geodesic spheres, and as described in [KV86], we may express these in terms of our Jacobi endomorphisms via

$$\begin{aligned}\operatorname{Hess}\big[\operatorname{dist}_g^{\gamma_0}(\cdot,z')\big]^\sharp\Big|_{N_{w_*}c^\flat} &= \dot{\mathbf{J}}^{\perp}(t_M)\cdot(\mathbf{J}^{\perp})^{-1}(t_M)\\ \operatorname{Hess}[\operatorname{dist}_g(z,\cdot)]^\sharp\Big|_{N_{w_*}c^\flat} &= -\dot{\mathbf{K}}^{\perp}(t_M)\cdot(\mathbf{K}^{\perp})^{-1}(t_M),\end{aligned} \tag{5.19}$$

where the minus sign arises from the different orientations of the two coordinate systems along $c^\flat$.

We conclude by putting these calculations together. Firstly, note that $\dot{\mathbf{K}}^{\perp}(t_M)\cdot(\mathbf{K}^{\perp})^{-1}(t_M)$ is a symmetric endomorphism since $\mathcal{W}\Big(\mathbf{K}^{\perp},\mathbf{K}^{\perp}\Big) = \mathbf{0}$. Therefore, by (5.18) and (5.19) we compute that the left-hand side of (5.16) is equal to

$$\begin{aligned}&\left|\det\Big[\dot{\mathbf{J}}^{\perp}(t_M)\cdot(\mathbf{J}^{\perp})^{-1}(t_M) - \dot{\mathbf{K}}^{\perp}(t_M)\cdot(\mathbf{K}^{\perp})^{-1}(t_M)\Big]\cdot\det\Big[\mathbf{J}^{\perp}(t_M)\cdot(\mathbf{K}^{\perp})^{\mathrm{t}}(t_M)\Big]\right|\\ &= \left|\det\Big[\dot{\mathbf{J}}^{\perp}(t_M) - \dot{\mathbf{K}}^{\perp}(t_M)\cdot(\mathbf{K}^{\perp})^{-1}(t_M)\cdot\mathbf{J}^{\perp}(t_M)\Big]\cdot\det\Big[(\mathbf{K}^{\perp})^{\mathrm{t}}(t_M)\Big]\right|\\ &= \left|\det\Big[\dot{\mathbf{J}}^{\perp}(t_M) - \Big((\mathbf{K}^{\perp})^{\mathrm{t}}\Big)^{-1}(t_M)\cdot(\dot{\mathbf{K}}^{\perp})^{\mathrm{t}}(t_M)\cdot\mathbf{J}^{\perp}(t_M)\Big]\cdot\det\Big[(\mathbf{K}^{\perp})^{\mathrm{t}}(t_M)\Big]\right|\\ &= \left|\det\Big[(\mathbf{K}^{\perp})^{\mathrm{t}}(t_M)\cdot\dot{\mathbf{J}}^{\perp}(t_M) - (\dot{\mathbf{K}}^{\perp})^{\mathrm{t}}(t_M)\cdot\mathbf{J}^{\perp}(t_M)\Big]\right|,\end{aligned}$$

[14]This makes sense as a tensor since the gradient of the sum of distances vanishes along $c^\flat$.

and this is exactly $\left|\det\left[-\mathcal{W}\left(\mathbf{K}^{\perp},\mathbf{J}^{\perp}\right)\Big|_{s=t_M}\right]\right|$. By constancy of the Wronskian, this is equal to $\left|\det\mathcal{W}\left(\mathbf{K}^{\perp},\mathbf{J}^{\perp}\right)\big|_{s=0}\right| = \left|\det\mathbf{K}^{\perp}(0)\right|$ and $\left|\det\mathcal{W}\left(\mathbf{K}^{\perp},\mathbf{J}^{\perp}\right)\big|_{s=t}\right| = \left|\det\mathbf{J}^{\perp}(t)\right|$, which is what we wanted to show. □

6. Proof of Theorem 4.3

Let $\gamma:[0,T]\longrightarrow {}^{\mathrm{b}}S^*X$ be a fixed broken geodesic with partition as described in Section 4.1, and let $\mathcal{U}^\gamma(t)$ be a corresponding γ-microlocalization of the half-wave group on X as shown in (4.2). We shall prove Theorem 4.3 by induction on k, the number of diffractions the geodesic γ undergoes. As in the smooth case, we begin by fixing an element $(t,z,z')\in\mathbb{R}_+\times U_1^\flat\times U_0^\flat$. To each point for which the Schwartz kernel $\boldsymbol{U}^\gamma\overset{\mathrm{def}}{=}\mathcal{K}[\mathcal{U}^\gamma(t)]$ is singular we associate a broken geodesic $c:[0,t]_s\longrightarrow {}^{\mathrm{b}}S^*Z$ with $c(0)=(z',\zeta')$ and $c(t)=(z,\zeta)$ which we use in the calculation. Note that we need only consider points (t,z,z') whose associated geodesics c satisfy

$$c(T_m)\in\mathrm{WF}'(A_m)$$

for all $m=1,\dots,2k+1$: if this fails, then $\boldsymbol{U}^\gamma$ will be smooth at (t,z,z'). By adjusting these microlocalizers if necessary, we may ensure that none of the points $c^\flat(t_m)$ are conjugate to one another since the points $\gamma^\flat(T_m)$ of the reference geodesic are not conjugate to one another. Lastly, we define the broken geodesic segments

$$c_m\overset{\mathrm{def}}{=}c\big|_{[T_m,T_{m+1}]}\text{ for }m=0,\dots,2k-1\text{ and }c_{2k}\overset{\mathrm{def}}{=}c\big|_{[T_{2k},t]}.$$

We start with the case of a single diffraction, $k=1$. Recall that the interim times are $t_m\overset{\mathrm{def}}{=}T_m-T_{m-1}$.

Lemma 6.1. *Let $\mathcal{U}^\gamma(t)=A_3\,\mathcal{U}(t-T_2)\,A_2\,\mathcal{U}(t_2)\,A_1\,\mathcal{U}(t_1)\,A_0$ be a γ-microlocalization of the half-wave group undergoing a single diffraction through the cone point Y_α. Then $\boldsymbol{U}^\gamma$ has a representation*

$$\boldsymbol{U}^\gamma(t,z,z')\equiv\int_{\mathbb{R}_\xi}e^{i\left[\mathrm{dist}_g^{\gamma_1}(z,Y_\alpha)+\mathrm{dist}_g^{\gamma_0}(Y_\alpha,z')-t\right]\xi}\,b(t,z,z',\xi)\,d\xi\;(\mathrm{mod}\;\mathcal{C}^\infty)\tag{6.1}$$

with amplitude $b\in S^0_{\mathrm{c}}\left(\mathbb{R}_t\times U_1^\flat\times U_0^\flat\times\mathbb{R}_\xi;|\Omega|^{\frac12}(U_1^\flat\times U_0^\flat)\right)$ given by

$$\boldsymbol{a}(z,z',\xi)\cdot\frac{i^{-m_{\gamma_0}}\,i^{-m_{\gamma_1}}}{2\pi i}\cdot\frac{\chi(\xi)\cdot\boldsymbol{D}_\alpha(\Pi_\alpha(z),\Pi_\alpha(z'))}{\mathrm{dist}_g^{\gamma_1}(z,Y_\alpha)^{\frac{n-1}{2}}\;\mathrm{dist}_g^{\gamma_0}(Y_\alpha,z')^{\frac{n-1}{2}}}\\ \times\Theta^{-\frac12}(z'\to Y_\alpha)\,\Theta^{-\frac12}(Y_\alpha\to z)\cdot\omega_g(z)\,\omega_g(z')\tag{6.2}$$

modulo elements of $S^{-\frac12+0}$, where $\boldsymbol{a}\in S^0_{\mathrm{c}}\left(U_1^\flat\times U_0^\flat\times\mathbb{R}_\xi;|\Omega|^{\frac12}(U_1^\flat\times U_0^\flat)\right)$ is the combined amplitude of the microlocalizers:

$$\boldsymbol{a}(z,z',\xi)\overset{\mathrm{def}}{=}\Big[a_3\big(z,\partial_z\,\mathrm{dist}_g^{c_2}(z,w)\cdot\xi\big)\;a_2(w,-\partial_w\,\mathrm{dist}_g(w,Y_\alpha)\cdot\xi)\\ \times a_1(w',\partial_{w'}\,\mathrm{dist}_g(Y_\alpha,w')\cdot\xi)\,a_0\big(z',-\partial_{z'}\,\mathrm{dist}_g^{c_0}(w',z')\cdot\xi\big)\Big]\Big|_{\substack{w=c_1^\flat(t_2)\\ w'=c_0^\flat(t_1)}}.$$

Proof. We shall apply the method of stationary phase to the oscillatory integral

$$\boldsymbol{U}^{\gamma} \equiv \int e^{i\psi}\, b_1(z,w,\theta)\, d(t_2,w,w',\xi)\, b_0(w',z',\theta')\, d\theta d\xi d\theta' \,(\mathrm{mod}\ \mathcal{C}^{\infty})$$

in the $(\theta, w; \theta', w')$-variables, where ψ is the phase function

$$\psi = \left[\mathrm{dist}_g^{c_2}(z,w) - (t - T_2)\right]\theta + \left[\mathrm{dist}_g(w,Y_\alpha) + \mathrm{dist}_g(Y_\alpha,w') - t_2\right]\xi \\ + \left[\mathrm{dist}_g^{c_0}(w',z') - t_1\right]\theta',$$

b_1 is the amplitude of $A_3\,\mathcal{U}(t-T_2)\,A_2$, d is the amplitude of the diffracting propagator $\mathcal{U}(t_2)$, and b_0 is the amplitude of $A_1\,\mathcal{U}(t_1)\,A_0$. Here, we identify $x(z) = \mathrm{dist}_g(z, Y_\alpha)$ for $z \in C_\alpha^\circ$, and we use the standardized oscillatory integral and amplitude formulations in Theorem 3.5 and Theorem 4.2.

The stationary phase calculations proceed, *mutatis mutandis*, as those of the interior case in Section 5. As before, the critical set in the (θ', w')-variables is $\left\{w'_* \overset{\mathrm{def}}{=} c_0^\flat(t_1) \text{ and } \theta' = \xi\right\}$. The signature and determinant of the Hessian of ψ in these variables are calculated following Lemmas 5.3 and 5.4 with the only change being notational—the smooth Jacobi fields become cone Jacobi fields. (Note in particular that the analogous expression to (5.18) yields the interesting relationship between the $\Theta(Y_\alpha \to Y_\beta)$ factors and determinants of differences of shape operators alluded to earlier.)

Following these calculations with those arising from the (θ, w)-variables then concludes the proof. □

To conclude the section, we go through the induction step of composing a propagator undergoing $k-1$ diffractions with a propagator undergoing a single diffraction as in the previous lemma:

$$\mathcal{U}^{\gamma}(t) = \left[A_{2k+1}\,\mathcal{U}(t-T_{2k})\,A_{2k}\right] \circ \mathcal{U}(t_{2k}) \circ \left[A_{2k-1}\,\mathcal{U}(t_{2k-1})\,A_{2k-2}\cdots A_1\,\mathcal{U}(t_1)\,A_0\right].$$

The Schwartz kernel of this γ-microlocalization may be represented as

$$\boldsymbol{U}^{\gamma} \equiv \int e^{i\psi}\, b_1(z,w,\theta)\, d(t_{2k},w,w',\xi)\, b_0(T_{2k-1},w',z',\theta')\, d\theta d\xi d\theta' \,(\mathrm{mod}\ \mathcal{C}^{\infty})$$

with the phase function being

$$\psi = \left[\mathrm{dist}_g^{c_{2k}}(z,w) - (t - T_{2k})\right]\theta + \left[\mathrm{dist}_g(w,Y_{\alpha_k}) + \mathrm{dist}_g(Y_{\alpha_k},w') - t_{2k}\right]\xi \\ + \left[\mathrm{dist}_g^{\tilde{\gamma}_{k-1}}(w',Y_{\alpha_{k-1}}) + \sum_{j=2}^{k-1}\mathrm{dist}_g^{\gamma_{j-1}}(Y_{\alpha_j},Y_{\alpha_{j-1}}) + \mathrm{dist}_g^{\gamma_0}(Y_{\alpha_1},z') - T_{2k-1}\right]\theta'$$

and with b_1 the amplitude of $A_{2k+1}\,\mathcal{U}(t-T_{2k})\,A_{2k}$, d the amplitude of the diffracting propagator $\mathcal{U}(t_{2k})$, and b_0 the amplitude of $A_{2k-1}\,\mathcal{U}(t_{2k-1})\,A_{2k-2}\cdots A_1\,\mathcal{U}(t_1)\,A_0$ as in (4.3). In this phase function, we set $\tilde{\gamma}_{k-1}^\flat$ to be the segment of $\gamma^\flat$ stretching from $Y_{\alpha_{k-1}}$ to $\gamma^\flat(T_{2k-1})$. We apply the method of stationary phase in the $(\theta, w; \theta', w')$-variables. The calculations for the (θ, w)-variables are exactly as before, and those coming from the (θ', w')-variables are different only in that the critical set forces the path taken from $Y_{\alpha_{k-1}}$ to Y_{α_k} to be the geodesic segment γ_{k-1} (which is the *unique* geodesic connecting the cone points) and the introduction of an overall factor of $e^{\frac{i\pi(n-1)}{4}} \cdot (2\pi)^{\frac{n+1}{2}}$. This ends the proof of Theorem 4.3.

7. A microlocal partition of unity

In this section, we develop a microlocal partition of unity on our conic manifold X which is adapted to the diffractive part of the geodesic flow; such methods have been previously used by Hillairet [Hil05] and by the second author [Wun02]. This is a preparatory step in the analysis of the wave trace at the length of strictly diffractive closed geodesic γ, ultimately allowing us to microlocalize $\mathcal{U}(t)$ in such a way that we may apply the calculation of Theorem 4.3. In particular, we will be able to microlocalize $\mathcal{U}(t)$ to the extent where we need only consider a single diffraction through one of the cone points Y_α at a time.

First, fix ℓ to be the minimal distance between the cone points of X:

$$\ell \stackrel{\text{def}}{=} \min\left\{\operatorname{dist}_g(Y_\alpha, Y_\beta) : \alpha, \beta = 1, \dots, N\right\}.$$

Let $\delta_{\text{cone}} > 0$ be a positive constant satisfying $\delta_{\text{cone}} < \min\left(\frac{x_*}{9}, \frac{1}{10}\,\ell\right)$.[15] For each $\alpha = 1, \dots, N$, we choose $\psi_\alpha \in \mathcal{C}_c^\infty(X)$ to be a nonnegative bump function such that $\psi_\alpha \equiv 1$ near the cone point Y_α and whose support is contained in the neighborhood $\{x_\alpha < \delta_{\text{cone}}\}$ of Y_α. Multiplying by ψ_α then localizes within this neighborhood of Y_α. We call these multiplication operators $\{\psi_\alpha\}_{\alpha=1}^N$ the *cone localizers*.

Next, let $\{A_j\}_{j=1}^J \subseteq \Psi^0(X^\circ)$ be a finite collection of pseudodifferential operators on the interior of X possessing the following four properties:

(i) Each A_j has compactly supported Schwartz kernel, i.e., $A_j \in \Psi_c^0(X^\circ)$;
(ii) For a fixed constant $\delta_{\text{int}} > 0$, the microsupport $\mathrm{WF}'(A_j) \subseteq S^*X^\circ$ of each A_j is contained in a ball of radius δ_{int} with respect to an overall fixed Finsler metric on the interior cosphere bundle; and
(iii) The A_j's complete the cone localizers to a microlocal partition of unity in the sense[16] that
$$\mathrm{Id} - \left[\sum_{\alpha=1}^N \psi_\alpha + \sum_{j=1}^J A_j\right] \in \Psi_c^{-\infty}(X^\circ).$$
(iv) The A_j have square roots modulo smoothing errors: there exists pseudodifferential operators, denoted $\sqrt{A_j} \in \Psi_c^0(X^\circ)$, such that $(\sqrt{A_j})^2 - A_j \in \Psi_c^{-\infty}(X^\circ)$. The cutoffs ψ_α also have smooth square roots.

We call these operators $\{A_j\}_{j=1}^J$ the *interior localizers*. Note that we may microlocalize in the interior more finely by adjusting the parameter δ_{int} or by choosing a finite number of the interior localizers as desired (while, of course, keeping in mind that they must form a microlocal partition of unity). In what follows, we shall write $B_\bullet$ for an operator which is either a cone localizer or an interior localizer.

Now, let us fix a strictly diffractive closed geodesic γ of period T; thus, γ is a piecewise smooth curve on X having jumps within some boundary component Y_α at each time of discontinuity. We subdivide its principal domain $[0, T]$ as

$$0 \stackrel{\text{def}}{=} T_0 < T_1 < \cdots < T_M \stackrel{\text{def}}{=} T,$$

requiring the lengths of these subintervals be sufficiently short:

$$t_m \stackrel{\text{def}}{=} T_m - T_{m-1} < \frac{\ell}{10} \text{ for each } m = 0, \dots, M.$$

[15] Thus the set $\{x < \delta_{\text{cone}}\}$ is well-defined.

[16] In particular, this error here is smoothing and compactly supported *in the interior*.

As $\sum_{k=0}^{m} t_k = T_m$, for all times $t \in \mathbb{R}$ we have

$$\mathcal{U}(t) = \mathcal{U}(t - T_{M-1})\,\mathcal{U}(t_{M-1})\,\mathcal{U}(t_{M-2}) \cdots \mathcal{U}(t_1).$$

To interweave our operators $B_\bullet$ and the above subdivision of the geodesic γ, let $\boldsymbol{w} = (\boldsymbol{w}_0, \ldots, \boldsymbol{w}_{M-1})$ be a word in the indices α and j for the cone and interior localizers respectively, i.e., either $\boldsymbol{w}_m = \alpha \in \{1, \ldots, N\}$ or $\boldsymbol{w}_m = j \in \{1, \ldots, J\}$ for each $m = 0, \ldots, M-1$. We write $\boldsymbol{W}$ for the collection of all such words. For each time $t \in \mathbb{R}$, the operator

$$\mathcal{U}(t) - \sum_{\boldsymbol{w} \in \boldsymbol{W}} \mathcal{U}(t - T_{M-1})\, B_{\boldsymbol{w}_{M-1}}\, \mathcal{U}(t_{M-1})\, B_{\boldsymbol{w}_{M-2}} \cdots B_{\boldsymbol{w}_1}\, \mathcal{U}(t_1)\, B_{\boldsymbol{w}_0}$$

then maps $\mathcal{D}_{-\infty}$ to $\mathcal{D}_{\infty}$ since the operators $B_\bullet$ make up the microlocal partition of unity above. Hence, taking the trace of this operator yields a smooth function on $\mathbb{R}$, implying that the singularities of $\operatorname{Tr} \mathcal{U}(t)$ are the same as the sum over $\boldsymbol{w} \in \boldsymbol{W}$ of the singularities of the microlocalized terms

$$\operatorname{Tr}\Big[\sqrt{B_{\boldsymbol{w}_0}}\, \mathcal{U}(t - T_{M-1})\, B_{\boldsymbol{w}_{M-1}}\, \mathcal{U}(t_{M-1})\, B_{\boldsymbol{w}_{M-2}} \cdots B_{\boldsymbol{w}_1}\, \mathcal{U}(t_1)\, \sqrt{B_{\boldsymbol{w}_0}}\Big];$$

here we have used cyclicity of the trace to move $\sqrt{B_{\boldsymbol{w}_0}}$ to the left. Let us now suppose γ is the only such purely diffractive closed geodesic of period T.[17] Given a word $\boldsymbol{w} \in \boldsymbol{W}$, for t close to T we have

$$\sqrt{B_{\boldsymbol{w}_M}}\, \mathcal{U}(t - T_{M-1})\, B_{\boldsymbol{w}_{M-1}}\, \mathcal{U}(t_{M-1})\, B_{\boldsymbol{w}_{M-2}} \cdots B_{\boldsymbol{w}_1}\, \mathcal{U}(t_1)\, \sqrt{B_{\boldsymbol{w}_M}} : \mathcal{D}_{-\infty} \longrightarrow \mathcal{D}_{\infty}$$

unless there is a parametrization of γ satisfying

$$\gamma(T_m) \in \mathrm{WF}'(B_{\boldsymbol{w}_m}) \text{ for all } m = 0, \ldots, M, \tag{7.1}$$

where we set $B_{\boldsymbol{w}_M} = B_{\boldsymbol{w}_0}$. Therefore, we need only consider terms satisfying (7.1) in calculating the trace.

For later use, we further refine our microlocal partition of unity to have the following two convenient properties, which we call Partition Properties 1 and 2.

Partition Property 1. *Suppose ψ_α and A_j are a cone localizer and an interior localizer respectively in the microlocal partition of unity, and let $p \in \mathrm{WF}'_{\mathrm{b}}(\psi_\alpha)$ and $q \in \mathrm{WF}'(A_j)$ be elements of their respective microsupports.*[18] *If there is a time $t \in \left[0, \frac{1}{10}\,\ell\right]$ such that $p \overset{t}{\underset{\mathbf{D}}{\sim}} q$ (or equivalently $q \overset{t}{\underset{\mathbf{D}}{\sim}} p$), then either*

(i) $p^\flat \in \{\psi_\alpha \equiv 1\}$, *or*
(ii) $x(p^\flat) > \frac{1}{100}\,\delta_{\mathrm{cone}}$.

Thus, the permissible alternatives are as follows: either the diffractive (forward or backward) geodesic flowout from $\mathrm{WF}'(A_j)$ all lies close to the cone point Y_α (i.e., within $\{\psi_\alpha \equiv 1\}$) or it stays slightly away from the cone point (so x is bounded below). This may be ensured by leaving the cone localizers ψ_α fixed and shrinking the support of the Schwartz kernels of the interior localizers A_j's—controlled by the constant δ_{int}—as needed.

We also make the following further requirement on the partition of unity: when flowing out from the microsupport of an interior localizer to that of a conic localizer and thence to the microsupport of another interior localizer, only one of these

[17] If there are multiple such geodesics, then the same argument will show that each contributes its own term of the same form to the wave trace.

[18] Here, $\mathrm{WF}'_{\mathrm{b}}(\psi_\alpha) = {}^{\mathrm{b}}S^*_{Y_\alpha} X \cup \mathrm{WF}'\big(\psi_\alpha|_{X^\circ}\big)$; the use of the b-microsupport is only to be precise.

flowouts may involve interaction with a cone point. To describe this, we use the notation

$$(7.2) \qquad p \overset{t}{\sim} q \Longleftrightarrow \begin{array}{l} p \text{ and } q \text{ are connected via a limit}^{19} \text{ of the ordinary geodesic} \\ \text{flow for time } t \text{ in the cosphere bundle } \textit{of the interior } S^*X^\circ. \end{array}$$

Partition Property 2. *Let A_{j_1} and A_{j_2} be interior localizers and ψ_α a cone localizer in our microlocal partition of unity, and let $p_1 \in \mathrm{WF}'(A_{j_1})$, $p_2 \in \mathrm{WF}'(A_{j_2})$, and $q \in \mathrm{WF}'_{\mathrm{b}}(\psi_\alpha)$ be points in their respective microsupports. Suppose that there are t and t' in $\left[0, \frac{1}{10}\,\ell\right]$ such that $p_1 \underset{\mathbf{D}}{\overset{t}{\sim}} q \underset{\mathbf{D}}{\overset{t'}{\sim}} p_2$. Then either*

$$p_1 \overset{t}{\sim} q \quad \textit{or} \quad q \overset{t'}{\sim} p_2$$

(with the same alternative holding for every choice of q).

Once again this property can be ensured by shrinking the microsupports of the A_j's and leaving the ψ_α's fixed; we simply rely on the fact that a diffractive geodesic cannot undergo two diffractions in time less than ℓ.

8. The wave trace along diffractive orbits

We now prove the Main Theorem. Recall that γ is a strictly diffractive closed geodesic in ${}^{\mathrm{b}}S^*X$ undergoing k diffractions through cone points $Y_{\alpha_1}, \ldots, Y_{\alpha_k}$, and these cone points are pairwise nonconjugate to one another. We denote the length of γ by L, and we decompose it as a concatenation of geodesic segments $\gamma_1, \ldots, \gamma_k$.

Proof of the Main Theorem. Using the microlocal partition of unity adapted to γ developed in Section 7, we may write $\operatorname{Tr}\mathcal{U}(t)$ as the sum

$$(8.1) \qquad \sum_{\boldsymbol{w} \in \boldsymbol{W}} \operatorname{Tr}\Big[\sqrt{B_{\boldsymbol{w}_0}}\,\mathcal{U}(t - T_{M-1})\,B_{\boldsymbol{w}_{M-1}}\,\mathcal{U}(t_{M-1})\,B_{\boldsymbol{w}_{M-2}} \cdots B_{\boldsymbol{w}_1}\,\mathcal{U}(t_1)\,\sqrt{B_{\boldsymbol{w}_0}}\Big]$$

modulo a smooth error. As discussed in the previous section, the terms in the sum which are microlocally nontrivial are those for which there is a partition of the domain of γ such that $\gamma(T_m) \in \mathrm{WF}'(B_{\boldsymbol{w}_m})$ for all $m = 0, \ldots, M$. If each of the microlocalizers $B_{\boldsymbol{w}_m}$ is an interior localizer, then we may compute the trace of the resulting term using the information from Theorem 4.3. Therefore, we need to massage the remaining terms in (8.1) into such a form, eliminating the cone localizers ψ_α from the expression. (Such a technique was previously employed in [Wun02] and in [Hil05].)

To start, we use cyclicity of the trace to ensure that $B_{\boldsymbol{w}_0}$ is not a cone localizer. Since $\delta_{\mathrm{cone}} < \frac{1}{10}\,\ell$ and each $t_m < \frac{1}{10}\,\ell$, any summand in (8.1) has at least two interior localizers appearing between any pair of cone localizers. Thus, a cyclic shift always suffices to ensure the outermost terms are $\sqrt{A_j}$'s rather than $\sqrt{\psi_\alpha}$'s, and by reparametrizing γ we may assume the propagators are evaluated as in (8.1).

We now deal with the internal copies of the cone localizers ψ_α, which appear as factors

$$A_{j_2}\,\mathcal{U}(t_{m+1})\,\psi_\alpha\,\mathcal{U}(t_m)\,A_{j_1}$$

corresponding to an internal diffraction. (There are also terms where one of the A_j is replaced by $\sqrt{A_j}$ and they are treated in exactly the same way.) To simplify

[19] In particular, if $p \overset{t}{\sim} q$, then either p or q may be a point of the boundary.

these terms and eliminate the cone localizers ψ_α, we appeal to the properties built into our microlocal partition of unity. If Partition Property 1(i) holds for $(q,t) \in \mathrm{WF}'(A_{j_2}) \times \{t_{m+1}\}$ or $(q,t) \in \mathrm{WF}'(A_{j_1}) \times \{t_m\}$, then we may replace ψ_α by the identity operator in the factor:

$$A_{j_2}\,\mathcal{U}(t_{m+1})\,\psi_\alpha\,\mathcal{U}(t_m)\,A_{j_1} \equiv A_{j_2}\,\mathcal{U}(t_{m+1})\,\mathrm{Id}\,\mathcal{U}(t_m)\,A_{j_1} \pmod{\mathcal{C}^\infty}.$$

On the other hand, if both of these compositions fall under Partition Property 1(ii), then the points in the support of ψ_α which are diffractively related to the projections of $\mathrm{WF}'(A_{j_1})$ and $\mathrm{WF}'(A_{j_2})$ to the base are at distance greater than $\frac{1}{100}\,\delta_{\text{cone}}$ from the boundary. Then by Partition Property 2, at most *one* of these compositions can involve a diffractive interaction with the boundary. Suppose the $\mathcal{U}(t_m)$ factor only propagates within the interior. Then there exists $Q \in \Psi^0_{\mathrm{c}}(X^\circ)$ where Q is microlocally the identity on the time-t_m geodesic flowout of $\mathrm{WF}'\,A_{j_1}$. Thus

$$A_{j_2}\,\mathcal{U}(t_{m+1})\,\psi_\alpha\,\mathcal{U}(t_m)\,A_{j_1} \equiv A_{j_2}\,\mathcal{U}(t_{m+1})\,\psi_\alpha\,Q\,\mathcal{U}(t_m)\,A_{j_1} \pmod{\mathcal{C}^\infty}.$$

We write

$$A_{j_2}\,\mathcal{U}(t_{m+1})\,\psi_\alpha\,Q\,\mathcal{U}(t_m)\,A_{j_1} = A_{j_2}\,\mathcal{U}(t_{m+1}+t_m)\,[\mathcal{U}(-t_m)\,\psi_\alpha\,Q\,\mathcal{U}(t_m)]\,A_{j_1}$$

and by applying the Egorov Theorem over X°, we conclude $[\mathcal{U}(-t_m)\,\psi_\alpha\,Q\,\mathcal{U}(t_m)]\,A_{j_1}$ is a pseudodifferential operator $\widetilde{A}_{j_1}$ in $\Psi^0_{\mathrm{c}}(X^\circ)$. Moreover, its principal symbol is $\sigma\big(\widetilde{A}_{j_1}\big) = \sigma(A_{j_1}) \cdot (G^{t_m})^*[\sigma(\psi_\alpha)]$. Hence, the microlocalized propagator collapses into an expression of the same form:

$$A_{j_2}\,\mathcal{U}(t_{m+1})\,\psi_\alpha\,\mathcal{U}(t_m)\,A_{j_1} = A_{j_2}\,\mathcal{U}(t_{m+1})\;\mathrm{Id}\;\mathcal{U}(t_m)\,\widetilde{A}_{j_2}.$$

In particular, the product of the principal symbols of these expressions remains unchanged. An analogous argument holds if $\mathcal{U}(t_{m+1})$ is the factor propagating within the interior.

Proceeding in this fashion, we may replace (8.1) with an analogous sum

$$\sum_{\boldsymbol{w}\in\boldsymbol{W}} \mathrm{Tr}\left[\sqrt{\widetilde{B}_{\boldsymbol{w}_0}}\,\mathcal{U}(t-T_{M-1})\,\widetilde{B}_{\boldsymbol{w}_{M-1}}\,\mathcal{U}(t_{M-1})\,\widetilde{B}_{\boldsymbol{w}_{M-2}}\cdots\widetilde{B}_{\boldsymbol{w}_1}\,\mathcal{U}(t_1)\,\sqrt{\widetilde{B}_{\boldsymbol{w}_0}}\right] \tag{8.2}$$

in which no cone localizers appear. Thus, each of the pseudodifferential operators $\widetilde{B}_{\boldsymbol{w}_m}$'s are either interior localizers A_j, modified interior localizers $\widetilde{A}_j$'s having essentially the same properties, or copies of the identity operator Id. In particular, the principal symbols of the pseudodifferential operators in each term still sum to 1 when evaluated along the geodesic, i.e.,

$$\sum_{\boldsymbol{w}\in\boldsymbol{W}}\prod_{m=0}^{M-1}\sigma\Big(\widetilde{B}_{\boldsymbol{w}_m}\Big)(\gamma(T_m)) = 1.$$

As each term in (8.2) falls under the description of Theorem 4.3, all that remains is to compute the trace of each term.

We now perform the method of stationary phase in the base variables of the expression for the propagator in Theorem 4.3, restricted to the diagonal and integrated over the base manifold X—we do not apply stationary phase in the phase variable ξ. In order to compute the trace of each term, we once again use cyclicity to permute so that the outermost factors of $\widetilde{B}_{w_m}$ are not identity terms but rather are interior microlocalizers $\sqrt{A_j}$ or $\sqrt{\widetilde{A}_j}$. (Note that this requires switching the roles of $t - T_M$ and of one of the t_j's, which amounts to a simple relabeling.) The resulting phase function is then

$$\phi \overset{\text{def}}{=} \left[\operatorname{dist}_g^{\gamma_k}(z, Y_{\alpha_{k-1}}) + \sum_{j=2}^{k-1} \operatorname{dist}_g^{\gamma_{j-1}}(Y_{\alpha_j}, Y_{\alpha_{j-1}}) + \operatorname{dist}_g^{\gamma_0}(Y_{\alpha_1}, z) - t \right] \xi,$$

where the variable z is supported in a compact subset of the interior of X owing to the support of the amplitude. The phase ϕ is critical in the z-variable precisely when

$$\partial_z \left[\operatorname{dist}_g^{\gamma_k}(z, Y_{\alpha_{k-1}}) + \operatorname{dist}_g^{\gamma_0}(Y_{\alpha_1}, z) \right] = 0,$$

and this forces z to lie along the unique geodesic $\gamma_k^\flat$ connecting the cone points Y_{α_1} and Y_{α_k}, as before. (Note that $\gamma_0 = \gamma_k$ in this calculation.) Unlike the previous calculations, though, there is no integration in the phase variables, and as a result there is no particular point along the segment which is fixed by stationarity. Thus, the entire segment of $\gamma_k^\flat$ within the support of the amplitude is part of the critical set, i.e., it is a Morse-Bott stationary manifold for the phase. Therefore, we applying this version of the method of stationary phase in Fermi normal coordinates (ν, ℓ) along $\gamma_k^\flat$, and we compute that

$$\operatorname{Tr}\left[\sqrt{\widetilde{B}_{\boldsymbol{w}_M}}\, \mathcal{U}(t - T_{M-1})\, \widetilde{B}_{\boldsymbol{w}_{M-1}}\, \mathcal{U}(t_{M-1})\, \widetilde{B}_{\boldsymbol{w}_{M-2}} \cdots \widetilde{B}_{\boldsymbol{w}_1}\, \mathcal{U}(t_1) \sqrt{\widetilde{B}_{\boldsymbol{w}_0}} \right]$$

has the oscillatory integral representation

$$\int_{\mathbb{R}_\xi} e^{-i(t-L)\xi}\, e^{\frac{i\pi(n-1)}{4}}\, (2\pi)^{\frac{n+1}{2}}\, i^{-m_{\gamma_k}} \left\{ \int_{\ell=0}^{\operatorname{dist}_g^{\gamma_k}(Y_{\alpha_1}, Y_{\alpha_k})} b(t; \nu_*, \ell; \nu_*, \ell; \xi) \right\} d\xi, \tag{8.3}$$

where b is the amplitude (4.9) from Theorem 4.3 and ν_* is the critical point in the ν-variables. In particular, the signature and Hessian factors coming from criticality in the ν-variables are the same as before. We may thus write (8.3) as

$$\int_{\mathbb{R}_\xi} e^{-i(t-L)\xi}\, a(t, \xi)\, d\xi, \tag{8.4}$$

where the amplitude $a \in S_{\mathrm{c}}^{-\frac{k(n-1)}{2}}(\mathbb{R}_t)$ is

$$\begin{aligned} &(2\pi)^{\frac{kn}{2}}\, e^{\frac{ik\pi(n-3)}{4}} \cdot \chi(\xi)\, \xi^{-\frac{k(n-1)}{2}} \cdot \left\{ \int_{\ell=0}^{\operatorname{dist}_g^{\gamma_k}(Y_{\alpha_1}, Y_{\alpha_k})} \boldsymbol{a}(\nu_*, \ell; \nu_*, \ell; \xi) \right\} \\ &\quad \times \left[\prod_{j=1}^{k} i^{-m_{\gamma_j}} \cdot \boldsymbol{D}_{\alpha_j}(q_j, q_j') \cdot \operatorname{dist}_g^{\gamma_j}(Y_{\alpha_{j+1}}, Y_{\alpha_j})^{-\frac{n-1}{2}} \cdot \Theta^{-\frac{1}{2}}(Y_{\alpha_j} \to Y_{\alpha_{j+1}}) \right] \end{aligned} \tag{8.5}$$

modulo elements of $S^{-\frac{k(n-1)}{2} - \frac{1}{2} + 0}$ and we treat the labels for γ_j and Y_{α_j} as being cyclic in $\{1, \ldots, k\}$. Adding up these contributions yields the asserted expression for the trace once we note that the combined amplitudes $\boldsymbol{a}$ sum to the amplitude of the identity operator. This concludes the proof. $\square$

Appendix A. Lagrangian distributions and their amplitudes

In this appendix, we briefly collect the various facts we need in the body of the work about Lagrangian distributions and their differing kinds of amplitudes. The bulk of these facts are due to Hörmander—see, e.g. [Hör71], but we approach Lagrangian distributions through the iterated regularity perspective of Melrose [Mel81] (see [Hör09] for a unified exposition).

Let Z be a smooth n-dimensional manifold, and suppose $\Lambda \subseteq T^*Z \setminus \mathbf{0}$ is a fiber-homogeneous Lagrangian submanifold of its cotangent bundle with the zero section removed. Writing $\mathfrak{X}^s_{\mathrm{loc}}$ for either the local Sobolev space $H^s_{\mathrm{loc}}(X)$ or the local Besov space $B^s_{2,\infty,\mathrm{loc}}(X)$, we say u is a Lagrangian distribution with respect to Λ of class $I\mathfrak{X}^s(Z,\Lambda)$ if and only if

$$A_N \cdots A_1 u \in \mathfrak{X}^s_{\mathrm{loc}} \text{ for all finite compositions of } A_j \in \Psi^1(Z) \text{ with } \sigma_1(A_j)\big|_\Lambda \equiv 0.$$

In particular, the property of a distribution $u \in \mathcal{C}^{-\infty}(Z)$ being Lagrangian with respect to Λ and $\mathfrak{X}^s$ is entirely local.

When u is an element of $IB^{-m-\frac{n}{4}}_{2,\infty}(Z,\Lambda)$, we also write $u \in I^m(Z,\Lambda)$; these are Hörmander's classes of Lagrangian distributions. As shown in [Hör09], one of their characterizing properties is the existence of local oscillatory integral representations: given $u \in I^m(Z,\Lambda)$ there is a covering of Z by open sets U on which

$$u(x) = \int_{\mathbb{R}^N_\theta} e^{i\phi(z,\theta)}\, a(z,\theta)\, d\theta. \tag{A.1}$$

Here, ϕ is a phase function locally parametrizing Λ over $U \subseteq Z$, and the amplitude a is a Kohn-Nirenberg symbol in the class $S^{m-\frac{n+2N}{4}}(U \times \mathbb{R}^N_\theta)$, i.e., a smooth function on $U \times \mathbb{R}^N_\theta$ satisfying the estimates

$$\left\| \langle\theta\rangle^{-m+|\beta|}\, D^\alpha_z D^\beta_\theta a(z,\theta) \right\|_{L^\infty(U\times\mathbb{R}^N)} \leqslant C_{\alpha,\beta}$$

for all multi-indices α and β, where $\langle\theta\rangle \overset{\text{def}}{=} \left(1+|\theta|^2\right)^{\frac{1}{2}}$.

When u is an element of a Sobolev-based iterated regularity space $IH^s(Z,\Lambda)$, it also has an oscillatory integral representation like (A.1), but in this case the amplitude a is an L^2*-based symbol* of class $S^{-s}L^2(U \times \mathbb{R}^n_\theta)$. In the general case, we define this class of symbols as follows.

Definition A.1. The space $S^m L^2\left(\mathbb{R}^n_z \times \mathbb{R}^N_\theta\right)$ of L^2*-based symbols of order* m on $\mathbb{R}^n_z \times \mathbb{R}^N_\theta$ are those smooth functions a satisfying the estimates

$$\left\| \langle\theta\rangle^{-m+|\beta|}\, D^\alpha_z D^\beta_\theta a(z,\theta) \right\|_{L^2(\mathbb{R}^n\times\mathbb{R}^N)} \leqslant C_{\alpha,\beta}$$

for all multi-indices α and β.

The Sobolev embedding theorem shows that these L^2-based symbols give an alternative filtration of the Kohn-Nirenberg symbol spaces:

$$S^m\left(\mathbb{R}^n_z \times \mathbb{R}^N_\theta\right) \subsetneq S^{m+\frac{N}{2}+0} L^2\left(\mathbb{R}^n_z \times \mathbb{R}^N_\theta\right) \subsetneq S^{m+0}\left(\mathbb{R}^n_z \times \mathbb{R}^N_\theta\right). \tag{A.2}$$

At the level of iterated regularity, this chain of inclusions just reflects the similar inclusions of regularity spaces $B^s_{2,\infty}(\mathbb{R}^n) \supsetneq H^{s+0}(\mathbb{R}^n) \supsetneq B^{s+0}_{2,\infty}(\mathbb{R}^n)$, in turn yielding the chain of inclusions

$$IB^s_{2,\infty}(\mathbb{R}^n,\Lambda) \supsetneq IH^{s+0}(\mathbb{R}^n,\Lambda) \supsetneq IB^{s+0}_{2,\infty}(\mathbb{R}^n,\Lambda)$$

Furthermore, we may use (A.2) to conclude the following rule for multiplication of L^2-based symbols:
(A.3)
$$a \in S^m L^2\big(\mathbb{R}^n_z \times \mathbb{R}^N_\theta\big) \text{ and } b \in S^{m'} L^2\big(\mathbb{R}^n_z \times \mathbb{R}^N_\theta\big) \Longrightarrow ab \in S^{m+m'-\frac{N}{2}+0}\big(\mathbb{R}^n_z \times \mathbb{R}^N_\theta\big).$$

We now state and prove our main technical lemma relating the two types of amplitudes, used in the proof of Lemma 3.3 above.

Lemma A.2. *Let $a \in S^m_{\mathrm{c}} L^2\Big(\mathbb{R}^{p+q}_{(w,z)} \times \mathbb{R}^N_\theta\Big)$ be an L^2-symbol which is compactly supported in the base variables (w,z). Suppose that there exist $\ell \in \mathbb{R}$ and $\delta > 0$ such that for all compactly supported $b \in S^\ell_{\mathrm{c}} L^2\big(\mathbb{R}^q_z \times \mathbb{R}^N_\theta\big)$,*

$$\int_{\mathbb{R}^q_z} a(w,z,\theta)\, b(z,\theta)\, dz \in S^{m+\ell-\frac{N}{2}-\delta} L^2\big(\mathbb{R}^p_w \times \mathbb{R}^N_\theta\big)$$

Then
$$a \in S^{m-\delta+0}_{\mathrm{c}} L^2\Big(\mathbb{R}^{p+q}_{(w,z)} \times \mathbb{R}^N_\theta\Big) \subsetneq S^{m-\frac{N}{2}-\delta+0}_{\mathrm{c}}\Big(\mathbb{R}^{p+q}_{(w,z)} \times \mathbb{R}^N_\theta\Big).$$

Proof. For simplicity we fix $\ell = 0$; the general proof is similar.

We begin by showing the undifferentiated estimate $a \in \langle\theta\rangle^{m-\delta+0} L^2(\mathbb{R}^{n+N})$. Let φ be an element of $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^q_z)$, and choose $b(z,\theta) = \varphi(z)\,\langle\theta\rangle^{-\frac{N}{2}+\delta-\varepsilon}$. Noting that $\langle\theta\rangle^{-(m-\delta)-\varepsilon}$ is an element of $S^{m-\delta-\frac{N}{2}} L^2(\mathbb{R}^N_\theta)$ for any $\varepsilon > 0$, we observe that the map

$$\varphi \longmapsto c(w,\theta) \overset{\text{def}}{=} \langle\theta\rangle^{-(m-\delta)-\varepsilon} \int_{\mathbb{R}^q_z} a(w,z,\theta)\,\varphi(z)\,dz$$

is continuous from $H^\infty_{\mathrm{c}}(\mathbb{R}^q_z)$ to $S^0 L^2\big(\mathbb{R}^p_w \times \mathbb{R}^N_\theta\big)$. Thus, by the Closed Graph Theorem there is an $A \in \mathbb{Z}_+$ such that

$$\|c\|_{L^2(\mathbb{R}^p_w \times \mathbb{R}^N_\theta)} \lesssim \left(\sum_{|\alpha| \leqslant A} \|D^\alpha_z \varphi\|^2_{L^2(\mathbb{R}^q_z)}\right)^{\frac{1}{2}}. \tag{A.4}$$

This implies that $\langle\theta\rangle^{-(m-\delta)-\varepsilon}\, a(w,z,\theta)$ must be an element $L^2(\mathbb{R}^p_w \times \mathbb{R}^N_\theta)$ with values in the dual to the Hilbert space whose norm is defined by the right-hand side of (A.4), i.e.,

$$a \in \langle\theta\rangle^{m-\delta+\varepsilon} L^2\big(\mathbb{R}^p_w \times \mathbb{R}^N_\theta; H^{-A}(\mathbb{R}^q_z)\big). \tag{A.5}$$

On the other hand, we know a priori that a satisfies symbol estimates and may, in particular, be differentiated without loss:

$$a \in \langle\theta\rangle^{m} L^2\big(\mathbb{R}^p_w \times \mathbb{R}^N_\theta; H^{\infty}(\mathbb{R}^q_z)\big). \tag{A.6}$$

Interpolating (A.5) and (A.6) yields that a is an element of

$$\langle\theta\rangle^{m-\delta+2\varepsilon} L^2\big(\mathbb{R}^p_w \times \mathbb{R}^N_\theta; H^{M}(\mathbb{R}^q_z)\big)$$

for every $M \in \mathbb{Z}_+$ and every $\varepsilon > 0$, and this is precisely the first estimate required to show that a is an element of $S^{m-\delta+0} L^2\Big(\mathbb{R}^{p+q}_{(w,z)} \times \mathbb{R}^N_\theta\Big)$. The higher-order symbol estimates are proved analogously, estimating the higher-order symbol norms in (A.4) instead of just the L^2-norm. The lemma then follows once we use the inclusion from (A.2). □

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Stanford University
E-mail address: `austin.ford@math.stanford.edu`

Northwestern University
E-mail address: `jwunsch@math.northwestern.edu`