THE TRACE OF THE GENERALIZED HARMONIC OSCILLATOR

JARED WUNSCH

ABSTRACT. We study a geometric generalization of the time-dependent Schrödinger equation for the harmonic oscillator

$$(D_t + \frac{1}{2}\Delta + V)\psi = 0 \tag{0.1}$$

where Δ is the Laplace-Beltrami operator with respect to a "scattering metric" on a compact manifold M with boundary (the class of scattering metrics is a generalization of asymptotically Euclidian metrics on $\mathbb{R}^n$, radially compactified to the ball) and V is a perturbation of $\frac{1}{2}\omega^2 x^{-2}$, with x a boundary defining function for M (e.g. $x = 1/r$ in the compactified Euclidian case). Using the *quadratic-scattering wavefront set*, a generalization of Hörmander's wavefront set that measures oscillation at ∂M as well as singularities, we describe a propagation of singularities theorem for solutions of (0.1). This enables us to prove the following trace theorem: let

$$S_\omega = \left\{ \frac{L}{\omega} : \text{there exists a closed geodesic in } \partial M \text{ of length } \pm L \right\}$$
$$\cup \left\{ \frac{n\pi}{\omega} : \text{ there exists a geodesic } n\text{-gon in } M \text{ with vertices in } \partial M \right\} \cup \{0\}.$$

Let $U(t) = e^{-it(\frac{1}{2}\Delta + V)}$ be the solution operator to the the Cauchy problem for (0.1). Then under a non-trapping assumption for the geodesic flow on $\overset{\circ}{M}$, we have

$$\operatorname{sing\,supp} \operatorname{Tr} U(t) \subset S_\omega,$$

where $\operatorname{Tr} U(t)$ is the distribution given by integrating the Schwartz kernel of $U(t)$ over the diagonal in $M \times M$ or, alternatively, by $\sum_j e^{-it\lambda_j}$, where λ_j are the eigenvalues of $\frac{1}{2}\Delta + V$.

Author's address:

Department of Mathematics
Columbia University
2990 Broadway, Mailcode 4406
New York NY 10027 USA

email: `jwunsch@math.columbia.edu`

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LA TRACE DE L'OSCILLATEUR HARMONIQUE GENERALISE

JARED WUNSCH

PRÉCIS. On étudie une généralisation géométrique de l'équation de Schrödinger dépendante du temps pour l'oscillateur harmonique

$$(D_t + \frac{1}{2}\Delta + V)\psi = 0 \tag{0.1}$$

où Δ est l'opérateur de Laplace-Beltrami associé à une "métrique scattering" sur une variété compacte M à bord (la classe de métriques scattering est une generalisation des métriques asymptotiquement euclidiennes sur $\mathbb{R}^n$, compactifié radialement à la boule) et V est une perturbation de $\frac{1}{2}\omega^2 x^{-2}$, où x est une function qui définit le bord de M (e.g. $x = 1/r$ dans le cas euclidien compactifié). En employant le *front d'onde quadratique-scattering*, une généralisation du front d'onde de Hörmander qui mesure l'oscillation en ∂M ainsi que les singularités, on décrit un théorème de propagation des singularités pour les solutions de (0.1). Ceci permet de démontrer le théorème de trace suivant: soit

$$S_\omega = \left\{ \frac{L}{\omega} : \text{ il existe une géodésique fermée dans } \partial M \text{ de longueur } \pm L \right\}$$
$$\cup \left\{ \frac{n\pi}{\omega} : \text{ il existe un } n\text{-gone géodésique dans } M \text{ dont les sommets appartiennent à } \partial M \right\} \cup \{0\}.$$

Soit $U(t) = e^{-it(\frac{1}{2}\Delta + V)}$ l'opérateur de solution du problème de Cauchy pour (0.1). Alors sous une hypothèse de non-captivité pour le flôt géodésique sur $\overset{\circ}{M}$, on a

$$\operatorname{supp} \operatorname{sing} \operatorname{Tr} U(t) \subset S_\omega,$$

où $\operatorname{Tr} U(t)$ est la distribution qu'on obtient en intégrant le noyau de Schwartz de $U(t)$ sur la diagonale de $M \times M$ où alternativement, de $\sum_j e^{-it\lambda_j}$, où λ_j sont les valeurs propres de $\frac{1}{2}\Delta + V$.

1. Introduction

Let M be a compact manifold with boundary endowed with a *scattering metric* g as defined by Melrose [9]. Thus in a neighborhood of ∂M, we can write

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2} \tag{1.1}$$

where x is a boundary defining function for ∂M, i.e. is smooth, nonnegative, and vanishes exactly at ∂M with $dx \neq 0$ at ∂M, and where $h \in \mathcal{C}^\infty(\mathrm{Sym}^2(T^*M))$ restricts to be a metric on ∂M. Scattering metrics form a class of complete, asymptotically flat metrics that includes asymptotically Euclidian metrics on $\mathbb{R}^n$, radially compactified to the n-ball; this class also includes metrics on $\mathbb{R}^n$ that are not asymptotically Euclidian but that look like arbitrary, non-round metrics on the sphere at infinity (see [9] for details).

We consider a generalization of the quantum-mechanical harmonic oscillator on the manifold M: let x be a boundary-defining function for ∂M with respect to which g has the form (1.1), e.g. $|z|^{-1}$ on flat $\mathbb{R}^n$ (modified to be a smooth function at $z = 0$). For any $\omega \in \mathbb{R}_+$, we consider the associated time-dependent Schrödinger equation

$$\left(D_t + \frac{1}{2}\Delta + \frac{\omega^2}{2x^2} + v\right)\psi = 0 \tag{1.2}$$

where v is a formally self-adjoint perturbation term that can include both magnetic and electric potential terms. We will take v to be an error term in a sense to be made precise later on; potentials of the form $v \in \mathcal{C}^\infty(M)$ are certainly allowed. Note that for such a v, $\frac{1}{2}\Delta + \frac{\omega^2}{2x^2} + v$ is semi-bounded, hence the Friedrichs extension gives a self-adjoint operator on $L^2(M)$ (with respect to the metric dg). Our class of operators thus includes compactly supported metric and potential perturbations of the standard harmonic oscillator on $\mathbb{R}^n$.

Perturbations of the free-particle Schrödinger equation on manifolds with scattering metrics were studied in [13] using a calculus of pseudodifferential operators on manifolds with boundary called the *quadratic-scattering* (or qsc) calculus and denoted $\Psi_{\mathrm{qsc}}(M)$. This calculus is a microlocalization of the Lie algebra of "quadratic-scattering vector fields" on M, given by

$$\mathcal{V}_{\mathrm{qsc}}(M) = x^2\mathcal{V}_{\mathrm{b}}(M) \tag{1.3}$$

where

$$\mathcal{V}_{\mathrm{b}}(M) = \{\text{vector fields on } M \text{ tangent to } \partial M\}. \tag{1.4}$$

Near ∂M, $\mathcal{V}_{\mathrm{qsc}}(M)$ is locally spanned over $\mathcal{C}^\infty(M)$ by vector fields of the form $x^3\partial_x$, $x^2\partial_{y_j}$ where x, y_j are product-type coordinates on M near ∂M, i.e. the y_j's are coordinates on ∂M. The Lie algebra $\mathcal{V}_{\mathrm{qsc}}(M)$ can be written as the space of sections of a vector bundle:

$$\mathcal{V}_{\mathrm{qsc}}(M) = \mathcal{C}^\infty(M; {}^{\mathrm{qsc}}TM);$$

we call ${}^{\mathrm{qsc}}TM$ the *quadratic scattering tangent bundle* of M. Let ${}^{\mathrm{qsc}}T^*M$ be the dual bundle (the *quadratic scattering cotangent bundle*). Let ${}^{\mathrm{qsc}}\overline{T}^*M$ be the unit-ball bundle over M obtained by radially compactifying the fibers of ${}^{\mathrm{qsc}}T^*M$ (see [9] or [13]). This is a manifold with corners. The principal symbols of operators in the qsc-calculus are conormal distributions on ${}^{\mathrm{qsc}}\overline{T}^*M$ with respect to the boundary (a precise definition of such distributions will be given in §2). There is an associated wavefront set, $\mathrm{WF}_{\mathrm{qsc}}$, which is a closed subset of $\partial({}^{\mathrm{qsc}}\overline{T}^* M)$.

In [13], propagation of $\mathrm{WF}_{\mathrm{qsc}}$ was described for perturbations of the free particle Schrödinger equation on M. In this paper, we discuss the analogous results for the harmonic oscillator, referring to [13] for all technical details. We can conclude from the propagation results that if there are no trapped geodesics on $\overset{\circ}{M}$, then except at a certain set of times

$$S_\omega = \left\{ \frac{L}{\omega} : \text{there exists a closed geodesic in } \partial M \text{ of length } \pm L \right\} \cup \left\{ \frac{n\pi}{\omega} : \text{ there exists a geodesic } n\text{-gon in } M \text{ with vertices in } \partial M \right\} \cup \{0\}, \tag{1.5}$$

there is no recurrence of $\mathrm{WF}_{\mathrm{qsc}}$ for solutions to (1.2). In the above definition of S_ω we adopt the convention that the sides of a geodesic n-gon in M with vertices in ∂M are maximally extended geodesics in $\overset{\circ}{M}$ (which automatically have infinite length) and geodesics in ∂M of length π; the latter geodesics appear naturally as limits of geodesics through $\overset{\circ}{M}$—cf. Proposition 1 of [10]. Using very general properties of the qsc calculus, in §5 we use the non-recurrence result to conclude that if $U(t)$ is the solution operator for the Cauchy problem for (1.2) then

$$\operatorname{sing\,supp} \operatorname{Tr} U(t) \subset S_\omega. \tag{1.6}$$

For example, if we have a compactly-supported potential perturbation of the standard harmonic oscillator on $\mathbb{R}^n$, $S_\omega = 2\pi\mathbb{Z}$: if M is the radial compactification of $\mathbb{R}^n$, ∂M is the unit $n-1$-sphere. Geodesics on $\overset{\circ}{M}$ connect antipodal points on ∂M and geodesics in ∂M are great circles, hence consecutive vertices of a geodesic n-gon are antipodal points and there exist geodesic n-gons iff n is even; closed geodesics in ∂M also only occur with lengths in $2\pi\mathbb{Z}$. Hence for a potential perturbation of the harmonic oscillator on $\mathbb{R}^n$, the trace of the solution operator can only be singular at multiples of 2π. One can deduce this easily from Mehler's formula in the unperturbed case.

The trace theorem (1.6) closely resembles a result of Chazarain [1] and Duistermaat-Guillemin [6] which says that on a compact Riemannian manifold without boundary,

$$\operatorname{sing\,supp} \operatorname{Tr} e^{it\sqrt{\Delta}} \subset \{\text{lengths of closed geodesics}\} \cup \{0\};$$

related results of Colin de Verdière using heat kernels can be found in [3] and [4]. Chazarain [2] has also proved a semi-classical trace theorem for the time-dependent Schrödinger equation, in which the lengths of closed bicharacteristics of the total symbol appear. By contrast, the trace theorem of this paper is a non-semi-classical result, and over $S^*\overset{\circ}{M}$, the relevant bicharacteristic flow is that of the symbol $|\xi|^2/2$ rather than the full symbol as in [2]. Results on singularities of perturbations of the harmonic oscillator have been obtained by Zelditch [15], Weinstein [12], Fujiwara [7], Yajima [14], Kapitanski-Rodnianski-Yajima [8], and Treves [11]. Periodic recurrence of singularities for perturbations of the harmonic oscillator on $\mathbb{R}^n$ was demonstrated by Zelditch [15] and Weinstein [12], and the trace theorem (1.6) was proven by Zelditch for perturbations of the harmonic oscillator in $\mathbb{R}^n$ by potentials in $\mathcal{B}(\mathbb{R}^n)$.

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2. The quadratic-scattering calculus

In this section, we briefly review the properties of the algebra $\Psi_{\mathrm{qsc}}(M)$, which was constructed in [13], and is closely related to the "scattering algebra" of Melrose [9].

Let $\mathcal{V}_{\mathrm{qsc}}(M)$ and $\mathcal{V}_{\mathrm{b}}(M)$ be defined by (1.3) and (1.4), and let $\mathrm{Diff}_{\mathrm{qsc}}(M)$ and $\mathrm{Diff}_{\mathrm{b}}(M)$ be the order-filtered algebras of smooth linear combinations of products of elements of $\mathcal{V}_{\mathrm{qsc}}(M)$ and $\mathcal{V}_{\mathrm{b}}(M)$ respectively. There exists a bi-filtered star-algebra $\Psi_{\mathrm{qsc}}(M)$, the "quadratic-scattering calculus" of pseudodifferential operators on M such that

- $\mathrm{Diff}^m_{\mathrm{qsc}}(M) \subset \Psi^{m,0}_{\mathrm{qsc}}(M)$
- $\Psi^{m,l}_{\mathrm{qsc}}(M) = x^l \Psi^{m,0}_{\mathrm{qsc}}(M) = \Psi^{m,0}_{\mathrm{qsc}}(M) x^l$
- $\Psi^{m,l}_{\mathrm{qsc}}(M) \subset \Psi^{m',l'}_{\mathrm{qsc}}(M)$ if $m \leqslant m'$ and $l' - m' \leqslant l - m$.
- $\bigcap_{m,l} \Psi^{m,l}_{\mathrm{qsc}}(M) \equiv \Psi^{-\infty,\infty}_{\mathrm{qsc}}(M)$ consists of operators whose Schwartz kernels are smooth functions on $M \times M$, vanishing to infinite order at $\partial(M \times M)$.
- Elements of $\Psi^{0,0}_{\mathrm{qsc}}(M)$ are bounded operators on $L^2(M)$.
- Given a sequence $A_j \in \Psi^{m-j,l+j}_{\mathrm{qsc}}(M)$ for $j = 0, 1, 2, \dots$, there exists an "asymptotic sum" $A \in \Psi^{m,l}_{\mathrm{qsc}}(M)$, uniquely determined modulo $\Psi^{-\infty,\infty}_{\mathrm{qsc}}(M)$, such that $A - \sum_0^{N-1} A_j \in \Psi^{m-N,l+N}_{\mathrm{qsc}}(M)$.

Let $C_{\mathrm{qsc}}M = \partial({}^{\mathrm{qsc}}\overline{T}^*M)$. Let σ be a boundary defining function for the boundary face ${}^{\mathrm{qsc}}S^*M$ of ${}^{\mathrm{qsc}}\overline{T}^*M$ created by the fiber compactification. Let x be the lift of a boundary defining function on M to ${}^{\mathrm{qsc}}\overline{T}^*M$—thus x defines the boundary face ${}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M$. Let $\dot{\mathcal{C}}^\infty(M)$ denote smooth functions on M vanishing to infinite order at ∂M and $\mathcal{C}^{-\infty}(M)$ the dual space to $\dot{\mathcal{C}}^\infty(M)$-valued densities. Following Melrose [9], we define conormal distributions on ${}^{\mathrm{qsc}}\overline{T}^*M$ with respect to $C_{\mathrm{qsc}}M$ as follows:

$$\mathcal{A}^{p,q}({}^{\mathrm{qsc}}\overline{T}^*M) = \{u \in \mathcal{C}^{-\infty}({}^{\mathrm{qsc}}\overline{T}^*M) : \mathrm{Diff}^k_{\mathrm{b}}({}^{\mathrm{qsc}}\overline{T}^*M)u \subset \sigma^p x^q L^\infty({}^{\mathrm{qsc}}\overline{T}^*M) \text{ for all } k\};$$

here $\mathrm{Diff}^k_{\mathrm{b}}$ is defined on the manifold with corners ${}^{\mathrm{qsc}}\overline{T}^*$ exactly as it was defined on manifolds with boundary: as the span of products of vector fields tangent to (all faces of) the boundary. Let

$$\mathcal{A}^{[m,l]}(C_{\mathrm{qsc}}M) = \mathcal{A}^{m,l}({}^{\mathrm{qsc}}\overline{T}^*M)/\mathcal{A}^{m-1,l+2}({}^{\mathrm{qsc}}\overline{T}^*M).$$

There exists a symbol map

$$j_{\mathrm{qsc},m,l} : \Psi^{m,l}_{\mathrm{qsc}}(M) \to \mathcal{A}^{[-m,l-m]}(C_{\mathrm{qsc}}M)$$

such that

- There is a short exact sequence

$$0 \to \Psi_{\mathrm{qsc}}^{m-1,l+1}(M) \to \Psi_{\mathrm{qsc}}^{m,l}(M) \xrightarrow{j_{\mathrm{qsc},m,l}} \mathcal{A}^{[-m,l-m]}(C_{\mathrm{qsc}}M) \to 0. \tag{2.1}$$

- The symbol map is multiplicative.
- The Poisson bracket extends continuously from the usual bracket defined on the interior of ${}^{\mathrm{qsc}}\overline{T}^*M$ to $\mathcal{A}^{[\cdot,\cdot]}$, and

$$j_{\mathrm{qsc},m_1+m_2-1,l_1+l_2}([P,Q]) = \frac{1}{i}\left\{j_{\mathrm{qsc},m_1,l_1}(P), j_{\mathrm{qsc},m_2,l_2}(Q)\right\}.$$

Furthermore, if $a \in \mathcal{A}^{m,l}({}^{\mathrm{qsc}}\overline{T}^*M)$, $\{a,b\} = H_a(b)$ where H_a is the extension of the usual Hamilton vector field on the interior of ${}^{\mathrm{qsc}}\overline{T}^*M$ to an element of $\sigma^{-m+1}x^{l+2}\mathcal{V}_{\mathrm{b}}({}^{\mathrm{qsc}}\overline{T}^*M)$. (We refer to the flow along H_a or $\sigma^{m-1}x^{-l-2}H_a$ as bicharacteristic flow.)
- There exists a (non-unique) "quantization map"

$$\mathrm{Op} : \mathcal{A}^{-m,l-m}({}^{\mathrm{qsc}}\overline{T}^*M) \to \Psi_{\mathrm{qsc}}^{m,l}(M)$$

such that

$$j_{\mathrm{qsc},m,l}(\mathrm{Op}(a)) = [a] \in \mathcal{A}^{[-m,l-m]}(C_{\mathrm{qsc}}M).$$

Definition 2.1. An operator $P \in \Psi_{\mathrm{qsc}}^{m,l}(M)$ is said to be *elliptic* at a point $p \in C_{\mathrm{qsc}}M$ if $j_{\mathrm{qsc},m,l}$ is locally invertible near p. The set of points at which P is elliptic is denoted $\mathrm{ell}\, P$. If P is elliptic everywhere, it is simply said to be *elliptic.*

Definition 2.2. Let $P \in \Psi_{\mathrm{qsc}}^{m,l}(M)$. A point $p \in C_{\mathrm{qsc}}M$ is in the complement of $\mathrm{WF}'_{\mathrm{qsc}}P$ (the *operator wavefront set* or *microsupport* of P) if there exists $Q \in \Psi_{\mathrm{qsc}}^{-m,-l}(M)$ such that Q is elliptic at p and $PQ \in \Psi_{\mathrm{qsc}}^{-\infty,\infty}(M)$.

We can now define the *qsc wavefront set* of $u \in \mathcal{C}^{-\infty}(M)$ as the subset $\mathrm{WF}_{\mathrm{qsc}}u$ of $C_{\mathrm{qsc}}M$ such that $p \notin \mathrm{WF}_{\mathrm{qsc}}u$ if and only if there exists $A \in \Psi_{\mathrm{qsc}}^{0,0}(M)$ with $p \in \mathrm{ell}\, A$ such that $Au \in \dot{\mathcal{C}}^{\infty}(M)$.

The qsc wavefront set and microsupport enjoy the following properties:

- For $A, B \in \Psi_{\mathrm{qsc}}(M)$, $\mathrm{WF}'_{\mathrm{qsc}}AB \subset \mathrm{WF}'_{\mathrm{qsc}}A \cap \mathrm{WF}'_{\mathrm{qsc}}B$ and $\mathrm{WF}'_{\mathrm{qsc}}A^* = \mathrm{WF}'_{\mathrm{qsc}}A$.
- Microlocal parametrices exist at elliptic points: if $P \in \Psi_{\mathrm{qsc}}^{m,l}(M)$ is elliptic at $p \in C_{\mathrm{qsc}}M$ then there exists $Q \in \Psi_{\mathrm{qsc}}^{-m,-l}(M)$ such that

$$p \notin \mathrm{WF}'_{\mathrm{qsc}}(PQ - I) \text{ and } p \notin \mathrm{WF}'_{\mathrm{qsc}}(QP - I).$$

- Microlocality: let $P \in \Psi_{\mathrm{qsc}}(M)$ and $u \in \mathcal{C}^{-\infty}(M)$. Then

$$\mathrm{WF}_{\mathrm{qsc}}Pu \subset \mathrm{WF}'_{\mathrm{qsc}}P \cap \mathrm{WF}_{\mathrm{qsc}}u.$$

- Microlocal elliptic regularity: Let $P \in \Psi_{\mathrm{qsc}}(M)$ and $u \in \mathcal{C}^{-\infty}(M)$. Then

$$\mathrm{WF}_{\mathrm{qsc}}(u) \subset \mathrm{WF}_{\mathrm{qsc}}(Pu) \cup (\mathrm{ell}\, P)^c.$$

- We can (and do) choose the map Op in such a way that

$$\mathrm{WF}'_{\mathrm{qsc}}\,\mathrm{Op}(a) \subset \operatorname{ess\,supp} a$$

  ($\operatorname{ess\,supp} a$ is the set of points in $C_{\mathrm{qsc}}M$ near which a does not vanish to infinite order).

We will also require a notion of qsc wavefront set that is uniform in a parameter.

Definition 2.3. Let $u \in \mathcal{C}(\mathbb{R}; \mathcal{C}^{-\infty}(M))$. For $S \subset \mathbb{R}$ compact, we say that $p \notin \mathrm{WF}^S_{\mathrm{qsc}}(u)$ if there exists a smooth family $A(t) \in \Psi^{0,0}_{\mathrm{qsc}}(M)$ such that $A(t)$ is elliptic at p for all $t \in S$ and $Au \in \mathcal{C}(S\,; \dot{\mathcal{C}}^\infty(M))$.

Associated to $\Psi_{\mathrm{qsc}}(M)$ is a family of Sobolev spaces

$$H^{m,l}_{\mathrm{qsc}}(M) = \left\{ u \in \mathcal{C}^{-\infty}(M) : \Psi^{m,-l}_{\mathrm{qsc}}(M) u \subset L^2(M) \right\}$$

such that

- If $A \in \Psi^{m',l'}_{\mathrm{qsc}}(M)$ then

$$A : H^{m,l}_{\mathrm{qsc}}(M) \to H^{m-m',l+l'}_{\mathrm{qsc}}(M)$$

  is continuous for any m, l.
- For any $l \in \mathbb{R}$,

$$\bigcap_m H^{m,l}_{\mathrm{qsc}}(M) = \dot{\mathcal{C}}^\infty(M)$$

  and

$$\bigcup_m H^{m,l}_{\mathrm{qsc}}(M) = \mathcal{C}^{-\infty}(M).$$

- If a_n is a bounded sequence in $\mathcal{A}^{-m,l-m}(M)$ and $a_n \to a$ in some $\mathcal{A}^{p,q}(M)$, then $\mathrm{Op}(a_n) \to \mathrm{Op}(a)$ in the strong operator topology on

$$\mathcal{B}\left(H^{M,L}_{\mathrm{qsc}}(M), H^{M-m,M+l}_{\mathrm{qsc}}(M) \right)$$

  for all M, L.

3. The propagation of $\mathrm{WF}_{\mathrm{qsc}}$

For details of all computations in this section, see [13], especially §11.

We consider the symbol and corresponding bicharacteristic flow for the operator

$$\mathcal{H} = \frac{1}{2}\Delta + \frac{\omega^2}{2x^2} + v$$

where

$$v \in \mathrm{Diff}^{1,1}_{\mathrm{qsc}}(M)$$

is formally self-adjoint and x is a boundary-defining function with respect to which g takes the form (1.1).

Let $\lambda dx/x^3 + \mu \cdot dy/x^2$ be the canonical one-form on ${}^{\mathrm{qsc}}T^*M$. The joint symbol of $\mathcal{H}$ is represented in $\mathcal{A}^{[-2,-2]}(C_{\mathrm{qsc}}M)$ by a conormal distribution of the form

$$(3.1)\quad j_{\mathrm{qsc},2,0}(\mathcal{H}) = \frac{1}{2x^2}\left(\lambda^2 + |\mu|^2 + \omega^2 + xr(\lambda,\mu)\right); \qquad r(\lambda,\mu) \in \lambda^2 x\, \mathcal{C}^\infty(x,y) + \lambda\mu\, \mathcal{C}^\infty(x,y) + \mu^2\, \mathcal{C}^\infty(x,y)$$

where $|\mu|$ denotes the norm of μ with respect to the metric $\bar{h} = h|_{\partial M}$. Note that (3.1) shows that $\mathcal{H}$ is an elliptic element of $\Psi^{2,0}_{\mathrm{qsc}}(M)$; the perturbation v does not enter into the expression (3.1) as it has lower order than $\frac{1}{2}\Delta + \frac{\omega^2}{2x^2}$ in both indices. The Hamilton vector field of $\mathcal{H}$ is

$$X = \tilde{X} + P$$

where

$$(3.2)\qquad \tilde{X} = \lambda x \partial_x + (\lambda^2 - |\mu|^2 + \omega^2)\partial_\lambda + \langle \mu, \partial_y\rangle + 2\lambda\mu\cdot\partial_\mu - \frac{1}{2}\partial_y|\mu|^2\cdot\partial_\mu$$

is the Hamilton vector field for the symbol $\frac{1}{2x^2}(\lambda^2 + |\mu|^2 + \omega^2)$, and

$$(3.3)\qquad P = p_1 x^2\partial_x + p_2 x\partial_y + q_1 x\partial_\lambda + q_2 x\partial_\mu$$

is the Hamilton vector field for the "error term" $\frac{1}{2}x^{-1}r(\lambda,\mu)$. Here we adopt the convention that $\langle a, b\rangle = \sum a_i b_j \bar{h}^{ij}(y)$, and $a \cdot b = \sum a_i b_i$. The vector field P is identically zero if h is a function of y only, and always vanishes at $x = 0$.

Under the flow along $\tilde{X}$,

$$\frac{d}{dt}(\lambda + i|\mu|) = (\lambda + i|\mu|)^2 + \omega^2,$$

hence

$$(3.4)\qquad \lambda + i|\mu| = \omega\frac{\sin\omega(t-t_0) + iR\cos\omega(t-t_0)}{\cos\omega(t-t_0) - iR\sin\omega(t-t_0)}$$

for some $R \in [0,1]$. For $R > 0$, this gives a periodic orbit with period π/ω. On $\{\mu \neq 0\}$ (i.e. $R > 0$), we set $\hat{\mu} = \mu/|\mu|$, and introduce the rescaled time parameter $s = \int |\mu| dt$ to rewrite the flow along $\tilde{X}$ as

$$(3.5)\qquad \frac{dy_i}{ds} = \bar{h}^{ij}\hat{\mu}_j \qquad\qquad \frac{d\hat{\mu}_i}{ds} = -\frac{1}{2}\hat{\mu}_j\hat{\mu}_k\partial_{y_i}\bar{h}^{jk}$$

$$(3.6)\qquad \frac{d\lambda}{ds} = \frac{\lambda^2 - |\mu|^2}{|\mu|} + \omega^2 \qquad\qquad \frac{d|\mu|}{ds} = 2\lambda$$

$$(3.7)\qquad \frac{dx}{ds} = \frac{\lambda x}{|\mu|}.$$

As the set $\mu = 0$ plays an important role in the geometry of $\tilde{X}$, we give it a name:

Definition 3.1. Let $\mathcal{N} \subset {}^{\mathrm{qsc}}\overline{T}^*M$ be the set given in our coordinates by $\{x = \mu = 0\}$. Let $\mathcal{N}_\pm \subset \mathcal{N}$ be the subsets on which $\pm\lambda \geqslant 0$. Let $\mathcal{N}^c_\pm = \mathcal{N}_\pm \cap {}^{\mathrm{qsc}}S^*M$ (i.e. $\mathcal{N}^c$ is the intersection of $\mathcal{N}$ with the corner). We refer to $\mathcal{N}$ as the "normal set," with $\mathcal{N}_+$ being the "incoming normal set" and $\mathcal{N}_-$ the "outgoing normal set."

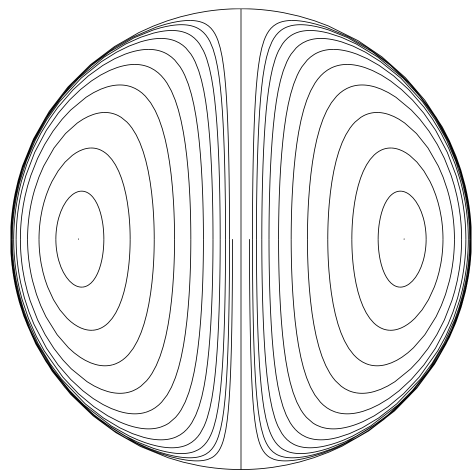

FIGURE 1. Integral curves of $\tilde{X}$, projected onto the (λ, μ) plane and radially compactified. The vertical line is the solution $\mu = 0$.

While $(\lambda, |\mu|)$ are undergoing a flow described by (3.4) (see Figure 1), then provided $R \neq 0$, (3.5) shows that $(y, \hat{\mu})$ are undergoing *unit speed geodesic flow* in ∂M with rescaled time parameter s. For $R = 0$, μ is identically zero, y is constant, and λ blows up at $t - t_0 = \pm\pi/2\omega$, i.e. the flow crosses $\mathcal{N}$ from $\mathcal{N}_-^c$ to $\mathcal{N}_+^c$ in time π/ω. More generally the integral curve starting at $\mu = 0$, $\lambda = \lambda_0$, reaches the corner at time $t = \omega^{-1}\arctan(\omega/\lambda_0)$.

Note that all terms in X are homogeneous of degree 1 in (λ, μ) except the term $\omega^2\partial_\lambda$, which is homogeneous of degree -1. If we let σ be the defining function for ${}^{\mathrm{qsc}}S^*M$ in ${}^{\mathrm{qsc}}\overline{T}^*M$ given by

$$\sigma = (\lambda^2 + |\mu|^2)^{-\frac{1}{2}}$$

and set

$$\bar{\lambda} = \sigma\lambda,$$
$$\bar{\mu} = \sigma\mu$$

then the vector field σX is tangent to the boundary of ${}^{\mathrm{qsc}}\overline{T}^*M$, and we have

$$\sigma X = \bar{\lambda}x\partial_x - |\bar{\mu}|^2\partial_{\bar{\lambda}} + \langle\bar{\mu}, \partial_y\rangle + (\bar{\lambda}\bar{\mu} - \frac{1}{2}\partial_y|\bar{\mu}|^2)\cdot\partial_{\bar{\mu}} - \bar{\lambda}\sigma\partial_\sigma + O(\sigma^2) + O(x) \tag{3.8}$$

where $O(\sigma^2)$ and $O(x)$ denote error terms of the form $\sigma^2 Y_1$ and xY_2, with Y_i tangent to $\partial({}^{\mathrm{qsc}}\overline{T}^*M)$; the $O(\sigma^2)$ term is just $\sigma\omega^2\partial_\lambda$, while the $O(x)$ term is what has above been denoted P.

The vector field X differs from the free-particle Hamilton vector-field X_{fp} described in [13][1] only in the term $\omega^2\partial_\lambda$, hence since this term is $O(\sigma)$, we have

$$\sigma X|_{{}^{\mathrm{qsc}}S^*M} = \sigma X_{\mathrm{fp}}|_{{}^{\mathrm{qsc}}S^*M}. \tag{3.9}$$

[1]Unfortunately, this vector field is called X as well in [13].

Definition 3.2. A maximally extended integral curve of σX on ${}^{\mathrm{qsc}}S^*M$ is said to be *non-trapped forward/backward* if

$$\lim_{t\to\pm\infty} x(t) = 0.$$

A point in ${}^{\mathrm{qsc}}S^*M\backslash\mathcal{N}^c$ is said to be non-trapped forward/backward if the integral curve through it is non-trapped. A point in $\mathcal{N}^c$ is said to be non-trapped forward/backward if it is not in the closure of any forward-/backward-trapped integral curves. Let $\mathcal{T}_\pm$ denote the set of forward-/backward-trapped points in ${}^{\mathrm{qsc}}S^*M$.

The only zeros of σX on ${}^{\mathrm{qsc}}S^*M$ are on the manifolds $\mathcal{N}^c_-$ (attracting) and $\mathcal{N}^c_+$ (repelling), so we can define

$$N_{\pm\infty} : {}^{\mathrm{qsc}}S^*M\backslash(\mathcal{N}_\pm \cup \mathcal{T}_\pm) \to \mathcal{N}^c_\mp$$

by

$$p \mapsto \lim_{t\to\pm\infty} \exp(t\sigma X)[p],$$

We extend this definition of $N_{\pm\infty}$ to ${}^{\mathrm{qsc}}\overline{T}^*M\backslash(\mathcal{N}_\pm \cup \mathcal{T}_\pm)$ by homogeneity. We further define

$$Y_{\pm\infty} : {}^{\mathrm{qsc}}\overline{T}^*M\backslash(\mathcal{N}_\pm \cup \mathcal{T}_\pm) \to \partial M$$

to be the projection of $N_{\pm\infty}$ to ∂M.

Theorem 3.3. *$N_{\pm\infty}$ and $Y_{\pm\infty}$ are smooth maps.*

If we let $C^\epsilon_\pm$ be the submanifold of ${}^{\mathrm{qsc}}S^*M$ given by

$$C^\epsilon_\pm = \left\{x^2 + |\bar\mu|^2 = \epsilon, \bar\lambda \gtrless 0\right\}$$

then for ϵ sufficiently small, $C^\epsilon_\mp$ is a fibration over ∂M with projection map $Y_{\pm\infty}$, and every integral curve of σX which is not trapped forward/backward passes through $C^\epsilon_\mp$;

The sets $\mathcal{T}_\pm\backslash\mathcal{N}^c_\pm$ are closed subsets of ${}^{\mathrm{qsc}}S^*M\backslash\mathcal{N}^c_\pm$.

By (3.9), this theorem follows from Theorem 11.6 of [13].

We can thus define the scattering relation:

Definition 3.4. Let $\mathcal{S} \subset \mathcal{N}^c_-\backslash\mathcal{T}_-$. The *scattering relation* on $\mathcal{S}$ is

$$\mathsf{Scat}(\mathcal{S}) = N_{-\infty}\left(N^{-1}_{+\infty}(\mathcal{S})\right) \subset \mathcal{N}^c_+.$$

It is shown in [13] that Scat takes closed sets to closed sets and Scat^{-1} takes open sets to open sets.

Example 3.5. If M is the radial compactification of $\mathbb{R}^n$ with an asymptotically Euclidian metric, we can identify the manifolds $\mathcal{N}^c_\pm$ with $S^{n-1} = \partial M$. Then for $\theta \in S^{n-1}$, $\mathsf{Scat}\,\theta$ consists of all $\theta' \in S^{n-1}$ such that there exists a geodesic γ in (uncompactified) $\mathbb{R}^n$ with $\lim_{t\to-\infty}\gamma'(t) = -\theta'$ and $\lim_{t\to+\infty}\gamma'(t) = \theta$. In other words, Scat consists of all directions in $\mathbb{R}^n$ that can scatter to the direction θ. In the Euclidian case, Scat is the antipodal map on S^{n-1}.

We now state theorems on propagation of $\mathrm{WF}_{\mathrm{qsc}}$ that will suffice to obtain results on $\operatorname{sing\,supp}\operatorname{Tr} U(t)$. (Slightly more sophisticated theorems, corresponding to Theorems 12.1-12.5 of [13], in fact hold here as well.)

Theorem 3.6 (Propagation over the boundary). *Let* $p \in ({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^\circ$ *and assume*

$$\exp(TX)[p] \in ({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^\circ.$$

Then $p \notin \mathrm{WF}_{\mathrm{qsc}}\psi(0)$ *iff there exists* $\delta > 0$ *such that* $\exp(TX)[p] \notin \mathrm{WF}^{[T-\delta,T+\delta]}_{\mathrm{qsc}}\psi$.

Theorem 3.7 (Propagation into the interior). *Let* $p \in {}^{\mathrm{qsc}}S^*M \backslash \mathcal{N}^c_-$ *be non-backward-trapped and let* $T \in (0, \pi/\omega)$. *If* $\exp(-TX)[N_{-\infty}(p)] \notin \mathrm{WF}_{\mathrm{qsc}}\psi(0)$ *then there exists* $\delta > 0$ *such that* $p \notin \mathrm{WF}^{[T-\delta,T+\delta]}_{\mathrm{qsc}}\psi$.

Theorem 3.8 (Scattering across the interior). *Let* $q \in \mathcal{N}^c_-$ *be non-backward-trapped. If*

$$\exp(-T_0X)\,[\mathsf{Scat}(q)] \cap \mathrm{WF}_{\mathrm{qsc}}\psi(0) = \emptyset$$

for some $T_0 \in (0, \pi/\omega)$, *then for every* $T \in (T_0, T_0 + \pi/\omega)$, *there exists* $\delta > 0$ *such that* $\exp((T - T_0)X)[q] \notin \mathrm{WF}^{[T-\delta,T+\delta]}_{\mathrm{qsc}}\psi$.

Theorem 3.9 (Global propagation into the boundary). *Let* $q \in \mathcal{N}^c_-$ *be non-backward-trapped. If*

$$\overline{N^{-1}_{+\infty}(q)} \cap \mathrm{WF}_{\mathrm{qsc}}\psi(0) = \emptyset$$

(*closure taken in* ${}^{\mathrm{qsc}}S^*M$) *then for* $T \in (0, \pi/\omega)$, *there exists* $\delta > 0$ *such that*

$$\exp(TX)[q] \notin \mathrm{WF}^{[T-\delta,T+\delta]}_{\mathrm{qsc}}\psi.$$

The proofs are by the same positive-commutator arguments used in [13] (which were in turn adapted from Craig-Kappeler-Strauss [5]), although the symbol constructions need to be slightly modified from those in [13] because the maps $Y_{\pm\infty}$ are not exactly constant along the flow of X; we discuss these issues in an appendix.

4. Non-recurrence of singularities

Throughout this section, we assume that there are no trapped geodesics in $\overset{\circ}{M}$.

This section is devoted proving

Theorem 4.1. *Let* S_ω *be defined by* (1.5). *For* $T \notin S_\omega$ *and for any* $p \in C_{\mathrm{qsc}}M$, *there exists an open neighborhood* $\mathcal{O}$ *of* p *and* $\epsilon > 0$ *such that if*

$$\mathrm{WF}_{\mathrm{qsc}}\psi(0) \subset \mathcal{O}$$

then

$$\mathrm{WF}^{[T-\epsilon,T+\epsilon]}_{\mathrm{qsc}}\psi \cap \mathcal{O} = \emptyset.$$

In order to deduce this theorem from Theorems 3.6–3.9, we first define a relation on $C_{\mathrm{qsc}}M$ which describes from what points singularities may reach a point $p \in C_{\mathrm{qsc}}M$:

Definition 4.2. Let $p, q \in C_{\mathrm{qsc}}M$. We write $p \overset{t}{\sim} q$ if there exists a continuous path γ from p to q in $C_{\mathrm{qsc}}M$ that is a concatenation of maximally extended integral curves of σX such that

$$\sum \Big(\text{lengths of integral curves in } {}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M\Big) = t, \tag{4.1}$$

where we define the length of an integral curve in ${}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M$ to be its length as an integral curve of X (and hence a finite number).

Then for $S \subset C_{\mathrm{qsc}}M$, let

$$\mathcal{G}_t(S) = \left\{ p \in C_{\mathrm{qsc}}M : p \overset{t}{\sim} q \text{ for some } q \in S \right\}.$$

If $p \overset{s}{\sim} q$ and $q \overset{t}{\sim} r$, then $q \overset{s+t}{\sim} r$, hence

$$\mathcal{G}_{s+t}(S) = \mathcal{G}_s \circ \mathcal{G}_t(S). \tag{4.2}$$

We also have

$$\mathcal{G}_t(S \cup T) = \mathcal{G}_t(S) \cup \mathcal{G}_t(T). \tag{4.3}$$

The relation $p \overset{t}{\sim} q$ is closed in the following sense:

Lemma 4.3. *Let $R \subset C_{\mathrm{qsc}}M \times C_{\mathrm{qsc}}M \times \mathbb{R}$ be defined by*

$$(p, q, t) \in R \text{ iff } p \overset{t}{\sim} q.$$

Then R is a closed subset of $C_{\mathrm{qsc}}M \times C_{\mathrm{qsc}}M \times \mathbb{R}$.

Proof. Suppose $p_i \to p$, $q_i \to q$, and $t_i \to t$ as $i \to \infty$, and that $(p_i, q_i, t_i) \in R$. We will show that $(p, q, t) \in R$.

For simplicity, we reformulate (4.1) as follows: let k be a Riemannian metric on the manifold $({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^\circ$ such that the norm of X with respect to k is one. (As $X = O(\sigma^{-1})$, k vanishes at ${}^{\mathrm{qsc}}S^*_{\partial M}M$.) Let $\theta = k(\cdot, X) \in \Omega^1(({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^\circ)$; extend θ to be zero on the interior of the boundary face ${}^{\mathrm{qsc}}S^*M$. Then the condition (4.1) is equivalent to

$$\int_\gamma \theta = t. \tag{4.4}$$

Now by hypothesis there exists a sequence γ_i of paths as in Definition 4.2 such that $\gamma_i(0) = p_i$, $\gamma_i(1) = q_i$, and $\int_{\gamma_i} \theta = t_i$ for all i. As the γ_i are all integral curves of σX, we apply Ascoli-Arzelà to obtain a path γ between p and q, made up of integral curves of σX with $\int_\gamma \theta = t$. □

Definition 4.4. Let

$$\mathcal{G}_t^{-1} S = \{p : \mathcal{G}_t(p) \subset S\}.$$

We now prove that $\mathcal{G}_t$ is, in an appropriate sense, a continuous set map.

Lemma 4.5. *If $K \subset \mathbb{R}$ is compact then*

$$\bigcup_{t \in K} \mathcal{G}_t$$

takes closed sets to closed sets, and

$$\bigcap_{t \in K} \mathcal{G}_t^{-1}$$

takes open sets to open sets.

Proof. Let π_L and π_R denote the projections of $C_{\mathrm{qsc}}M \times C_{\mathrm{qsc}}M \times \mathbb{R}$ onto the "left" and "right" factors of C_{qsc} and let π_t denote projection to $\mathbb{R}$. Then we can write

$$\bigcup_{t\in K} \mathcal{G}_t(S) = \pi_L(\pi_R^{-1} S \cap \pi_t^{-1} K \cap R)$$

and

$$\bigcap_{t\in K} \mathcal{G}_t^{-1}(S) = \left[\pi_R(\pi_L^{-1}(S^c) \cap \pi_t^{-1} K \cap R)\right]^c$$

hence the result follows from Lemma 4.3. □

Theorems 3.6–3.9 can now be conveniently recast as

Main propagation theorem. *If $S \subset C_{\mathrm{qsc}}M$ and*

$$\mathcal{G}_t(S) \cap \mathrm{WF}_{\mathrm{qsc}}\psi(0) = \emptyset$$

then there exists $\epsilon > 0$ such that

$$S \cap \mathrm{WF}_{\mathrm{qsc}}^{[T-\epsilon,T+\epsilon]}\psi = \emptyset.$$

Proof. By (4.3), it suffices to prove the result for $S = \{p\}$, a single point in $C_{\mathrm{qsc}}M$. By (4.2), it suffices to prove the result for small t; we take $t < \pi/\omega$ for simplicity. If

$$p \in \left({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M\right)^\circ \backslash \mathcal{N},$$

then for any t, as discussed in §3, $\mathcal{G}_t(p)$ is a single point in $({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^\circ$, and the result follows from Theorem 3.6.

Let $\arctan_+$ denote the branch of arctan taking values in $[0, \pi)$. If $p \in \mathcal{N}^\circ$, then for $t \in (0, \omega^{-1}\arctan_+(\lambda(p)/\omega))$, $\mathcal{G}_t(p)$ is again a point in $\mathcal{N}^\circ$, and again the theorem follows from Theorem 3.6. At $t = \omega^{-1}\arctan_+(\lambda(p)/\omega)$, $\exp(-tX)[p] \in \mathcal{N}_-^c$, and

$$\mathcal{G}_t(p) = \overline{N_{+\infty}^{-1}(\exp(-tX)[p])} \subset {}^{\mathrm{qsc}}S^*M,$$

hence Theorem 3.9 takes care of this case. For

$$\omega^{-1}\arctan_+(\lambda(p)/\omega) < t < \pi/\omega,$$

we once again have $\mathcal{G}_t(p) \subset ({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^\circ$, and Theorem 3.8 finishes the proof.

If, on the other hand, $p \in {}^{\mathrm{qsc}}S^*M$, $\mathcal{G}_t(p) \subset ({}^{\mathrm{qsc}}\overline{T}^*M)^\circ$ for $t \in (0, \pi/\omega)$: $\mathcal{G}_t(p)$ is a single point if $p \notin \mathcal{N}_+^c$, or a whole set, given by the scattering relation, if $p \in \mathcal{N}_+^c$. The theorem then follows from Theorem 3.7 in the former case, and Theorem 3.8 in the latter. □

The relation $\mathcal{G}_t$ is non-recurrent except at certain times:

Lemma 4.6. *For $T \notin S_\omega$ and any $p \in C_{\mathrm{qsc}}M$, there exists an open neighborhood $\mathcal{O}$ of p and $\epsilon > 0$ such that*

$$\mathcal{G}_t(\mathcal{O}) \cap \mathcal{O} = \emptyset \quad \text{for all } t \in [T-\epsilon, T+\epsilon].$$

Proof. By compactness of ∂M, S_ω is closed. Hence if $T \notin S_\omega$, there exists $\epsilon > 0$ such that

$$K = [T-\epsilon, T+\epsilon] \subset \mathbb{R}\backslash S_\omega.$$

By Lemma 4.5, $\cup_{t\in K}\mathcal{G}_t(p)$ is closed. If this set does not contain p then we can choose an open set $\mathcal{U}$ containing $\cup_{t\in K}\mathcal{G}_t(p)$ but such that $p \notin \overline{\mathcal{U}}$. By Lemma 4.5, we can then set

$$\mathcal{O} = \bigcap_{t\in K} \mathcal{G}_t^{-1}(\mathcal{U})\backslash\overline{\mathcal{U}}.$$

Thus it will suffice to prove that for $t \notin S_\omega$, $p \notin \mathcal{G}_t(p)$.

First we take the case $p \in {}^{\text{qsc}}S^*M\backslash\mathcal{N}_-^c$. Then for $t \in (0, \pi/\omega)$,

$$\mathcal{G}_t(p) = \exp(-tX)[N_{-\infty}(p)] \subset ({}^{\text{qsc}}\overline{T}^*_{\partial M}M)^\circ,$$

and this set certainly doesn't contain p. Let $\mathcal{I}$ be the involution of $\mathcal{N}^c$ swapping $\mathcal{N}_+^c$ and $\mathcal{N}_-^c$. Then

$$\mathcal{G}_{\pi/\omega}(p) = \overline{N_{+\infty}^{-1}\circ\mathcal{I}\circ N_{-\infty}(p)},$$

and this set doesn't contain p unless $Y_{+\infty}(p) = Y_{-\infty}(p)$, i.e. unless p lies on a geodesic 1-gon with vertex in ∂M. For $t \in (\pi/\omega, 2\pi/\omega)$,

$$\mathcal{G}_t(p) = \exp(-(t-\pi/\omega)X)[\mathsf{Scat}\circ\mathcal{I}\circ N_{-\infty}(p)],$$

again a subset of $({}^{\text{qsc}}\overline{T}^*_{\partial M}M)^\circ$. The set

$$\mathcal{G}_{2\pi/\omega}(p) = \overline{N_{+\infty}^{-1}\circ\mathcal{I}\circ\mathsf{Scat}\circ\mathcal{I}\circ N_{-\infty}(p)},$$

and this set certainly does contain p. Continuing in this manner, we find that if $t = n\pi/\omega + r$ with $r \in (0, \pi/\omega)$ then

$$\mathcal{G}_t(p) = \exp(-rX)(\mathsf{Scat}\circ\mathcal{I})^n N_{-\infty}(p) \subset ({}^{\text{qsc}}\overline{T}^*_{\partial M}M)^\circ,$$

while

$$\mathcal{G}_{n\pi/\omega}(p) = \overline{N_{+\infty}^{-1}\circ\mathcal{I}\circ(\mathsf{Scat}\circ\mathcal{I})^n\circ N_{-\infty}(p)},$$

hence $p \in \mathcal{G}_t(\omega)$ iff there exists a geodesic n-gon passing through p with vertices in ∂M (this is always the case for n even, as we are allowed to repeat edges).

Now we take the case $p \in ({}^{\text{qsc}}\overline{T}^*_{\partial M}M)^\circ\backslash\mathcal{N}$. The flow of X in $({}^{\text{qsc}}\overline{T}^*_{\partial M}M)^\circ\backslash\mathcal{N}$ is, as discussed in §3, given by unit speed geodesic flow in ∂M with time parameter $s = \int|\mu|dt$, while $(\lambda, |\mu|)$ undergo the motion (3.4). The only fixed-point of the $(\lambda, |\mu|)$ flow is given by $\lambda = 0$, $|\mu| = \omega$; all other orbits are periodic with period π/ω. Hence if $(\lambda(p), |\mu(p)|) \neq (0, \omega)$ and $t \notin (\pi/\omega)\mathbb{Z}$ then $p \notin \mathcal{G}_t(p)$, since the $(\lambda, |\mu|)$ coordinates distinguish between these two points. If, on the one hand, $t = n\pi/\omega$, we have by (3.4)

$$\begin{aligned} s &= \int_0^{n\pi/\omega} |\mu|\,dt \\ &= \Im\int_0^{n\pi/\omega} \omega\frac{\sin\omega(t-t_0) + iR\cos\omega(t-t_0)}{\cos\omega(t-t_0) - iR\sin\omega(t-t_0)}\,dt \\ &= n\omega\Im\int_{-\pi/2\omega}^{\pi/2\omega}\frac{\tan\omega t - iR}{1+iR\tan\omega t}\,dt \\ &= n\pi \end{aligned} \tag{4.5}$$

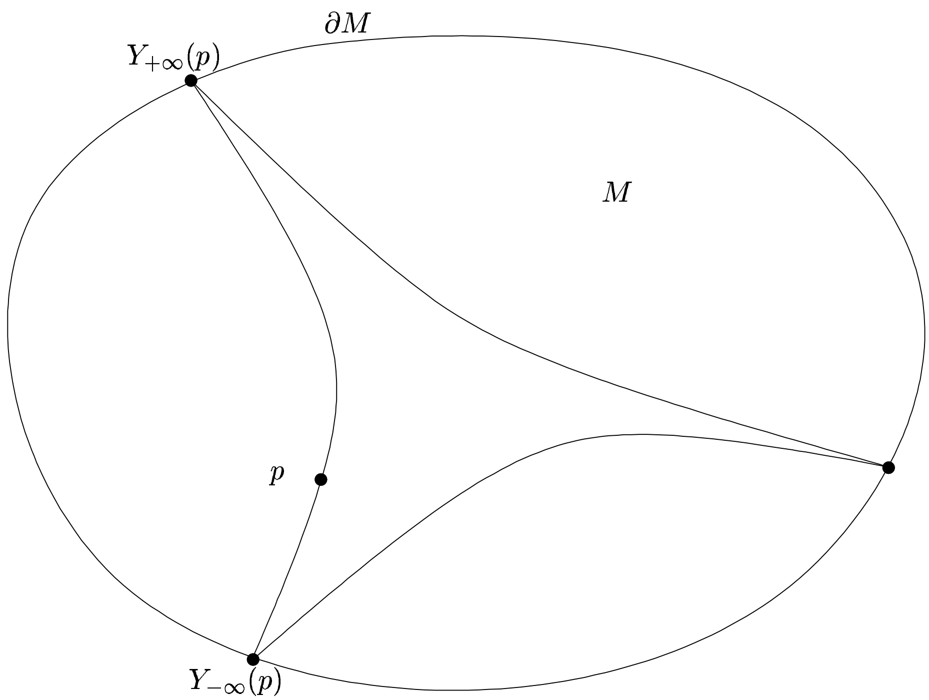

FIGURE 2. A point p on a geodesic triangle with vertices in ∂M.

(recall that $R = 0$ only on $\mathcal{N}$). Thus by (3.5), for $(\lambda, |\mu|) \neq (0, \omega)$, $p = \mathcal{G}_{n\pi/\omega}(p)$ only if there is a closed geodesic of length $n\pi$ in ∂M. On the other hand, if $(\lambda(p), |\mu(p)|) = (0, \omega)$, $(\lambda, |\mu|)$ remains constant along the flow, so $p = \mathcal{G}_t(p)$ only if there is a closed geodesic in ∂M of length ωt. This proves the result for $p \in ({}^{\text{qsc}}\overline{T}^*_{\partial M} M)^\circ \backslash \mathcal{N}$.

The proof for $p \in \mathcal{N}$ (including $\mathcal{N}^c$) proceeds like the proof for $p \in {}^{\text{qsc}}S^*M \backslash \mathcal{N}_-$; certainly if $t \notin (\pi/\omega)\mathbb{Z}$, $p \notin \mathcal{G}_t(p)$, as λ is constant on $\mathcal{G}_t(p)$ at fixed t, and equals $\lambda(p)$ only for $t \in (\pi/\omega)\mathbb{Z}$. The same geometrical discussion used in the proof for points in $({}^{\text{qsc}}S^*M)^\circ$ also shows that $p \notin \mathcal{G}_{n\pi/\omega}(p)$ unless there is a geodesic n-gon with vertices in ∂M, with one vertex at $y(p)$. □

Proof of Theorem 4.1. The theorem follows directly from the Main Propagation Theorem and Lemma 4.6. □

From Theorem 4.1, we deduce the following, which is the key result for our trace theorem.

Corollary 4.7. *Given $T \notin S_\omega$, there exists $\epsilon > 0$, $k \in \mathbb{Z}_+$, and $A_i \in \Psi^{0,0}_{\text{qsc}}(M)$, $i = 1, \dots, k$ such that*

$$A_i U_\omega(t) A_i \in \mathcal{C}^\infty([T - \epsilon, T + \epsilon]\,;\, \Psi^{-\infty,\infty}_{\text{qsc}}(M))$$

and

$$I = \sum_{i=1}^{k} A_i^2 + R$$

(I denotes the identity operator) with $R \in \Psi_{\mathrm{qsc}}^{-\infty,\infty}(M)$.

Proof. By Theorem 4.1, we can find a partition of unity $(b_{1,i})^2$, subordinate to a cover $\mathcal{O}_i$ of $C_{\mathrm{qsc}}M$, such that $\mathrm{WF}_{\mathrm{qsc}}\psi(0) \subset \mathcal{O}_i$ implies that $\mathrm{WF}_{\mathrm{qsc}}^{[T-\epsilon,T+\epsilon]}\psi \cap \mathcal{O}_i = \emptyset$. Extend the $b_{1,i}$ to be smooth functions on ${}^{\mathrm{qsc}}\overline{T}^*M$ with $\operatorname{ess\,supp} b_{1,i} \subset \mathcal{O}_i$. Set $B_{1,i} = \mathrm{Op}(b_{1,i})$. Then

$$\sum_i B_{1,i}^2 - I = C_1 \in \Psi_{\mathrm{qsc}}^{-1,1}(M).$$

Let c_1 denote a representative of the symbol of C_1 in $\mathcal{A}^{1,2}({}^{\mathrm{qsc}}\overline{T}^*M)$. Setting $b_{2,i} = -c_1 b_{1,i}/2$ and $B_{2,i} = \mathrm{Op}(b_{2,i})$, we have

$$\sum_i (B_{1,i} + B_{2,i})^2 - I = C_2 \in \Psi_{\mathrm{qsc}}^{-2,2}(M).$$

Now let c_2 represent the symbol of C_2, set $b_{3,i} = -c_2 b_{1,i}/2$ and $B_{3,i} = \mathrm{Op}(b_{3,i})$, and continue in this manner, defining $B_{j,i}$ inductively. Then use asymptotic summation to obtain $A_i \sim \sum_j B_{j,i}$, with $\mathrm{WF}'_{\mathrm{qsc}} A_i \subset \mathcal{O}_i$ and $I = \sum A_i^2 + R$ with $R \in \Psi_{\mathrm{qsc}}^{-\infty,\infty}(M)$.

By our construction of $\mathcal{O}_j$, for all $i = 1, \dots, k$ we have

$$\mathrm{WF}'_{\mathrm{qsc}} A_i \cap \mathrm{WF}_{\mathrm{qsc}}^{[T-\epsilon,T+\epsilon]} U(t) A_i \psi(0) = \emptyset$$

for $t \in [T-\epsilon, T+\epsilon]$, hence by microlocality, $\mathrm{WF}_{\mathrm{qsc}} A_i U(t) A_i \psi(0) = \emptyset$ for any $\psi(0) \in \mathcal{C}^{-\infty}(M)$, i.e. $A_i U(t) A_i \in \mathcal{C}([T-\epsilon, T+\epsilon]\,;\, \Psi_{\mathrm{qsc}}^{-\infty,\infty}(M))$. Smoothness in t follows similarly, as

$$D_t^k A_i U(t) A_i = A_i (-\mathcal{H})^k U(t) A_i,$$

and since $\mathcal{H} \in \Psi_{\mathrm{qsc}}(M)$,

$$\mathrm{WF}_{\mathrm{qsc}}^{[T-\epsilon,T+\epsilon]} (-\mathcal{H})^k U(t) A_i \psi(0) \subset \mathrm{WF}_{\mathrm{qsc}}^{[T-\epsilon,T+\epsilon]} U(t) A_i \psi(0). \quad \square$$

5. The trace

We begin the study of $\operatorname{Tr} U(t)$ by showing that it exists as a distribution:

Proposition 5.1. *For* $\phi \in \mathcal{S}(\mathbb{R})$,

$$\int \phi(t) U(t)\, dt \in \Psi_{\mathrm{qsc}}^{-\infty,\infty}(M)$$

and

$$\phi \mapsto \operatorname{Tr} \int \phi(t) U(t)\, dt$$

is a tempered distribution on $\mathbb{R}$.

Proof. The structure of the argument is standard—see, for example, part II of [2]. We reproduce it only owing to the slight novelty of the Sobolev spaces involved.

Choose $\kappa \in \mathbb{R}$ below the spectrum of $\mathcal{H}$. Then by ellipticity of $\mathcal{H}$,

$$(\kappa + \mathcal{H})^{-k} : H_{\mathrm{qsc}}^{0,0}(M) \to H_{\mathrm{qsc}}^{2k,0}(M).$$

Since

$$U(t) = (\kappa + \mathcal{H})^k (\kappa + \mathcal{H})^{-k} U(t) = (\kappa - D_t)^k (\kappa + \mathcal{H})^{-k} U(t),$$

we can write

$$\int \phi(t)U(t)\,dt = \int (\kappa - D_t)^k \phi(t)(\kappa + \mathcal{H})^{-k} U(t)\,dt. \tag{5.1}$$

$U(t)$ is unitary on $H^{0,0}_{\mathrm{qsc}}(M)$, so

$$(\kappa + \mathcal{H})^{-k} U(t) : H^{0,0}_{\mathrm{qsc}}(M) \to H^{2k,0}_{\mathrm{qsc}}(M)$$

is bounded uniformly in t. Since $\cap_k H^{2k,0}_{\mathrm{qsc}}(M) = \dot{\mathcal{C}}^\infty(M)$, (5.1) shows that

$$\int \phi(t)U(t)\,dt : \mathcal{C}^{-\infty}(M) \to \dot{\mathcal{C}}^\infty(M),$$

i.e.

$$\int \phi(t)U(t)\,dt \in \Psi^{-\infty,\infty}_{\mathrm{qsc}}(M).$$

Furthermore, if we take k large enough so that $(\kappa + \mathcal{H})^{-k}U(t)$ is trace-class, we see that $\phi \mapsto \operatorname{Tr} \int \phi(t)U(t)\,dt$ is a tempered distribution of order at most k. □

We are now in a position to prove our main theorem:

Theorem 5.2. *If there are no trapped geodesics in* $\overset{\circ}{M}$ *then*

$$\operatorname{sing\,supp} \operatorname{Tr} U(t) \subset S_\omega.$$

Proof. Let $\phi \in \mathcal{C}^\infty(\mathbb{R})$ be 0 for $x > 2$ and 1 for $x < 1$. Set

$$W_n = \operatorname{Op}[(1 - \phi(nx))(1 - \phi(n\sigma))] \in \Psi^{0,0}_{\mathrm{qsc}}(M);$$

then $W_n \to I$ strongly on $L^2(M)$. We regularize $\operatorname{Tr} U(t)$ by examining instead

$$\operatorname{Tr} U(t)W_n;$$

this is a smooth function on $\mathbb{R}$ since $D_t^p \operatorname{Tr} U(t)W_n = \operatorname{Tr}(-\mathcal{H})^p U(t)W_n$.

Given $T \notin S_\omega$, we choose A_i, $i = 1, \dots, k$ as in Corollary 4.7, and write

$$\operatorname{Tr} U(t)W_n = \operatorname{Tr} IU(t)W_n = \sum_{i=1}^k \operatorname{Tr} A_i^2 U(t)W_n + \operatorname{Tr} RU(t)W_n.$$

$A_iU(t)W_n$ is trace-class, so we may now rewrite

$$\operatorname{Tr} U(t)W_n = \sum_{i=1}^k \operatorname{Tr} A_iU(t)W_nA_i + \operatorname{Tr} RU(t)W_n.$$

As $n \to \infty$, $D_t^p RU(t)W_n$ converges to $D_t^p RU(t)$ in the norm topology on operators $H^{m,l}_{\mathrm{qsc}}(M) \to H^{m',l'}_{\mathrm{qsc}}(M)$ for any m, l, m', l', and any $p \in \mathbb{Z}_+$; thus $\operatorname{Tr} RU(t)W_n$ approaches a smooth function as $n \to \infty$. Thus, if we can also show that

1. $\lim_{n\to\infty} \operatorname{Tr} U(t)W_n = \operatorname{Tr} U(t)$, and
2. $\lim_{n\to\infty} \operatorname{Tr} A_iU(t)W_nA_i = \operatorname{Tr} A_iU(t)A_i$ for all $i = 1, \dots, k$,

in the sense of distributions, we will have $\operatorname{Tr} U(t) \in \mathcal{C}^\infty([T-\epsilon, T+\epsilon])$ for some $\epsilon > 0$, and we will be done.

Both (1) and (2) follow from the following identity, which holds, in the distributional sense, for any $A \in \Psi^{p,q}_{\mathrm{qsc}}(M)$ (and any p, q):

$$\lim_{n\to\infty} \operatorname{Tr} AU(t)W_nA = \operatorname{Tr} AU(t)A.$$

To prove this, let $\phi \in \mathcal{S}(\mathbb{R})$ be a test function, let κ lie below the spectrum of $\mathcal{H}$, and write

$$\begin{aligned}
\lim_{n\to\infty} \int \phi(t) \operatorname{Tr} AU(t)W_nA\,dt \\
&= \lim_{n\to\infty} \operatorname{Tr} \int \phi(t)U(t)W_nA^2\,dt \\
&= \lim_{n\to\infty} \operatorname{Tr} \int \phi(t)(\kappa - D_t)^m(\kappa+\mathcal{H})^{-m}U(t)W_nA^2\,dt \\
&= \lim_{n\to\infty} \int [(\kappa - D_t)^m\phi(t)] \operatorname{Tr}\left[(\kappa+\mathcal{H})^{-m}U(t)W_nA^2\right]\,dt \\
&= \lim_{n\to\infty} \int [(\kappa - D_t)^m\phi(t)] \operatorname{Tr}\left[A(\kappa+\mathcal{H})^{-m}U(t)W_nA\right]\,dt \\
&= \int [(\kappa - D_t)^m\phi(t)] \operatorname{Tr}\left[A(\kappa+\mathcal{H})^{-m}U(t)A\right]\,dt \\
&= \int \phi(t) \operatorname{Tr} AU(t)A\,dt;
\end{aligned}$$

here we take m large enough that $(\kappa+\mathcal{H})^{-m}U(t)$ is trace-class; the penultimate equality follows from the norm convergence

$$A(\kappa+\mathcal{H})^{-m}U(t)W_nA \to A(\kappa+\mathcal{H})^{-m}U(t)A$$

as operators $H^{0,0}_{\mathrm{qsc}}(M) \to H^{2m-2p-\epsilon,2q-\epsilon}_{\mathrm{qsc}}(M)$ for all p, q and all $\epsilon > 0$. □

Appendix: the propagation theorems

As noted above, the only obstacle to proving Theorems 3.6–3.9 in exactly the same manner as Theorems 12.1–12.5 of [13] is the fact that $(Y_{\pm\infty})_*X \neq 0$ in the harmonic oscillator case; we merely have

$$(Y_{\pm\infty})_*X = O(\sigma).$$

This makes no difference in proving Theorems 3.6 or 3.8, but we must modify the constructions of the symbols $a_\pm$ and $\tilde{a}_\pm$ used to prove the other three theorems.

We modify the symbols $a_+^{m,l}$ and $\tilde{a}_+^{m,l}$ defined in §13 of [13] by replacing the factor $\psi_{-\infty} = \phi(d(Y_{-\infty}(p), y_0))$ (ϕ is a cutoff function) by

$$\tilde{\psi}_{-\infty} = \phi(d(Y_{-\infty}(p), y_0)^2 - \epsilon\sigma).$$

Since $X\sigma = -\bar{\lambda} + O(\sigma^2) + O(x) = -1 + O(\sigma^2) + O(x) + O(|\bar{\mu}|^2)$ and since $(Y_{-\infty})_*X = O(\sigma)$,

$$-X(\tilde{\psi}_{-\infty}) = -\phi'(d(Y_{-\infty}(p), y_0)^2 - \epsilon\sigma)\left[O(\sigma) + \epsilon + O(\sigma^2) + O(x) + O(|\bar{\mu}|^2)\right].$$

The quantity in square brackets is strictly positive for $x, \sigma, \bar{\mu}$ sufficiently small, and the constructions of a_+ and $\tilde{a}_+$ in [13] go through as before, with $\tilde{\psi}_{-\infty}$ replacing $\psi_{-\infty}$, and b_+ constructed so as to ensure that σ is small on $\operatorname{supp} a_+$.

Similarly, in the construction of a_- and $\tilde{a}_-$, we replace $\psi_{+\infty}(q) = \phi(d(Y_{+\infty}(q)), y_0)$ with

$$\tilde{\psi}_{+\infty}(q) = \phi(d(Y_{+\infty}(q), y_0)^2 + \epsilon\sigma).$$

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E-mail address: `jwunsch@math.harvard.edu`

Department of Mathematics, Harvard University, 1 Oxford St. Rm. 325, Cambridge MA 02138