SPREADING OF LAGRANGIAN REGULARITY ON RATIONAL INVARIANT TORI

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Abstract. Let P_h be a self-adjoint semiclassical pseudodifferential operator on a manifold M such that the bicharacteristic flow of the principal symbol on T^*M is completely integrable and the subprincipal symbol of P_h vanishes. Consider a semiclassical family of eigenfunctions, or, more generally, quasimodes u_h of P_h. We show that on a nondegenerate rational invariant torus, Lagrangian regularity of u_h (regularity under test operators characteristic on the torus) propagates both along bicharacteristics, and also in an additional "diffractive" manner. In particular, in addition to propagating along null bicharacteristics, regularity fills in the interiors of small annular tubes of bicharacteristics.

1. Introduction

It is a well-known fact of semiclassical microlocal analysis, that the analogue of Hörmander's theorem on propagation of singularities for operators of real principal type [7] holds for the semiclassical wavefront set (also known as "frequency set"): it propagates along null bicharacteristics of operators with real principal symbol [6, 12]. Given a Lagrangian submanifold $\mathcal{L}$ of T^*M, we may introduce a finer notion of regularity, the local *Lagrangian* regularity along $\mathcal{L}$. We show here that on rational invariant tori in integrable systems, local Lagrangian regularity not only propagates along bicharacteristics, but spreads in additional ways as well.

Let P_h be a semiclassical pseudodifferential operator on a manifold M, with real principal symbol p (this is automatic if P is self-adjoint). Assume that the bicharacteristic flow of p is completely integrable. (In fact we only need to assume integrability *locally,* near one invariant torus.) Let u_h be a family of quasimodes of P_h, i.e. assume that $\|(P_h - \lambda)u_h\|_{L^2} = O(h^N)$ for some $N \in \mathbb{N}$, as $h \downarrow 0$ either through a discrete sequence or continuously. (Note that this certainly includes the possibility of letting u_h be a sequence of actual eigenfunctions). Let $\mathcal{L}$ be an invariant torus in the characteristic set $\{p = \lambda\}$. Then the bicharacteristic flow is by definition tangent to $\mathcal{L}$, and we show (even in the absence of the integrability hypothesis) that Lagrangian regularity propagates along bicharacteristics—this is Theorem A below. If a single trajectory is dense in $\mathcal{L}$, then this is the whole story for propagation, as the set on which Lagrangian regularity holds is open, hence the whole torus either enjoys Lagrangian regularity or none of it does. At

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the opposite extreme, if $\mathcal{L}$ is a torus on which all frequencis of the motion are rationally related, we may ask the finer question: what subsets of the space of all orbits may carry Lagrangian regularity? The answer (assuming a nondegeneracy condition holds) turns out to be somewhat constrained: given a single orbit, Lagrangian regularity along a small tube around it implies Lagrangian regularity along the orbit itself. This is our Theorem B. (In the special case of two-dimensional tori, we can go further: again, either the whole torus enjoys Lagrangian regularity or no points on it do.) The order of regularity up to which our result holds is constrained by the order of the quasimode. We speculate that a finer theorem may be obtainable by more authentically "second-microlocal" methods.

Example 1. As a simple example of our main result, Theorem B, we consider the case $M = S^1_x \times S^1_y$, $P_h = h^2\Delta = -h^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$; we consider Lagrangian regularity on the Lagrangian torus $\mathcal{L} = \{\xi = 0, \eta = 1\}$ for quasimodes satisfying

$$(h^2\Delta - 1)u_h \in h^{k+1}L^2(S^1 \times S^1).$$

Lagrangian regularity on this particular $\mathcal{L}$ is special in that we may test for it using powers of the *differential* operator $D_x = i^{-1}(\partial/\partial x)$. The theorem tells us the following in this case: let $\Upsilon(x)$ be a smooth cutoff function supported on $\{|x| \in [\epsilon, 3\epsilon]\}$ and nonzero at $\pm 2\epsilon$. Let ϕ be another cutoff, nonzero at the origin and supported in $[-2\epsilon, 2\epsilon]$. If, for all $k' \leq k$, we have

$$\left\| D_x^{k'}(\Upsilon(x)u_h) \right\| \leq C < \infty,$$

then for all $k' \leq k$,

$$\left\| D_x^{k'}(\phi(x)u_h) \right\| \leq \tilde{C} < \infty,$$

i.e. the D_x^k regularity fills in the "hole" in the support of Υ. In this special case, the result can be proved directly by employing a positive commutator argument using only differential operators; the positive commutator will arise from the usual commutant $h^{-1}xD_x$.

A less trivial example, that of the spherical pendulum, is discussed in §3 below.

The methods of proof (and the idea of the paper) arose from work of Burq-Zworski [4, 5] and a subsequent refinement by Burq-Hassell-Wunsch [3] on the spreading of L^2 mass for quasimodes on the Bunimovich stadium. The central argument here is a generalization of the methods used to prove that a quasimode cannot concentrate too heavily in the interior of the rectangular part of the stadium (which is essentially the example discussed above on $M = S^1 \times S^1$).

We remark that our hypotheses in this paper are quite far from those in the study of "quantum integrable systems" where one examines eigenfunctions of a system of n commuting operators on an n-manifold. For instance, if we take $P_h = h^2\Delta + h^2 V$ on the torus, with V a real valued, smooth bump-function, then the operator P_h satisfies the hypotheses of our Theorems A and B, and yet there does not exist a system of $n-1$ other operators commuting with P_h, with independent symbols. Moreover, even in the completely integrable case, given that we study eigenfunctions of a single operator, it may be possible to use the degeneracy of the system to construct non- or partially-Lagrangian quasimodes. Little seems to be known in this direction.

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2. Lagrangian Regularity

We begin by setting some notation and recalling some concepts of semiclassical analysis. For detailed background on this subject, we refer the reader to [6, 12].

Let M^n be a smooth manifold and fix $\mathcal{L} \subset T^*M$ a Lagrangian submanifold. Throughout the rest of the paper, we assume[1] $u_h \in L^2(M; \Omega^{1/2})$, with $h \in (0, h_0)$; here $\Omega^{1/2}$ denotes the bundle of *half-densities* on M, i.e. the square root of the density bundle $|\Lambda^n M|$. We will in future, however, suppress the half-density nature of u_h as well as its h-dependence, writing simply $u \in L^2(M)$; similarly, all operators will tacitly be semiclassical families of operators, operating on half-densities. The hypothesis that our operators act on half-densities ensures that if $A = \operatorname{Op}_h(a)$ with $a(x, \xi; h) \sim a_0(x, \xi) + h a_1(x, \xi) + \dots$, the terms a_0 (principal symbol) and a_1 (subprincipal symbol) are both invariantly defined as functions on T^*M (see [6]).

Furthermore, we will deal with an operator P rather than $P - \lambda$, absorbing the constant term into the definition of the operator.

We begin by defining a notion of Lagrangian regularity of a family of functions along $\mathcal{L}$, following the treatment of the "homogeneous" case in [9].

Definition 2. Let $\mathcal{M}$ denote the module (over $\Psi_h(M)$) of semiclassical pseudodifferential operators with symbols vanishing on $\mathcal{L}$.

Let $q \in \mathcal{L}$, $k \in \mathbb{N}$, and $u \in L^2(M)$. We say that u has Lagrangian regularity of order k at q, and write $q \in S^k_{\mathcal{L}}(u)$, if and only if there is a neighborhood U of q in T^*M such that for all $k' = 0, 1, \dots, k$ and all $A_1, \dots, A_{k'} \in \mathcal{M}$ with $\mathrm{WF}' A_j \subset U$, $h^{-k'} A_1 \cdots A_{k'} u \in L^2(M)$.

Proposition 3. *Fix $q \in \mathcal{L}$, and let A_i ($i = 1, \dots, n$) be a collection of elements of $\mathcal{M}$ with $d\sigma(A_i)$ spanning $N^*_q\mathcal{L}$. We have*

$$q \in S_{\mathcal{L}}(u) \Longleftrightarrow h^{-k'} A_{i_1} \cdots A_{i_{k'}} u \in L^2 \quad \forall (i_1, \dots, i_{k'}) \in \{1, \dots, n\}^{k'},\ k' \le k.$$

[1] We may just as well assume that $h \downarrow 0$ through a discrete sequence; this will make no difference in what follows.

Proof. We begin with the case $k = 1$. Given any B characteristic on $\mathcal{L}$ and microsupported sufficiently close to q, we may factor $\sigma(B) = \sum c_i \sigma(A_i)$ by Taylor's theorem. Thus, letting C_i be operators with symbol c_i, we obtain

$$h^{-1}Bu = \sum h^{-1}C_iA_iu + Ru$$

for some semiclassical operator R, hence we obtain the desired estimate on $h^{-1}Bu$ since R is uniformly (in h) L^2-bounded.

More generally, if $B_{\alpha_1}, \dots, B_{i_k}$ is a k-tuple of operators characteristic on $\mathcal{L}$, we have

$$h^{-k}B_{i_1}\cdots B_{i_k}u = h^{-k}\prod_{j=1}^{k}(C_{i_j}A_{i_j} + hR_{i_j})u;$$

We then obtain the desired estimate inductively, using the fact that each commutator of the form $[C, A]$ or $[R, A]$ produces a further factor of h. $\square$

We note that it follows from the work of Alexandrova [1] that $S_{\mathcal{L}} = \mathcal{L}$ if and only if we can actually write u in the form of an oscillatory integral

$$\int a(x, \theta, h)e^{i\phi(x,\theta)/h}\,d\theta$$

plus a term with semiclassical wavefront set away from $\mathcal{L}$; here ϕ is a phase function parametrizing the Lagrangian $\mathcal{L}$ in the sense introduced by Hörmander. This is the semiclassical analog of a central result in the Hörmander-Melrose theory of conic Lagrangian distributions [9, Chapter 25].

We now observe that the analogue of Hörmander's theorem on propagation of singularities for operators of real principal type is easy to prove in our setting.

Theorem A. *Let $P \in \Psi_h(M)$ have real principal symbol p. Let $\mathcal{L} \subset \{p = 0\}$ be a Lagrangian submanifold of T^*M. Then $Pu \in h^{k+1}L^2(M)$ implies that $S^k_{\mathcal{L}}(u)$ is invariant under the Hamilton flow of p.*

The author is grateful to M. Zworski for suggesting the following brief proof.

Sketch. By [8, Theorem 21.1.6], there is a local symplectomorphism taking p to ξ_1 and $\mathcal{L}$ to $\mathcal{L}_0 \equiv \{\xi = 0\}$. Following the development in [1], we may quantize this to a semiclassical FIO that conjugates P to hD_{x^1} modulo $O(h^\infty)$ (cf. [9, Theorem 26.1.3] in the non-semiclassical setting). Lagrangian regularity along $\mathcal{L}_0$ is iterated regularity under $h^{-1}(hD_{x^i})$, i.e. is just classical Sobolev regularity, uniform in h. The theorem thus reduces to the statement that Sobolev regularity for solutions to $D_{x^1}u \in h^kL^2(M)$ propagates along the lines $(x^1 \in \mathbb{R}, x' = \text{const})$, which is easily verified. $\square$

3. Integrable flow

We continue to assume that $P \in \Psi_h(M)$ has real principal symbol. We now further assume that $p = \sigma(P)$ has *completely integrable* bicharacteristic flow, i.e. that there exist functions $f_2, \dots, f_n$ on T^*M, Poisson commuting with p and with each other, and with $dp, df_2, \dots, df_n$ pointwise linearly independent. We again emphasize that we in fact only require the f_i's to exist in some open subset of interest in T^*M. Let Σ denote the characteristic set in T^*M. Let $(I_1, \dots, I_n, \theta_1, \dots, \theta_n)$ be action-angle variables and let $\omega_i = \partial p/\partial I_i$ be the frequencies. We also let $\omega_{ij} = \partial^2 I/\partial I_i \partial I_j$. (We refer the reader to [2] for an account of the theory of integrable systems, and in particular for a treatment of action-angle variables.)

Let $\mathcal{L} \subset \Sigma$ be a *rational* invariant torus, i.e. one on which $\omega_i/\omega_j \in \mathbb{Q}$ for all $i, j = 1, \dots, n$. We further assume that $\mathcal{L}$ is *nondegenerate* in the following sense: we assume that the matrix

$$\begin{pmatrix} \omega_{11} & \dots & \omega_{1n} & \omega_1 \\ \vdots & \ddots & \vdots & \vdots \\ \omega_{n1} & \dots & \omega_{nn} & \omega_n \\ \omega_1 & \dots & \omega_n & 0 \end{pmatrix} \tag{1}$$

is invertible on $\mathcal{L}$. This is precisely the condition of *isoenergetic nondegeneracy* often used in KAM theory (see [2], Appendix 8D). It is easy to verify that the condition is equivalent to the condition that the map from the energy surface to the projectivization of the frequencies

$$\{p = 0\} \ni I \mapsto [\omega_1(I) : \dots : \omega_n(I)] \in \mathbb{RP}^n$$

be a local diffeomorphism.

For later convenience, we introduce special notation for the frequencies and their derivatives on $\mathcal{L}$: we let

$$\overline{\omega}_i = \omega|_{\mathcal{L}}, \quad \overline{\omega}_{ij} = \omega_{ij}|_{\mathcal{L}}.$$

On $\mathcal{L}$, we of course only know from Theorem A that $S^k_{\mathcal{L}}(u)$ is a union of orbits of H_p, which, being rational, are not dense in $\mathcal{L}$. There are, however, further constraints on $S^k_{\mathcal{L}}(u)$.

Definition 4. An *annular neighborhood* of a closed orbit ρ is an open set $U = V \backslash K \subset \mathcal{L}$ such that $\rho \subset K \subset V$ with K compact and V open in $\mathcal{L}$.

We can now state our main result.

Theorem B. *Suppose $Pu = f \in h^{k+1}L^2$. Let ρ be a null bicharacteristic for p on the rational invariant torus $\mathcal{L}$. If a small enough annular neighborhood of ρ is in $S^k_{\mathcal{L}}(u)$, then so is ρ.*

The meaning of "small enough" depends only on the $\overline{\omega}_i$'s.

If $n = 2$ then either $S_{\mathcal{L}}(u) = \mathcal{L}$ or $S_{\mathcal{L}}(u) = \emptyset$.

Thus, conormal regularity propagates "diffusively" to fill in annular neighborhoods.

Example 5. Horozov [10, 11] has studied the *spherical pendulum*, i.e. the system on T^*S^2 with Hamiltonian $h = (1/2)|\xi|^2 + x_3$ on T^*S^2 (with x_3 one of the Euclidean coordinates on $S^2 \subset \mathbb{R}^3$). Integrals of motion are h and p_θ, the angular momentum. Horozov showed that when $h \in (-1, 1] \cup [7/\sqrt{17}, \infty)$, all values of p_θ lead to isoenergetically nondegenerate invariant tori, while for $h \in (1, 7/\sqrt{17})$, there are exactly two values of p_θ for which isoenergetic nondegeneracy fails. Thus our results show that if we consider quasimodes for the operator

$$P_h = (1/2)h^2 \Delta_{S^2} + x_3$$

then for any torus $\mathcal{L}$ not associated to the one of the exceptional pairs of (h, p_θ) identified by Horozov, either $S_{\mathcal{L}} = \mathcal{L}$ or $S_{\mathcal{L}} = \emptyset$.

Example 6. We now illustrate with an example the necessity of the isoenergetic nondegeneracy condition. As in the introduction, let $M = S^1 \times S^1$, but now let $P = hD_x$; it is easy to verify that *no* Lagrangian torus is isoenergetically nondegenerate in this case. Let $\mathcal{L} = \{\xi = \eta = 0\}$, the zero-section of T^*M. Lagrangian regularity in this setting is, as noted above, just Sobolev regularity, uniform in h.

Let $\psi(y)$ be a bump function supported near $y = 0$. Then

$$u(x, y) = e^{i\psi(y)/\sqrt{h}}$$

has wavefront set only in $\mathcal{L}$. It is manifestly Lagrangian on the complement of $\operatorname{supp}\psi$, which forms an annular neighborhood of the orbit $\{x \in S^1, y = 0, \xi = \eta = 0\} \subset \operatorname{supp}\psi$. It is not Lagrangian, however, on $\operatorname{supp}\psi$, as it lacks iterated regularity under $h^{-1}(hD_y)$.

4. Symbol Construction

By shifting coordinates, we may assume that ρ is the orbit passing through $\{\theta = 0\}$.

For each i, j let

$$\gamma_{ij} = \widetilde{\min}_{k,l\in\mathbb{Z}}((\theta_i + 2\pi k)\overline{\omega}_j - (\theta_j + 2\pi l)\overline{\omega}_i),$$

where $\widetilde{\min}$ denotes the value with the smallest norm, i.e. may be positive or negative. Each γ_{ij} then takes values in an interval determined by $\overline{\omega}_i$, $\overline{\omega}_j$, and is smooth where it takes on values in the interior of the interval. (If $\overline{\omega}_i = p/q$ and $\overline{\omega}_j = p'/q'$ then γ_{ij} takes values in $[-\pi a, \pi a]$ where $a = \gcd(qp', pq')/qq'$.) The "small enough" condition in the statement of Theorem B is just the following: each γ_{ij} should be smooth on the annular neighborhood of ρ where we assume Lagrangian regularity.

Note that $\gamma_{ij}(\theta) = 0$ for all i, j exactly when there exists $\tilde{\theta} \in \mathbb{R}^n$, equivalent to θ modulo $2\pi\mathbb{Z}^n$, such that $[\tilde{\theta}_1 : \dots \tilde{\theta}_n] = [\overline{\omega}_1 : \dots : \overline{\omega}_n]$. Thus the functions γ_{ij} define ρ on $\mathcal{L}$: we have $\{I = \overline{I},\ \gamma_{ij} = 0\ \forall i, j\} = \rho$; indeed, the vanishing of each $\gamma_{i,i+1}$ and of $\gamma_{n,1}$ suffices to define ρ, and these n functions may be taken as coordinates on $\mathcal{L}$ in a neighborhood of ρ. The central point of our argument will be that the γ_{ij} are "propagating variables" with derivatives along the flow that, taken together, will suffice to give Lagrangian regularity.

Since the γ_{kl} define ρ and are smooth on the annular neighborhood U where we have assumed regularity, there is a smooth cutoff function

$$\psi := \psi(\gamma_{12},, \gamma_{23}, \dots, \gamma_{n-1,n}, \gamma_{n,1})$$

with $\psi = 1$ on ρ and $\nabla\psi$ having its support on $\mathcal{L}$ contained in U. We may also arrange for ψ to be the square of a smooth function. Let ϕ_ϵ be a cutoff supported in $[-\epsilon, \epsilon]$, with smooth square root.

Let

$$a_{ij}(x) = \psi \cdot \phi_\epsilon(\left|I - \overline{I}\right|) \cdot \gamma_{ij} \cdot (\omega_i \overline{\omega}_j - \omega_j \overline{\omega}_i) \tag{2}$$

We compute first that, where $\gamma_{ij} \in \mathcal{C}^\infty$,

$$\{p, \gamma_{ij}\} = (\omega_i \overline{\omega}_j - \omega_j \overline{\omega}_i) \tag{3}$$

(since γ_{ij} is locally given by expressions of the form $((\theta_i + 2\pi k)\overline{\omega}_j - (\theta_j + 2\pi l)\overline{\omega}_i)$ with k, l *fixed*) and hence that

$$\{p, a_{ij}\} = \{p, \psi\}\phi_\epsilon(\left|I - \overline{I}\right|)\gamma_{ij}(\omega_i \overline{\omega}_j - \omega_j \overline{\omega}_i) + \psi\phi_\epsilon(\left|I - \overline{I}\right|)(\omega_i \overline{\omega}_j - \omega_j \overline{\omega}_i)^2.$$

We further note that as ψ is a function of the γ_{ij}'s, by (3) the first term in this expression is a sum of terms divisible by

$$(\omega_{k_1}\overline{\omega}_{l_1} - \omega_{l_1}\overline{\omega}_{k_1})(\omega_{k_2}\overline{\omega}_{l_2} - \omega_{l_2}\overline{\omega}_{k_2})$$

for various k_i, l_i. Thus we may write

$$\{p, a_{ij}\} = \sum e_k f_l + \psi\phi_\epsilon(\left|I - \overline{I}\right|)(\omega_i \overline{\omega}_j - \omega_j \overline{\omega}_i)^2, \tag{4}$$

where each e_k and f_l vanishes on $\mathcal{L}$ and with support intersecting $\mathcal{L}$ only in U.

We will also employ a symbol that is *invariant* under the flow: for each $j = 1, \dots, n$, set

$$w_j = \phi_\epsilon(\left|I - \overline{I}\right|)I_j.$$

5. Nondegeneracy

Using a positive commutator argument, we will find that we can control operators whose symbols are multiples of $(\omega_i \overline{\omega}_j - \omega_j \overline{\omega}_i)$. These quantities vanish on $\mathcal{L}$, but our nondegeneracy hypothesis permits us to use them to control Lagrangian regularity on $\mathcal{L}$. To see this, rewrite

$$(\omega_i \overline{\omega}_j - \omega_j \overline{\omega}_i) = (\omega_i - \overline{\omega}_i)\overline{\omega}_j - (\omega_j - \overline{\omega}_j)\overline{\omega}_i$$

and expand about $\mathcal{L}$ in the I variables, to rewrite this as

$$\sum_k (\overline{\omega}_{ik}\overline{\omega}_j - \overline{\omega}_{jk}\overline{\omega}_i)(I_k - \overline{I}_k) + O((I - \overline{I})^2)$$

We now prove a key algebraic lemma:

Lemma 7. *Let $v_1, \dots, v_n$, and v_{ij}, $i, j = 1, \dots, n$ be real numbers, with $v_{ij} = v_{ji}$. The functionals $\alpha_{ij}(x) = \sum_k (v_{ik}v_j - v_{jk}v_i)x_k$ (for $i, j = 1, \dots, n$) together with the covector $(v_1, \dots, v_n)$ span $(R^n)^*$ iff the matrix*

$$\begin{pmatrix} v_{11} & \dots & v_{1n} & v_1 \\ \vdots & \ddots & \vdots & \vdots \\ v_{n1} & \dots & v_{nn} & v_n \\ v_1 & \dots & v_n & 0 \end{pmatrix} \tag{5}$$

is nondegenerate.

Proof of Lemma. We may assume that not all of the v_i's are zero, as the result is trivial in that case.

Let

$$\vec{\zeta}_{ij} = \begin{pmatrix} v_{1i}v_j - v_{1j}v_i \\ \vdots \\ v_{ni}v_j - v_{nj}v_i \end{pmatrix}.$$

Letting A be the matrix with entries v_{ij} and

$$\vec{u}_{ij} = v_j e_i - v_i e_j$$

where e_i is the standard basis for $\mathbb{R}^n$, we have

$$\vec{\zeta}_{ij} = A\vec{u}_{ij}.$$

Let U denote the span of the $\vec{u}_{ij}$'s. Thus,

$$U^\perp = \bigcap_{i,j} \vec{u}_{ij}^\perp = \bigcap_{i,j} \mathbb{R} \cdot \{\vec{w} \in \mathbb{R}^n | [w_i : w_j] = [v_i : v_j] \ \forall i, j\} = \mathbb{R}\vec{v}$$

where $\vec{v} = (v_1, \dots v_n)^t$. Thus, $U = \vec{v}^\perp$. Hence the span of the $\vec{\zeta}_{ij}$ is of $A(\vec{v}^\perp)$. The assertion of the lemma is then that $A(\vec{v}^\perp)$ and $\vec{v}$ are complementary iff the matrix (5) is nondegenerate. This equivalence follows from the observation that

$$\begin{pmatrix} A & \vec{v} \\ \vec{v}^t & 0 \end{pmatrix} \cdot \begin{pmatrix} \vec{w} \\ z \end{pmatrix} = \begin{pmatrix} A\vec{w} + z\vec{v} \\ \langle \vec{v}, \vec{w} \rangle, \end{pmatrix}$$

hence (5) has nontrivial nullspace iff there exists a nonzero $\vec{w} \in \vec{v}^\perp$ with $A\vec{w} \in \mathbb{R}\vec{v}$. □

6. Proof of Theorem B

We note, first of all, that in the special case when $n = 2$, a neighborhood of any closed orbit $\rho' \neq \rho$ is itself an annular neighborhood of ρ. Hence the special result for $n = 2$ follows directly from the general one.

We now prove Theorem B by induction on k; we suppose it true for $k \leq K - 1$ (and note that for $k = 0$ it is vacuous).

Let $A_{ij} \in \Psi_h(M)$ be self-adjoint, with symbol a_{ij} constructed above and vanishing subprincipal symbol. Then we have by (4),

$$ih^{-3}[P, A_{ij}] = h^{-2}B_{ij}^2 + \sum_{k,l} h^{-2}E_k F_l + R \tag{6}$$

with B_{ij} self-adjoint with vanishing subprincipal symbols, and

$$\sigma(B_{ij}) = b_{ij} = (\psi\phi_\epsilon(|I - \overline{I}|))^{1/2} \cdot (\omega_i\overline{\omega}_j - \omega_j\overline{\omega}_i), \tag{7}$$

and with E_k, F_l characteristic on $\mathcal{L}$ with the supports of $\sigma(E_k)$, $\sigma(F_l)$ intersecting $\mathcal{L}$ only in U. (R, E_k, and F_l of course depend on i, j but we suppress these extra indices.)

Let W_j have symbol w_j constructed above, and be self-adjoint with vanishing subprincipal symbol. Then

$$ih^{-3}[P, W_j] \in \Psi_h(M).$$

For a multi-index α with $|\alpha| = K - 1$, set

$$Q_{ij} = A_{ij} W_1^{2\alpha_1} \dots W_n^{2\alpha_n},$$

We will also need the operator denoted in multi-index notation

$$W^\alpha = W_1^{\alpha_1} \dots W_n^{\alpha_{K-1}}$$

Now we examine

$$ih^{-2K-1}\langle (P^*Q_{ij} - Q_{ij}Pu, u\rangle = ih^{-2K-1}(\langle Q_{ij}u, f\rangle - \langle f, Q_{ij}^*u\rangle). \tag{8}$$

For any $\delta > 0$, we may estimate the RHS by

$$C_\delta \|h^{-K-1}f\|^2 + \delta(\|h^{-K}Q_{ij}u\|^2 + \|h^{-K}Q_{ij}^*u\|^2).$$

Note that both Q_{ij} and Q_{ij}^* are $(2K+1)$-fold products of operators vanishing on M, and that each contains the factors A_{ij} and W^α. By (2) and (7), $\sigma(A_{ij})$ is divisible by $\sigma(B_{ij})$; thus, by elliptic regularity we may estimate the RHS by

$$C_\delta \|h^{-K-1}f\|^2 + C\delta \sum_\alpha \|h^{-K}B_{ij}W^\alpha u\|^2 + \sum_{j=0}^{2K-2} h^{-j}\langle D_j u, u\rangle$$

where C is independent of δ, and each D_j is a sum of products of j elements of $\mathcal{M}$, all microsupported on $\operatorname{supp} A_{ij}$; these arise from commutator terms in which we have reordered products of elements of $\mathcal{M}$.

Now we recall that $P^* - P \in h^2\Psi_h(M)$, hence, by (6) we may write

$$\begin{aligned} &\langle ih^{-2K-1}(P^*Q_{ij} - Q_{ij}P)u, u\rangle \\ &= \|h^{-K}B_{ij}W^\alpha u\|^2 + \sum_{k,l} \langle h^{-K}E_kW^\alpha u, h^{-K}F_lW^\alpha u\rangle + \sum_{j=0}^{2K-2} h^{-j}\langle \tilde{D}_j u, u\rangle \end{aligned} \tag{9}$$

with the $\tilde{D}_j$ sharing the properties of the D_j above.

Putting together the information from our commutator, we now have, for all $\delta > 0$,

$$(10)\quad (1-C\delta)\big\|h^{-2K}B_{ij}W^\alpha u\big\|^2$$

$$\leq C_\delta\big\|h^{-K-1}f\big\|^2+\sum_{k,l}(\big\|h^{-K}E_kW^\alpha u\big\|^2+\big\|h^{-K}F_lW^\alpha u\big\|^2)+\sum_{j=0}^{2K-2}h^{-j}\Big|\Big\langle\tilde{\tilde{D}}_j u,u\Big\rangle\Big|,$$

with the $\tilde{\tilde{D}}_j$'s satisfying the same properties as D_j above. Each of the E_k and F_l terms is controlled by our hypothesis on $S^k_{\mathcal{L}}u$, while the $\tilde{\tilde{D}}_j$ terms are bounded by the inductive assumption.

Now we use our nondegeneracy hypothesis as reflected in Lemma 7. Recall that $\mathcal{L}\subset\Sigma$, hence the operator P is characteristic on $\mathcal{L}$; moreover, we have $dp|_{\mathcal{L}}=\sum\overline{\omega}_k dI_k$, hence Lemma 7 tells us that P and B_{ij}, for $i,j=1,\dots,n$, are a collection of operators fulfilling the hypotheses of Proposition 3. Thus, adding together equations (10) for all possible values of i,j, and multi-index α, together with terms involving P rather than B_{ij} (which vanish up to commutators of P with W's), we obtain the desired estimate, by Proposition 3. □

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