Schrödinger evolution and geometry

Jared Wunsch
Northwestern University

MFO Oberwolfach
Analysis and Geometric Singularities 2007

Advertisement

MSRI Program "Analysis of Singular Spaces"
August 18, 2008 to December 19, 2008

Organized By: G. Carron, E. Hunsicker, R. Melrose, M. Taylor, A. Vasy, J. Wunsch

Positions available for:
Postdocs (7), deadline Dec. 1
Research Professors, Oct. 1
Members, Dec. 1

www.msri.org

Also workshops (open to grad students):

Broader Connections: Analysis of Singular Spaces
August 28, 2008 to August 29, 2008

Introductory Workshop on Analysis of Singular Spaces
September 2, 2008 to September 5, 2008

Elliptic and Hyperbolic Equations on Singular Spaces
October 27, 2008 to October 31, 2008

Schrödinger equation

Consider the Schrödinger equation

$$(\frac{1}{i}\partial_t + (1/2)\Delta + V)\psi = 0$$
$$\psi|_{t=0} = \psi_0 \in L^2(X).$$

on $\mathbb{R} \times X$, where (X, g) is a Riemannian manifold, V is real-valued, and

$$\Delta = \Delta_g = d^* d$$

is (nonnnegative) Laplace-Beltrami operator on X.
Describes (nonrelativistic) quantum evolution of a particle moving on X.
Question: how do geometry of X and behavior of V influence qualitative behavior of solutions? (Will focus mostly on $V = 0$.)

Schrödinger equation

Consider the Schrödinger equation

$$(\frac{1}{i}\partial_t + (1/2)\Delta + V)\psi = 0$$
$$\psi|_{t=0} = \psi_0 \in L^2(X).$$

on $\mathbb{R} \times X$, where (X, g) is a Riemannian manifold, V is real-valued, and

$$\Delta = \Delta_g = d^* d$$

is (nonnnegative) Laplace-Beltrami operator on X.
Describes (nonrelativistic) quantum evolution of a particle moving on X.
Question: how do geometry of X and behavior of V influence qualitative behavior of solutions? (Will focus mostly on $V = 0$.)

$\mathbb{R}^n$

To see that *global* geometry plays a strong role, consider the fundamental solution in two settings. Easy to write down fundamental solution on $\mathbb{R}^n$:

$$K_{\mathbb{R}^n}(t,x,y) = (2\pi i t)^{-n/2} e^{i(x-y)^2/2t}.$$

Some obvious observations:

- K is in C^∞ for $t > 0$.
- K is oscillatory for $t > 0$ at spatial infinity.

Less obvious:

- H^s and L^p mapping properties in spacetime
- propagation of singularities

$\mathbb{R}^n$

To see that *global* geometry plays a strong role, consider the fundamental solution in two settings. Easy to write down fundamental solution on $\mathbb{R}^n$:

$$K_{\mathbb{R}^n}(t,x,y) = (2\pi i t)^{-n/2} e^{i(x-y)^2/2t}.$$

Some obvious observations:

- K is in $\mathcal{C}^\infty$ for $t > 0$.
- K is oscillatory for $t > 0$ at spatial infinity.

Less obvious:

- H^s and L^p mapping properties in spacetime
- propagation of singularities

$\mathbb{R}^n$

To see that *global* geometry plays a strong role, consider the fundamental solution in two settings. Easy to write down fundamental solution on $\mathbb{R}^n$:

$$K_{\mathbb{R}^n}(t, x, y) = (2\pi i t)^{-n/2} e^{i(x-y)^2/2t}.$$

Some obvious observations:

- K is in $\mathcal{C}^\infty$ for $t > 0$.
- K is oscillatory for $t > 0$ at spatial infinity.

Less obvious:

- H^s and L^p mapping properties in spacetime
- propagation of singularities

S^1

By contrast, consider fundamental solution on S^1 :

$$K_{S^1}(t,x,y) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{2\pi^2 in^2 t + 2\pi in(x-y)} = \vartheta(x-y; 2\pi t)$$

i.e. Jacobi theta function evaluated on boundary of halfplane of definition ($\vartheta(z;\tau)$ analytic on $\operatorname{Im}\tau > 0$).

Not smooth in spacetime anywhere; neither is restriction to diagonal $x = y$.
Cf. Kapitanski-Rodnianski, 1997 *Does a quantum particle know the time?* for subtle changes in Besov regularity of restrictions to different times.

One moral of this story: compact manifolds are harder than noncompact, and in particular, trapped geodesics are hard.

S^1

By contrast, consider fundamental solution on S^1 :

$$K_{S^1}(t,x,y) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{2\pi^2 in^2 t + 2\pi in(x-y)} = \vartheta(x-y; 2\pi t)$$

i.e. Jacobi theta function evaluated on boundary of halfplane of definition ($\vartheta(z;\tau)$ analytic on $\operatorname{Im}\tau > 0$).

Not smooth in spacetime anywhere; neither is restriction to diagonal $x = y$.
Cf. Kapitanski-Rodnianski, 1997 *Does a quantum particle know the time?* for subtle changes in Besov regularity of restrictions to different times.

One moral of this story: compact manifolds are harder than noncompact, and in particular, trapped geodesics are hard.

S^1

By contrast, consider fundamental solution on S^1 :

$$K_{S^1}(t,x,y) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{2\pi^2 i n^2 t + 2\pi i n(x-y)} = \vartheta(x-y; 2\pi t)$$

i.e. Jacobi theta function evaluated on boundary of halfplane of definition ($\vartheta(z;\tau)$ analytic on $\operatorname{Im}\tau > 0$).

Not smooth in spacetime anywhere; neither is restriction to diagonal $x = y$.
Cf. Kapitanski-Rodnianski, 1997 *Does a quantum particle know the time?* for subtle changes in Besov regularity of restrictions to different times.

One moral of this story: compact manifolds are harder than noncompact, and in particular, trapped geodesics are hard.

Maybe can hope to generalize some of what we know on Euclidean space to other manifolds. A candidate situation might be scattering or asymptotically conic manifolds.

Introduced by Melrose in context of geometric scattering theory.

Noncompact manifold X with compactification $\overline{X}$, ends that look like large ends of cones:
Neighborhood of $\partial\overline{X}$ is parametrized by $(r_0, \infty)_r \times \partial\overline{X}$ with metric

$$g = dr^2 + r^2 h(r^{-1}, \theta, d\theta)$$

θ are coordinates in $\partial\overline{X}$, h a smooth family of metrics on $\partial\overline{X}$.

Includes asymptotically Euclidean space ($r = |x|$, $\theta \in S^{n-1}$)! Also allows $\mathbb{R}^n$ with a metric that's asymptotically non-round on "sphere at infinity."

Maybe can hope to generalize some of what we know on Euclidean space to other manifolds. A candidate situation might be scattering or asymptotically conic manifolds.

Introduced by Melrose in context of geometric scattering theory.

Noncompact manifold X with compactification $\overline{X}$, ends that look like large ends of cones:
Neighborhood of $\partial\overline{X}$ is parametrized by $(r_0, \infty)_r \times \partial\overline{X}$ with metric

$$g = dr^2 + r^2 h(r^{-1}, \theta, d\theta)$$

θ are coordinates in $\partial\overline{X}$, h a smooth family of metrics on $\partial\overline{X}$.

Includes asymptotically Euclidean space ($r = |x|$, $\theta \in S^{n-1}$)! Also allows $\mathbb{R}^n$ with a metric that's asymptotically non-round on "sphere at infinity."

Maybe can hope to generalize some of what we know on Euclidean space to other manifolds. A candidate situation might be scattering or asymptotically conic manifolds.

Introduced by Melrose in context of geometric scattering theory.

Noncompact manifold X with compactification $\overline{X}$, ends that look like large ends of cones:
Neighborhood of $\partial\overline{X}$ is parametrized by $(r_0, \infty)_r \times \partial\overline{X}$ with metric

$$g = dr^2 + r^2 h(r^{-1}, \theta, d\theta)$$

θ are coordinates in $\partial\overline{X}$, h a smooth family of metrics on $\partial\overline{X}$.

Includes asymptotically Euclidean space ($r = |x|$, $\theta \in S^{n-1}$)! Also allows $\mathbb{R}^n$ with a metric that's asymptotically non-round on "sphere at infinity."

Trapping

Since S^1 seemed a bit pathological, probably we don't like closed geodesics.

More generally, let γ be a geodesic.

Definition

γ is *forward/backward trapped* if $\lim_{t\to\pm\infty} r\circ\gamma(t) \neq \infty$.
γ is *trapped* if it is both forward and backward trapped.

Estimates

On any manifold, of course,

$$\|\psi\|_{L^2(X)}$$

is conserved under evolution.

More generally, for A a "reasonable" operator,

$$\begin{aligned}\partial_t \langle A\psi, \psi\rangle &= \left\langle A\dot{\psi}, \psi\right\rangle + \left\langle A\psi, \dot{\psi}\right\rangle \\ &= \langle A(-i/2)\Delta\psi, \psi\rangle + \langle A\psi, (-i/2)\Delta\psi\rangle \\ &= (i/2)\langle [\Delta, A]\psi, \psi\rangle\end{aligned}$$

("Heisenberg equation" for time-evolution of expected value of A.)

Estimates

On any manifold, of course,

$$\|\psi\|_{L^2(X)}$$

is conserved under evolution.

More generally, for A a "reasonable" operator,

$$\begin{aligned}\partial_t \langle A\psi, \psi\rangle &= \left\langle A\dot{\psi}, \psi \right\rangle + \left\langle A\psi, \dot{\psi} \right\rangle \\ &= \langle A(-i/2)\Delta\psi, \psi\rangle + \langle A\psi, (-i/2)\Delta\psi\rangle \\ &= (i/2)\langle [\Delta, A]\psi, \psi\rangle\end{aligned}$$

("Heisenberg equation" for time-evolution of expected value of A.)

Heisenberg, continued

$$\partial_t \langle A\psi, \psi\rangle = (i/2)\langle [\Delta, A]\psi, \psi\rangle$$

With $A = 1$, this gives norm conservation. $A = \Delta^s$ gives conservation of norm in $\dot{H}^s$.

Can also use if know we control $\langle A\psi, \psi\rangle$ to get

$$(1/2)\left|\int_0^T \langle [\Delta, A]\psi, \psi\rangle \, dt\right| \le |\langle A\psi, \psi\rangle_T| + |\langle A\psi, \psi\rangle_0|$$

Heisenberg, continued

$$\partial_t \langle A\psi, \psi \rangle = (i/2)\langle [\Delta, A]\psi, \psi \rangle$$

With $A = 1$, this gives norm conservation. $A = \Delta^s$ gives conservation of norm in $\dot{H}^s$.

Can also use if know we control $\langle A\psi, \psi \rangle$ to get

$$(1/2)\left| \int_0^T \langle [\Delta, A]\psi, \psi \rangle \, dt \right| \leq |\langle A\psi, \psi \rangle_T| + |\langle A\psi, \psi \rangle_0|$$

Flat space Morawetz

On $\mathbb{R}^n$ (say, with $n > 3$) use test operator $A = \Delta^{-1/4} D_r \Delta^{-1/4}$ (with $D_r = i^{-1}\partial_r$):
Write

$$\Delta \approx D_r^2 + \frac{\Delta_\theta}{r^2}$$

So compute

$$i[\Delta, D_r] = 2\frac{\Delta_\theta}{r^3} + \text{ lower order}$$

Main term is positive operator!
Hence

$$\langle A\psi, \psi\rangle|_0^T = \int_0^T \left\| r^{-3/2} \Delta_\theta^{1/2} (\Delta^{-1/4}\psi) \right\|^2 dt + \text{ error terms.}$$

Flat space Morawetz

On $\mathbb{R}^n$ (say, with $n > 3$) use test operator $A = \Delta^{-1/4} D_r \Delta^{-1/4}$ (with $D_r = i^{-1}\partial_r$):
Write

$$\Delta \approx D_r^2 + \frac{\Delta_\theta}{r^2}$$

So compute

$$i[\Delta, D_r] = 2\frac{\Delta_\theta}{r^3} + \text{ lower order}$$

Main term is positive operator!
Hence

$$\langle A\psi, \psi\rangle|_0^T = \int_0^T \left\| r^{-3/2}\Delta_\theta^{1/2}(\Delta^{-1/4}\psi)\right\|^2 dt + \text{ error terms.}$$

Flat space Morawetz

On $\mathbb{R}^n$ (say, with $n > 3$) use test operator $A = \Delta^{-1/4} D_r \Delta^{-1/4}$ (with $D_r = i^{-1}\partial_r$):
Write

$$\Delta \approx D_r^2 + \frac{\Delta_\theta}{r^2}$$

So compute

$$i[\Delta, D_r] = 2\frac{\Delta_\theta}{r^3} + \text{ lower order}$$

Main term is positive operator!
Hence

$$\langle A\psi, \psi\rangle|_0^T = \int_0^T \left\| r^{-3/2} \Delta_\theta^{1/2} (\Delta^{-1/4}\psi) \right\|^2 dt + \text{ error terms.}$$

Morawetz continued

Let $\nabla\!\!\!/\psi = r^{-1}\nabla_\theta$.

Since A is of order 0, $\langle A\psi, \psi\rangle$ is controlled by $\|\psi_0\|^2$, so we obtain

$$\int_0^T \left\| r^{-1/2} \nabla\!\!\!/ (\Delta^{-1/4}\psi) \right\|^2 dt \lesssim \|\psi_0\|^2.$$

(Cf. Morawetz (1968) for analogous identity for Klein-Gordon.)

Morawetz continued

Let $\not\nabla \psi = r^{-1} \nabla_\theta$.

Since A is of order 0, $\langle A\psi, \psi \rangle$ is controlled by $\|\psi_0\|^2$, so we obtain

$$\int_0^T \left\| r^{-1/2} \not\nabla (\Delta^{-1/4} \psi) \right\|^2 dt \lesssim \|\psi_0\|^2.$$

(Cf. Morawetz (1968) for analogous identity for Klein-Gordon.)

Local smoothing, etc.

By translation invariance, any derivative is a tangential derivative (locally)!
So in fact for any $W \Subset \mathbb{R}^n$,

$$\int_0^T \|\psi\|^2_{H^{1/2}(W)}\, dt \lesssim \|\psi_0\|^2.$$

(Cf. Kato—"local smoothing estimate" for KdV; for Schrödinger, Constantin-Saut, Sjölin, Vega.)

To put it concisely,

$$\psi_0 \in L^2 \Longrightarrow \psi \in L^2_{\text{loc}} H^{1/2}_{\text{loc}}.$$

Note in particular, that ψ must be in $H^{1/2}_{\text{loc}}$ for almost every t, but certainly not for every t!

Local smoothing, etc.

By translation invariance, any derivative is a tangential derivative (locally)!
So in fact for any $W \Subset \mathbb{R}^n$,

$$\int_0^T \|\psi\|^2_{H^{1/2}(W)}\, dt \lesssim \|\psi_0\|^2.$$

(Cf. Kato—“local smoothing estimate” for KdV; for Schrödinger, Constantin-Saut, Sjölin, Vega.)

To put it concisely,

$$\psi_0 \in L^2 \Longrightarrow \psi \in L^2_{\mathrm{loc}} H^{1/2}_{\mathrm{loc}}.$$

Note in particular, that ψ must be in $H^{1/2}_{\mathrm{loc}}$ for almost every t, but certainly not for every t!

More robust

For a global $H^{1/2}$ estimate, and for a more robust proof, have to work harder:

Note that the symbolic version of the above computation is:

$$\{(1/2)|\xi|^2, \sigma(A)\} \geq 0$$

since $H_{(1/2)|\xi|^2} = \xi \cdot \partial_x$, with $a = \hat{\xi} \cdot \hat{x}$, we find that the Poisson bracket is exactly

$$\frac{|\xi|}{|x|}(1 - (\hat{\xi} \cdot \hat{x})^2)$$

which is (barely) nonnegative, vanishing on radial set.

This is of course geometrically delicate!

Smoothing on manifolds

More robust: for $\epsilon > 0$,

$$\{(1/2)|\xi|^2, |x|^{-\epsilon}(\hat{\xi} \cdot \hat{x})\}$$

has a mixed sign, but is positive on incoming set where $\hat{\xi} \cdot \hat{x} \approx -1$.

It turns out that we can microlocalize this argument on the incoming set, and then propagate it across to outgoing.

Theorem (Craig-Kappeler-Strauss, 1995)

On asymptotically Euclidean manifolds, microlocally away from trapped rays,

$$\int_0^T \left\| \langle r \rangle^{-1/2-\epsilon} \Delta^{1/4} \psi \right\|^2 dt \lesssim \|\psi_0\|^2.$$

Also holds in asymptotically conic setting.

Smoothing on manifolds

More robust: for $\epsilon > 0$,

$$\{(1/2)|\xi|^2, |x|^{-\epsilon}(\hat{\xi} \cdot \hat{x})\}$$

has a mixed sign, but is positive on incoming set where $\hat{\xi} \cdot \hat{x} \approx -1$.

It turns out that we can microlocalize this argument on the incoming set, and then propagate it across to outgoing.

Theorem (Craig-Kappeler-Strauss, 1995)

On asymptotically Euclidean manifolds, microlocally away from trapped rays,

$$\int_0^T \left\| \langle r \rangle^{-1/2-\epsilon} \Delta^{1/4} \psi \right\|^2 dt \lesssim \|\psi_0\|^2.$$

Also holds in asymptotically conic setting.

Other geometric settings

Theorem (Doi, 1996)

Local smoothing holds on nontrapping manifolds with a wide variety of end structures, including asymptotically hyperbolic manifolds. But $L^2 H^{1/2}$ local smoothing fails microlocally near any trapped ray, i.e.

$$\psi \notin L^2_{loc} H^{1/2}_{loc} \text{ there.}$$

Of course, on a compact manifold, we cannot be in $L^2 H^s$ for any $s > 0$ with L^2 initial data.

What happens, e.g., on asymptotically Euclidean space with *mild* trapping? Do we get some intermediate $L^2 H^s$ estimate?

Other geometric settings

Theorem (Doi, 1996)

Local smoothing holds on nontrapping manifolds with a wide variety of end structures, including asymptotically hyperbolic manifolds. But $L^2 H^{1/2}$ local smoothing fails microlocally near any trapped ray, i.e.

$$\psi \notin L^2_{loc} H^{1/2}_{loc} \text{ there.}$$

Of course, on a compact manifold, we cannot be in $L^2 H^s$ for any $s > 0$ with L^2 initial data.

What happens, e.g., on asymptotically Euclidean space with *mild* trapping? Do we get some intermediate $L^2 H^s$ estimate?

Trapping

Theorem (Burq, 2003)

Let $\Omega \subset \mathbb{R}^n$ be union of two strictly convex obstacles. Let ψ solve the Schrödinger equation with Dirichlet boundary conditions in $\mathbb{R}^n \backslash \Omega$. Then

$$\int_0^T \|\psi\|_{H^{1/2-\epsilon}_{loc}} \leq C\|\psi_0\|^2.$$

i.e. local smoothing holds with epsilon derivative loss. (Can do several obstacles, with some extra hypotheses guaranteeing hyperbolicity of flow.) Proof uses cut-off resolvent estimate of Ikawa.

Theorem (Christianson, 2007)

Same holds on asymptotically Euclidean space with a single trapped hyperbolic orbit. Also weighted estimate ($\langle r\rangle^{-1/2-\epsilon}$ weight as before).

More complicated trapping

Consider now more complicated trapping situation: assume (X, g) is asymptotically Euclidean, analytic near ∞, and assume that the geodesic flow is hyperbolic on the energy surface, and that the topological pressure $P(1/2)$ is negative.

Hypotheses tell us that trapped set is a rather filamentary fractal. In the case $n = 2$, $P(1/2) < 0 \iff$ the Hausdorff dimension of the fixed-energy trapped set is less than 3.

Theorem (Christianson (+others), 2007)

Then there is still $L^2 H^{1/2-\epsilon}$ local smoothing for the Schrödinger equation.

Main ingredient is a resolvent estimate of Nonnenmacher-Zworski, following techniques of Anantharaman and Anantharaman-Nonnenmacher developed for studying entropy of limit measures (quantum chaos).

More complicated trapping

Consider now more complicated trapping situation: assume (X, g) is asymptotically Euclidean, analytic near ∞, and assume that the geodesic flow is hyperbolic on the energy surface, and that the topological pressure $P(1/2)$ is negative.

Hypotheses tell us that trapped set is a rather filamentary fractal. In the case $n = 2$, $P(1/2) < 0 \iff$ the Hausdorff dimension of the fixed-energy trapped set is less than 3.

Theorem (Christianson (+others), 2007)

Then there is still $L^2 H^{1/2-\epsilon}$ local smoothing for the Schrödinger equation.

Main ingredient is a resolvent estimate of Nonnenmacher-Zworski, following techniques of Anantharaman and Anantharaman-Nonnenmacher developed for studying entropy of limit measures (quantum chaos).

More complicated trapping

Consider now more complicated trapping situation: assume (X, g) is asymptotically Euclidean, analytic near ∞, and assume that the geodesic flow is hyperbolic on the energy surface, and that the topological pressure $P(1/2)$ is negative.

Hypotheses tell us that trapped set is a rather filamentary fractal. In the case $n = 2$, $P(1/2) < 0 \iff$ the Hausdorff dimension of the fixed-energy trapped set is less than 3.

Theorem (Christianson (+others), 2007)

Then there is still $L^2 H^{1/2-\epsilon}$ local smoothing for the Schrödinger equation.

Main ingredient is a resolvent estimate of Nonnenmacher-Zworski, following techniques of Anantharaman and Anantharaman-Nonnenmacher developed for studying entropy of limit measures (quantum chaos).

More complicated trapping

Consider now more complicated trapping situation: assume (X, g) is asymptotically Euclidean, analytic near ∞, and assume that the geodesic flow is hyperbolic on the energy surface, and that the topological pressure $P(1/2)$ is negative.

Hypotheses tell us that trapped set is a rather filamentary fractal. In the case $n = 2$, $P(1/2) < 0 \iff$ the Hausdorff dimension of the fixed-energy trapped set is less than 3.

Theorem (Christianson (+others), 2007)

Then there is still $L^2 H^{1/2-\epsilon}$ local smoothing for the Schrödinger equation.

Main ingredient is a resolvent estimate of Nonnenmacher-Zworski, following techniques of Anantharaman and Anantharaman-Nonnenmacher developed for studying entropy of limit measures (quantum chaos).

Global in time

Global in time (Tataru, Rodnianski-Tao): if have compactly supported smooth, nontrapping perturbation of $\mathbb{R}^n$, and $V = 0$, then

$$\int_{-\infty}^{\infty} \left\| \langle r \rangle^{-1/2-\epsilon} \nabla \psi \right\|^2 + \left\| \langle r \rangle^{-3/2-\epsilon} \psi \right\|^2 dt \lesssim \|\psi_0\|_{\dot{H}^{1/2}}.$$

Uses absence of resonance/eigenvalue at zero for Δ (which leads to a Poincaré inequality), as well as absence of imbedded eigenvalues.

Sharp weights

Sharp weights (Sugimoto, Hassell-Tao-Wunsch). Remember that

$$\int_0^T \left\| r^{-1/2} \nabla\!\!\!/ \psi \right\|^2 dt \lesssim \|\psi_0\|^2_{H^{1/2}};$$

but if we want radial derivatives (globally) then need weight $r^{-1/2-\epsilon}$.

Can do a *bit* better though. On asymptotically conic manifold, for each i, j,

$$\int_0^T \left\langle r^{-1/2} \nabla\!\!\!/_i \psi, r^{-1/2} \nabla_j \psi \right\rangle dt \lesssim \|\psi_0\|^2_{H^{1/2}};$$

Sharp weights

Sharp weights (Sugimoto, Hassell-Tao-Wunsch). Remember that

$$\int_0^T \left\| r^{-1/2} \not\nabla \psi \right\|^2 dt \lesssim \|\psi_0\|_{H^{1/2}}^2;$$

but if we want radial derivatives (globally) then need weight $r^{-1/2-\epsilon}$.

Can do a *bit* better though. On asymptotically conic manifold, for each i, j,

$$\int_0^T \left\langle r^{-1/2} \not\nabla_i \psi, r^{-1/2} \nabla_j \psi \right\rangle dt \lesssim \|\psi_0\|_{H^{1/2}}^2;$$

Inverse square potentials, spatial weights

On $\mathbb{R}^n$, $V = r^{-2}a(\theta)$ has same scaling of Δ, and cannot be regarded as a perturbation. (Is, among other things, a simplified model of what happens on conic manifolds; expect diffractive effects.)

Theorem (Burq-Planchon-Stalker-Tavildar-Zadeh)

Assume $\inf a > -(n-2)^2/4$*; take Friedrichs extension of* $(1/2)\Delta + V$*. Then usual local smoothing estimate holds.*

Also, the authors observe that

$$\int_0^T \left\| r^{-1}\psi \right\|^2_{L^2_{\mathrm{loc}}} \, dt \lesssim \|\psi_0\|^2$$

(i.e. we are permitted singular weight at origin).

Inverse square potentials, spatial weights

On $\mathbb{R}^n$, $V = r^{-2}a(\theta)$ has same scaling of Δ, and cannot be regarded as a perturbation. (Is, among other things, a simplified model of what happens on conic manifolds; expect diffractive effects.)

Theorem (Burq-Planchon-Stalker-Tavildar-Zadeh)

Assume $\inf a > -(n-2)^2/4$; *take Friedrichs extension of* $(1/2)\Delta + V$. *Then usual local smoothing estimate holds.*

Also, the authors observe that

$$\int_0^T \left\| r^{-1}\psi \right\|^2_{L^2_{\mathrm{loc}}} \, dt \lesssim \|\psi_0\|^2$$

(i.e. we are permitted singular weight at origin).

Inverse square potentials, spatial weights

On $\mathbb{R}^n$, $V = r^{-2}a(\theta)$ has same scaling of Δ, and cannot be regarded as a perturbation. (Is, among other things, a simplified model of what happens on conic manifolds; expect diffractive effects.)

Theorem (Burq-Planchon-Stalker-Tavildar-Zadeh)

Assume $\inf a > -(n-2)^2/4$; *take Friedrichs extension of* $(1/2)\Delta + V$. *Then usual local smoothing estimate holds.*

Also, the authors observe that

$$\int_0^T \|r^{-1}\psi\|^2_{L^2_{\text{loc}}}\, dt \lesssim \|\psi_0\|^2$$

(i.e. we are permitted singular weight at origin).

Generalizations

Generalization to manifolds with no trapping, *single cone point* (Planchon-Stalker-Wunsch).

Several poles (Duyckaerts). Involves new ideas, as there are trapped rays, undergoing successive diffractive interactions with the poles.

Questions:

- How to generalize to other singular spaces, hence deal with significant diffractive effects in non-Euclidean background?
- When is L^2H^s with $s \in (0, 1/2)$ sharp? (I.e. a real, non-epsilonic loss). Use a thicker trapped set?

Generalizations

Generalization to manifolds with no trapping, *single cone point* (Planchon-Stalker-Wunsch).

Several poles (Duyckaerts). Involves new ideas, as there are trapped rays, undergoing successive diffractive interactions with the poles.

Questions:

- How to generalize to other singular spaces, hence deal with significant diffractive effects in non-Euclidean background?
- When is L^2H^s with $s \in (0, 1/2)$ sharp? (I.e. a real, non-epsilonic loss). Use a thicker trapped set?

Generalizations

Generalization to manifolds with no trapping, *single cone point* (Planchon-Stalker-Wunsch).

Several poles (Duyckaerts). Involves new ideas, as there are trapped rays, undergoing successive diffractive interactions with the poles.

Questions:

- How to generalize to other singular spaces, hence deal with significant diffractive effects in non-Euclidean background?
- When is $L^2 H^s$ with $s \in (0, 1/2)$ sharp? (I.e. a real, non-epsilonic loss). Use a thicker trapped set?

The Strichartz estimates

Original proofs due to Strichartz on $\mathbb{R}^n$ were via restriction theorems (1977 for wave equation).
Work in $\mathbb{R}^n$, $n > 2$. Exponents (q, p) are *admissible* if

$$\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{p}\right)$$

and

$$p \in [2, 2n/(n-2)].$$

Then a solution to homogeneous Schrödinger equation satisfies, for (q, p) admissible,

$$\|\psi\|_{L^q(\mathbb{R}_t; L^p(\mathbb{R}^n))} \lesssim \|\psi_0\|_{L^2}.$$

The Strichartz estimates

Original proofs due to Strichartz on $\mathbb{R}^n$ were via restriction theorems (1977 for wave equation).
Work in $\mathbb{R}^n$, $n > 2$. Exponents (q, p) are *admissible* if

$$\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{p}\right)$$

and

$$p \in [2, 2n/(n-2)].$$

Then a solution to homogeneous Schrödinger equation satisfies, for (q, p) admissible,

$$\|\psi\|_{L^q(\mathbb{R}_t; L^p(\mathbb{R}^n))} \lesssim \|\psi_0\|_{L^2}.$$

- $q = \infty$, $p = 2$ is conservation of norm.
- $q = 2$, $p = 2n/(n-2)$ is called the "endpoint" and this estimate is due to Keel-Tao (1998), and is false in $n = 2$.
- Inhomogeneous version: $(i\partial_t + (1/2)\Delta\psi)u = f$ with zero initial data implies

$$\|\psi\|_{L^q L^p} \lesssim \|f\|_{L^{s'} L^{r'}}$$

if (q, p) and (s, r) admissible.

- $q = \infty$, $p = 2$ is conservation of norm.
- $q = 2$, $p = 2n/(n-2)$ is called the “endpoint” and this estimate is due to Keel-Tao (1998), and is false in $n = 2$.
- Inhomogeneous version: $(i\partial_t + (1/2)\Delta\psi)u = f$ with zero initial data implies

$$\|\psi\|_{L^q L^p} \lesssim \|f\|_{L^{s'} L^{r'}}$$

if (q, p) and (s, r) admissible.

- $q = \infty$, $p = 2$ is conservation of norm.
- $q = 2$, $p = 2n/(n-2)$ is called the "endpoint" and this estimate is due to Keel-Tao (1998), and is false in $n = 2$.
- Inhomogeneous version: $(i\partial_t + (1/2)\Delta\psi)u = f$ with zero initial data implies

$$\|\psi\|_{L^q L^p} \lesssim \|f\|_{L^{s'} L^{r'}}$$

if (q, p) and (s, r) admissible.

Proof of inhomogeneous estimate with $(q, p) = (s, r)$

Let $U(t)$ denote the solution operator to the homogeneous problem

$$(U(t)\psi_0)(x) = \int K(t, x, y)\psi_0(y)\, dy$$

and $T(t)$ the inhomogeneous

$$T(t)f = \int_{s<t} U(t-s)f(s)\, ds = \int_{s<t} \int K(t-s, x, y) f(s, y)\, dy\, ds.$$

We have $U(t) : L^2 \to L^2$ and $L^1 \to t^{-n/2} L^\infty$ so by interpolation, for $p \in [2, \infty]$,

$$U(t) : L^{p'} \to t^{-\frac{2}{q}} L^p.$$

Proof of inhomogeneous estimate with $(q, p) = (s, r)$

Let $U(t)$ denote the solution operator to the homogeneous problem

$$(U(t)\psi_0)(x) = \int K(t, x, y)\psi_0(y)\,dy$$

and $T(t)$ the inhomogeneous

$$T(t)f = \int_{s<t} U(t-s)f(s)\,ds = \int_{s<t}\int K(t-s, x, y)f(s, y)\,dy\,ds.$$

We have $U(t) : L^2 \to L^2$ and $L^1 \to t^{-n/2}L^\infty$ so by interpolation, for $p \in [2, \infty]$,

$$U(t) : L^{p'} \to t^{-\frac{2}{q}}L^p.$$

Proof (continued)

Then

$$\|Tf(t)\|_{L^p(\mathbb{R}^n)} \leq \int_{s<t} \|U(t-s)f(s)\|_p \, ds$$
$$\lesssim \int |t-s|^{-\frac{2}{q}} \|f(s)\|_{p'} \, ds$$
$$= |t|^{-\frac{2}{q}} * \|f(t)\|_{p'}.$$

Now Hardy-Littlewood-Sobolev says: convolution with $|t|^{-\frac{2}{q}}$ maps $L^{q'}(\mathbb{R}) \to L^q(\mathbb{R})$ provided $q' < q$
(i.e. on $p < 2n/(n-2)$, i.e. off of endpoint!).
Hence

$$\|Tf\|_{L^q L^p} \lesssim \|f\|_{L^{q'} L^{p'}}$$

(Ginibre-Velo, 1985).

Proof (continued)

Then

$$\|Tf(t)\|_{L^p(\mathbb{R}^n)} \leq \int_{s<t} \|U(t-s)f(s)\|_p \, ds \lesssim \int |t-s|^{-\frac{2}{q}} \|f(s)\|_{p'} \, ds = |t|^{-\frac{2}{q}} * \|f(t)\|_{p'}.$$

Now Hardy-Littlewood-Sobolev says: convolution with $|t|^{-\frac{2}{q}}$ maps $L^{q'}(\mathbb{R}) \to L^q(\mathbb{R})$ provided $q' < q$
(i.e. on $p < 2n/(n-2)$, i.e. off of endpoint!).
Hence

$$\|Tf\|_{L^qL^p} \lesssim \|f\|_{L^{q'}L^{p'}}$$

(Ginibre-Velo, 1985).

Proof (continued)

Then

$$\begin{aligned}\|Tf(t)\|_{L^p(\mathbb{R}^n)} &\leq \int_{s<t} \|U(t-s)f(s)\|_p \, ds \\ &\lesssim \int |t-s|^{-\frac{2}{q}} \|f(s)\|_{p'} \, ds \\ &= |t|^{-\frac{2}{q}} * \|f(t)\|_{p'}.\end{aligned}$$

Now Hardy-Littlewood-Sobolev says: convolution with $|t|^{-\frac{2}{q}}$ maps $L^{q'}(\mathbb{R}) \to L^q(\mathbb{R})$ provided $q' < q$
(i.e. on $p < 2n/(n-2)$, i.e. off of endpoint!).
Hence

$$\|Tf\|_{L^q L^p} \lesssim \|f\|_{L^{q'} L^{p'}}$$

(Ginibre-Velo, 1985).

Remarks

Homogeneous estimates follow via a duality argument.

Note that all we used about the propagator was:

- Uniform $L^2 \to L^2$ bound.
- $L^1 \to L^\infty$ estimate, i.e. sup bound on K

$$|K(t,x,y)| \lesssim t^{-n/2}.$$

Remarks

Homogeneous estimates follow via a duality argument.

Note that all we used about the propagator was:

- Uniform $L^2 \longrightarrow L^2$ bound.
- $L^1 \longrightarrow L^\infty$ estimate, i.e. sup bound on K

$$|K(t,x,y)| \lesssim t^{-n/2}.$$

Application to NLS

Immediate applications to nonlinear Schrödinger equations (which arise in nonlinear optics; superconductivity; quantum field theory)

The equation

$$\partial_t \psi = -i \Delta \psi + \lambda |\psi|^{p-1} \psi$$

is locally wellposed in H^s with

$$s \geq \max(0, n/2 - 2/(p-1))$$

and p odd or $\geq \lfloor s \rfloor + 1$. (cf. Cazenave-Weissler, 1990).

Application to NLS

Immediate applications to nonlinear Schrödinger equations (which arise in nonlinear optics; superconductivity; quantum field theory)

The equation

$$\partial_t \psi = -i\Delta\psi + \lambda|\psi|^{p-1}\psi$$

is locally wellposed in H^s with

$$s \geq \max(0, n/2 - 2/(p-1))$$

and p odd or $\geq \lfloor s \rfloor + 1$. (cf. Cazenave-Weissler, 1990).

- So ingredients for Strichartz estimate (including endpoint) are $L^\infty L^2$ estimate, plus "dispersive" estimate $K \in t^{-n/2} L^\infty$.
- But Strichartz may still hold when the latter fails!
  On asymptotically Euclidean space, for instance,
  $K \notin t^{-n/2} L^\infty$ due to existence of conjugate points: near diagonal, the propagator has form

$$t^{-n/2} a e^{id(x,y)^2/2t}$$

  (cf. Hassell-Wunsch, 2005) but when x, y are conjugate to each other, takes on a more complicated oscillatory integral form, with phase variables and correspondingly more negative power of t (focusing effect).
- Nevertheless: (local-in-time) Strichartz still holds on asymptotically conic spaces as long as there are no trapped rays. (Cf. Staffilani-Tataru, Burq, Hassell-Tao-Wunsch, Robbiano-Zuily, Bouclet-Tzvetkov.)

- So ingredients for Strichartz estimate (including endpoint) are $L^\infty L^2$ estimate, plus "dispersive" estimate $K \in t^{-n/2} L^\infty$.
- But Strichartz may still hold when the latter fails!
  On asymptotically Euclidean space, for instance,
  $K \notin t^{-n/2} L^\infty$ due to existence of conjugate points: near diagonal, the propagator has form

$$t^{-n/2} a e^{i d(x,y)^2/2t}$$

  (cf. Hassell-Wunsch, 2005) but when x, y are conjugate to each other, takes on a more complicated oscillatory integral form, with phase variables and correspondingly more negative power of t (focusing effect).
- Nevertheless: (local-in-time) Strichartz still holds on asymptotically conic spaces as long as there are no trapped rays. (Cf. Staffilani-Tataru, Burq, Hassell-Tao-Wunsch, Robbiano-Zuily, Bouclet-Tzvetkov.)

- So ingredients for Strichartz estimate (including endpoint) are $L^\infty L^2$ estimate, plus "dispersive" estimate $K \in t^{-n/2} L^\infty$.
- But Strichartz may still hold when the latter fails! On asymptotically Euclidean space, for instance, $K \notin t^{-n/2} L^\infty$ due to existence of conjugate points: near diagonal, the propagator has form

$$t^{-n/2} a e^{id(x,y)^2/2t}$$

  (cf. Hassell-Wunsch, 2005) but when x, y are conjugate to each other, takes on a more complicated oscillatory integral form, with phase variables and correspondingly more negative power of t (focusing effect).
- Nevertheless: (local-in-time) Strichartz still holds on asymptotically conic spaces as long as there are no trapped rays. (Cf. Staffilani-Tataru, Burq, Hassell-Tao-Wunsch, Robbiano-Zuily, Bouclet-Tzvetkov.)

Relationship to local smoothing

If have a parametrix for Schrödinger equation in some small set, satisfying $L^2 \to L^2$ and $L^1 \to t^{-n/2} L^\infty$ estimates, local smoothing allows us to localize our problem and get a local Strichartz estimate.

Say $G(t)$ is a local parametrix (i.e. approximation for $U(t)$ in some region Ω). Let φ be a cutoff in Ω.

Then

$$(i\partial_t + (1/2)\Delta)(\varphi\psi) = [(1/2)\Delta, \varphi]\psi$$

Hence by Duhamel

$$\begin{aligned}
\varphi\psi(t) &= \int_0^t U(t-s)[(-i/2)\Delta, \varphi]\psi(s)\, ds \\
&= U(t) \int_0^t U(-s)[(-i/2)\Delta, \varphi]\psi(s)\, ds \\
&\approx G(t) \int_0^t U(-s)[(-i/2)\Delta, \varphi]\psi(s)\, ds
\end{aligned}$$

Relationship to local smoothing

If have a parametrix for Schrödinger equation in some small set, satisfying $L^2 \to L^2$ and $L^1 \to t^{-n/2}L^\infty$ estimates, local smoothing allows us to localize our problem and get a local Strichartz estimate.

Say $G(t)$ is a local parametrix (i.e. approximation for $U(t)$ in some region Ω). Let φ be a cutoff in Ω.

Then

$$(i\partial_t + (1/2)\Delta)(\varphi\psi) = [(1/2)\Delta, \varphi]\psi$$

Hence by Duhamel

$$\begin{aligned}\varphi\psi(t) &= \int_0^t U(t-s)[(-i/2)\Delta, \varphi]\psi(s)\, ds \\ &= U(t)\int_0^t U(-s)[(-i/2)\Delta, \varphi]\psi(s)\, ds \\ &\approx G(t)\int_0^t U(-s)[(-i/2)\Delta, \varphi]\psi(s)\, ds\end{aligned}$$

Relationship to local smoothing

If have a parametrix for Schrödinger equation in some small set, satisfying $L^2 \to L^2$ and $L^1 \to t^{-n/2}L^\infty$ estimates, local smoothing allows us to localize our problem and get a local Strichartz estimate.

Say $G(t)$ is a local parametrix (i.e. approximation for $U(t)$ in some region Ω). Let φ be a cutoff in Ω.

Then

$$(i\partial_t + (1/2)\Delta)(\varphi\psi) = [(1/2)\Delta, \varphi]\psi$$

Hence by Duhamel

$$\begin{aligned}\varphi\psi(t) &= \int_0^t U(t-s)[(-i/2)\Delta, \varphi]\psi(s)\,ds \\ &= U(t)\int_0^t U(-s)[(-i/2)\Delta, \varphi]\psi(s)\,ds \\ &\approx G(t)\int_0^t U(-s)[(-i/2)\Delta, \varphi]\psi(s)\,ds\end{aligned}$$

Strichartz and smoothing continued

Local smoothing: $\psi(s) \in L^2 H^{1/2}_{\text{loc}}$.
So

$$[(-i/2)\Delta, \varphi]\psi(s) \in L^2 H^{-1/2}$$

i.e. for each t_0,

$$\int_{t_0}^{T} U(-s)[(-i/2)\Delta, \varphi]\psi(s)\, ds \in L^2$$

(by local smooothing *again* and duality argument).

If we know that $G : L^2 \to L^q L^p$ and that can control error terms in parametrix, we get local Strichartz, i.e. estimate on $\|\varphi\psi\|_{L^q L^p}$. (We have used Christ-Kiselev Lemma here!)

Strichartz and smoothing continued

Local smoothing: $\psi(s) \in L^2 H^{1/2}_{\mathrm{loc}}$.
So

$$[(-i/2)\Delta, \varphi]\psi(s) \in L^2 H^{-1/2}$$

i.e. for each t_0,

$$\int_{t_0}^{T} U(-s)[(-i/2)\Delta, \varphi]\psi(s)\, ds \in L^2$$

(by local smooothing *again* and duality argument).

If we know that $G : L^2 \to L^q L^p$ and that can control error terms in parametrix, we get local Strichartz, i.e. estimate on $\|\varphi\psi\|_{L^q L^p}$. (We have used Christ-Kiselev Lemma here!)

Geometric Strichartz results

The sensitivity of Strichartz to geometry (in particular, to trapping) is considerably more mysterious than that of local smoothing.

Theorem (Burq-Gérard-Tzvetkov, 2002)

If X is a compact manifold of dimension n, and (q, p) are admissible,

$$\|\psi\|_{L^q_{loc}L^p(X)} \lesssim \|\psi_0\|_{H^{\frac{1}{q}}(X)}$$

(i.e. Strichartz hold with derivative loss).

- Proof uses parametrix for *semiclassical* Strichartz $ih\partial_t + (1/2)h^2\Delta$ (i.e., in parametrix for very short times, decreasing with frequency).
- Known to be sharp on the sphere.
- Better estimates by Bourgain on flat tori, e.g. $\|\psi\|_{L^4_{\text{loc}}L^4} \lesssim \|\psi_0\|_{H^\epsilon}$ on T^2.

Geometric Strichartz, continued

- On asymptotically Euclidean space with hyperbolic trapped set, negative pressure: Strichartz with epsilon derivative loss (Christianson, 2007).
- Long times (various results on nontrapping asymptotically Euclidean spaces; cf. Rodnianski-Tao, Tataru).
- Inverse-square potentials: Strichartz with no loss (Burq-Planchon-Stalker-Tavildar-Zadeh, 2006).
- Multiple poles, likewise (Duyckaerts). (Recall this is a trapping situation.)
- In 1D, step functions with finite number of jumps in "metric" (Banica 2003), no loss. (This is also a kind of trapping.)

Geometric Strichartz, continued

- On asymptotically Euclidean space with hyperbolic trapped set, negative pressure: Strichartz with epsilon derivative loss (Christianson, 2007).
- Long times (various results on nontrapping asymptotically Euclidean spaces; cf. Rodnianski-Tao, Tataru).
- Inverse-square potentials: Strichartz with no loss (Burq-Planchon-Stalker-Tavildar-Zadeh, 2006).
- Multiple poles, likewise (Duyckaerts). (Recall this is a trapping situation.)
- In 1D, step functions with finite number of jumps in "metric" (Banica 2003), no loss. (This is also a kind of trapping.)

Geometric Strichartz, continued

- On asymptotically Euclidean space with hyperbolic trapped set, negative pressure: Strichartz with epsilon derivative loss (Christianson, 2007).
- Long times (various results on nontrapping asymptotically Euclidean spaces; cf. Rodnianski-Tao, Tataru).
- Inverse-square potentials: Strichartz with no loss (Burq-Planchon-Stalker-Tavildar-Zadeh, 2006).
- Multiple poles, likewise (Duyckaerts). (Recall this is a trapping situation.)
- In 1D, step functions with finite number of jumps in "metric" (Banica 2003), no loss. (This is also a kind of trapping.)

Geometric Strichartz, continued

- On asymptotically Euclidean space with hyperbolic trapped set, negative pressure: Strichartz with epsilon derivative loss (Christianson, 2007).
- Long times (various results on nontrapping asymptotically Euclidean spaces; cf. Rodnianski-Tao, Tataru).
- Inverse-square potentials: Strichartz with no loss (Burq-Planchon-Stalker-Tavildar-Zadeh, 2006).
- Multiple poles, likewise (Duyckaerts). (Recall this is a trapping situation.)
- In 1D, step functions with finite number of jumps in "metric" (Banica 2003), no loss. (This is also a kind of trapping.)

Hyperbolic spaces

Get full estimates on hyperbolic spaces (Banica, 2004; cf. Tataru for wave equation).

For radial solutions, there are *improved* estimates (Banica, 2004): follows since dispersive estimate ($L^1 \to L^\infty$ bound) has improved weight at infinity.
Various geometric generalizations in radially symmetric spaces by Banica, Banica-Duyckaerts, Pierfelice (all to radial data, in this strong sense).

Open questions

- Is the B-G-T estimate optimal on compact manifolds of negative curvature, or do the semiclassical propagator estimates of Anantharaman, Anantharaman-Nonnenmacher allow some kind of improvement?
- B-G-T also observe that defocusing NLS enjoys global strong existence with data in H^1 on T^3, $S^1 \times S^2$, and S^3, but in some sense for different reasons in each case! (Fourier analysis on T^3, bilinear Strichartz on S^3, bilinear plus trilinear on $S^1 \times S^2$.)
- What about singular spaces? Symmetric spaces of higher rank?

Propagation of singularities

Consider fixed-time restriction of solution $\psi(t)$.
Question: How is $\mathrm{WF}\psi(t)$ determined by ψ_0?

Fundamental solution on $\mathbb{R}^n$ shows that smooth ψ_0 may later develop singularities, and conversely singular ψ_0 may become smoothed. The trick is to track the "missing" singularities.
Craig-Kappeler-Strauss: for $t > 0$, the regularity of $\psi(t)$ at $(x, \hat{\xi}) \in S^*X$ is determined by the behavior in a conic neighborhood of the backward geodesic through this point.

Theorem (Hassell-Wunsch, 2003)

On an asymptotically conic space X, $\mathrm{WF}\psi(t)$ is determined by oscillatory behavior of ψ_0, and in particular by $\mathrm{WF}_{\mathrm{sc}}(e^{ir^2/2t}\psi_0)$.

$\mathrm{WF}_{\mathrm{sc}}$ is "scattering WF set" (Melrose).

Propagation of singularities

Consider fixed-time restriction of solution $\psi(t)$.
Question: How is WF$\psi(t)$ determined by ψ_0?

Fundamental solution on $\mathbb{R}^n$ shows that smooth ψ_0 may later develop singularities, and conversely singular ψ_0 may become smoothed. The trick is to track the "missing" singularities.
Craig-Kappeler-Strauss: for $t > 0$, the regularity of $\psi(t)$ at $(x, \hat{\xi}) \in S^*X$ is determined by the behavior in a conic neighborhood of the backward geodesic through this point.

Theorem (Hassell-Wunsch, 2003)

On an asymptotically conic space X, WF$\psi(t)$ is determined by oscillatory behavior of ψ_0, and in particular by $\mathrm{WF}_{\mathrm{sc}}(e^{ir^2/2t}\psi_0)$.

$\mathrm{WF}_{\mathrm{sc}}$ is "scattering WF set" (Melrose).

Propagation of singularities

Consider fixed-time restriction of solution $\psi(t)$.
Question: How is $\mathrm{WF}\psi(t)$ determined by ψ_0?

Fundamental solution on $\mathbb{R}^n$ shows that smooth ψ_0 may later develop singularities, and conversely singular ψ_0 may become smoothed. The trick is to track the "missing" singularities. Craig-Kappeler-Strauss: for $t > 0$, the regularity of $\psi(t)$ at $(x, \hat{\xi}) \in S^*X$ is determined by the behavior in a conic neighborhood of the backward geodesic through this point.

Theorem (Hassell-Wunsch, 2003)

On an asymptotically conic space X, $\mathrm{WF}\psi(t)$ is determined by oscillatory behavior of ψ_0, and in particular by $\mathrm{WF}_{\mathrm{sc}}(e^{ir^2/2t}\psi_0)$.

$\mathrm{WF}_{\mathrm{sc}}$ is "scattering WF set" (Melrose).

Propagation of singularities

Consider fixed-time restriction of solution $\psi(t)$.
Question: How is $\mathrm{WF}\psi(t)$ determined by ψ_0?

Fundamental solution on $\mathbb{R}^n$ shows that smooth ψ_0 may later develop singularities, and conversely singular ψ_0 may become smoothed. The trick is to track the "missing" singularities. Craig-Kappeler-Strauss: for $t > 0$, the regularity of $\psi(t)$ at $(x, \hat{\xi}) \in S^*X$ is determined by the behavior in a conic neighborhood of the backward geodesic through this point.

Theorem (Hassell-Wunsch, 2003)

On an asymptotically conic space X, $\mathrm{WF}\psi(t)$ is determined by oscillatory behavior of ψ_0, and in particular by $\mathrm{WF}_{\mathrm{sc}}(e^{ir^2/2t}\psi_0)$.

$\mathrm{WF}_{\mathrm{sc}}$ is "scattering WF set" (Melrose).

Scattering wavefront set

Consider $X = \mathbb{R}^n$, with compactification B^n.
Measure behavior at infinity ($= S^{n-1}$) with *scattering wavefront set*.

Lives in ${}^{sc}T^*_{\partial\overline{X}}X \approx S^{n-1} \times \mathbb{R}^n$, the (rescaled) cotangent bundle over the boundary at infinity.

Scattering WF (Melrose): on $\mathbb{R}^n$, let $a(x)$, $\phi(x) \sim \phi_0(\hat{x}) + \dots$ be symbols of order 0. Then $(\hat{x}_0, \xi_0) \in \mathrm{WF}_{\mathrm{sc}} a(x)e^{ir\phi(x)}$ iff

- $\hat{x}_0$ is in cone support of a,
- $\xi_0 = \phi_0(\hat{x}_0)\hat{x}_0 + \nabla\phi_0(\hat{x}_0)$.

(SC Wavefront set is graph of $(\phi_0, d\phi_0)$ over support of a on sphere at infinity.)

Scattering wavefront set

Consider $X = \mathbb{R}^n$, with compactification B^n.
Measure behavior at infinity ($= S^{n-1}$) with *scattering wavefront set*.

Lives in ${}^{sc}T^*_{\partial \overline{X}} X \approx S^{n-1} \times \mathbb{R}^n$, the (rescaled) cotangent bundle over the boundary at infinity.

Scattering WF (Melrose): on $\mathbb{R}^n$, let $a(x)$, $\phi(x) \sim \phi_0(\hat{x}) + \dots$ be symbols of order 0. Then $(\hat{x}_0, \xi_0) \in \mathrm{WF}_{\mathrm{sc}} a(x) e^{ir\phi(x)}$ iff

- $\hat{x}_0$ is in cone support of a,
- $\xi_0 = \phi_0(\hat{x}_0)\hat{x}_0 + \nabla \phi_0(\hat{x}_0)$.

(SC Wavefront set is graph of $(\phi_0, d\phi_0)$ over support of a on sphere at infinity.)

Euclidean fundamental solution

Example

- On $\mathbb{R}^1$, solution $(t-1)^{-1/2}e^{i(x-x_0)^2/2(t-1)}$ (1D fundamental solution, shifted in space and time).
- $\psi_0 = Ce^{-ix^2/2+ix\cdot x_0}$
- $\mathrm{WF}_{\mathrm{sc}}(e^{ix^2/2t}\psi_0)$ is infinite unless $t=1$ owing to quadratic oscillation in phase, i.e. form $e^{i\alpha x^2}$.
- $\mathrm{WF}_{\mathrm{sc}}(e^{ix^2/2}\psi_0) = \mathrm{WF}_{\mathrm{sc}}(Ce^{ix\cdot x_0}) = \{(\pm 1, \pm x_0)\}$.
- Correspondingly $\psi(t) \in \mathcal{C}^\infty$ unless $t=1$; $\mathrm{WF}\psi(1) = N^*\{x = x_0\}$. Can recover these data from $\mathrm{WF}_{\mathrm{sc}}(e^{ix^2/2}\psi_0)$.

Sojourn relation

Define a map $S^*\mathbb{R}^n = \mathbb{R}^n \times S^{n-1} \to S^{n-1} \times \mathbb{R}^n$ as follows:
Given $(x, \hat{\xi}) \in \mathbb{R}^n \times S^{n-1}$ let γ be the unit-speed geodesic through $(x, \hat{\xi})$.

Set

$$S(x, \hat{\xi}) = (\theta, \zeta) \in S^{n-1} \times \mathbb{R}^n, \text{ with } \zeta = \lambda\theta + \mu,$$

$$\theta = \lim_{t \to -\infty} \frac{\gamma(t)}{|\gamma(t)|},$$

$$\lambda = \lim_{t \to -\infty} -(t - |\gamma(t)|)$$

$$\mu = \lim_{t \to -\infty} -|\gamma(t)| \left(\theta - \frac{\gamma(t)}{|\gamma(t)|} \right).$$

θ : asymptotic direction; $-\lambda$: "sojourn time" (cf. Guillemin, Majda). On Euclidean space, $-\lambda = x \cdot \hat{\xi}$.

S is a contact diffeomorphism: $S^*(X^\circ) \to {}^{sc}T^*_{\partial X}(X)$

Sojourn relation

Define a map $S^*\mathbb{R}^n = \mathbb{R}^n \times S^{n-1} \to S^{n-1} \times \mathbb{R}^n$ as follows:
Given $(x, \hat{\xi}) \in \mathbb{R}^n \times S^{n-1}$ let γ be the unit-speed geodesic through $(x, \hat{\xi})$.
Set

$$S(x, \hat{\xi}) = (\theta, \zeta) \in S^{n-1} \times \mathbb{R}^n, \text{ with } \zeta = \lambda\theta + \mu,$$

$$\theta = \lim_{t \to -\infty} \frac{\gamma(t)}{|\gamma(t)|},$$

$$\lambda = \lim_{t \to -\infty} -(t - |\gamma(t)|)$$

$$\mu = \lim_{t \to -\infty} -|\gamma(t)| \left(\theta - \frac{\gamma(t)}{|\gamma(t)|} \right).$$

θ : asymptotic direction; $-\lambda$: "sojourn time" (cf. Guillemin, Majda). On Euclidean space, $-\lambda = x \cdot \hat{\xi}$.

S is a contact diffeomorphism: $S^*(X^\circ) \to {}^{sc}T^*_{\partial X}(X)$

Sojourn relation

Define a map $S^*\mathbb{R}^n = \mathbb{R}^n \times S^{n-1} \to S^{n-1} \times \mathbb{R}^n$ as follows:
Given $(x, \hat{\xi}) \in \mathbb{R}^n \times S^{n-1}$ let γ be the unit-speed geodesic through $(x, \hat{\xi})$.
Set

$$S(x, \hat{\xi}) = (\theta, \zeta) \in S^{n-1} \times \mathbb{R}^n, \text{ with } \zeta = \lambda\theta + \mu,$$

$$\theta = \lim_{t \to -\infty} \frac{\gamma(t)}{|\gamma(t)|},$$

$$\lambda = \lim_{t \to -\infty} -(t - |\gamma(t)|)$$

$$\mu = \lim_{t \to -\infty} -|\gamma(t)| \left(\theta - \frac{\gamma(t)}{|\gamma(t)|} \right).$$

θ : asymptotic direction; $-\lambda$: "sojourn time" (cf. Guillemin, Majda). On Euclidean space, $-\lambda = x \cdot \hat{\xi}$.

S is a contact diffeomorphism: $S^*(X^\circ) \to {}^{\mathrm{sc}}T^*_{\partial \overline{X}}(X)$

Propagation of nonuniform WF

Theorem (Hassell-W., 2003)

$$(x, \hat{\xi}) \in \mathrm{WF}\,\psi(T)$$

iff

$$\frac{1}{T} S(x, \hat{\xi}) \in \mathrm{WF}_{\mathrm{sc}}(e^{ir^2/2T}\psi_0).$$

- Proof relies on parametrix construction in category of "scattering fibered Legendrians" (cf. Melrose-Zworski, Hassell-Vasy). Show that for $x \in K \Subset X^\circ$ and (r, θ) near $\partial\overline{X}$, can write

$$e^{-ir^2/2t} K(t, (r, \theta), x) \approx t^{-\frac{n}{2}-\frac{k}{2}} \int_{U \Subset \mathbb{R}^k} a(\dots) e^{i\phi(r^{-1}, \theta, x, v) r/t} \, dv$$

  Can regard RHS as a "scattering FIO" taking scattering wavefront set to ordinary wavefront set via sojourn relation (parametrized by ϕ).
  In simple nondegenerate case, no integral necessary, and phase is just $-rt^{-1}$ times "sojourn time" λ from above.
- More recent work of Nakamura gives alternative proof of same theorem, phrased in terms of "isotropic WF" and includes long-range case.
- Different (slightly cruder) story for uniform-in-time WF; cf. Lascar, Wunsch, Robbiano-Zuily, Martinez-Nakamura-Sordoni, Ito, Szeftel.

- Proof relies on parametrix construction in category of "scattering fibered Legendrians" (cf. Melrose-Zworski, Hassell-Vasy). Show that for $x \in K \Subset X^\circ$ and (r, θ) near $\partial \overline{X}$, can write

$$e^{-ir^2/2t} K(t, (r, \theta), x) \approx t^{-\frac{n}{2}-\frac{k}{2}} \int_{U \Subset \mathbb{R}^k} a(\dots) e^{i\phi(r^{-1},\theta,x,v)r/t}\, dv$$

  Can regard RHS as a "scattering FIO" taking scattering wavefront set to ordinary wavefront set via sojourn relation (parametrized by ϕ).
  In simple nondegenerate case, no integral necessary, and phase is just $-rt^{-1}$ times "sojourn time" λ from above.
- More recent work of Nakamura gives alternative proof of same theorem, phrased in terms of "isotropic WF" and includes long-range case.
- Different (slightly cruder) story for uniform-in-time WF; cf. Lascar, Wunsch, Robbiano-Zuily, Martinez-Nakamura-Sordoni, Ito, Szeftel.

- Proof relies on parametrix construction in category of "scattering fibered Legendrians" (cf. Melrose-Zworski, Hassell-Vasy). Show that for $x \in K \Subset X^\circ$ and (r, θ) near $\partial \overline{X}$, can write

$$e^{-ir^2/2t} K(t, (r, \theta), x) \approx t^{-\frac{n}{2}-\frac{k}{2}} \int_{U \Subset \mathbb{R}^k} a(\dots) e^{i\phi(r^{-1},\theta,x,v)r/t}\, dv$$

  Can regard RHS as a "scattering FIO" taking scattering wavefront set to ordinary wavefront set via sojourn relation (parametrized by ϕ).
  In simple nondegenerate case, no integral necessary, and phase is just $-rt^{-1}$ times "sojourn time" λ from above.
- More recent work of Nakamura gives alternative proof of same theorem, phrased in terms of "isotropic WF" and includes long-range case.
- Different (slightly cruder) story for uniform-in-time WF; cf. Lascar, Wunsch, Robbiano-Zuily, Martinez-Nakamura-Sordoni, Ito, Szeftel.

Harmonic oscillator

Theorem (Zelditch, 1983)

On flat $\mathbb{R}^n$*, let* $V = (1/2)r^2 + e(x)$ *with* $e(x) \in S^0(\mathbb{R}^n)$*.*
Then $\mathrm{WF}(\psi(n\pi)) = (-1)^n \mathrm{WF}(\psi_0)$*.*

Theorem (Doi, 2004)

If $e(x) \in S^1(\mathbb{R}^n)$ *there is a finite-speed-of-propagation relationship between* $\mathrm{WF}(\psi(n\pi))$ *and* $\mathrm{WF}(\psi_0)$ *depending on* $e(x)$*.*

Can also prove a trace theorem (see below).

Harmonic oscillator

Theorem (Zelditch, 1983)

On flat $\mathbb{R}^n$*, let* $V = (1/2)r^2 + e(x)$ *with* $e(x) \in S^0(\mathbb{R}^n)$*.*
Then $\mathrm{WF}(\psi(n\pi)) = (-1)^n \mathrm{WF}(\psi_0)$*.*

Theorem (Doi, 2004)

If $e(x) \in S^1(\mathbb{R}^n)$ *there is a finite-speed-of-propagation relationship between* $\mathrm{WF}(\psi(n\pi))$ *and* $\mathrm{WF}(\psi_0)$ *depending on* $e(x)$*.*

Can also prove a trace theorem (see below).

Geometric harmonic oscillator

Let X be an asymptotically conic manifold with no trapped rays.
Let $V = r^2/2 + O(r^{-1})$.

Let

$$S = \{\pm L : \text{there exists a closed geodesic in } \partial X \text{ of length } L\}$$
$$\cup \{n\pi : \text{ there exists a geodesic } n\text{-gon in } X \text{ with vertices in } \partial X\} \cup \{0\}.$$

Then

$$\operatorname{singsupp} \operatorname{Tr} U(t) \subset S,$$

(This is a spectral quantity: $\operatorname{Tr} U(t) = \sum e^{i\lambda_j t}$, sum over eigenvalues.)

Question: Is there a trace *formula* à la Duistermaat-Guillemin in this case or the 1-symbol perturbation of Euclidean case?
Can we hear the geometry of a harmonic oscillator?

Geometric harmonic oscillator

Let X be an asymptotically conic manifold with no trapped rays.
Let $V = r^2/2 + O(r^{-1})$.

Let

$$S = \{\pm L : \text{there exists a closed geodesic in } \partial X \text{ of length } L\}$$
$$\cup\{n\pi : \text{ there exists a geodesic } n\text{-gon in } X \text{ with vertices in } \partial X\} \cup \{0\}.$$

Then

$$\operatorname{singsupp} \operatorname{Tr} U(t) \subset S,$$

(This is a spectral quantity: $\operatorname{Tr} U(t) = \sum e^{i\lambda_j t}$, sum over eigenvalues.)

Question: Is there a trace *formula* à la Duistermaat-Guillemin in this case or the 1-symbol perturbation of Euclidean case?
Can we hear the geometry of a harmonic oscillator?

Geometric harmonic oscillator

Let X be an asymptotically conic manifold with no trapped rays. Let $V = r^2/2 + O(r^{-1})$.

Let

$$S = \{\pm L : \text{there exists a closed geodesic in } \partial X \text{ of length } L\}$$
$$\cup\{n\pi : \text{ there exists a geodesic } n\text{-gon in } X \text{ with vertices in } \partial X\} \cup \{0\}.$$

Then

$$\operatorname{singsupp} \operatorname{Tr} U(t) \subset S,$$

(This is a spectral quantity: $\operatorname{Tr} U(t) = \sum e^{i\lambda_j t}$, sum over eigenvalues.)

Question: Is there a trace *formula* à la Duistermaat-Guillemin in this case or the 1-symbol perturbation of Euclidean case?

Can we hear the geometry of a harmonic oscillator?

Geometric harmonic oscillator

Let X be an asymptotically conic manifold with no trapped rays.
Let $V = r^2/2 + O(r^{-1})$.

Let

$$S = \{\pm L : \text{there exists a closed geodesic in } \partial X \text{ of length } L\}$$
$$\cup\{n\pi : \text{ there exists a geodesic } n\text{-gon in } X \text{ with vertices in } \partial X\} \cup \{0\}.$$

Then

$$\operatorname{singsupp} \operatorname{Tr} U(t) \subset S,$$

(This is a spectral quantity: $\operatorname{Tr} U(t) = \sum e^{i\lambda_j t}$, sum over eigenvalues.)

Question: Is there a trace *formula* à la Duistermaat-Guillemin in this case or the 1-symbol perturbation of Euclidean case?
Can we hear the geometry of a harmonic oscillator?