NEWTON POLYGONS AND RESONANCES OF MULTIPLE DELTA-POTENTIALS

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Abstract. We prove explicit asymptotics for the location of semiclassical scattering resonances in the setting of h-dependent delta-function potentials on $\mathbb{R}$. In the cases of two or three delta poles, we are able to show that resonances occur along specific lines of the form $\operatorname{Im} z \sim -\gamma h \log(1/h)$. More generally, we use the method of Newton polygons to show that resonances near the real axis may only occur along a finite collection of such lines, and we bound the possible number of values of the parameter γ. We present numerical evidence of the existence of more and more possible values of γ for larger numbers of delta poles.

1. Introduction

We consider certain "leaky" semiclassical quantum systems where most of the energy escapes to infinity but some h-dependent fraction is trapped. In such settings, it has often been observed that strings of resonances occur along curves $\operatorname{Im} z \sim -\gamma h \log(1/h)$ for certain values of γ related to the geometry. This has been observed, with varying degrees of precision, in scattering with nonsmooth potentials on the real line [14], [18]; scattering by multiple delta singularities in $\mathbb{R}^3$ [17]; scattering between a corner and an analytic obstacle [5]; scattering on a manifold with conic singularities [7], [10], [13]; and scattering by thin barriers, modeled by h-dependent δ-potentials [6], [11]. In some of these settings where the geometry of trapping is relatively simple, e.g., [5], [6], the structure of all resonances near the real axis can be precisely understood, with one or more strings of resonances occurring at

$$\operatorname{Im} z \sim -\gamma h \log(1/h), \tag{1}$$

for certain values of γ and no others present. More generally, however, the picture is muddier, with some information known about $O(h \log(1/h))$-width resonance-free regions near $\mathbb{R}$ and in some cases about existence of a limited region in which the resonances are distributed as in (1).

Here we analyze a situation in which the geometry of trapping is complicated enough to generate multiple strings of resonances of the form (1), and moreover for that structure to vary interestingly as we tune the parameters of the problem. This is the situation of several thin barriers on $\mathbb{R}$, modeled by potentials of the

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form $h^{1+\beta}\delta(x)$, $\beta > 0$. One dimensional problems with delta function barriers have been studied before in [2, Section II.2], [6], [8], [12], [15], [16] but only the second reference considered our asymptotic regime, and that only in a very special case. In the case of two and three delta poles, we are able to analyze the distribution of resonances very precisely: in the former case, there is a single curve of resonances near the real axis with $\operatorname{Im} z \sim -\gamma h \log(1/h)$ (Theorem 1); in the latter, there may be either one or two such families instead (Theorem 3). In particular, in the latter case there is one family if the deltas all have equal strength. In [17, Appendix A], Zerzeri computes resonances of multiple delta poles in $\mathbb{R}^3$ and finds analogously that they are all asymptotic to a single logarithmic curve.

In the more general case of N δ-potentials, we are able to constrain the locations of resonances by analyzing the secular determinant that governs their locations in terms of its Newton polygon. We show (Theorem 7) that in any set $\operatorname{Im} z \geq -Mh\log(1/h)$ there may be no more than $2^{N-1} - 1$ possible values for the parameter γ in (1), and that all such possible values may be simply expressed in terms of the various strengths β of the potential poles and differences of distances among them.

2. General setup

Consider the semiclassical Hamiltonian on the real line

$$P = -h^2\partial_x^2 + V(x), \qquad V(x) = \sum_{j=1}^{N} V_j \delta(x - x_j), \qquad h > 0,$$

where $x_1 < \cdots < x_N$, and each $V_j = C_j h^{1+\beta_j}$ for some $C_j \in \mathbb{R} \setminus \{0\}$ and $\beta_j > 0$.

A *resonant state* u is an outgoing distributional solution to

$$(-h^2\partial_x^2 + V - z^2)u = 0, \tag{2}$$

and a *resonance* is a value of $z \in \mathbb{C}$ for which a resonant state exists. More explicitly, define I_j for $j = 0, \ldots, N$, by $I_0 = (-\infty, x_1]$, $I_j = [x_j, x_{j+1}]$ when $1 \leq j \leq N-1$, and $I_N = [x_N, +\infty)$. If (2) holds in the sense of distributions, then $u = v_j^+ e^{izx/h} + v_j^- e^{-izx/h}$ on I_j, with appropriate continuity and jump conditions (which we state in (3) and (4) below) at each x_j. Such a solution u is *outgoing* if it is not identically zero and if $v_N^- = v_0^+ = 0$. See Section 2.1 et seq. of [9] for an introduction to resonances.

For (2) to hold we need u to be continuous at each x_j, i.e., the continuity condition is

$$-v_{j-1}^- e^{-ix_j z/h} - v_{j-1}^+ e^{+ix_j z/h} + v_j^- e^{-ix_j z/h} + v_j^+ e^{+ix_j z/h} = 0. \tag{3}$$

Moreover, u' must have a jump at each x_j so that $(h^2\partial_x^2 + z^2)u$ contains a multiple of $\delta(x - x_j)$ which equals $V_j u(x_j)\delta(x - x_j)$. That leads to the jump condition

$$\begin{aligned}\frac{hz}{i}\left(v_{j-1}^- e^{-izx_j/h} - v_{j-1}^+ e^{+izx_j/h} - v_j^- e^{-izx_j/h} + v_j^+ e^{+izx_j/h}\right)& \\ + V_j(v_j^- e^{-ix_j z/h} + v_j^+ e^{+ix_j z/h}) &= 0.\end{aligned} \tag{4}$$

To bring the continuity and jump conditions (3) and (4) to a more manageable form, we now require $z \neq 0$, set

$$\Upsilon_j = \frac{V_j}{2izh} = \frac{C_j h^{\beta_j}}{2iz}, \tag{5}$$

and take

$$w = e^{-iz/h}, \quad y_j^+ = v_j^+ e^{ix_j z/h}, \quad y_j^- = v_j^- e^{-ix_{j+1} z/h}.$$

These are the values of the amplitudes immediately following interaction with the potential poles.

Let $\ell_j = x_{j+1} - x_j = |I_j|$. Our continuity and jump equations (3) and (4) now read

$$\begin{cases} -y_{j-1}^- - y_{j-1}^+ e^{i\ell_{j-1}z/h} + y_j^- e^{i\ell_j z/h} + y_j^+ = 0, \\ y_{j-1}^- - y_{j-1}^+ e^{i\ell_{j-1}z/h} + y_j^- e^{i\ell_j z/h}(-1 - 2\Upsilon_j) + y_j^+(1 - 2\Upsilon_j) = 0. \end{cases}$$

Adding these equations yields

$$y_j^+ = T_j e^{i\ell_{j-1}z/h} y_{j-1}^+ + R_j e^{i\ell_j z/h} y_j^- \tag{6}$$

with

$$T_j = \frac{1}{1 - \Upsilon_j}, \ R_j = \frac{\Upsilon_j}{1 - \Upsilon_j}. \tag{7}$$

Subtracting $1 - 2\Upsilon_j$ times the first from the second yields likewise

$$y_{j-1}^- = T_j e^{i\ell_j z/h} y_j^- + R_j e^{i\ell_{j-1}z/h} y_{j-1}^+. \tag{8}$$

In the extreme cases $j = 0$ or N we simply get the special cases where there is no reflection:

$$y_N^+ = T_N e^{i\ell_{N-1}z/h} y_{N-1}^+$$

and

$$y_0^- = T_1 e^{i\ell_1 z/h} y_1^-.$$

Note that these components are completely determined by the others.

3. Logarithmic strings for two and three deltas

In this section we consider the simpler cases $N = 2$ and $N = 3$, in which our description of the resonances is more complete.

3.1. Two deltas.

Let $N = 2$, and put $\ell_1 = \ell$.

Theorem 1. *All resonances obeying $1/2 \le |z| \le 2$ and $\operatorname{Re} z > 0$ are given by*

$$z_k = \frac{\pi h k}{\ell} - i\frac{\beta_1 + \beta_2}{2\ell} h \log(1/h) + O(h), \tag{9}$$

for some positive integers k. Moreover, for any δ such that $\delta < 1$ and $\delta \le \min(\beta_1, \beta_2)$ we have

$$\operatorname{Re} z_k = \frac{\pi h}{\ell}\Big(k + \frac{H(C_1C_2)}{2} + O(h^\delta)\Big), \tag{10}$$

where H is the Heaviside function, and

$$\operatorname{Im} z_k = \frac{h}{2\ell}\Bigg(-(\beta_1 + \beta_2)\log(1/h) + \log\Big(\frac{|C_1C_2|\ell^2}{4\pi^2h^2k^2}\Big) + O(h^\delta)\Bigg). \tag{11}$$

Proof. In this case we have

$$y_1^+ = R_1 e^{i\ell z/h} y_1^-, \qquad y_1^- = R_2 e^{i\ell z/h} y_1^+,$$

and so resonances occur if and only if

$$e^{-2i\ell z/h} = R_1 R_2. \tag{12}$$

Take the logarithm of both sides of (12) and multiply through by $ih/2\ell$ to get

$$z = \frac{ih}{2\ell}\log(R_1R_2) + \frac{\pi h k}{\ell}, \tag{13}$$

where k is an integer. Substituting

$$\log(R_1R_2) = \log\Big(\frac{-C_1C_2h^{\beta_1+\beta_2}}{4z^2(1-\Upsilon_1)(1-\Upsilon_2)}\Big) = -(\beta_1+\beta_2)\log(1/h) + O(1), \tag{14}$$

into (13) gives

$$z = \frac{\pi h k}{\ell} - i\frac{\beta_1+\beta_2}{2\ell} h\log(1/h) + O(h).$$

It is clear that if $\pi hk \le \ell/3$ or $\pi hk \ge 3\ell$ then the right hand side is not in $\{z \in \mathbb{C}\colon \operatorname{Re} z > 0 \text{ and } 1/2 \le |z| \le 2\}$ for h small. Hence, to establish (9), is enough to prove that, for h small enough, if k is such that $\ell/3 \le \pi hk \le 3\ell$, then (13) has a unique solution z in the half-annulus $A = \{z \in \mathbb{C}\colon \operatorname{Re} z \ge 0 \text{ and } 1/4 \le |z| \le 4\}$.

For this we apply Rouché's theorem (the Corollary of Section 5.2 of [1]) with $f(z) = z - \frac{1}{\ell}\pi hk$ and $g(z) = \frac{ih}{2\ell}\log(R_1R_2)$ on the half-annulus A (note that g is analytic on A by Corollary 2 of Section 4.4 of [1]). Since $f(z) = 0$ obviously has a unique solution in A, it is enough to check that $|g(z)| < |f(z)|$ on ∂A. For that, note that on ∂A we have $|f(z)| \ge 1/12$ and use (14).

Finally, to get (10) and (11), note that (12), (14) yield $z^2 = \frac{\pi^2h^2k^2}{\ell^2} + O(h\log(1/h))$; since $\Upsilon_j = O(h^{\beta_j})$,

$$\begin{aligned}\log(R_1R_2) &= \log\Big(\frac{-C_1C_2h^{\beta_1+\beta_2}}{4z^2(1-\Upsilon_1)(1-\Upsilon_2)}\Big)\\ &= -(\beta_1+\beta_2)\log(1/h) + \log\Big(\frac{-C_1C_2\ell^2}{4\pi^2h^2k^2}\Big) + O(h^\delta).\end{aligned}$$

Inserting this into (13) yields (10), (11). □

3.2. **Three deltas.** For $N = 3$ we use $w = e^{-iz/h}$ and write

$$y_2^- = R_3 w^{-\ell_2} y_2^+, \qquad y_1^+ = R_1 w^{-\ell_1} y_1^-,$$

and plugging those into the equations for y_2^+ and y_1^- gives

$$\begin{aligned}y_2^+ &= R_1T_2w^{-2\ell_1}y_1^- + R_2R_3w^{-2\ell_2}y_2^+,\\ y_1^- &= R_1R_2w^{-2\ell_1}y_1^- + T_2R_3w^{-2\ell_2}y_2^+.\end{aligned}$$

These equations have a nontrivial solution if and only if

$$R_1R_2^2R_3w^{-2\ell_1-2\ell_2} - R_1R_2w^{-2\ell_1} - R_1T_2^2R_3w^{-2\ell_1-2\ell_2} - R_2R_3w^{-2\ell_2} + 1 = 0.$$

Since $T_2 = R_2 + 1$, another way to write this is

$$w^{2\ell_1+2\ell_2} - R_1R_2w^{2\ell_2} - R_2R_3w^{2\ell_1} - R_1(1+2R_2)R_3 = 0. \tag{15}$$

If the delta functions are equally spaced, this can be solved using the quadratic formula and works out similarly to the case of two deltas.

Theorem 2. *If $\ell_1 = \ell_2 = \ell$, then there are positive real numbers γ_+ and γ_- (which may or may not be distinct, depending on ℓ, β_1, β_2, β_3), such that all resonances obeying $1/2 \le |z| \le 2$ and $\operatorname{Re} z > 0$ are given by*

$$z_k^+ = \frac{\pi h k}{\ell} - i\gamma_+ h\log(1/h) + O(h), \qquad z_k^- = \frac{\pi h k}{\ell} - i\gamma_- h\log(1/h) + O(h),$$

for some positive integers k.

Simple explicit formulas for the $\gamma_\pm$ can be obtained either by elaborating the calulation in the proof of Theorem 2 (which is a more complicated version of the one in Theorem 1), or as special cases of the ones in Theorem 3 below. More precise asymptotics for the real and imaginary parts of $z_k^\pm$, as in (10) and (11), follow as in the proof of Theorem 1.

Proof. By the quadratic formula, (15) is equivalent to

$$(w^{2\ell} - r_-)(w^{2\ell} - r_+) = 0,$$

where

$$r_\pm = \frac{1}{2}\Big((R_1 + R_3)R_2 \pm \sqrt{(R_1+R_3)^2R_2^2 + 4R_1(1+2R_2)R_3}\ \Big).$$

By the same argument as in the proof of Theorem 1 we get strings of resonances

$$z_k^\pm = \frac{\pi h k}{\ell} + \frac{ih}{2\ell}\log r_\pm,$$

where $\log r_\pm = -\gamma_\pm \ell \log(1/h) + O(1)$ for some $\gamma_\pm > 0$.

There are various ways to choose the β_j so as to make either $\gamma_+ \ne \gamma_-$ or $\gamma_+ = \gamma_-$. For example, if $\beta_1 + 2\beta_2 < \beta_3$, then $R_3 = O(R_1R_2^2h^\delta)$ for some $\delta > 0$, and thus $r_\pm = \frac{1}{2}(R_1R_2 \pm R_1R_2 + O(R_1R_2h^\delta))$, and $\gamma_+ \ne \gamma_-$.

On the other hand, if for example $\beta_1 < \beta_3 < \beta_1 + 2\beta_2$, then we get $r_\pm \sim \pm\sqrt{R_1R_3}$ and hence $\gamma_+ = \gamma_-$. ☐

Our next theorem gives necessary conditions on the logarithmic curves the resonances can approach when ℓ_1 is not necessarily equal to ℓ_2.

Theorem 3. *Let ℓ_1, ℓ_2, β_1, β_2, β_3 be given. Let $h_1, h_2, \ldots$ be a sequence of positive numbers tending to 0. Let $z = z(h_j)$ be a sequence of resonances such that $z = h^{o(1)}$ (i.e. such that $z(h_j) = e^{f(h_j)}$ for some $f\colon (0, h_1] \to \mathbb{C}$ obeying $|f(h)| = o(\log(1/h))$) and $\operatorname{Im} z \ge -Mh\log(1/h)$ for some positive M. Then this sequence has a subsequence such that*

$$\frac{\operatorname{Im} z}{h\log(1/h)} \to -\gamma, \tag{16}$$

for some $\gamma \in \{\gamma_+, \gamma_-\}$, where γ_+ and γ_- are determined as follows:

(1) *If $\beta_3\ell_1 - \beta_2\ell_1 - \beta_2\ell_2 \le \beta_1\ell_2 \le \beta_2\ell_1 + \beta_2\ell_2 + \beta_3\ell_1$, then*

$$\gamma_+ = \gamma_- = \frac{\beta_1 + \beta_3}{2\ell_1 + 2\ell_2}.$$

(2) *If $\beta_3\ell_1 - \beta_2\ell_1 - \beta_2\ell_2 > \beta_1\ell_2$, then*

$$\gamma_+ = \frac{\beta_3 - \beta_2}{2\ell_2} > \gamma_- = \frac{\beta_1 + \beta_2}{2\ell_1}.$$

(3) *If* $\beta_2\ell_1 + \beta_2\ell_2 + \beta_3\ell_1 < \beta_1\ell_2$, *then*

$$\gamma_+ = \frac{\beta_1 - \beta_2}{2\ell_1} > \gamma_- = \frac{\beta_2 + \beta_3}{2\ell_2}.$$

Remark 4. Note that because the resonances of $-h^2\partial_x^2 + V(-x)$ are the same as the resonances of $-h^2\partial_x^2 + V(x)$, it is no loss of generality to make the simplifying assumption

$$\beta_3\ell_1 \le \beta_1\ell_2. \tag{17}$$

Then the three cases of the theorem reduce to the following two:

(1) If $\beta_1\ell_2 \le \beta_2\ell_1 + \beta_2\ell_2 + \beta_3\ell_1$, then

$$\gamma_+ = \gamma_- = \frac{\beta_1 + \beta_3}{2\ell_1 + 2\ell_2}.$$

(2) If $\beta_2\ell_1 + \beta_2\ell_2 + \beta_3\ell_1 < \beta_1\ell_2$, then

$$\gamma_+ = \frac{\beta_1 - \beta_2}{2\ell_1} > \gamma_- = \frac{\beta_2 + \beta_3}{2\ell_2}.$$

One can interpret (2) as corresponding to the case in which the middle delta is strong enough to split the interval (x_1, x_3) at x_2, and (1) as corresponding to the case in which it is not.

Note also that in each of the limiting situations $\beta_1 \to \infty$ or $\beta_2 \to \infty$, which each correspond to one of the delta functions becoming vanishingly small, the resonances converge to those of a two-delta problem as in Theorem 1.

Proof. As noted above, we may without loss of generality proceed under the assumption (17), and show the simpler version of the theorem in Remark 4.

After passing to a subsequence, we have $\operatorname{Im} z/h \log(1/h) \to -\gamma$ for some $\gamma \in [-\infty, M]$, and so $w = h^{\gamma+o(1)}$. By the reflection coefficient formulas (5) and (7) and using $z = h^{o(1)}$, we have $\Upsilon_j = \frac{C_j}{2i} h^{\beta_j+o(1)}$ and hence $R_j = h^{\beta_j+o(1)}$, and thus the resonance equation (15) implies that $\gamma > 0$.

Next, we eliminate y_1^- and y_2^+ from (6) and (8), the equations for the $y_j^\pm$, by writing

$$y_2^- = R_3 w^{-\ell_2} y_2^+, \qquad y_1^+ = R_1 w^{-\ell_1} y_1^-,$$

which gives

$$\begin{aligned} y_2^+ &= R_1 T_2 w^{-2\ell_1} y_1^- + R_2 R_3 w^{-2\ell_2} y_2^+, \\ y_1^- &= R_1 R_2 w^{-2\ell_1} y_1^- + T_2 R_3 w^{-2\ell_2} y_2^+. \end{aligned}$$

We now substitute $w = h^{\gamma+o(1)}$, $T_j = h^{o(1)}$, and $R_j = h^{\beta_j+o(1)}$.

That gives

$$y_2^+ = h^{\beta_1 - 2\ell_1\gamma + o(1)} y_1^- + h^{\beta_2+\beta_3-2\ell_2\gamma+o(1)} y_2^+, \tag{18}$$

$$y_1^- = h^{\beta_1+\beta_2-2\ell_1\gamma+o(1)} y_1^- + h^{\beta_3-2\ell_2\gamma+o(1)} y_2^+. \tag{19}$$

We now consider three cases according to whether the y_2^+ terms on the left and on the right of (18) have comparable sizes or whether one dominates the other.

Case I. If the sizes are comparable, i.e., if

$$\beta_2 + \beta_3 - 2\ell_2\gamma = 0 \tag{20}$$

then we use (15), and observe that the $w^{2\ell_1+2\ell_2}$ and $R_2R_3w^{2\ell_1}$ terms both equal $h^{\beta_2+\beta_3+2\ell_1\gamma+o(1)}$, $R_1R_2w^{2\ell_2} = h^{\beta_1+2\beta_2+\beta_3+o(1)}$, and $R_1(1+2R_2)R_3 = h^{\beta_1+\beta_3+o(1)}$. So the $w^{2\ell_1+2\ell_2}$ and $R_2R_3w^{2\ell_1}$ terms need to be at least as big as the $R_1(1+2R_2)R_3$ term, and they need to cancel one another; the former condition means we need $\beta_2+\beta_3+2\ell_1\gamma \le \beta_1+\beta_3$, i.e., by (20), $\beta_2\ell_2+\beta_2\ell_1+\beta_3\ell_1 \le \beta_1\ell_2$.

Case II. If the term on the right is dominant, i.e., if $\beta_2+\beta_3-2\ell_2\gamma < 0$, then (18) becomes

$$y_2^+ = h^{\beta_1-\beta_2-\beta_3+2\ell_2\gamma-2\ell_1\gamma+o(1)}y_1^-,$$

which, inserted into (19), gives

$$y_1^- = h^{\beta_1+\beta_2-2\ell_1\gamma+o(1)}y_1^- + h^{\beta_1-\beta_2-2\ell_1\gamma+o(1)}y_1^- = h^{\beta_1-\beta_2-2\ell_1\gamma+o(1)}y_1^-.$$

Hence $0 = \beta_1-\beta_2-2\ell_1\gamma$, or $\gamma = \frac{\beta_1-\beta_2}{2\ell_1}$. This requires $(\beta_2+\beta_3)\ell_1 < (\beta_1-\beta_2)\ell_2$.

Case III: If the term on the left is dominant, i.e., if $\beta_2+\beta_3-2\ell_2\gamma > 0$, then (18) becomes

$$y_2^+ = h^{\beta_1-2\ell_1\gamma+o(1)}y_1^-,$$

which, inserted into (19), gives

$$y_1^- = h^{\beta_1+\beta_2-2\ell_1\gamma+o(1)}y_1^- + h^{\beta_1+\beta_3-2\ell_1\gamma-2\ell_2\gamma+o(1)}y_1^-.$$

Of these three terms, two must be of the same size and the other must be no bigger. We accordingly have three subcases.

Subcase 1: If the term on the left is the small one, then $\beta_1+\beta_2-2\ell_1\gamma = \beta_1+\beta_3-2\ell_1\gamma-2\ell_2\gamma \le 0$. That means $\gamma = \frac{\beta_3-\beta_2}{2\ell_2}$ and we require $\beta_1\ell_2+\beta_2\ell_2+\beta_2\ell_1 \le \beta_3\ell_1$. This contradicts (17).

Subcase 2: If the first term on the right is the small one, then $0 = \beta_1+\beta_3-2\ell_1\gamma-2\ell_2\gamma \le \beta_1+\beta_2-2\ell_1\gamma$. That means $\gamma = \frac{\beta_1+\beta_3}{2\ell_1+2\ell_2}$, and then we need $(\beta_2+\beta_3)(\ell_1+\ell_2) > (\beta_1+\beta_3)\ell_2 \ge (\beta_3-\beta_2)(\ell_1+\ell_2)$ which is equivalent to $\beta_2\ell_1+\beta_2\ell_2+\beta_3\ell_1 > \beta_1\ell_2 \ge \beta_3\ell_1-\beta_2\ell_1-\beta_2\ell_2$.

Subcase 3: If the second term on the right is the small one, then $0 = \beta_1+\beta_2-2\ell_1\gamma \le \beta_1+\beta_3-2\ell_1\gamma-2\ell_2\gamma$. That means $\gamma = \frac{\beta_1+\beta_2}{2\ell_1}$ and we require $\ell_2(\beta_1+\beta_2) \le (\beta_3-\beta_2)\ell_1$. This contradicts (17).

In summary, under the assumption (17), we have three possible values of γ, each with a corresponding necessary condition on the coefficients:

- If $\gamma = \frac{\beta_2+\beta_3}{2\ell_2}$, then $\beta_2\ell_1+\beta_2\ell_2+\beta_3\ell_1 \le \beta_1\ell_2$.
- If $\gamma = \frac{\beta_1-\beta_2}{2\ell_1}$, then $\beta_2\ell_1+\beta_2\ell_2+\beta_3\ell_1 < \beta_1\ell_2$.
- If $\gamma = \frac{\beta_1+\beta_3}{2\ell_1+2\ell_2}$, then $\beta_1\ell_2 < \beta_2\ell_1+\beta_2\ell_2+\beta_3\ell_1$.

The conclusions of the theorem follow from these. □

4. N deltas

In this section we generalize the observations in the special cases of two and three delta-poles to the general case of N poles: the main tool, as before, is simply examination of the leading terms in the (generally transcendental) equations determining their location. To aid in understanding those terms, we begin by introducing the machinery of Newton polygons, a traditional tool in the study of resolution of plane algebraic curves which also applies in the setting studied here.

4.1. **Newton polygons.** Here we explore how Newton polygons apply to the analysis of equations of a form generalizing e.g. (15) which (as we will show below) arise in the study of the more general case. In particular, let

$$p(h,w) = \sum_{j=0}^{N} h^{\nu_j + o(1)} w^{\lambda_j},$$

where all exponents ν_j, λ_j are real and nonnegative. Note, we take the polygon to contain the semi-infinite horizontal and vertical segments, which seems to differ slightly from standard conventions in the Newton Polygon literature.

Definition 5. The Newton polygon is of p is the boundary of the convex hull of the union of the first quadrants displaced to have vertices at the points (ν_j, λ_j):

$$\partial \operatorname{Conv} \bigcup_{j=0}^{N} ((\nu_j, \lambda_j) + [0,\infty)^2).$$

See Figure 1 for an example, and see [4, Section 8.3] for the classical algebro-geometric theory of Newton polygons.

Lemma 6. *Consider the equation*

$$p(h,w) \equiv \sum_{j=0}^{N} h^{\nu_j + o(1)} w^{\lambda_j} = 0, \tag{21}$$

where all exponents ν_j, λ_j are real and nonnegative. Suppose that, for all $j \geq 1$, we have $\nu_j > \nu_0$ and $\lambda_j < \lambda_0$. Fix any $M > 0$. Then any sequence of roots of the equation $w = w(h)$ for $h \in (0,1)$ with $|w|^{-1} = O(h^{-M})$ has a subsequence that asymptotically satisfies (as $h \downarrow 0$)

$$\log |w| \sim \gamma \log h$$

where $-1/\gamma$ is one of the finitely many nonzero slopes occurring in the Newton polygon of p.

Proof. Say we have a family of solutions $w = w(h)$ with $h \downarrow 0$ and (without loss of generality) with $|w|^{-1} \leq h^{-M}$. Then since $\log |w| \geq M \log h$, the ratio $\log |w| / \log h$ lies in $(-\infty, M)$, hence along a subsequence, $\log |w| / \log h$ converges to $\gamma \in [-\infty, M]$.

We first rule out the case $\gamma \leq 0$, much as in the proof of Theorem 3. If $\gamma \leq 0$ then for any $\epsilon > 0$, $|w| > h^{\epsilon}$ for h sufficiently small. Hence for all $j \geq 1$, choosing ϵ sufficiently small yields

$$\frac{h^{\nu_j + o(1)} |w|^{\lambda_j}}{h^{\nu_0 + o(1)} |w|^{\lambda_0}} \leq h^{\nu_j - \nu_0 + o(1)} h^{-\epsilon(\lambda_0 - \lambda_j)} \to 0.$$

Thus the term in p given by $h^{\nu_0 + o(1)} w^{\lambda_0}$ is dominant, and it cannot be cancelled by the other terms and hence (21) cannot hold. Hence we may take $\gamma > 0$ finite and assume, passing to our subsequence, that $\log |w| = (\gamma + o(1)) \log h$.

Then

$$\sum_{j=0}^{N} h^{\nu_j + \gamma \lambda_j + o(1)} = 0. \tag{22}$$

If the minimum exponent $\nu_j + \gamma\lambda_j$ occurring in the sum is unique, then as $h \downarrow 0$, the term $h^{\nu_j+\gamma\lambda_j+o(1)}$ dominates all other terms in the sum for h sufficiently small, hence (21) again cannot hold. So the minimum exponent in (22) must occur in at least two terms, say j and k; in particular, then,

$$\nu_j + \gamma\lambda_j = \nu_k + \gamma\lambda_k,$$

and $\gamma = -(\nu_k - \nu_j)/(\lambda_k - \lambda_j)$ is the negative reciprocal of the slope of the line connecting these two points.

We claim that the minimality of the exponent

$$\rho \equiv \nu_j + \gamma\lambda_j = \nu_k + \gamma\lambda_k$$

further entails that the segment $\overline{(\nu_j,\lambda_j)(\nu_k,\lambda_k)}$ is in the Newton polygon, which will complete our characterization of γ as the negative reciprocal of the slope of a segment of the Newton polygon. To see this, we first observe that minimality of ρ means for every i, $\rho \leq \nu_i + \gamma\lambda_i$. Since for every $s \in \mathbb{R}$,

$$\rho = s\nu_j + (1-s)\nu_k + \gamma(s\lambda_j + (1-s)\lambda_k)$$

the point $s(\nu_j,\lambda_j) + (1-s)(\nu_k,\lambda_k)$ cannot lie in the quadrant $(\nu_i,\lambda_i) + (0,\infty)^2$, as this would imply $\rho > \nu_i + \gamma\lambda_i$. Thus we have shown that minimality of ρ means that the interior of every quadrant $(\nu_i,\lambda_i) + [0,\infty)^2$ lies above the line

$$L \equiv \{s(\nu_j,\lambda_j) + (1-s)(\nu_k,\lambda_k)\colon s \in \mathbb{R}\},$$

hence the convex hull of the quadrants $\underline{(\nu_i,\lambda_i) + [0,\infty)^2}$ lies entirely in the closed half-space above L. Since the segment $\overline{(\nu_j,\lambda_j)(\nu_k,\lambda_k)}$ of L does lie in the convex hull of the vertices, it must be in the Newton polygon, as asserted. □

4.2. **Analysis of the secular determinant.** We now employ the method of Newton polygons introduced above to analyze the case of N delta poles; the main problem is to find a good description of the secular determinant arising in the equations for a putative resonant state.

In the following, we employ multiindex notation for combinations of exponents β_j $(j = 1,\dots,N)$ and lengths ℓ_j $(j = 1,\dots,N)$, e.g. writing $\sigma\cdot\beta = \sum_j \sigma_j\beta_j$.

Note that our result on this general case of N deltas, like our Theorem 3 on three arbitrarily spaced deltas, focuses on resonances in a narrower region of $\mathbb{C}$ than in Theorems 1 and 2: the imaginary part is a priori $O(h\log(1/h))$.

Theorem 7. *Consider the Hamiltonian on the real line*

$$P = -h^2\partial_x^2 + V(x), \qquad V(x) = \sum_{j=1}^{N} V_j\delta(x - x_j),$$

where $x_1 < \cdots < x_N$, and each $V_j = C_j h^{1+\beta_j}$ for some $C_j \in \mathbb{R}$ and $\beta_j > 0$.

Let $z = z(h_j)$ be a sequence of resonances such that $z = h^{o(1)}$ (as in Theorem 3) and $\operatorname{Im} z \geq -Mh\log(1/h)$ for some positive M. Then this sequence has a subsequence such that $\operatorname{Im} z \sim -\gamma h\log(1/h)$ where γ is one of at most $2^{N-1} - 1$ values. All possible values of γ are positive numbers of the form

$$\frac{\sigma^+\cdot\beta - \sigma^-\cdot\beta}{2(\alpha^+\cdot\ell - \alpha^-\cdot\ell)}$$

for some $\sigma^\pm \in \{0,1,2\}^N$ and $\alpha^\pm \in \{0,1\}^{N-1}$, where $\ell_j = x_{j+1} - x_j$.

Remark 8. As of the publication of this paper, Theorem 7 has been considerably refined in work of Brady [3], who obtains sharp estimates on the number of values of γ that may arise.

Proof. Setting $w = e^{-iz/h}$, we recall that the condition $\operatorname{Im} z \geq -Mh\log(1/h)$ yields $|w|^{-1} \leq h^{-M}$.

We collect the components $(y_1^-, y_1^+, \dots, y_{N-1}^-, y_{N-1}^+)$ into a vector, which lies in the nullspace of $A_N - I$ where A_N is the $2(N-1) \times 2(N-1)$ matrix given by the equations (6), (8):

$$\begin{pmatrix} 0 & R_2 w^{-\ell_1} & T_2 w^{-\ell_2} & 0 & 0 & \dots & 0 & 0 & 0 \\ R_1 w^{-\ell_1} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & R_3 w^{-\ell_2} & T_3 w^{-\ell_3} & \dots & 0 & 0 & 0 \\ 0 & T_2 w^{-\ell_1} & R_2 w^{-\ell_2} & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & R_N w^{-\ell_{N-1}} \\ 0 & 0 & 0 & 0 & 0 & \dots & T_{N-1} w^{-\ell_{N-2}} & R_{N-1} w^{-\ell_{N-1}} & 0 \end{pmatrix}.$$

The general pattern is that of a pentadiagonal matrix with zeros on the diagonal and overlapping blocks

$$\begin{pmatrix} 0 & 0 & R_{j+1} w^{-\ell_j} & T_{j+1} w^{-\ell_{j+1}} \\ T_j w^{-\ell_{j-1}} & R_j w^{-\ell_j} & 0 & 0 \end{pmatrix},$$

which arises in the rows y_j^-, y_j^+ and columns $y_{j-1}^+, y_j^-, y_j^+, y_{j+1}^-$.

Note that in the base case $N = 2$ we get the matrix

$$\begin{pmatrix} 0 & R_2 w^{-\ell_1} \\ R_1 w^{-\ell_1} & 0 \end{pmatrix}$$

and

$$\det(A_2 - I) = 1 - R_1 R_2 w^{-2\ell_1}. \tag{23}$$

We claim that just as in this example, we always get only even powers of $w^{-\ell_j}$, i.e. that we may, more generally, express

$$\det(A_N - I) = \sum_{\alpha \in \{0,1\}^{N-1}} a_\alpha w^{-2\alpha \cdot \ell} \tag{24}$$

where the coefficients a_α are composed of (unspecified) sums of products of T_j and R_j's. This will follow from the following more general lemma.

Lemma 9. *Let W_N be a $2(N-1) \times 2(N-1)$ matrix of the form*

$$\begin{pmatrix} \sigma_1 & R_2 w^{-\ell_1} & T_2 w^{-\ell_2} & 0 & 0 & \dots & 0 & 0 & 0 \\ R_1 w^{-\ell_1} & \sigma_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & R_3 w^{-\ell_2} & T_3 w^{-\ell_3} & \dots & 0 & 0 & 0 \\ 0 & T_2 w^{-\ell_1} & R_2 w^{-\ell_2} & \sigma_4 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \sigma_{2N-3} & R_N w^{-\ell_{N-1}} \\ 0 & 0 & 0 & 0 & 0 & \dots & T_{N-1} w^{-\ell_{N-2}} & R_{N-1} w^{-\ell_{N-1}} & \sigma_{2N-2} \end{pmatrix}$$

where each $\sigma_j \in \{0, -1\}$. Then $\det W_N$ is of the form.

$$\sum_{\alpha \in \{0,1\}^{N-1}} a_\alpha w^{-2\alpha \cdot \ell} \tag{25}$$

where each a_α is a sum of products of T_j and R_j's.

The greater generality of taking σ_j terms on the diagonal rather than all -1's is of no interest except that it enables the following inductive proof to work.

Proof of Lemma. The result holds for $N = 2$ since we get $\sigma_1\sigma_2 - R_1R_2w^{-2\ell_1}$. We now proceed inductively. For brevity we denote an entry of the form $R_iw^{-\ell_j}$ or $T_iw^{-\ell_j}$ simply L_j (as we will never employ any cancellation among terms, the ambiguity in the index i and the difference between T_i and R_i are of no importance); we also write $\pm$ to be independent and completely unimportant signs in the following computation. We simply need to show that each L_j appears in each summand in the determinant either not at all or as L_j^2.

In our abbreviated notation, we now have

$$W_N = \begin{pmatrix} \sigma_1 & L_1 & L_2 & 0 & 0 & \dots & 0 & 0 & 0 \\ L_1 & \sigma_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & L_2 & L_3 & \dots & 0 & 0 & 0 \\ 0 & L_1 & L_2 & \sigma_4 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \sigma_{2N-3} & L_{N-1} \\ 0 & 0 & 0 & 0 & 0 & \dots & L_{N-2} & L_{N-1} & \sigma_{2N-2} \end{pmatrix}.$$

Decomposing W_N by cofactors in the first column yields

$$\begin{aligned} &\sigma_1 \det \begin{pmatrix} \sigma_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \sigma_3 & L_2 & L_3 & \dots & 0 & 0 & 0 \\ L_1 & L_2 & \sigma_4 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \sigma_{2N-3} & L_{N-1} \\ 0 & 0 & 0 & 0 & \dots & L_{N-2} & L_{N-1} & \sigma_{2N-2} \end{pmatrix} \\ &\quad - L_1 \det \begin{pmatrix} L_1 & L_2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \sigma_3 & L_2 & L_3 & \dots & 0 & 0 & 0 \\ L_1 & L_2 & \sigma_4 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \sigma_{2N-3} & L_{N-1} \\ 0 & 0 & 0 & 0 & \dots & L_{N-2} & L_{N-1} & \sigma_{2N-2} \end{pmatrix} \\ &\quad \equiv \sigma_1 \det B_N - L_1 \det C_N. \end{aligned} \tag{26}$$

We deal with these terms as follows. Decomposing B_N further by cofactors in its first row gives a single term that equals $\sigma_1\sigma_2$ times the determinant of a matrix of the form W_{N-1}, which by the inductive hypothesis is a sum of terms of the form coefficient times $L_2^{2\alpha_2} \dots L_{N-1}^{2\alpha_{N-1}}$ with $\alpha_j \in \{0, 1\}$; hence this term is of the desired form.

Likewise, decomposing $\det C_N$ by cofactors in the first column gives, from the top left L_1 entry, a term L_1^2 times a term of the form $\det W_{N-1}$, hence yields a sum of terms $L_1^2 L_2^{2\alpha_2} \dots L_{N-1}^{2\alpha_{N-1}}$ by the inductive hypothesis. Finally the L_1 entry in position $(3,1)$ gives a term

$$L_1^2 \det \begin{pmatrix} L_2 & 0 & 0 & \dots & 0 & 0 & 0 \\ \sigma_3 & L_2 & L_3 & \dots & 0 & 0 & 0 \\ 0 & 0 & \sigma_5 & L_3 & L_4 & \dots & 0 \\ 0 & L_2 & L_3 & \sigma_6 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \sigma_{2N-3} & L_{N-1} \\ 0 & 0 & 0 & \dots & L_{N-2} & L_{N-1} & \sigma_{2N-2} \end{pmatrix} \equiv L_1^2 \det D_N.$$

Now exchanging the first two rows of D_N gives another matrix of the form W_{N-1} but where the $(2,2)$ entry is necessarily 0 (rather than allowed to be -1). Thus by induction, this term is also of the desired form. Note that it was this last case that necessitated allowing the more general σ_j entries on the diagonal in the inductive hypothesis. This completes the proof of the lemma. $\square$

We have now established (24). Recall that the coefficients a_α are of the form of $\pm$ products of reflection and transmission coefficients T_j and R_j given by (7), with Υ_j given by (5). In the region $z = h^{o(1)}$, we have $\Upsilon_j = h^{\beta_j + o(1)}$, hence

$$\begin{aligned} T_j &= 1 + h^{\beta_j + o(1)}, \\ R_j &= h^{\beta_j + o(1)}, \end{aligned} \tag{27}$$

which implies that the terms a_α are all of the form $h^{\mu + o(1)}$ for some values of μ given by sums of powers β_j occurring in the reflection coefficients R_j. Since each R_j appears in at most two rows, we note that the only possibilities for the appearance of R_j in a coefficient a_α are as $R_j^{\sigma_j}$ with $\sigma_j \in \{0,1,2\}$.

We have now established that the equation

$$\det(A_N - I) = 0$$

is of the form

$$\sum_{\alpha \in \{0,1\}^{N-1}} h^{\mu_\alpha + o(1)} w^{-2\alpha\cdot\ell} = 0. \tag{28}$$

We now claim further that all terms except the term 1 are of the form $h^{\mu_\alpha + o(1)} w^{-2\alpha\cdot\ell}$ where $\alpha \neq 0$ and $\mu_\alpha > 0$. By (27), this follows from the following lemma. As above we use the notation L_i to be either $T_i w^{-\ell_j}$ for some i, j, or $R_i w^{-\ell_j}$.

Lemma 10. *Every term in the secular determinant* $\det(A_N - I)$ *except the diagonal term* 1 *is of the form* $R_j E w^{-\alpha\cdot\ell}$ *where* E *is some product of reflection and transmission coefficients and* $\alpha \neq 0$.

In other words, each nonconstant term has at least one reflection coefficient and a negative power of w.

Proof. We again work by induction. By (23), the result certainly holds for $N = 2$. Cofactor decomposition as above in the first column then yields

$$\det(A_N - I) = (-1)\det B_N - R_1 w^{-\ell_1} \det C_N.$$

Since $\det C_N$ is a sum of product of reflection and transmission coefficients and negative powers of w, the second term certainly satisfies the desired conclusion, so we need only examine the first. The matrix B_N is given by

$$B_N = \begin{pmatrix} -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & R_3 w^{-\ell_2} & T_3 w^{-\ell_3} & \dots & 0 & 0 & 0 \\ T_2 w^{-\ell_1} & R_2 w^{-\ell_2} & -1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & R_N w^{-\ell_{N-1}} \\ 0 & 0 & 0 & 0 & \dots & T_{N-1} w^{-\ell_{N-2}} & R_{N-1} w^{-\ell_{N-1}} & -1 \end{pmatrix}.$$

Cofactor expansion in the first row now allows us to write $\det B_N = -\det(B_{N-1})$, a secular determinant of the same form as $\det(A_N - I)$, hence the lemma now follows by induction. □

Now we return to the representation (28) of the secular determinant. Multiplying through by $w^{2|\ell|}$ (with $|\ell| \equiv \sum \ell_j$ in multiindex notation) gives an equation with positive powers of w :

$$\sum_{\alpha \in \{0,1\}^{N-1}} h^{\mu_\alpha + o(1)} w^{2(|\ell| - \alpha\cdot\ell)} = 0. \tag{29}$$

Here the leading powers μ_α are all sums of powers arising in the delta potentials of the form

$$\mu = \sigma \cdot \beta$$

for $\sigma \in \{0,1,2\}^N$. Moreover by the preceding lemma there is a "leading" term $h^0 w^{2|\ell|}$, with all other terms having *both* a higher power of h and a lower power of w.

Thus Lemma 6 applies to show that any sequence of solutions to this equation with $|w|^{-1} = O(h^{-M})$ has a subsequence with $\log|w| \sim \gamma \log h$ for γ a strictly negative reciprocal slope arising in the Newton polygon associated to the function (29). Since there are at most 2^{N-1} distinct powers of w in this equation there are at most $2^{N-1} - 1$ nonzero finite slopes in the Newton polygon, and γ may only take the negative reciprocal of one of these values. Note further that owing to our characterization of the exponents of h and w, all γ's are thus of the form

$$\frac{\sigma^+ \cdot \beta - \sigma^- \cdot \beta}{2(\alpha^+ \cdot \ell - \alpha^- \cdot \ell)}$$

for some $\sigma^\pm \in \{0,1,2\}^N$ and $\alpha^\pm \in \{0,1\}^{N-1}$.

Now given

$$\log|w| \sim \gamma \log h$$

and $w = e^{-iz/h}$ we of course get

$$\operatorname{Im} z \sim -\gamma h \log(1/h)$$

as desired. □

Remark 11. It is instructive to compare the general result of Theorem 7 to the special cases of two and three poles analyzed above. In the case of two delta poles, Theorem 7 correctly implies that as $h \downarrow 0$ there can be at most a single curve of resonances $\operatorname{Im} z \sim -\gamma h \log(1/h)$ within any set $\operatorname{Im} z > -Mh\log(1/h)$ with M fixed. In the case of three deltas, however, the bound given by this theorem is that there can be at most 3 such curves, while Theorem 3 shows that 2 is in fact the sharp maximum number of resonance lines. This discrepancy is clearer if we examine the Newton polygon for (15): recalling that $R_j \sim C_j h^{\beta_j}$ we see that the vertices involved are

$$(\beta_1 + \beta_3, 0), \ (\beta_2 + \beta_3, 2\ell_1), \ (\beta_1 + \beta_2, 2\ell_2), \ (0, 2\ell_1 + 2\ell_2).$$

A priori, this many vertices could yield a Newton diagram with 3 nonvanishing finite slopes, hence we conclude naively from Theorem 7 that there could be at most 3 possible values of γ. Note, encouragingly, that the form of the secular determinant established in the proof of Theorem 7 is indeed giving the sharp overall form of the equation (15). But it turns out on closer inspection of the equation that not every possible Newton polygon can arise here. In particular, under the assumption (17) (which we recall is always valid up to reversing the x-coordinate), the vertex $(\beta_1 + \beta_2, 2\ell_2)$ always lies strictly above the line $\overline{(0, 2\ell_1 + 2\ell_2, 0)(\beta_1 + \beta_3, 0)}$, hence cannot lie in the Newton polygon. Thus there can be either two or one nonzero finite slopes in the Newton polygon, depending on whether $(\beta_2 + \beta_3, 2\ell_1)$ lies below or above this line; this is determined by the condition

$$\beta_1 \ell_2 \gtrless \beta_2 \ell_1 + \beta_2 \ell_2 + \beta_3 \ell_1,$$

i.e. agrees with the analysis of the cases in Remark 4. (See Figure 1.)

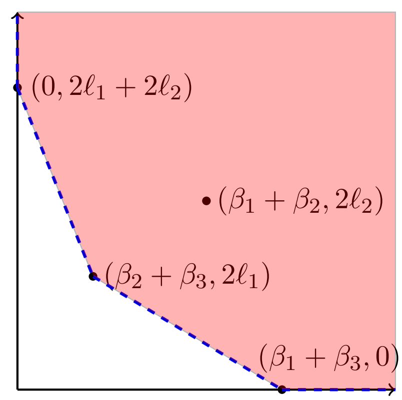

FIGURE 1. Newton polygon for the case $N = 3$. The Newton polygon is the union of the dashed lines forming the boundary of the shaded region. The point $(\beta_1 + \beta_2, 2\ell_2)$ does not lie on the boundary of the shaded region, i.e., is not in the Newton polygon. This depicts the case $\beta_2\ell_1 + \beta_2\ell_2 + \beta_3\ell_1 < \beta_1\ell_2$, which guarantees that $(\beta_2 + \beta_3, 2\ell_1)$ does lie in the Newton polygon, hence two distinct nonzero finite slopes arise.

5. Some numerical studies and discussion of the results

We can program the secular determinant matrix $A_N = A_N(z)$ from the proof of Theorem 7 into the software program *Mathematica*, and study the resulting complex equations

$$\det(I - A_N(z)) = 0. \tag{30}$$

Resonances occur at solutions to (30). It is particularly informative to plot the argument of the left hand side of (30); then poles become clear points about which the phase angle winds. Such plots allow us to numerically observe the results in Theorems 1, 2, and 3 in the case of 2 or 3 delta functions, and to test the bounds of what we can prove in the general case in Theorem 7. Our findings are presented in Figure 2 and Figure 3 respectively. Throughout, we have taken $h = 10^{-6}$ and plotted the argument on a region of the complex plane such that $1 - 3h < \operatorname{Re} z < 1 + 3h$ and $-3h < \operatorname{Im} z < 0$.

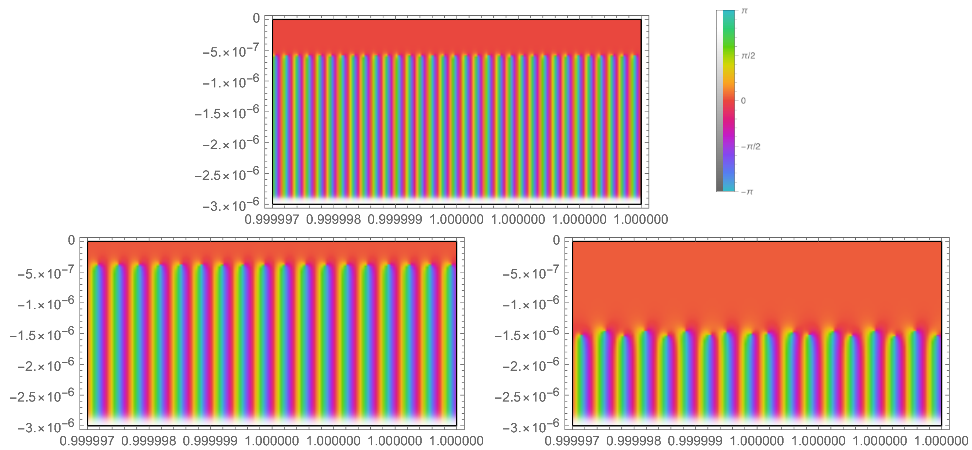

FIGURE 2. (Top) A plot showing the resonances arising in the setting of $N = 2$ delta functions, and a legend for the plot showing the correspondence between colors and complex arguments of the left hand side of (30). (Bottom) The cases of $N = 3$ delta functions in the setting of one line of resonances from Theorem 3, Case (1) (Left) and two lines of resonances from Theorem 3, Case (2) (Right).

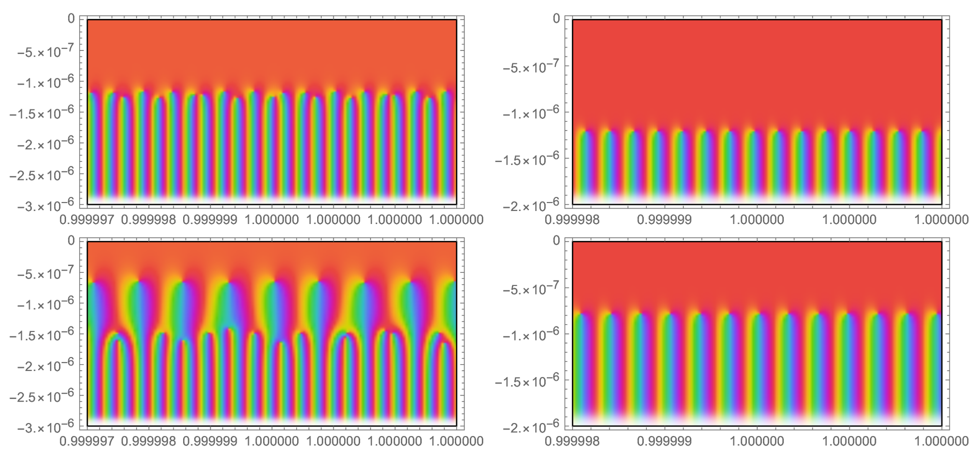

FIGURE 3. (Top) A plot showing the resonances arising in the setting of $N = 5$. (Bottom) The cases of $N = 6$. In both cases, by varying values of β and ℓ, we can generate either multiple resonance lines or only one line.

In Figure 2, we observe that for $N = 2$, we have one line of resonances as in Theorem 1. This line is demonstrated in the top image with $\beta_1 = 1$, $\beta_2 = .5$, $x_1 = -10$ and $x_2 = 5\sqrt{2}$, which give

$$\operatorname{Im} z \sim -\frac{\beta_1 + \beta_2}{2(x_2 - x_1)} h \log(1/h) \approx -6 \cdot 10^{-7}.$$

Meanwhile for $N = 3$, we can choose $\beta_1, \beta_2, \beta_3$ such that we are either in the setting of Theorem 3, Case (1) (bottom left) or Theorem 3, Case (2) (bottom right). In these cases, we took $x_1 = -5$, $x_2 = 0$, $x_3 = 3\sqrt{2}$, with $\beta_1 = \beta_2 = \beta_3 = 1$ and hence

$$\operatorname{Im} z \sim -\frac{\beta_1 + \beta_3}{2(x_3 - x_1)} h \log(1/h) \approx -3 \cdot 10^{-7}.$$

for the image on the left and $\beta_1 = .9, \beta_2 = .1, \beta_3 = 1$ and hence

$$\operatorname{Im} z_+ \sim -\frac{\beta_3 - \beta_2}{2(x_3 - x_2)} h \log(1/h) \approx -1.5 \cdot 10^{-6},$$

$$\operatorname{Im} z_- \sim -\frac{\beta_1 + \beta_2}{2(x_2 - x_1)} h \log(1/h) \approx -1.4 \cdot 10^{-6},$$

for the image on the right. These plots match the results of our theorem perfectly.

In Figure 3, we demonstrate that in the case of either $N = 5$ or $N = 6$, we can achieve a variety of outcomes. Indeed, setting $\beta_j = 1$ for all j, we observe what appears to be a single line of resonances looking at the figures on the left. Selecting β values that lead to different interaction strengths, we convincingly observe three resonance lines in the top right plot computed with $x_1 = -5$, $x_2 = -\sqrt{2}$, $x_3 = 0$, $x_4 = 2\sqrt{2}$, $x_5 = 7$ and $\beta_1 = 1$, $\beta_2 = .6$, $\beta_3 = .1$, $\beta_4 = .6$, $\beta_5 = 1$. A similar result holds for 6 δ functions in the bottom right plot using $x_1 = -7$, $x_2 = -2\sqrt{2}$, $x_3 = -\pi/4$, $x_4 = \sqrt{2}$, $x_5 = e$, $x_6 = 5$ and $\beta_1 = 1$, $\beta_2 = .1$, $\beta_3 = .5$, $\beta_4 = .2$, $\beta_5 = .5$, $\beta_6 = 1$.

We thus observe that while it does appear possible to generate multiple strings of resonances, the $2^{N-1} - 1$ upper bound of Theorem 7 may be far from optimal. Indeed, it is intriguing that in the case when all β values are equal, the numerical findings are that there is only one string of resonances. Hence there may, for instance, be symmetry reductions that allow us to dramatically improve the bounds on the number of resonance lines.

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