CAUSTICS OF WEAKLY LAGRANGIAN DISTRIBUTIONS

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Abstract. We study semiclassical sequences of distributions u_h associated to a Lagrangian submanifold of phase space $\mathcal{L} \subset T^*X$. If u_h is a semiclassical Lagrangian distribution, which concentrates at a maximal rate on $\mathcal{L}$, then the asymptotics of u_h are well-understood by work of Arnol'd, provided $\mathcal{L}$ projects to X with a stable simple Lagrangian singularity. We establish sup-norm estimates on u_h under much more general hypotheses on the rate at which it is concentrating on $\mathcal{L}$ (again assuming a stable simple projection). These estimates apply to sequences of eigenfunctions of integrable and KAM Hamiltonians.

1. Introduction

Let X be a smooth n-dimensional manifold. Let $p(x, \xi) \in \mathcal{C}^\infty(T^*X; \mathbb{R})$ be a Hamiltonian function, and $P_h \in \Psi_h(X)$ a self-adjoint pseudodifferential operator with principal symbol p. If the Hamilton flow associated to p is integrable, the phase space T^*X is foliated by invariant Arnol'd–Liouville Lagrangian tori on which the flow is quasi-periodic [5]; if p is a perturbation of an integrable Hamiltonian, the KAM theorem [16], [2], [18] ensures that certain invariant tori on which the frequencies of motion satisfy a Diophantine condition still survive the perturbation.

Now let u_h be a sequence of eigenfunctions of P_h, i.e., $P_h u_h = E_h u_h$ with $h \downarrow 0$, and where $E_h = E + O(h)$. We recall that the *semiclassical wavefront set* $\mathrm{WF}_h\, u_h$ is a measure of where, in phase space, a sequence of eigenfunctions may concentrate as $h \downarrow 0$, and that it is known to lie in the characteristic set $\{p = E\}$, and to be invariant under the Hamilton flow of p. $\mathrm{WF}_h\, u_h$ may thus concentrate on a single Arnol'd-Liouville torus in integrable or near-integrable systems, and in the case of the Diophantine tori in the latter setting, may not concentrate on any proper subset (as it is closed and invariant under an irrational flow). Sequences of eigenfunctions of this type are thus the quantum analogue of classical states that have well-defined values of the commuting variables, in the integrable case, or that remain in quasi-periodic motion in the KAM setting. Some research has been devoted to understanding the properties of these sequences of eigenfunctions concentrating on Lagrangian tori; for instance Galkowski–Toth [11] studied sup-norm estimates in the case in which the system is *quantum* completely integrable, with the eigenfunctions being joint eigenfunctions of a family of

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commuting operators whose symbols cut out the invariant torus. Very little is known in the KAM case, however.

In this paper, we study the most general setting in which a family of eigenfunctions u_h may concentrate along a Lagrangian submanifold $\mathcal{L}$ of T^*X. In particular, *we do not assume that u_h is a Lagrangian distribution,* i.e. it does not necessarily enjoy semiclassical Lagrangian regularity; this notion (defined below) would presuppose that the rate of concentration of u_h along $\mathcal{L}$ occurs at maximal possible rate. By contrast, we will only assume that there is *some* quantitative rate of concentration on $\mathcal{L}$, and our results reflect this rate explicitly. The sup norm estimates also depend (as is well-known in the case of Lagrangian distributions) on the *singularities* of the projection to the base of the Lagrangian in question. The critical values of the projection map $\pi : \mathcal{L} \to X$ are referred to as a *caustic*, and the concentration of mass of u_h near such points is a familiar phenomenon from everyday life, for instance in the brighter image of a light source on the surface of one's tea at points where rays are focused by the side of cup. The study of such phenomena has a long history—see, e.g., [7, f.87]. While in general the critical values of π may be quite wild, we confine our attention here to the finite list of *stable simple singularities* developed by Arnol'd [4, Corollary 11.5]; in dimension not exceeding 5, every Lagrangian projection can be perturbed to have a singularity in this list [4, Corollary 11.7]. In the case of actual Lagrangian distributions, our results reduce to the classical descriptions of the asymptotics of caustics in [3], [9], [13]. By contrast, our results are nontrivial even in the case where $\mathcal{L}$ projects diffeomorphically onto the base (see §2 below), as the rate of concentration on the torus affects the rate of growth strongly in every case.

We measure the rate of concentration of u_h along h by an *iterated regularity* definition. Let us suppose that we normalize to $\|u_h\|_{L^2} = 1$. If the Lagrangian were simply $\mathcal{L} \equiv \{x = 0\} \subset T^*\mathbb{R}^n$, the rate at which a family of distributions concentrates on $\mathcal{L}$ could be given by asking how much smaller $x^\alpha u_h$ is than u_h as $h \downarrow 0$; we might, for instance, ask that

$$\|x^\alpha u_h\|_{L^2} = O(h^{(1-\delta)|\alpha|}),$$

for some $\delta \in [0, 1]$. This is a special case of the following general definition. In what follows, $\Psi_h^{-\infty}(X)$ denotes the algebra of semiclassical pseudodifferential operators on X with rapidly-decreasing symbols, and $\sigma_h \colon \Psi_h^{-\infty}(X) \to \mathcal{C}^\infty(T^*X)$ denotes the principal symbol map [21, Chapter 14].

Definition 1.1. Let $\mathcal{L} \subset T^*X$ be a compact Lagrangian submanifold and let $\delta \in [0, 1]$. We say that u_h is a δ-Lagrangian distribution with respect to $\mathcal{L}$, if for all N and all $A_1, \dots A_N \in \Psi_h^{-\infty}(X)$ such that $\sigma_h(A_j) = 0$ on $\mathcal{L}$, u_h enjoys the iterated regularity property

$$\|A_1 \dots A_N u_h\|_{L^2(X)} \leq C_N h^{N(1-\delta)}, \quad h \in (0, 1).$$

When $\delta = 0$ this is the usual definition of semiclassical Lagrangian regularity—cf. [1]. When $\delta = 1$ the definition is satisfied for any $u_h \in L^2(X)$. For intermediate values of δ we thus have a notion of partial Lagrangian regularity, encoding a concentration of the states in question on a Lagrangian submanifold at a variable rate. (We do not consider $\delta > 1$, as this would not be achievable with u_h compactly microsupported, by the uncertaintly principle.)

Our main results are local sup-norm estimates for a semiclassical family of distributions u_h that are δ-Lagrangian with respect to $\mathcal{L}$, where $\mathcal{L}$ has a singular projection given by one of the stable simple singularities listed in Table 2 below. There are two versions of these estimates: in the first, we make no further assumptions, but in the second, stronger, estimate, we additionally assume that u_h satisfies an approximate eigenfunction equation (where we have now absorbed the eigenparameter into the operator)

$$P_h u_h = O_{L^2}(h)$$

where $\sigma(P_h) = 0$ on $\mathcal{L}$. Our estimates all involve a constraint on $\delta :$ it cannot exceed a threshold δ_0 that depends on the form of the caustic (but is equal to 1 in the nonsingular case). Beyond this threshold, the phenomenology seems intriguingly different, and for the special case of the fold singularity, we also give estimates for $\delta > \delta_0$, and see that there is indeed a change of qualitative behavior of extremizers (§6).

In the next section, we describe our results in the special case of the rectangular flat torus. In this setting, they are far from sharp, with improvements available using number-theoretic tools. We then recall the general geometric setting of stable simple Lagrangian singularities, and proceed to the proofs of the main theorems. The main ingredients here are, first, a recapitulation of the Hörmander–Melrose theory of Lagrangian distributions in the setting considered here, with limited regularity. This allows us to write a δ-Lagrangian distribution u_h as an oscillatory integral in which the amplitude function is not uniformly smooth as $h \downarrow 0$ but rather lies in an h-dependent symbol class satisfying

$$h^{-\delta|\alpha|} \partial^\alpha a \in h^{-\gamma} L^\infty$$

for some γ. We then estimate the size of the function on the caustic by estimating the resulting oscillatory integral. This integral estimate is well-known when $\delta = 0$ (i.e., the standard Lagrangian case)—see [3], [9], [13]. In the case at hand, however, the usual proof of this classical result fails to yield a sharp result: it employs the Malgrange Preparation Theorem in an essential way, and this entails a hard-to-quantify number of derivatives falling on the amplitude, incurring $h^{-\delta}$ penalties each time. We thus employ a different, cruder method that so far as we know is novel, where we split the integral into pieces to estimate sup-norms rather than obtaining the precise asymptotics along the caustic that are part of the classical theory.

Our main result is as follows.

Type	Order κ	Threshold δ_0
A_{m+1}	$\frac{1}{2}-\frac{1}{m+2}$	$\frac{1}{m+2}$, $(m>0)$; 1, $(m=0)$
D_{m+1} (m even), D^-_{m+1} (m odd)	$\frac{1}{2}-\frac{1}{2m}$	$\frac{1}{m+1}$
D^+_{m+1} (m odd)	$\frac{1}{2}-\frac{1}{2m}$	$\frac{1}{m}$
E_6	$\frac{5}{12}$	$\frac{1}{6}$
E_7	$\frac{4}{9}$	$\frac{1}{7}$
E_8	$\frac{7}{15}$	$\frac{1}{8}$

TABLE 1. Orders of caustics and thresholds of Lagrangian regularity.

Theorem 1.2. *Let u_h be a δ-Lagrangian distribution with respect to a Lagrangian $\mathcal{L}$, microsupported in a set where the projection of $\mathcal{L}$ has a singularity that is Lagrange-equivalent to one of the stable simple singularities listed in Table 1. Assume that $\delta < \delta_0$ for the corresponding threshold δ_0 listed in the table. Then there exists C such that for all $h \in (0,1)$,*

$$\frac{\|u_h\|_{L^\infty}}{\|u_h\|_{L^2}} \leq Ch^{-\kappa-n\delta/2}$$

where κ is the order listed in Table 1.

If it is further the case that

$$Pu = O(h)$$

where P is an operator of real principal type whose principal symbol vanishes on $\mathcal{L}$, then for all $\epsilon > 0$ there exists C_ϵ such that for all $h \in (0,1)$,

$$\frac{\|u_h\|_{L^\infty}}{\|u_h\|_{L^2}} \leq C_\epsilon h^{-\kappa-(n-1)\delta/2-\epsilon}.$$

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2. Flat tori

As an illustration of the effects of weak Lagrangian regularity on sup-norm estimates in a geometrically simple setting, we directly prove our main results in the special case of square flat tori: $X = \mathbb{R}^n/2\pi\mathbb{Z}^n$. For each $\alpha \in (\mathbb{R}^n)^*$, let $e_\alpha(x) = e^{-i\alpha x}$ denote the corresponding complex exponential.

Fix a frequency vector $\omega \in (\mathbb{R}^n)^*$. Employing canonical coordinates (x, ξ) on T^*X, we will consider the Lagrangian

$$\mathcal{L} = \{\xi = \omega\} \subset T^*X.$$

A normalized δ-Lagrangian sequence is thus a sequence of functions u_j on $\mathbb{T}^n$ such that

$$\|u_j\|_{L^2} = 1$$

and such that for appropriately chosen $h \equiv h_j \downarrow 0$ and any N and choice of indices $k_1, \dots k_N \in \{1, \dots, n\}$,

$$(1) \quad \big(h^{-1+\delta}(hD_{k_1} - \omega_{k_1})\big) \dots \big(h^{-1+\delta}(hD_{k_N} - \omega_{k_N})\big) u_j = O_{L^2}(1) \text{ as } j \to \infty.$$

We return to the notation u_h for the sequence of functions, bearing in mind that $h = h_j \downarrow 0$ through a discrete sequence of values. (Note that the general definition of Lagrangian regularity would allow *any* operators characteristic on $\mathcal{L}$, rather than the specific operators $hD_j - \alpha_j$ used here; however by elliptic regularity, it suffices to consider just this set of test operators whose symbols are a set of defining functions for $\mathcal{L}$.) Note that one immediate consequence of the assumption (1) is a crude L^∞ estimate based on Sobolev embedding: this estimate yields $D^\alpha u_h = O_{L^2}(h^{-|\alpha|})$, hence certainly

$$(2) \qquad \sup |u_h| = O(h^{-n/2+\epsilon}) \|u_h\|_{L^2}$$

for all $\epsilon > 0$.

We now write u_h as the Fourier series

$$\sum_{\alpha \in \mathbb{Z}^n} a_\alpha(h) e_\alpha(x).$$

Fixing any $\delta' > \delta$, we split

$$u_h = v_h + w_h$$

where

$$v_h = \sum_{|\alpha - h^{-1}\omega| < h^{-\delta'}} a_\alpha(h) e_\alpha(x),$$

$$w_h = \sum_{|\alpha - h^{-1}\omega| \geq h^{-\delta'}} a_\alpha(h) e_\alpha(x).$$

Since they are orthogonal, the estimate (1) applies to both v_h and w_h separately. Taking $k_j = k$ all the same, this yields for the Fourier series of w_h the estimate (for each k)

$$\sum_{|\alpha - h^{-1}\omega| \geq h^{-\delta'}} \big[h^\delta(\alpha_k - \omega_k/h)\big]^N |a_\alpha|^2 = O(1);$$

adding up the estimates for $k = 1, \dots, n$ and using the comparability of $\sum_1^n |x_j|^N$ and $|x|^N$ yields

$$\sum_{|\alpha - h^{-1}\omega| \geq h^{-\delta'}} \big[h^\delta|\alpha - \omega/h|\big]^N |a_\alpha|^2 = O(1),$$

i.e.,

$$\sum_{|\alpha-h^{-1}\omega|\geq h^{-\delta'}} h^{N(\delta-\delta')}|a_\alpha|^2 = O(1),$$

hence

$$\|w_h\|_{L^2} = O(h^\infty).$$

By (2), then

$$\|w_h\|_{L^\infty} = O(h^\infty),$$

and we need only consider v_h in our estimates henceforth.

To estimate v_h, we let

$$N_\mu(h) = \#\{\alpha \in \mathbb{Z}^n \colon |\alpha - h^{-1}\omega| < h^{-\mu}\}$$

for $\mu \in (0, 1]$. From the leading term in the Gauss circle problem, we have $N_\mu(h) \sim Ch^{-n\mu}$ for a constant $C > 0$ that depends only on n. Thus, since u_h is L^2-normalized, we easily see by Cauchy–Schwarz that

$$\|v_h\|_{L^\infty} \leq \sqrt{N_{\delta'}(h)} = O(h^{-n\delta'/2}).$$

We have thus obtained

$$\|u_h\|_{L^\infty} \leq \sqrt{N_{\delta'}(h)} = O(h^{-n\delta/2-\epsilon})$$

for any $\epsilon > 0$, as $\delta' > \delta$ can be chosen arbitrarily. This bound is achieved (up to an epsilon power) by taking all $a_\alpha = N_{\delta'}(h)^{-1/2}$ for α such that $|\alpha - \omega/h| \leq Ch^{-\delta'}$, and zero otherwise.

This is, up to a loss of $h^{-\epsilon}$, precisely the special case of Theorem 1.2 for projectable Lagrangians (the case A_1). When $\delta = 1$ we essentially get the counting function for eigenfunctions in a large ball, but when $\delta = 0$ we get $O(1)$, the estimate for actual Lagrangian distributions associated to a projectable Lagrangian.

Note that we could recover the ϵ lost here relative to the sharp statement of Theorem 1.2 by using Cauchy–Schwarz, somewhat as in Lemma 4.1 below. We have preferred to give a treatment that emphasizes the role of simply counting lattice points in domains in $\mathbb{R}^n$, however; in particular, this point of view makes the improvement in the result very clear when we assume that the u_{h_j} are Laplace eigenfunctions, i.e.,

$$(h_j^2\Delta - 1)u_{h_j} = 0.$$

The point is that this gives us more precise localization in one direction (conormal to the characteristic set). In that case, v_h now consists only of a sums as above with the further constraint $|\alpha| = h^{-1}$, hence the L^∞ estimate is replaced by $\sqrt{\widetilde{N}_{\delta'}(h)}$ where $\delta' > \delta$ and

$$\widetilde{N}_\mu(h) = \#\{\alpha \in \mathbb{Z}^n \colon |\alpha| = h^{-1},\ |\alpha - h^{-1}\omega| \leq Ch^{-\mu}\} \tag{3}$$

for $\mu \in (0, 1]$. (Now of course we take ω only with $|\omega| = 1$.) This quantity is a little subtler to estimate than $N_\mu(h)$.

To obtain an improved upper bound on $\widetilde{N}_\mu(h)$, we note that just as with the usual Gauss method for the circle problem, we may bound it by the sum of volumes of unit boxes centered at all lattice points in the set on the right side of (3), and that this is in turn bounded by the volume of the set

$$\{\alpha \in \mathbb{R}^n \colon ||\alpha| - h^{-1}| < C,\ |\alpha - h^{-1}\omega| \leq Ch^{-\mu}\}.$$

(Indeed, this estimate applies even if u_h is an $O(h)$ quasimode of $h^2\Delta - 1$.) The result is comparable to the volume of the subset of the sphere of radius h^{-1} on which $|\alpha - h^{-1}\omega| \leq Ch^{-\mu}$, i.e. we get

$$\widetilde{N}_\mu(h) = O(h^{-(n-1)\mu}). \tag{4}$$

Thus, using this estimate for $\widetilde{N}_{\delta'}$ on the function v_h in our splitting, yields a sup-norm estimate for eigenfunctions (which would also apply for $O(h)$ quasimodes) as follows:

$$\|u_h\|_{L^\infty} \leq \sqrt{\widetilde{N}_{\delta'}(h)} = O(h^{-(n-1)\delta/2-\epsilon}) \tag{5}$$

for any $\epsilon > 0$, as $\delta' > \delta$ can be chosen arbitrarily. Again this recovers a special case of Theorem 1.2. But this result is not, in this special case, optimal. We motivate the optimal result by a crude lower bound.

Lemma 2.1. *For any $\delta \in (0, 1]$ and in any dimension $n \geq 1$, there exists a sequence of $h \downarrow 0$ such that*

$$\widetilde{N}_\delta(h) \geq Ch^{1-(n-1)\delta}.$$

Setting

$$f_h = \sum_{\substack{|\alpha| = h^{-1} \\ |\alpha - h^{-1}\omega| \leq Ch^{-\delta}}} e_\alpha$$

yields

$$\|f_h\|_{L^\infty} = f_h(0) = \widetilde{N}_\delta(h)$$

and, by orthogonality,

$$\|f_h\|_{L^2} = \sqrt{\widetilde{N}_\delta(h)}.$$

Thus, setting $u_h = f_h/\|f_h\|_{L^2}$, Lemma 2.1 shows that for an L^2-normalized δ-Lagrangian sequence of Laplace eigenfunctions on the torus we can achieve

$$\|u_h\|_{L^\infty} \geq Ch^{1/2-(n-1)\delta/2}. \tag{6}$$

Proof of lemma. For $j \in \mathbb{N}$, let

$$M(j) = \#\{\alpha \in \mathbb{Z}^n \colon |\alpha|^2 = j,\ |\alpha - j^{1/2}\omega| \leq j^{\delta/2}\}.$$

Thus,

$$\widetilde{N}(j^{-1/2}) = M(j).$$

Now

$$\sum_{J\leq j\leq 2J} M(j) = \#\mathbb{Z}^n \cap \Omega_J \tag{7}$$

where

$$\Omega_J \equiv \big\{r\theta \in \mathbb{R}^n \colon r \in [\sqrt{J}, \sqrt{2J}],\ |\theta - r\omega| < r^\delta\big\}.$$

The quantity (7) is comparable to the volume of the solid in question (again by counting enclosed unit cubes), hence

$$\sum_{J\leq j\leq 2J} M(j) \geq C \int_{\sqrt{J}}^{\sqrt{2J}} (r^\delta)^{n-1}\, dr \sim C J^{1/2+(n-1)\delta/2}.$$

On the other hand there are J terms in the sum, so one of them must be at least

$$J^{-1/2+(n-1)\delta/2}.$$

Using this procedure to pick a sequence of $h = j^{-1/2}$ in the dyadic intervals $(J, 2J) = (2^k, 2^{k+1})$ gives the desired sequence. □

In dimension $n \geq 5$, if for $m \in \mathbb{N}$ we let $r_n(m)$ denote the number of integer lattice points on the sphere of radius $m^{1/2}$, it is known that there exist positive constants c_n, C_n such that

$$c_n m^{n/2-1} \leq r_n(m) \leq C_n m^{n/2-1}$$

Thus, the number of lattice points on the sphere of radius h^{-1} is comparable to h^{-n+2} for $n \geq 5$. If we then multiply by the fraction of the volume of the sphere that is occupied by the cap of size $h^{-\delta}$ we obtain a heuristic estimate exactly of order $h^{1-(n-1)\delta}$. This is indeed also known to be essentially an upper bound, for sufficiently large δ: Bourgain–Rudnick [6, Proposition 1.4] show that for $n \geq 5$, for $\delta \in [1/2, 1]$, for all $\epsilon > 0$ there exists $C = C_\epsilon$ such that for all h

$$\widetilde{N}_\delta(h) \leq C h^{1-(n-1)\delta-\epsilon}.$$

(Similar results for the special cases $n = 3, 4$ are also obtained in [6].) Optimal lower bounds on $\widetilde{N}_\delta(h)$ of the form of the first equation in Lemma 2.1 (uniform in radius, rather than along a subsequence as deduced above) have recently been obtained by Sardari [19, Corollary 1.9]; see also the celebrated work of Duke [10] and Iwaniec [15] in the special case of dimension 3.

3. Stable simple singularities of Lagrangian projections

We now return to the general geometric setting of a non-projectable Lagrangian (i.e., the projection map is not assumed to be a diffeomorphism), and recall the normal forms of *stable simple* singularities of Lagrangian projections as developed by Arnol'd [4, Corollary 11.8], [3]. We will in fact use the alternative parametrizations of the Lagrangians given by Duistermaat [9, Theorems 3.1.1 and 3.2.1]. We recall first the notion of local *Lagrange-equivalence*: two Lagrangians in T^*X are locally equivalent if they can be mapped one to another by a fiber-preserving local symplectomorphism of

T^*X. *Stability* of a Lagrangian projection means that nearby (in the $\mathcal{C}^\infty$ topology) Lagrangians are locally Lagrange-equivalent to the original. The *simple* singularities are those that under perturbation can be locally equivalent to only a finite list of singularities at nearby points [4, Definition 11.1]. Stability does not imply simplicity nor conversely in general, but stability does imply simplicity in dimension up to 5. Thus the classification is in fact an exhaustive list of the stable singularities in these dimensions; moreover every Lagrangian in dimension up to 5 can be locally perturbed to be equivalent to one in this list (stable Lagrangians are dense). We refer the reader to [9] and to [4], [3] for further details on the notions of stability and simplicity, and the classification.

We recall that every Lagrangian manifold $\mathcal{L}$ of $T^*\mathbb{R}^n$ may locally be parametrized in the following form:

$$\mathcal{L} = \big\{(x, \phi'_x(x,\theta))\colon \phi'_\theta(x,\theta) = 0\big\}.$$

Two phase functions ϕ and $\widetilde{\phi}$ are easily seen to parametrize Lagrange-equivalent Lagrangians if

$$\widetilde{\phi}(x,\theta) = \phi(x',\theta') + \psi(x') \tag{8}$$

for some fiber-preserving local diffeomorphism

$$(x,\theta) \mapsto (x'(x), \theta'(x,\theta)),$$

and $\psi \in \mathcal{C}^\infty$. In [9] this is referred to as *equivalence of unfoldings* of the Lagrangian singularities, and it is as a classification of unfoldings up to the equivalence (8) that the classification is phrased in that work and in this form that we will employ it: every phase function parametrizing a stable simple singularity is locally equivalent to one in the Table 2 (whose entries we explain below) in the sense (8).

Duistermaat [9] parametrizes the stable simple singularities in $\mathbb{R}^n$ with phase functions

$$\phi(x,\theta) = \sum_{j=1}^n x_j f_j(\theta) + f(\theta)$$

where f_j, f are given by Table 2 (taken from [9, Theorem 3.1.1 and Theorem 3.2.1]); here n is the dimension, and k is the number of phase variables θ (whose least possible value for each singularity is listed in the table); the f_j's beyond those enumerated ($f_1, \dots, f_m$ for the A_{m+1} and $D^\pm_{m+1}$) are taken to equal 0; the variables θ' are the remaining $\theta \in \mathbb{R}^k$ variables beyond those appearing explicitly ($\theta_2, \dots, \theta_k$ for A_{m+1}; $\theta_3, \dots, \theta_k$ for $D^\pm_{m+1}$ and E_6).

The virtue, from the point of view of our analysis, of the parametrizations in Table 2 is that the functions f are always weighted homogeneous, as are the $x_j f_j$ if we consider a joint homogeneity in x, θ. We will employ these facts below in our analysis of the asymptotics.

Type	$f(\theta)$	$f_1(\theta), \ldots, f_n(\theta)$	
A_{m+1}	$\pm\theta_1^{m+2} + (\theta')^2$	$\theta_1, \ldots, \theta_1^m$	$n \geq m \geq 0,\ k \geq 1$
$D_{m+1}^{\pm}$	$\theta_1^2\theta_2 \pm \theta_2^m + (\theta')^2$	$\theta_1, \theta_2, \ldots, \theta_2^{m-1}$	$n \geq m \geq 3,\ k \geq 2$
E_6	$\theta_1^3 \pm \theta_2^4 + (\theta')^2$	$\theta_1, \theta_2, \theta_2^2, \theta_1\theta_2, \theta_1\theta_2^2$	$n \geq 5,\ k \geq 2$
E_7	$\theta_1^3 + \theta_1\theta_2^3 + (\theta')^2$	$\theta_1, \theta_2, \theta_2^2, \theta_2^3, \theta_2^4, \theta_1\theta_2$	$n \geq 6,\ k \geq 2$
E_8	$\theta_1^3 + \theta_2^5 + (\theta')^2$	$\theta_1, \theta_2, \theta_2^2, \theta_2^3, \theta_1\theta_2, \theta_1\theta_2^2, \theta_1\theta_2^3$	$n \geq 7, k \geq 2$

TABLE 2. Classification of stable simple singularities with parametrizations.

Which of these singularities appear in "real-life" Hamiltonian systems seems to be an intriguing open question. We may easily find the fold singularity (A_2) arising in integrable systems: a one-dimensional harmonic oscillator

$$p = x^2 + \xi^2$$

has a fold singularity at each turning point of the Lagrangian torus $p = E$ for every $E > 0$. In two dimensions, we may also find fold singularities in the geodesic flow on convex surfaces of rotation: on the surface

$$\{(x, f(x)\cos\theta, f(x)\sin\theta) \colon x \in [a,b],\ \theta \in S^1\},$$

the Clairaut integral constrains the projection of a Lagrangian torus to be a cylinder lying between two extremal values of the x variable, where the torus projection has a fold.

More complex singularities seem harder to come by in simple examples of integrable systems; examples are known, at least numerically, for invariant tori in nonintegrable settings, however. For instance, the Hénon–Heiles Hamiltonian has been shown to have invariant tori with cusps (A_3) [20]; Section 5 of [20] also refers to the existence of swallowtails in analogous computations for $n = 3$. The notion of stability employed in Arnol'd's classification is probably *not* the physically relevant one for KAM systems where we have a Hamiltonian of the form $|\xi|^2 + V(x)$: corners, for instance, arise naturally and stably in these settings—see [8] and further discussion in [17]. Likewise, it is natural in exploring extremizing sequences of eigenfunctions to explore the *blowdown singularity*, as this is the (unstable) singularity to which is associated the extremizing sequence of spherical harmonics on S^n. We furthermore do not consider degenerate Lagrangian tori, such as the equatorial orbits on surfaces of rotation on which Gaussian beams may concentrate. We focus here on Arnol'd's stable simple singularities merely on the grounds that they are the first natural case to consider.

4. The Hörmander–Melrose theory for δ-Lagrangians

In this section, we show that δ-Lagrangian distributions can be obtained as Fourier integrals with symbols in a suitable symbol class. This is a semiclassical version of the Hörmander–Melrose theory (previously worked out in [1] in the case $\delta = 0$), adapted to the case of δ-Lagrangian regularity.

The results in this section are local in nature and so it suffices to work in Euclidean space. More precisely, the results may also be microlocalized: if $B \in \Psi_h(X)$ has compact microsupport then Bu_h is δ-Lagrangian whenever u_h is (since we can just replace A_N by $A_N B$ in verifying the oscillatory testing definition. Thus, we may always restrict our analysis to distributions u_h microsupported in arbitrarily small sets.

We introduce for $\delta \in [0, 1]$, a symbol class consisting of families of smooth functions whose higher derivatives satisfy sup norm estimates that worsen by powers of h:

$$S^k_\delta(\mathbb{R}^n \times \mathbb{R}^N) = \{a(x, \theta; h) \colon |\partial^\alpha_{(x,\theta)} a(x, \theta; h)| \leq C_\alpha h^{-k-\delta|\alpha|} \text{ for all } \alpha \in \mathbb{N}^{n+N},\ h \in (0, 1)\}. \tag{9}$$

We will use the convention on the semiclassical Fourier transform from [21], with

$$\mathcal{F}_h u_h(\xi) \equiv \int e^{-ix\xi/h} u_h(x)\, dx.$$

As it occurs frequently in what follows, we employ the shorthand $+0$ for "$+\epsilon$ for all $\epsilon > 0$." We will revert to writing the definition out in full where important quantities may depend on the choice of ϵ, however.

We will require, in what follows, a sharp version of Sobolev embedding associated to distributions that are δ-Lagrangian with respect to the zero section $o \subset T^*\mathbb{R}^n$. (Note that such distributions are in fact exactly the symbols we will be dealing with, since the zero section is parametrized by the phase function $\phi = 0$, and the distribution is its own amplitude.

Lemma 4.1. *Let $a(x; h)$ be a δ-Lagrangian distribution with respect to the zero section. Then $a \in S_\delta^{\frac{n\delta}{2}}$, with estimates depending on only finitely many δ-Lagrangian seminorms.*

Note that Lemma 4.1 is sharp, as shown by the example

$$a(x; h) = h^{-\delta/2} e^{-x^2/h^{2\delta}} \tag{10}$$

in one dimension.

Proof. For any semiclassical family of functions u_h, let

$$T^\delta_h u_h(\xi) = (2\pi h)^{-n\delta/2} \int u_h(x) e^{-i\xi x/h^\delta}\, dx$$

denote the semiclassical Fourier transform on scale h^δ; note that we have scaled T_h^δ to be unitary, with

$$(T_h^\delta)^{-1} v_h(x) = (2\pi h)^{-n\delta/2} \int v_h(\xi) e^{i\xi x/h^\delta}\, d\xi.$$

Thus by integration by parts, for all α and β,

$$\xi^\alpha T_h^\delta (h^\delta D_x)^\beta a = T_h^\delta (h^\delta D_x)^{\alpha+\beta} a \in L^2,$$

uniformly as $h \downarrow 0$. In particular, then,

$$\langle \xi \rangle^{n/2+1} T_h^\delta (h^\delta D_x)^\beta a \in L^2,$$

hence by Cauchy–Schwarz applied to the inverse transform

$$\begin{aligned} \sup |(h^\delta D_x)^\beta a| &\leq (2\pi h)^{-n\delta/2} \| \langle \xi \rangle^{-n/2-1} \|_{L^2} \| \langle \xi \rangle^{n/2+1} T_h^\delta (h^\delta D_x)^\beta a \|_{L^2} \\ &\leq C_\beta h^{-n\delta/2} \end{aligned}$$

for all β.

□

Fix a Lagrangian submanifold $\mathcal{L} \subset T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$ and let ϕ be a phase function that locally parametrizes $\mathcal{L}$ with N phase variables as described in §3. In particular we assume that

$$\mathcal{L} \cap U = \{(x, \phi_x(x,\theta)) \subset \mathbb{R}^{2n} : (x,\theta) \in V \text{ and } \phi_\theta(x,\theta) = 0\}$$

where $U \subset \mathbb{R}^{2n}$ and $V \subset \mathbb{R}^n \times \mathbb{R}^N$ are open and bounded.

Given a symbol a and phase function ϕ, we will employ the standard oscillatory integral notation

$$I(a,\phi)[x] \equiv \int_{\mathbb{R}^N} a(x,\theta) e^{i\phi(x,\theta)/h}\, d\theta.$$

Proposition 4.2. *Let $\delta \in [0, 1/2)$.*

(1) *Let u_h be a δ-Lagrangian distribution with respect to $\mathcal{L}$, with $\|u_h\|_{L^2} = 1$ and $\mathrm{WF}_h(u_h) \subset U$. For every point $\gamma = (x_0, \xi_0) \in U \cap \mathcal{L}$, we can find a symbol $a(x,\theta)$ in the class $S_\delta^{\frac{N}{2}+\frac{n\delta}{2}}(\mathbb{R}^{n+N})$ such that*
$$u_h = I(a,\phi).$$
microlocally near γ.

(2) *Conversely, let $a(x,\theta)$ be a symbol in the class $S_\delta^{\frac{N}{2}}$ supported in V. Then*
$$u_h = I(a,\phi)$$
is a δ-Lagrangian distribution u_h with $\mathrm{WF}_h(u_h) \subset U$ and $\|u_h\|_{L^2}$ is bounded.

We remark that the discrepancy in symbol orders in the two parts of this proposition is necessary even in the model case where $\mathcal{L}$ is the zero-section, as shown by the example (10).

Proof. We closely follow the proof of Theorem 4.4 of [1] and begin by assuming that $\mathcal{L}$ is transverse to the constant section $\xi = \xi_0$ at γ. In particular, this implies that we can write

$$\mathcal{L} \cap U = \{(\partial_\xi H(\xi), \xi) \in \mathbb{R}^{2n} : x \in W\} \tag{11}$$

for some open bounded $W \subset \mathbb{R}^N$ and some smooth function $H \in \mathcal{C}_b^\infty(W; \mathbb{R})$ which we extend to $\mathbb{R}^n$. The symbols

$$b_j \equiv x_j - \partial_{\xi_j} H(\xi)$$

generate the module of $A \in \Psi_h^{-\infty}$ characteristic to $\mathcal{L} \cap U$. Hence u_h with $\mathrm{WF}_h(u_h) \subset U$ has δ-Lagrangian regularity with respect to $\mathcal{L}$ if and only if we have

$$\|(x - \partial_\xi H(hD))^\alpha u_h\|_{L^2} = O(h^{(1-\delta)|\alpha|})$$

for all α. Taking the semiclassical Fourier transform in x and applying Plancherel, we obtain

$$\|(-hD - \partial_\xi H(\xi))^\alpha \mathcal{F}_h u_h\|_{L^2} = O(h^{n/2+(1-\delta)|\alpha|}). \tag{12}$$

Setting

$$v_h(\xi) = e^{iH(\xi)/h} \mathcal{F}_h u_h(\xi) \tag{13}$$

we obtain

$$\|\partial^\alpha v_h\|_{L^2} = h^{-|\alpha|} \|(-hD - \partial_\xi H)^\alpha \mathcal{F}_h u_h\|_{L^2} = O(h^{n/2-\delta|\alpha|}).$$

Hence we have established that for u_h with $\mathrm{WF}_h(u_h) \subset U$ and $\mathcal{L}$ transverse to the constant section locally parametrized as (11) that

$$u_h \in L^2 \text{ is } \delta\text{-Lagrangian} \Longleftrightarrow \|\partial_\xi^\alpha (e^{iH/h} \mathcal{F}_h u_h)\|_{L^2} = O(h^{n/2-\delta|\alpha|}) \text{ for all } \alpha. \tag{14}$$

Under the assumption that u_h is δ-Lagrangian, Sobolev embedding yields

$$\|\partial^\alpha v_h\|_{L^\infty} = O(h^{n/2-\delta(|\alpha|+n/2)}) \tag{15}$$

and so we have $v_h \in S_\delta^{n(\delta-1)/2}$, and by (13), this shows that we may write u_h as an oscillatory integral parametrized by the special phase function $H(\xi) - x \cdot \xi$. Note that the order of the amplitude, which comes out to $n/2 + n\delta/2$, includes a contribution from the factor of h^{-n} in the inverse Fourier transform.

In order to establish the proposition for an arbitrary phase ϕ parametrizing a Lagrangian transverse to the constant section satisfying (11), we consider the more general oscillatory integral

$$\mathcal{F}_h(I(a, \phi))(\xi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} a(x, \theta) e^{i(\phi(x,\theta) - x\cdot\xi)/h} \, d\theta \, dx$$

for an arbitrary symbol $a \in S_\delta^r(\mathbb{R}^n \times \mathbb{R}^N)$.

As in [1], from the implicit function theorem and the nondegeneracy of the phase function ϕ, shrinking U and W if necessary, we can find smooth functions $\bar{x}\in \mathcal{C}_b^\infty(W;\mathbb{R}^n), \bar{\theta} \in \mathcal{C}_b^\infty(W;\mathbb{R}^N)$ such that for fixed $\xi \in W$, the phase

$$\Phi(x,\theta;\xi) = \phi(x,\theta) - x\cdot\xi$$

is stationary precisely in (x,θ) at $(\bar{x}(\xi),\bar{\theta}(\xi);\xi)$, and this stationary point is nondegenerate. Furthermore, if a is compactly supported close to $(\bar{x}(\xi_0),\bar{\theta}(\xi_0))$, then $\mathcal{F}_h(I(a,\phi))$ is $O(h^\infty)$ for $\xi \notin W$ by nonstationary phase, and $\operatorname{sgn}(\partial^2\Phi)$ can be assumed to be constant on the support of a.

For $\xi \in W$ we have the stationary phase expansion

$$\mathcal{F}_h(I(a,\phi))(\xi) = e^{i\Phi(\bar{x}(\xi),\bar{\theta}(\xi);\xi)/h} \sum_{k=0}^{K-1} h^{n/2+N/2+k}(P_{2k}(D)a)(\bar{x}(\xi),\bar{\theta}(\xi)) + R_K(\xi)$$

where P_{2k} is a differential operator of order $2k$,

$$P_0 = (2\pi)^{(n+N)/2}|\det(\partial^2\Phi)|^{-1/2}\cdot e^{i\pi\operatorname{sgn}(\partial^2\Phi)/4}$$

and

$$\sup|R_K| \leq C_K h^{n/2+N/2+K}\sum_{|\alpha|\leq 2K+n+N+1}\sup|\partial^\alpha a| = O(h^{-r-\delta+(n/2+N/2+K)(1-2\delta)}).$$

Since $(\partial_\xi H(\xi),\xi) = (\bar{x}(\xi),\xi)\in\mathcal{L}$, we obtain

$$\partial_\xi\Phi(\bar{x}(\xi),\bar{\theta}(\xi);\xi) = -\partial_\xi H(\xi)$$

and so by adding a suitable constant to H we may assume that

$$\Phi(\bar{x}(\xi),\bar{\theta}(\xi);\xi) = -H(\xi)$$

for $\xi\in W$.

Recalling that $\delta < 1/2$, we can choose K sufficiently large so that

$$\sup|R_K| = O(h^{-r+n/2+N/2+M(1-2\delta)})$$

for arbitrary $M\in\mathbb{N}$, giving

(16)
$$e^{iH/h}\mathcal{F}_h(I(a,\phi)) = \sum_{k=0}^{M-1} h^{n/2+N/2+k}(P_{2k}a)(\bar{x}(\xi),\bar{\theta}(\xi)) + O_{L^\infty}(h^{-r+n/2+N/2+M(1-2\delta)}).$$

To estimate the derivatives of $e^{iH/h}\mathcal{F}_h(I(a,\phi))$ we compute

$$hD_{\xi_k}(e^{iH/h}\mathcal{F}_h(I(a,\phi))) = e^{iH/h}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N}(\partial_{\xi_k}H(\xi)-x_k)a(x,\theta)e^{i(\phi(x,\theta)-x\cdot\xi)/h}\,d\theta\,dx.$$

From the nondegeneracy of the stationary points $(\bar{x}(\xi),\bar{\theta}(\xi);\xi)$, the map $(x,\theta,\xi)\mapsto(\partial_x\Phi,\partial_\theta\Phi,\xi)$ is a local diffeomorphism in a neighbourhood of $\{(\bar{x}(\xi),\bar{\theta}(\xi),\xi) : \xi\in W\}$. As the factor $\partial_{\xi_k}H(\xi)-x_k$ vanishes at $(\partial_x\Phi,\partial_\theta\Phi,\xi) = (0,0,\xi)$, Taylor expansion gives

(17)
$$(\partial_{\xi_k}H(\xi)-x_k)e^{i\Phi/h} = h\left(\sum_{i=1}^n b_i^{(k)}(x,\theta,\xi)D_{x_i} + \sum_{j=1}^N c_j^{(k)}(x,\theta,\xi)D_{\theta_j}\right)e^{i\Phi/h}$$

for $b_i^{(k)}, c_j^{(k)} \in \mathcal{C}_b^\infty(\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n)$. Integration by parts in the operator

$$L_k = \sum_i b_i^{(k)} D_{x_i} + \sum_j c_j^{(k)} D_{\theta_j}$$

thus shows that

$$D_{\xi_k}(e^{iH/h}\mathcal{F}_h(I(a,\phi))) = e^{iH/h}\mathcal{F}_h(I(L_k^T a,\phi))$$

with L_k^T a first order differential operator and so $L_k^T a \in S_\delta^{r+\delta}$. By iterating this integration by parts we obtain

$$D_\xi^\alpha(e^{iH/h}\mathcal{F}_h(I(a,\phi))) = e^{iH/h}\mathcal{F}_h(I(L^\alpha a,\phi))$$

where L^α is a differential operator of order α, only involving differentiation in (x,θ), and with coefficients smooth in (x,θ,ξ). By utilising (16), we obtain

$$\|\partial_\xi^\alpha(e^{iH/h}\mathcal{F}_h I(a,\phi))\|_{L^\infty} = O(h^{-r+n/2+N/2-\delta|\alpha|}). \tag{18}$$

Equation (18) implies that $e^{iH/h}\mathcal{F}_h(I(a,\phi)) \in S_\delta^{r-n/2-N/2}$, and so from (16) and a semiclassical analogue of [14, Proposition 1.1.10] we deduce the expansion

$$e^{iH/h}\mathcal{F}_h(I(a,\phi)) \sim \sum_{k=0}^\infty h^{n/2+N/2+k}(P_{2k}a)(\bar{x}(\xi),\bar{\theta}(\xi)) \tag{19}$$

in the sense that

(20)

$$e^{iH/h}\mathcal{F}_h(I(a,\phi)) - \sum_{k=0}^{M-1} h^{n/2+N/2+k}(P_{2k}a)(\bar{x}(\xi),\bar{\theta}(\xi)) \in S_\delta^{r-n/2-N/2-M(1-2\delta)}.$$

As $\mathcal{F}_h I(a,\phi)$ is $O(h^\infty)$ outside the bounded set W, we can combine (19) and (14) to show that $I(a,\phi)$ has δ-Lagrangian regularity and is bounded in L^2, proving part (2) of the proposition in the case where $\mathcal{L}$ is transverse to the constant section.

We now complete the proof of part (1), under the same transversality assumption. The idea is to use the expansion (19) to construct a symbol $a(x,\theta)$ such that $v_h = e^{iH/h}\mathcal{F}_h(I(a,\phi)) + O_{S_\delta}(h^\infty)$, where v_h is as in (13). We write $\psi(x,\theta) = \partial_x\phi(x,\theta)$. This function is smooth in a neighbourhood V of $(\bar{x}(\xi_0),\bar{\theta}(\xi_0))$ and satisfies $\psi(\bar{x}(\xi),\bar{\theta}(\xi)) = \xi \in \mathbb{R}^n$ as Φ is stationary in (x,θ) at $(\bar{x}(\xi),\bar{\theta}(\xi))$.

We begin by taking

$$a_0 = (2\pi h)^{-(n+N)/2}\left(|\det(\partial^2\Phi)|^{1/2}\cdot e^{-i\pi \mathrm{sgn}(\partial^2\Phi)/4} v_h\right)\circ\psi$$

for (x,θ) near $(\bar{x}(\xi_0),\bar{\theta}(\xi_0))$ and cutting off smoothly away from V^c, we have $a_0 \in S_\delta^{(n\delta+N)/2}$ and by truncating the expansion(19) after the leading term, we obtain

$$e^{iH/h}\mathcal{F}_h(I(a_0,\phi)) - v_h \in S_\delta^{n(\delta-1)/2-(1-2\delta)}.$$

Proceeding iteratively, we can construct a sequence of symbols

$$a_k \in S_\delta^{(n\delta+N)/2-(1-2\delta)k}$$

supported in V such that

$$e^{iH/h}\mathcal{F}_h\Big(I\big(\sum_{k=0}^{l-1} a_k h^{k(1-2\delta)}, \phi\big)\Big) - v_h \in S_\delta^{n(\delta-1)-l(1-2\delta)}.$$

Borel summation then yields a total symbol $a \in S_\delta^{(n\delta+N)/2}$ with

$$e^{iH/h}\mathcal{F}_h(I(a,\phi)) - v \in S_\delta^{-\infty}$$

which allows us to conclude that

$$u_h = I(a,\phi)$$

microlocally near (x_0, ξ_0), with a in the required symbol class.

It remains to establish parts (1) and (2) of the Proposition in the case where $\mathcal{L}$ is not transverse to the constant section $\xi = \xi_0$ at $\gamma = (x_0, \xi_0)$. We proceed as in [1] and apply a symplectic transformation to reduce to the transverse case as follows.

We can choose our coordinates $x = (x', x'')$ and $\xi = (\xi', \xi'')$ in $\mathbb{R}^k \times \mathbb{R}^{n-k}$ so that the tangent space $T_\gamma\mathcal{L}$ takes the form

$$T_\gamma\mathcal{L} = \{(0, x''; \xi', Bx'') : x'' \in \mathbb{R}^{n-k}, \xi' \in \mathbb{R}^k\}$$

where B is a symmetric matrix. Here we have identified $T_\gamma\mathcal{L}$ with a n-dimensional subspace of $\mathbb{R}^n \times \mathbb{R}^n$ in the natural way. If B were invertible, then this tangent space would be transverse to the constant section, so we choose a diagonal $(n-k)\times(n-k)$ matrix D such that $B+D$ is nonsingular.

Then the transformed Lagrangian

$$\tilde{\mathcal{L}} = \{(0, x''; \xi', (B+D)x'') : x'' \in \mathbb{R}^{n-k}, \xi' \in \mathbb{R}^k\}$$

is transverse to the constant section through $\tilde{\gamma} \equiv (x_0, \xi_0 + Dx_0'')$ and is parametrized by the phase function

$$\tilde{\phi}(x,\theta) = \phi(x,\theta) + \frac{1}{2}Dx'' \cdot x''.$$

Taking $A_j \in \Psi_h$ characteristic to $\mathcal{L}$ and compactly microlocalized near γ, partial Lagrangian regularity implies

$$\left(\prod_{j=1}^m e^{iDx''\cdot x''/2h} A_j e^{-iDx''\cdot x''/2h}\right) e^{iDx''\cdot x''/2h} u_h = O_{L^2}(h^{(1-\delta)m})$$

for L^2-normalised u_h with partial Lagrangian regularity with respect to $\mathcal{L}$. The operators

$$B_j = e^{iDx''\cdot x''/2h} A_j e^{-iDx''\cdot x''/2h}$$

are shown in [1] to be semiclassical pseudodifferential operators that are compactly microlocalized near $\tilde{\gamma}$ with principal symbols

$$\sigma(B_j)(x,\xi) = \sigma(A_j)(x, \xi - Dx'') \tag{21}$$

which are characteristic to $\tilde{\mathcal{L}}$.

From part (1) of the proposition in the case where $\mathcal{L}$ is transverse to the constant section, it follows that we can find a symbol $a \in S_\delta^{N/2+n\delta/2}$ with

$$e^{iDx''\cdot x''/2h}u_h = I(a, \tilde{\phi})$$

microlocally near $\tilde{\gamma}$ and so we can conclude that

$$u_h = I(a, \phi)$$

microlocally near γ. This completes the proof of part (1) of the proposition.

Similarly, if u_h is given by $I(a, \phi)$ for $a \in S_\delta^{\frac{N}{2}}$, then $e^{iDx''\cdot x''/2h}u_h = I(a, \tilde{\phi})$. From part (2) of the proposition in the case where $\mathcal{L}$ is transverse to the constant section, it follows that $e^{iDx''\cdot x''/2h}u_h$ is an L^2-bounded δ-Lagrangian distribution with respect to $\tilde{\mathcal{L}}$. As such, we have

$$\left(\prod_{j=1}^{m} B_j\right) e^{iDx''\cdot x''/2h}u_h = O_{L^2}(h^{(1-\delta)m})$$

for any collection of $B_j \in \Psi_h$ characteristic to $\tilde{\mathcal{L}}$ and compactly microlocalized near $\tilde{\gamma}$. In particular, by (21) this is true for $B_j = e^{iDx''\cdot x''/2h}A_j e^{-iDx''\cdot x''/2h}$ where $A_j \in \Psi_h$ is characteristic to $\mathcal{L}$ and compactly microlocalized near γ, and we obtain

$$\left(\prod_{j=1}^{m} A_j\right) u_h = O_{L^2}(h^{(1-\delta)m})$$

for arbitrary such A_j, which completes the proof of part (2) of the proposition. $\square$

In the case that the Lagrangian is *projectable* onto the base manifold i.e., that the projection map is a diffeomorphism, we can parametrize it using a phase function ϕ with 0 phase variables, and a simpler argument establishes the result in Proposition 4.2 without the restriction that $\delta < 1/2$.

Proposition 4.3. *Let $\delta \in [0, 1]$ and suppose u_h is a semiclassical distribution with δ-Lagrangian regularity with respect to an arbitrary Lagrangian $\mathcal{L} \subset T^*X$. For every point $\gamma = (x_0, \xi_0) \in U \cap \mathcal{L}$ at which $\mathcal{L}$ is projectable and parametrized by the phase function $\phi(x)$, we can find a symbol $a(x)$ in the class $S_\delta^{\frac{n\delta}{2}}(\mathbb{R}^n)$ such that*

$$u_h(x) = a(x)e^{i\phi(x)/h}$$

microlocally near γ.

Proof. From the assumptions on $\mathcal{L}$, we can find a bounded open set $W \subset \mathbb{R}^n$ with

$$\mathcal{L} \cap U = \{(x, \partial_x\phi(x)) \in \mathbb{R}^{2n} : x \in W\}.$$

The symbols

$$b_j := \xi_j - \partial_{x_j}\phi$$

are then characteristic to $\mathcal{L} \cap U$ and by partial Lagrangian regularity, we have

$$\|(hD - \partial_x \phi)^\alpha u_h\|_{L^2} = O(h^{(1-\delta)|\alpha|}).$$

Setting

$$a(x) = u_h(x) e^{-i\phi(x)/h}$$

we obtain

$$\|\partial^\alpha a\|_{L^2} = h^{-|\alpha|} \|(hD - \partial_x \phi)^\alpha u_h\|_{L^2} = O(h^{-\delta|\alpha|}).$$

Sobolev embedding yields

$$\|\partial^\alpha a\|_{L^\infty} = O(h^{-\delta(|\alpha|+n/2)})$$

and so we have $a \in S_\delta^{n\delta/2}(\mathbb{R}^n)$. □

More generally, we now show that we can also obtain Fourier integral representations for δ-Lagrangian distributions with $\delta \geq 1/2$, provided we restrict ourselves to a particular class of phase functions.

As in the proof of Proposition 4.2, it suffices to treat the case where $\mathcal{L} \cap U$ is transverse to the constant section $\xi = \xi_0$ at γ. Under this assumption, we can locally parametrize our Lagrangian as

$$\mathcal{L} \cap U = \{(\partial_\xi H(\xi), \xi) : \xi \in W\}$$

for some smooth function H and open set W. If the point $\gamma \in \mathcal{L}$ does not lie on the zero section, then we can always obtain this transversality condition by choosing coordinates on the base space appropriately [12, p. 102]. After choosing such coordinates, one possible choice of phase function to locally parametrize $\mathcal{L}$ is

$$\phi(x, \theta) = x \cdot \theta - H(\theta).$$

For this particular choice of phase function we have a simpler argument to arrive at the analogous result to Proposition 4.2, valid for all $\delta \in [0, 1]$. Recall that X denotes a smooth n-manifold.

Proposition 4.4. *Let $\delta \in [0, 1]$.*

(1) *Suppose u_h is a semiclassical distribution with δ-Lagrangian regularity with respect to an arbitrary Lagrangian $\mathcal{L} \subset T^*X$. For every point $\gamma \in \mathcal{L}$, we can choose local coordinates on X and find a symbol $a(\theta)$ in the class $S_\delta^{\frac{n}{2}+\frac{n\delta}{2}}(\mathbb{R}^{2n})$ and a function $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that*

$$u_h(x) = I(a, \phi)[x] = \int_{\mathbb{R}^n} a(\theta) e^{i(x\cdot\theta - H(\theta) - \psi(x))/h} \, d\theta$$

microlocally near γ.

If γ does not lie on the zero section of T^*X then we can take $\psi = 0$.

(2) *Conversely, for a Lagrangian locally parametrized as*

$$\mathcal{L} \cap U = \{(\partial_\xi H(\xi), \xi) : \xi \in W\}$$

and $a \in S_\delta^{\frac{n}{2}}$ *supported in* W*,*

$$u_h(x) = I(a, \phi)[x] = \int_{\mathbb{R}^n} a(\theta) e^{i(x\cdot\theta - H(\theta))/h} \, d\theta$$

determines a δ*-Lagrangian distribution with respect to* $\mathcal{L}$ *with* $\mathrm{WF}_h(u_h) \subset U$*; moreover* $\|u_h\|_{L^2}$ *is bounded.*

Proof. We begin by proving part (1) of the proposition. We may assume without loss of generality that u_h is compactly microlocalized in a neighbourhood U of γ by applying a microlocal cutoff.

First we suppose that γ does not lie in the zero section. Then again by choosing coordinates on the base space appropriately we can locally parametrize our Lagrangian $\mathcal{L}$ as

$$\mathcal{L} \cap U = \{(\partial_\xi H(\xi), \xi) : \xi \in W\}$$

in induced canonical coordinates (x, ξ), for some $H \in \mathcal{C}^\infty(\mathbb{R}^n)$ where $U \subset T^*X$ and $W \subset \mathbb{R}^n$ are open and bounded. Setting

$$a = (2\pi h)^{-n} \mathcal{F}_h u_h \cdot e^{iH/h}, \tag{22}$$

semiclassical Fourier inversion immediately yields the sought Fourier integral representation, and from (15), it follows that

$$\|\partial^\alpha a\|_{L^\infty} = O\left(h^{-\frac{n(1+\delta)}{2} - \delta|\alpha|}\right)$$

as required.

On the other hand, if $\gamma = (x_0, \xi_0)$ does lie in the zero section, we consider the distribution $\tilde{u}_h = e^{i\psi/h} u_h$ for an arbitrary smooth real-valued ψ with $\psi'(x_0) \neq 0$. Since u_h is δ-Lagrangian with respect to $\mathcal{L}$, for any collection of operators $A_j \in \Psi_h^{-\infty}$ that are characteristic to $\mathcal{L}$ we have the iterated regularity estimate

$$\left\| \left(\prod_{j=1}^N e^{i\psi/h} A_j e^{-i\psi/h} \right) \tilde{u}_h \right\|_{L^2} = O(h^{(1-\delta)N}).$$

By Egorov's theorem, each of the operators

$$\tilde{A}_j = e^{i\psi/h} A_j e^{-i\psi/h}$$

is itself a semiclassical pseudodifferential operator, with principal symbol

$$\sigma(\tilde{A}_j) = \sigma(A_j)(x, \xi - \psi'(x).)$$

It follows that $\tilde{u}_h$ enjoys δ-Lagrangian regularity with respect to

$$\tilde{\mathcal{L}} = \{(x, \xi - \psi'(x)) : (x, \xi) \in \mathcal{L}\}$$

with $\tilde{\gamma} \equiv \gamma + (0, \psi'(x))$ not lying in the zero section. We can now choose coordinates on the base space X such that in the associated canonical coordinates, the Lagrangian $\tilde{\mathcal{L}}$ is locally parametrized near $\tilde{\gamma}$ by

$$\tilde{\mathcal{L}} \cap \tilde{U} = \{(\partial_\xi H(\xi), \xi) : \xi \in \tilde{W}\}$$

for some $H \in \mathcal{C}^\infty(\mathbb{R}^n)$ and for some open set $\tilde{W}$.We can now treat $\tilde{u}_h$ as was done for γ off the zero section, obtaining the oscillatory integral representation

$$\tilde{u}_h(x) = \int_{\mathbb{R}^n} a(\theta) e^{i(x\cdot\theta - H(\theta))/h}\, d\theta$$

microlocally near $\tilde{\gamma}$ and consequently

$$u_h(x) = \int_{\mathbb{R}^n} a(\theta) e^{i(x\cdot\theta - H(\theta) - \psi(x))/h}\, d\theta$$

microlocally near γ.

Part (2) of the proposition follows immediately from (22) and (14). □

4.1. **Improvements for quasimodes.** We now additionally assume that the δ-Lagrangian distribution u_h satisfies

$$\|Pu_h\|_{L^2} = O(h)$$

for a semiclassical pseudodifferential operator P of real principal type, with principal symbol p characteristic to $\mathcal{L}$, i.e., vanishing on it. Note that this hypothesis can be localized, as if B is a pseudodifferential operator with compact microsupport, then we also have

$$\|PBu_h\|_{L^2} = O(h).$$

Under the hypotheses that u_h is such a quasimode, we will obtain an improvement to Proposition 4.2 and Proposition 4.4. The first step is obtaining a mixed iterated regularity estimate.

Lemma 4.5. *Suppose u_h is a compactly microlocalized δ-Lagrangian distribution with respect to $\mathcal{L}$ that additionally satisfies*

$$\|Pu_h\|_{L^2} = O(h), \quad \|u_h\|_{L^2} = 1$$

where P is a semiclassical pseudodifferential operator characteristic to $\mathcal{L}$. Then for any $\epsilon > 0$, u_h enjoys the mixed iterated regularity estimate

$$\|PA_1 \dots A_N u_h\|_{L^2} = O(h^{N(1-\delta)+1-\epsilon}) \tag{23}$$

for any $A_j \in \Psi^{-\infty}$ characteristic to $\mathcal{L}$.

Proof. We have

$$PA_1 \dots A_N u_h = A_1 \dots A_N P u_h + O(h^{N(1-\delta)+1}) \tag{24}$$

as each commutator $[P, A]$ has $O(h)$ principal symbol characteristic to $\mathcal{L}$. We now proceed inductively to show that

$$\|A_1 \dots A_N P u_h\|_{L^2} = O(h^{N(1-\delta)+1-2^{-k}}) \tag{25}$$

for every non-negative integer k. For $k = 0$, (25) follows from (24), δ-Lagrangian regularity and L^2-boundedness of P. Now if we have (25) for a particular k and any collection of characteristic operators, we can compute

$$\begin{aligned}\|A_1 \dots A_N P u_h\|_{L^2}^2 &= |\langle A_N^* \dots A_1^* A_1 \dots A_N P u_h, P u_h \rangle| \\ &= O(h^{2N(1-\delta)+1-2^{-k}}) \cdot O(h) \\ &= O(h^{2N(1-\delta)+2-2^{-k}}).\end{aligned}$$

Taking square roots completes the induction and and using (24) once more proves (23). ☐

Proposition 4.6. *Let $\delta \in [0, 1/2)$. Suppose u_h satisfies the assumptions of Lemma 4.5 and that P has real-valued principal symbol p satisfying $|\partial p| \neq 0$ on $p^{-1}(0)$. For every point $\gamma = (x_0, \xi_0) \in U \cap \mathcal{L}$, we can find a symbol $a(x, \theta)$ in the class $S_\delta^{\frac{N}{2}+\frac{(n-1)\delta}{2}+0}(\mathbb{R}^{n+N})$ such that*

$$u_h = I(a, \phi)$$

microlocally near γ.

Proof. Following the proof of Proposition 4.2, we can assume that $\mathcal{L}$ is transverse to the constant section at γ. It then suffices to prove the estimate

$$\|\partial^\alpha v_h\|_\infty = O(h^{n/2-\delta(|\alpha|+(n-1)/2)-\epsilon}),$$

where $v_h(\xi) = \mathcal{F}_h u_h(\xi) e^{iH(\xi)/h}$, improving on (15) by a factor of $h^{\delta/2-\epsilon}$.

We do this by computing

$$\begin{aligned}P(x - \partial_\xi H(hD))^\alpha u_h &= P(x - \partial_\xi H(hD))^\alpha \mathcal{F}_h^{-1}(e^{-iH/h} v_h) \\ &= P \mathcal{F}_h^{-1}(e^{-iH/h}(-hD)^\alpha v_h).\end{aligned}$$

From Lemma 25 and Plancherel's theorem, it follows that

$$\|Q D^\alpha v_h\|_{L^2} = O(h^{\frac{n}{2}-\delta|\alpha|+1-\epsilon}) \tag{26}$$

where $Q = e^{iH/h} \mathcal{F}_h P \mathcal{F}_h^{-1} e^{-iH/h}$, with the exponential functions being regarded as multiplication operators. The principal symbol of Q is given by

$$q(x, \xi) = \sigma(\mathcal{F}_h P \mathcal{F}_h^{-1})(x, \xi + \partial_x H) = p(-\xi + \partial_x H, x)$$

from Egorov's theorem, so Q is characteristic to the zero section. As P was of real principal type, and characteristic to the Lagrangian $\mathcal{L}$ which is locally projectable in ξ, we have $\partial_x p \neq 0$ and so $\partial_\xi q \neq 0$. Reordering indices, we can assume

$$q(x, \xi) = e(x, \xi)(\xi_1 - b(x, \xi')).$$

with $e, b \in \mathcal{C}^\infty$ and $e(x_0, \xi_0) \neq 0$, where we have split $\xi = (\xi_1, \xi')$. By shrinking the initial microlocal cutoff of u_h if necessary, the local ellipticity of e together with (26) implies

$$\|(hD_{x_1} - b(x, hD')) D^\alpha v_h\|_{L^2} = O(h^{\frac{n}{2}-\delta|\alpha|+1-\epsilon}). \tag{27}$$

Recall that we may microlocalize u_h as finely as we like at the outset without affecting the hypotheses of this proposition, hence we assume without loss of generality that $v_h = O(h^\infty)$ outside a small neighborhood of ξ_0. Consequently (27) together with [21, Lemma 7.11] implies

$$\|D^\alpha v_h(x_1,\cdot)\|_{L^2(\mathbb{R}^{n-1})} = O(h^{\frac{n}{2}-\delta|\alpha|-\epsilon}).$$

Again using the fact that v is compactly supported modulo residual terms, Sobolev embedding in the remaining $n-1$ variables yields

$$\|\partial^\alpha v_h\|_{L^\infty} = O(h^{\frac{n}{2}-\frac{\delta(n-1)}{2}-\delta|\alpha|-\epsilon}) \tag{28}$$

as required.

□

As in Proposition 4.3, we have a simpler argument in the case that the Lagrangian is projectable onto X, that parametrizes $\mathcal{L}$ using a phase function ϕ with 0 phase variables.

Proposition 4.7. *Suppose u_h satisfies the assumptions of Lemma 4.5 and that P has real-valued principal symbol p satisfying $|\partial p| \neq 0$ on $p^{-1}(0)$. For every point $\gamma = (x_0,\xi_0) \in \mathcal{L}\cap U$ at which $\mathcal{L}$ is projectable and parametrized by the phase function $\phi(x)$, we can find a symbol $a(x)$ in the class $S_\delta^{\frac{(n-1)\delta}{2}+0}(\mathbb{R}^n)$ and a function $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that*

$$u_h(x) = a(x)e^{i\phi(x)/h}$$

microlocally near γ.

Proof. Choosing $U \subset \mathbb{R}^{2n}$ a small neighbourhood of γ with $\mathcal{L}\cap U$ projectable, we write

$$\mathcal{L}\cap U = \{(x,\partial_x\phi(x)) \in \mathbb{R}^{2n} : x \in W\}$$

for a bounded open set W. The symbols

$$b_j = \xi_j - \partial_{x_j}\phi$$

are then characteristic to $\mathcal{L}\cap U$ and by Lemma 4.5 we have

$$\|P(hD-\partial_x\phi)^\alpha u_h\|_{L^2} = O(h^{(1-\delta)|\alpha|+1-0}).$$

Taking $a = u_h e^{-i\phi/h}$ as in the proof of Proposition 4.3, it follows that

$$\|Pe^{i\phi/h}D^\alpha a\|_{L^2} = O(h^{-\delta|\alpha|}). \tag{29}$$

As P is of real principal type and is characteristic to the Lagrangian $\mathcal{L}$, which is locally projectable, we have $\partial_\xi p \neq 0$ and by reordering indices we can write p in the form

$$p(x,\xi) = e(x,\xi)(\xi_1 - b(x,\xi'))$$

with $e, b \in \mathcal{C}^\infty$ and $e(x_0,\xi_0) \neq 0$, where we have split $\xi = (\xi_1,\xi')$. The local ellipticity of e and (29) together show that

$$\|(hD_{x_1} - b(x,hD'))e^{i\phi/h}D^\alpha a\|_{L^2} = O(h^{-\delta|\alpha|+1-0}).$$

As u_h can be assumed to be $O(h^\infty)$ outside a small neighbourhood of x_0, we can apply [21, Lemma 7.11] once again to obtain

$$\|D^\alpha a(x_1,\cdot)\|_{L^2(\mathbb{R}^{n-1})} = O(h^{-\delta|\alpha|-0}).$$

Sobolev embedding in the remaining $n-1$ variables yields

$$\|D^\alpha a\|_{L^\infty} = O(h^{-\frac{\delta(n-1)}{2}-\delta|\alpha|-0})$$

as required. ☐

As in Proposition 4.4, we may also dispense with the condition that $\delta < 1/2$ if we specialize to a simple class of phase functions.

Proposition 4.8. *Suppose u_h satisfies the assumptions of Lemma 4.5 and that P has real-valued principal symbol p satisfying $|\partial p| \neq 0$ on $p^{-1}(0)$. For every point $\gamma = (x_0, \xi_0) \in \mathcal{L} \cap U$, we can choose local coordinates on X and find a symbol $a(\theta)$ in the class $S_\delta^{\frac{n}{2}+\frac{(n-1)\delta}{2}+0}(\mathbb{R}^n)$ and functions H, $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that*

$$u_h(x) = I(a,\phi)[x] = \int_{\mathbb{R}^n} a(\theta) e^{i(x\cdot\theta - H(\theta) - \psi(x))/h}\, d\theta$$

microlocally near γ. If γ does not lie on the zero section of T^*X then we can take $\psi = 0$.

Proof. As in the proof of Proposition 4.4, we begin by microlocalizing u_h to a neighbourhood U of γ, making the assumption that γ does not lie on the zero section, and choosing canonical coordinates so that $\mathcal{L}$ is locally projectable in ξ. The estimate (28) then immediately implies

$$\|\partial^\alpha a\|_{L^\infty} = O(h^{-\frac{n}{2}-\frac{\delta(n-1)}{2}-\delta|\alpha|-\epsilon})$$

as required.

If γ lies on the zero section, then we can proceed as in the end of the proof of Proposition 4.4, noting that $\tilde{u}_h$ will necessarily be an $O(h)$ quasimode for for the conjugated operator $\tilde{P} = e^{i\psi/h} P e^{-i\psi/h}$. ☐

5. Duistermaat's degenerate stationary phase and L^∞ estimates below threshold

Let $\mathcal{L}$ be a Lagrangian with a stable simple singularity and u_h be a δ-Lagrangian distribution with respect to $\mathcal{L}$, microsupported in a small neighborhood of the singularity in question and with $\|u_h\|_{L^2} = 1$. Assume that $\delta \leq \delta_0$ where δ_0 is the threshold for the singularity type (as listed in Table 1). Fix a phase function $\phi_0 = x\cdot\theta - H(\theta)$ parametrizing the stable simple singularity. By Propositions 4.2 and 4.6 (which we may apply since all thresholds δ_0 in question are less than $1/2$),

$$u_h(x) = I(a_0, \phi_0)[x]$$

where $a_0 \in S_\delta^{\frac{n}{2}+\frac{n\delta}{2}}(\mathbb{R}^{2n})$ in general or $a_0 \in S_\delta^{\frac{n}{2}+\frac{(n-1)\delta}{2}+0}(\mathbb{R}^{2n})$ if u_h satisfies an equation as described the latter proposition. By the classification of stable simple singularities, there is $\phi(x,\theta) = \sum x_j f_j(\theta) + f(\theta)$ chosen from Table (2) above that is locally equivalent to ϕ_0 in the sense that

$$\phi_0(x,\theta) = \phi(x',\theta') + \psi(x')$$

for some local fiber-preserving diffeomorphism

$$(x,\theta) \mapsto (x(x'), \theta(x',\theta'))$$

and some $\psi \in \mathcal{C}^\infty$. We thus change coordinates in the integral $I(a_0,\phi_0)$ from θ to θ' and note that pullback under this coordinate change leaves a in the same symbol class. This results in an integral of the form $I(a,\phi)$ with a a symbol of the same type, times an overall phase factor $e^{i\psi/h}$, all pulled back by a local diffeomorphism in x. Consequently in order to prove Theorem 1.2, it suffices to show that an oscillatory integral with one of the phase functions in Table (2) and with an amplitude lying in $S_\delta^{k/2}$ (where k is the number of phase variables) is $O_{L^\infty}(h^{-\kappa})$; here we have multiplied through by $h^{n\delta/2}$ resp. $h^{((n-1+\epsilon)\delta/2}$ in the two cases of a general δ-Lagrangian or a quasimode in order to eliminate the δ-dependence of the symbol order. In other words, pulling out an explicit factor of $h^{-k/2}$ as part of the normalization of the integral, it will suffice to prove the following:

Theorem 5.1. *Let*

$$I(x) = h^{-k/2} \int_{\mathbb{R}^k} a(x,\theta) e^{i\phi/h}\, d\theta,$$

where ϕ is one of the phase functions arising in Table 2, and where

$$a \in S_\delta^0.$$

For $\delta \in [0,\delta_0]$, where δ_0 is the threshold value listed in Table 1, there exists C such that for all $h \in (0,1)$,

$$\left\| I(x) \right\|_{L^\infty} \leq Ch^{-\kappa}$$

where κ is the order of the caustic listed in Table 1.

A novelty of the approach here is that we are unable to employ the Malgrange Preparation Theorem/Mather Division Theorem as in the classic treatments with $\delta = 0$ [9, Lemma 2.1.4, Equation (4.1.3)] and [13, Theorem 9.1]: the trouble is that the use of the Preparation Theorem costs numerous derivatives which are hard to keep track of, and each of these derivatives hitting the amplitude costs us h^δ. Since we are trying to obtain a cruder result (estimates rather than full asymptotics) we are able to use simpler and more robust methods. We now describe the method of proof.

Recall that we work only with the simple stable caustics in Arnol'd's classification. Depending on the overall dimension, each of these caustics can have an "equisingularity manifold" along which the form of the singularity of the projection is unchanged. In the model cases under discussion this

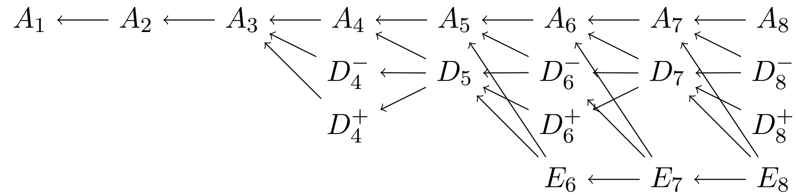

Figure 1. Subordination diagram of caustics (taken from [9, Figure 1]).

arises just because the phase is independent of some of the x variables, in particular, of all the x_j variables corresponding to vanishing f_j in Table 2. Let k_0 be the largest j with $f_j \neq 0$, so that $j = m$ for A_{m+1}, $j = m$ for $D^{\pm}_{m+1}$, and $j = 5, 6, 7$ for E_6, E_7, and E_8 respectively. Then near any point in the manifold $\{(x_1, \dots, x_{k_0}) = 0\} \subset \mathbb{R}^n \times \mathbb{R}^k$ the singularity is of the same type as at the origin, hence the term "equisingularity manifold." (A definition of equisingularity applicable in more general cases, but not needed here, is as the set of points where the germ of the phase is equivalent (via germs of mappings) to the singularity at a given point—see [9, p.243].)

Away from the equisingularity manifold, the Lagrangian projection will have a different singularity than that near the origin; the latter singularity is said to be *subordinate* to the one at the origin (see [9, p.255]). Arnol'd's classification comes with a characterization of subordinate singularities, encapsulated in the "subordination diagram" of caustics given in Figure 1. Arrows point from singularities to those types that may possibly arise in a small neighborhood of the origin in the complement of the equisingularity manifold $\{(x_1, \dots x_{k_0}) = 0\}$.

We will employ the subordination diagram to prove Theorem 5.1 by proceeding inductively to the right through the columns of the diagram, proving at each stage that the theorem holds for a new set of singularity types based on its validity for all subordinate types.

This means that the steps of our induction are:

(1) A_1 (no singularity)
(2) A_2
(3) A_3
(4) A_4, $D_4^{\pm}$
(5) A_5, D_5
(6) A_6, $D_6^{\pm}$, E_6
(7) A_7, D_7, E_7
(8) A_8, $D_8^{\pm}$, E_8.

Another ingredient in our arguments will be the quasi-homogeneity of the phase functions. We reproduce for the reader's convenience a table from Duistermaat [9, Table 4.3.2] showing the homogeneities of the parametrizing functions f_j and f from Table 2. These are the exponents r_j and s_j such

Type	$r_1, \dots r_k$	$s_1, \dots, s_k$
A_{m+1}	$\frac{1}{m+2}, \frac{1}{2}, \dots, \frac{1}{2}$	$\frac{1}{m+2}, \dots, \frac{m}{m+2}, 0, \dots, 0$
$D^{\pm}_{m+1}$	$\frac{1}{2} - \frac{1}{2m}, \frac{1}{m}, \frac{1}{2}, \dots, \frac{1}{2}$	$\frac{1}{2} - \frac{1}{2m}, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}, 0, \dots, 0$
E_6	$\frac{1}{3}, \frac{1}{4}, \frac{1}{2}, \dots, \frac{1}{2}$	$\frac{1}{3}, \frac{1}{4}, \frac{1}{2}, \frac{7}{12}, \frac{5}{6}, 0, \dots, 0$
E_7	$\frac{1}{3}, \frac{2}{9}, \frac{1}{2}, \dots \frac{1}{2}$	$\frac{1}{3}, \frac{2}{9}, \frac{4}{9}, \frac{6}{9}, \frac{8}{9}, \frac{5}{9}, 0, \dots 0$
E_8	$\frac{1}{3}, \frac{1}{5}, \frac{1}{2}, \dots \frac{1}{2}$	$\frac{1}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{8}{15}, \frac{11}{15}, \frac{14}{15}, 0, \dots 0$

TABLE 3. Homogeneities in the parametrizations of the stable singularities.

that

$$\phi(\lambda^{1-s_1} x_1, \dots, \lambda^{1-s_n} x_n, \lambda^{r_1}\theta_1, \dots, \lambda^{r_k}\theta_k) = \lambda\phi(x, \theta). \tag{30}$$

Note that these homogenities arise as follows in the parametrizations given above: the r_ℓ are simply the inverses of the homogeneities of the terms in $f(\theta)$ and then the s_ℓ are computed by writing the monomials $f_\ell(\theta)$ as $\theta_1^{s_{\ell 1}} \dots \theta_k^{s_{\ell k}}$ and then setting

$$s_\ell = \sum_{j=1}^k s_{\ell j} r_j.$$

These homogeneities lead directly to the orders of the relevant caustics, which are given by

$$\kappa = \frac{1}{2}k - \sum_{j=1}^k r_j$$

in each case—see Table 1 above. We are able to prove that our L^2-normalized Lagrangian distributions have sup-norm bounds given by $O(h^{-\kappa})$ in each case up to some threshold value of δ in our symbol regularity estimates, and this threshold, interestingly, seems to depend (at least in our proof) on not just the homogeneities of the caustic in question from the table above, but indeed on the homogeneities of caustics subordinate to it in Figure 1. Our threshold δ_0 for a singularity (see Table 1) is the *minimum* of the homogeneities r_j for all caustics encountered as we move to the left along arrows of the subordination diagram. Note the distinction in thresholds between $D^{\pm}_{m+1}$ for m odd occurs because A_m is subordinate to D^-_{m+1} but not to D^+_{m+1}. Meanwhile, the orders in our Table 1 simply match the orders of caustics in [9, Table 4.3.2].

With these preliminaries in hand, we proceed to the proof of the theorem.

Proof. We inductively show that if the result holds for all subordinate types to the singularity parametrized by ϕ then it holds for the singularity of ϕ as well. (The base case of the induction will be discussed at the end.)

Following the notation employed above (and in the proof of [9, Proposition 4.3.1]), we let k_0 denote the number of nonzero f_j in the parametrization of ϕ given in Table 2, which is equal to the codimension of the equisingularity manifold. Hence ϕ is in fact independent of all x variables except $x_1, \dots x_{k_0}$.

For $a \in (0, \infty)$, set

$$\Omega(a) = \{x : \sum_{j=1}^{k_0} |x_j|^{\frac{1}{1-s_j}} \leq a\}. \tag{31}$$

For simplicity of notation we will use multi-index notation for the scalings, so, e.g.

$$\mu^r \theta \equiv (\mu^{r_1}\theta_1, \dots, \mu^{r_k}\theta_k),$$

and

$$|r| \equiv \sum_{j=1}^{k} r_j.$$

We will also write

$$\mu^{1-s} x = (\mu^{1-s_1} x_1, \dots \mu^{1-s_{k_0}} x_{k_0}, x_{k_0+1}, \dots x_n), \tag{32}$$

i.e., the scaling in this case only applies to the first k_0 coordinates.

We first make a change of variables $\theta = h^r \eta$ (in the multiindex notation just introduced). By homogeneity (30),

$$f(\mu^r \eta) = \mu f(\eta), \qquad f_j(\mu^r \eta) = \mu^{s_j} f_j(\eta), \tag{33}$$

so that

$$I(x) = h^{-k/2+|r|} \int a(x, h^r \eta)\, e^{i \sum h^{s_j-1} x_j f_j(\eta)} e^{if(\eta)}\, d\eta.$$

Recall from [9, p.263] that $\nabla f(\eta) = 0$ only at $\eta = 0$. Hence we may obtain convergence of the integral by integrating by parts repeatedly using the first order differential operator

$$L_0 \equiv (1 + |\nabla f|^2)^{-1}(1 + \sum \partial_j f(\eta) D_j)$$

which has the property that $L_0 e^{if(\eta)} = e^{if(\eta)}$. Application of this operator to the remaining parts of the phase entails no loss in powers of h as long as $\delta \leq r_j$ (our standing assumption), and

$$|x_j| \leq h^{1-s_j} \text{ for all } j = 1, \dots, k_0,$$

i.e., as long as $x \in \Omega(h)$ (as defined in (31)). Moreover, provided $x \in \Omega(h)$, each factor of L_0 hitting the exponential term $e^{i \sum h^{s_k-1} x_k f_k(\eta)}$ is in fact a sum of terms each bounded by a multiple of one of the expressions

$$\frac{(\partial_j f)(\partial_j f_k)}{(\partial_j f)^2} \tag{34}$$

outside a large ball. From (33) we easily compute the homogeneities of derivatives:

$$(\partial_j f)(\mu^r \eta) = \mu^{1-r_j}(\partial_j f)(\eta), \quad (\partial_j f_k)(\mu^r \eta) = \mu^{s_k - r_j}(\partial_j f_k)(\eta), \tag{35}$$

hence (34) has negative homogeneities. We also note that terms where L_0 falls on a have increased decay for similar reasons. Thus, iteration of the integration by parts renders the integral convergent in η. Hence we obtain

$$I(x) = O(h^{-k/2+|r|}) \text{ for } x \in \Omega(h),$$

which suffices to prove the desired estimate for such values of x, since $\kappa = -k/2 + |r|$.

It thus remains to prove the desired estimate for $x \in \Omega(R) \backslash \Omega(h)$ for some R; without loss of generality we may do a fixed rescaling to take $R = 1$. For any $x \in \Omega(1) \backslash \Omega(h)$, there exists $\lambda \in [h, 1]$ such that if we set $x_j = \lambda^{1-s_j} y_j$ (for $j = 1, \ldots, k_0$) we now have $y \in \partial\Omega(1)$.

Thus, employing the change of variables $\theta = \lambda^r \eta$ (in the notation (32)), we obtain

$$\begin{aligned} I(\lambda^{1-s} y) &= h^{-k/2} \int a(\lambda^{1-s} y, \theta) e^{i\phi(\lambda^{1-s} y, \theta)/h} \, d\theta \\ &= \lambda^{|r|} h^{-k/2} \int a(\lambda^{1-s} y, \lambda^r \eta) e^{i\phi(\lambda^{1-s} y, \lambda^r \eta)/h} \, d\eta \\ &= \lambda^{|r|} h^{-k/2} \int a(\lambda^{1-s} y, \lambda^r \eta) e^{i(\lambda/h)\phi(y,\eta)} \, d\eta. \end{aligned}$$

We split $I(\lambda^{1-s} y)$ into two pieces by letting $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ equal 1 on $(-1, 1)$ and 0 on $\mathbb{R} \backslash (-2, 2)$, picking $R > 0$, and expressing

$$I(\lambda^{1-s} y) = J_<(\lambda^{1-s} y) + J_>(\lambda^{1-s} y)$$

with

$$\begin{aligned} J_<(\lambda^{1-s} y) &\equiv \lambda^{|r|} h^{-k/2} \int \chi(|\eta|/R) a(\lambda^{1-s} y, \lambda^r \eta) e^{i(\lambda/h)\phi(y,\eta)} \, d\eta, \\ J_>(\lambda^{1-s} y) &\equiv \lambda^{|r|} h^{-k/2} \int (1 - \chi(|\eta|/R)) a(\lambda^{1-s} y, \lambda^r \eta) e^{i(\lambda/h)\phi(y,\eta)} \, d\eta, \end{aligned}$$

We once again remark that by (35), since $\nabla f(\eta)$ is nonzero for $\eta \neq 0$ and has larger homogeneity than the ∇f_j's (since all $s_j < 1$), $\nabla_\eta \phi$ is nonzero on sufficiently large quasi-homogeneous balls, hence if R is sufficiently large, the denominator of the operator

$$L_1 = \frac{h}{\lambda} |\nabla_\eta \phi|^{-2} \sum (\partial_{\eta_j} \phi) D_{\eta_j},$$

is nonvanishing, and we may use it to integrate by parts in our expression for $J_>$. Derivatives falling on the a term each yield a factor bounded uniformly in $y \in \partial\Omega(1)$ by $(h/\lambda)\lambda^{r_j} h^{-\delta}$ for some j; since $\lambda < 1$ and $\delta \leq r_j$, this is bounded by $(h/\lambda)^{1-\delta}$ uniformly in $y \in \partial\Omega(1)$. Derivatives falling on the cutoff χ of course yield (h/λ), which is smaller yet since $h/\lambda \leq 1$. If we employ a high enough power of L_1, we moreover obtain convergence of the integral in η, again by considerations of homogeneity. Hence for all $N \geq N_0$,

$$J_>(\lambda^{1-s}y) = O(\lambda^{|r|}h^{-k/2}(h/\lambda)^{N(1-\delta)})$$

where N_0 only depends on the phase function ϕ. Recalling that $\kappa = k/2 - |r|$ and that $h/\lambda \leq 1$, we thus may choose $N \geq N_0$ to obtain

$$J_>(\lambda^{1-s}y) = O(h^{-\kappa}),$$

uniformly for $y \in \partial\Omega(1)$ and $\lambda \in [h,1]$ and hence uniformly for $x \in \Omega(1)\backslash\Omega(h)$.

We now turn to estimating $J_<(\lambda^{1-s}y)$. This term does have a stationary phase. To estimate it, we rewrite

$$\begin{aligned} J_<(\lambda^{1-s}y) &= \lambda^{|r|}h^{-k/2}\Big(\frac{h}{\lambda}\Big)^{+k/2}\Big(\frac{h}{\lambda}\Big)^{-k/2}\int \chi(|\eta|/R)a(\lambda^{1-s}y,\lambda^r\eta)e^{i(\lambda/h)\phi(y,\eta)}\,d\eta \\ &\equiv \lambda^{|r|}h^{-k/2}\Big(\frac{h}{\lambda}\Big)^{+k/2}K(\lambda^{1-s}y). \end{aligned}$$

Note then that the integral expression for $K(\lambda^{1-s}y)$ is once again of the type that our theorem applies to, but with (h/λ) replacing h as the small parameter, and where we are interested in taking y in $\partial\Omega(1)$, hence away from the set $\mathcal{E} = \{x : x_1 = x_2 = \ldots = x_{k_0} = 0\}$, where the phase is most singular. In particular, since $\lambda < 1$, we do still have $a(\lambda^{1-s}y, \lambda^r\eta)\chi(|\eta|/R) \in S^0_\delta$, compactly supported, uniformly for $\lambda \in [h,1]$. With y constrained to be near $\partial\Omega(1)$, and hence away from $\mathcal{E}$, the projection of the equisingularity manifold through the origin, we are guaranteed that the phase must parametrize a singularity strictly further down the subordination diagram (Figure 1); cf. [9, Proposition 4.3.1]. Thus the phase function ϕ is equivalent, locally near any (y_0, η_0) at which it is stationary with $y_0 \in \partial\Omega(1)$, to some other phase function $\widetilde{\phi}$ where $\widetilde{\phi}$ is one of the phase functions from Table 2 parametrizing a singularity subordinate to the one we started with. Since, as noted at the beginning of this section, we may change phase function to an equivalent one by making a change of phase variables (and a coordinate transformation in the base), we may use our inductive hypothesis to estimate

$$K(\lambda^{1-s}y) = O\big((h/\lambda)^{-\kappa'}\big).$$

Here $\kappa' \leq \kappa$, since moving down the subordination diagram reduces the order of the caustic.

Thus, recalling that $\kappa = k/2 - |r|$, and using the facts that $\lambda \geq h$ and $\kappa \geq \kappa'$, we reassemble our estimates for $J_{\gtrless}$ to obtain

$$\begin{aligned} |I(\lambda^{1-s}y)| &\leq C\lambda^{|r|}h^{-k/2}\Big(\frac{h}{\lambda}\Big)^{+k/2}\Big(\frac{h}{\lambda}\Big)^{-\kappa'} + Ch^{-\kappa} \\ &= C\lambda^{-\kappa+\kappa'}h^{-\kappa'} + Ch^{-\kappa} \\ &\leq Ch^{-\kappa}. \end{aligned}$$

To complete the induction, it suffices to establish Theorem 1.2 for a δ-Lagrangian distribution u_h that is microsupported on a projectable subset of the Lagrangian $\mathcal{L}$. That is, it remains to establish the case A_1. The claimed order of u_h in this case is $\kappa = 0$, with threshold $\delta = 1$. Due to the breakdown of stationary phase asymptotics for $\delta \geq 1/2$, it is simplest to use the particular representation $u_h = ae^{i\phi/h}$ obtained in Proposition 4.3 and Proposition 4.7 for δ-Lagrangian distributions and quasimodes respectively. In either case, we have $\|u_h\|_{L^\infty} = \|a\|_{L^\infty}$, and the desired estimates on $\|u_h\|_{L^\infty}$ follow. □

6. Beyond the δ_0 threshold

In this section, we determine the sharp L^∞ estimates for the situation described in Theorem 1.2, but now with $\delta \in [\delta_0, 1]$ beyond the threshold of that theorem. We work in the simplest nontrivial case, that of the fold caustic A_2 in $\mathbb{R}^1$. This caustic is famously associated with the asymptotics of the Airy function

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x\theta+\theta^3/3)}\, d\theta,$$

as the phase of the integral has an A_2 singularity.

Theorem 6.1. *Let $\delta \in [0,1]$. Let u_h be a compactly supported δ-Lagrangian distribution with respect to the Lagrangian*

$$\{x = \xi^2\} \subset T^*\mathbb{R}^1$$

Then there exists $C = C_\delta$ such that for $h \in (0,1)$, we have

$$\frac{\|u_h\|_{L^\infty}}{\|u_h\|_{L^2}} \leq \begin{cases} Ch^{-(1+3\delta)/6} & \text{if } \delta \in [0,1/3] \\ Ch^{-(1+\delta)/4} & \text{if } \delta \in [1/3,1] \end{cases} \tag{36}$$

and these estimates are sharp.

Proof. Theorem 1.2 for the singularity A_2 in dimension $n = 1$ gives the estimate

$$\frac{\|u_h\|_{L^\infty}}{\|u_h\|_{L^2}} = O(h^{-1/6-\delta/2}) \tag{37}$$

for $\delta \in [0,1/3]$, which is saturated at $x = 0$ by the example

$$u_h(x) = \int_{\mathbb{R}} \chi(\theta/h^\delta) e^{i(x\theta+\theta^3)/h}\, d\theta$$

where $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ is a h-independent bump function, nonvanishing at 0. To see this, we observe that $\|u_h\|_{L^2} = O(h^{1/2+\delta/2})$ by Plancherel, and u_h is a δ-Lagrangian distribution by (14). Direct computation then yields

$$u_h(0) = \int_{\mathbb{R}} \chi(\theta/h^\delta) e^{i\theta^3/h}\, d\theta = h^{1/3}\int_{\mathbb{R}} \chi(h^{1/3-\delta}\theta) e^{i\theta^3}\, d\theta \sim Ch^{1/3}$$

for $C \neq 0$ as $h \downarrow 0$. Hence

$$\frac{\|u_h\|_{L^\infty}}{\|u_h\|_{L^2}} \gtrsim \frac{h^{1/3}}{h^{1/2+\delta/2}} = h^{-1/6-\delta/2}. \tag{38}$$

At the threshold $\delta = 1/3$, (37) coincides with (36), and both give the bound

$$\frac{\|u_h\|_{L^\infty}}{\|u_h\|_{L^2}} = O(h^{-1/3}).$$

We now prove the estimate (36) in the case $\delta \geq 1/3$. Recall from the beginning of the proof of Proposition 4.2 that for *any* $\delta \in [0,1]$ if we parametrize our Lagrangian with the special phase function $\phi(x,\theta) = x\theta - \theta^3/3$, we arrive at the oscillatory integral representation

$$u_h(x;h) = \int_{\mathbb{R}} a(\theta;h) e^{i(x\theta - \theta^3/3)/h} \, d\theta, \tag{39}$$

where $a \in \mathcal{C}_c(\mathbb{R})$ satisfies the estimate $\|\partial^\alpha a\|_{L^2} = O(h^{-1/2-\delta|\alpha|})$ (12) as well as the Sobolev embedding estimate (15) $\|\partial^\alpha a\|_{L^\infty} = O(h^{-(1+\delta)/2})$. (Note that in the notation of (12), (15), we have $a = h^{-1} v_h$, with the factor of h^{-1} arising from the inverse semiclassical Fourier transform.)

We now integrate by parts in (39) using an h-dependent regularization of the operator $(x-\theta^2)^{-1} h D_\theta$ which stabilizes the exponential factor in the integrand, but is singular at the caustic. To this end, we introduce the differential operator

$$L = (x - \theta^2 + ih^{1-\delta})^{-1}(hD_\theta + ih^{1-\delta}).$$

This operator stabilizes the exponential factor in the integrand and has transpose

$$L^T = (x - \theta^2 + ih^{1-\delta})^{-1}(-hD_\theta + ih^{1-\delta}) - \frac{2h\theta}{(x-\theta^2+ih^{1-\delta})^2}.$$

Integration by parts shows $u_h(x)$ is bounded above by

$$h\int_{\mathbb{R}} \frac{|D_\theta a|}{|x-\theta^2+ih^{1-\delta}|}\, d\theta + h^{1-\delta}\int_{\mathbb{R}} \frac{|a|}{|x-\theta^2+ih^{1-\delta}|}\, d\theta + 2h\int_{\mathbb{R}} \frac{|\theta a|}{|x-\theta^2+ih^{1-\delta}|^2}\, d\theta$$

Using Cauchy–Schwarz, we obtain

$$\begin{aligned}
|u_h(x)| &\lesssim (h\|D_\theta a\|_{L^2} + h^{1-\delta}\|a\|_{L^2}) \left(\int_{\mathbb{R}} \frac{1}{(x-\theta^2)^2 + h^{2-2\delta}}\, d\theta\right)^{1/2} \\
&\quad + h\|a\|_{L^\infty} \int_{\mathbb{R}} \frac{|\theta|}{(x-\theta^2)^2 + h^{2-2\delta}}\, d\theta \\
&\lesssim h^{1/2-\delta} \left(\int_{\mathbb{R}} \frac{1}{(x-\theta^2)^2 + h^{2-2\delta}}\, d\theta\right)^{1/2} + h^{(1-\delta)/2} \int_{\mathbb{R}} \frac{|\theta|}{(x-\theta^2)^2 + h^{2-2\delta}}\, d\theta.
\end{aligned}$$

We now estimate these integrals as follows.

Lemma 6.2. *We have the following two integral estimates, uniform for $x \in \mathbb{R}$ as $\epsilon \to 0^+$.*

$$\int_{\mathbb{R}} \frac{1}{(x-\theta^2)^2+\epsilon^2}\, d\theta = O(\epsilon^{-3/2}) \tag{40}$$

$$\int_{\mathbb{R}} \frac{|\theta|}{(x-\theta^2)^2+\epsilon^2}\, d\theta = O(\epsilon^{-1}). \tag{41}$$

Proof. We evaluate the first integral by changing variables to set $\eta = \theta\epsilon^{-1/2}$. This yields

$$\epsilon^{-3/2} M(-x\epsilon^{-1}),$$

where

$$M(\alpha) \equiv \int_{-\infty}^{\infty} \frac{1}{(\eta^2+\alpha)^2+1}\, d\eta.$$

It thus suffices to show that $\sup_{\alpha\in\mathbb{R}} |M(\alpha)| < \infty$. Indeed M is manifestly uniformly bounded for $\alpha \geq 0$; to deal with negative α, we note that the integral can be evaluated explicitly by contour integration to yield $\pi \operatorname{Re}(\alpha + i)^{-1/2}$, which is indeed uniformly bounded for $\alpha \in \mathbb{R}$.

The integral (41) is simply

$$\begin{aligned}\int_{\mathbb{R}} \frac{|\theta|}{(x-\theta^2)^2+\epsilon^2}\, d\theta &= 2\int_0^{\infty} \frac{\theta}{(x-\theta^2)^2+\epsilon^2}\, d\theta \\ &= \epsilon^{-1}\left[\arctan\left(\frac{\theta^2-x}{\epsilon}\right)\right]_0^{\infty} \\ &\leq \pi\epsilon^{-1} \quad \square\end{aligned}$$

As a consequence of these estimates, and since we are taking $\delta \geq 1/3$, we now obtain

$$\begin{aligned}|u_h(x)| &\lesssim h^{1/2-\delta}\cdot h^{-3/4+3\delta/4} + h^{(1-\delta)/2}\cdot h^{\delta-1} \\ &= O(h^{-(1+\delta)/4})\end{aligned}$$

uniformly for $x \in \mathbb{R}$ for any $\delta \geq 1/3$. This is the desired upper bound.

To show that the estimate is sharp, we simply remark that our estimate is saturated by the δ-Lagrangian distribution given by (39) with amplitude

$$a = h^{(\delta-3)/4}\chi(\theta/h^{(1-\delta)/2})e^{i\theta^3/3h}$$

where χ is a h-independent bump function, nonvanishing at 0. This a has L^2 norm $O(h^{-1/2})$, hence $\|u_h\|_{L^2}$ is uniformly bounded, by Plancherel. Moreover, we have

$$\|\partial_\theta^\alpha a\|_{L^2} = O(h^{-1/2-\delta|\alpha|}) \tag{42}$$

as $\theta^2/h \leq h^{-\delta}$ in the support of a, hence u_h is indeed an L^2 bounded δ-Lagrangian distribution by (14). On the other hand, we may explicitly compute

$$u(0) = h^{(\delta-3)/4} \int \chi(\theta/h^{(1-\delta)/2}) \gtrsim h^{(\delta-3)/4} \cdot h^{(1-\delta)/2} = h^{-(1+\delta)/4},$$

thereby saturating our upper bound. □

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