PERIODIC DAMPING GIVES POLYNOMIAL ENERGY DECAY

JARED WUNSCH

ABSTRACT. Let u solve the damped Klein–Gordon equation
$$\big(\partial_t^2 - \sum \partial_{x_j}^2 + m\,\mathrm{Id} + \gamma(x)\partial_t\big)u = 0$$
on $\mathbb{R}^n$ with $m > 0$ and $\gamma \geq 0$ bounded below on a $2\pi\mathbb{Z}^n$-invariant open set by a positive constant. We show that the energy of a solution decays at a polynomial rate. This is proved via a periodic observability estimate on $\mathbb{R}^n$.

1. INTRODUCTION

Consider the damped Klein–Gordon equation on $[0, \infty) \times \mathbb{R}^n$:

$$(1) \qquad \big(\partial_t^2 - \sum \partial_{x_j}^2 + m\,\mathrm{Id} + \gamma(x)\partial_t\big)u = 0,$$

with $\gamma(x) \geq 0$ for all x, and $m > 0$. Burq–Joly [8] have recently proved that if there is *uniform geometric control* in the sense that there exist $T, \epsilon > 0$ such that $\int_0^T \gamma(x(t))\,dt \geq \epsilon$ along every straight line unit-speed trajectory, then u enjoys exponential energy decay, thus generalizing classic results of Bardos, Lebeau, Rauch, and Taylor [14], [3], [4] to a noncompact setting. By contrast, in the case of merely periodic γ (or, more generally, under the assumption that γ is strictly positive on a family of balls whose dilates cover $\mathbb{R}^n$) then Burq–Joly show that a logarithmic decay of energy still holds.

In this note, we show that in fact u enjoys at least a *polynomial* rate of energy decay (with derivative loss) provided that γ is nontrivial and periodic, or, more generally, strictly positive on a periodic set:

Theorem 1. *Assume that $m > 0$ and $0 \leq \gamma \in L^\infty$ and that there exist $\epsilon > 0$ and a $2\pi\mathbb{Z}^n$-invariant open set $\Omega \subset \mathbb{R}^n$ such that $\gamma(x) \geq \epsilon$ for a.e. $x \in \Omega$. Then there exists $C > 0$ such that for u solving* (1),

$$\|(u(t), u_t(t))\|_{H^1 \times L^2} \leq \frac{C}{\sqrt{1+t}} \|(u(0), u_t(0))\|_{H^2 \times H^1}.$$

Note that we do not require any hypothesis of geometric control.

We proceed by a standard route to this estimate by first proving an *observability estimate*, which then leads to a *resolvent estimate*.

Let
$$\Delta = -\sum \partial_{x_j}^2$$
denote the nonnegative Laplace operator. The observability estimate (which may be of independent interest owing to applications in control theory) is then as follows:

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty, open, $2\pi\mathbb{Z}^n$-invariant set. For all $\lambda \in \mathbb{R}$ we have the following estimate:*
$$(\Delta - \lambda)u = f \implies \|u\|_{L^2(\mathbb{R}^n)} \leq C\big(\|f\|_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\Omega)}\big)$$
with C independent of λ.

From the observability estimate, it is not difficult to obtain a *resolvent estimate* as follows:

Theorem 3. *Let γ be as in Theorem 1. Then*

(2) $(\Delta + m\operatorname{Id} + is\gamma(x) - s^2\operatorname{Id})u = f \implies \|u\|_{L^2} \lesssim C(1 + |s|)\|f\|_{L^2}.$

The strategy will be to prove Theorem 2 by reducing it to known observability estimates on the torus; this argument is the main novelty here. This leads to Theorem 3 by standard arguments (given below). The decay estimate Theorem 1 then follows by a functional-analytic argument due to Borichev–Tomilov [6].

We remark that *the decay rate obtained here is almost certainly not optimal* in the case of smooth damping. The work of Anantharaman-Léautaud [1] shows that on $\mathbb{T}^2$ one can obtain better estimates for damping with better than L^∞ smoothness by moving the damping to the left-hand side of the estimates, treating it as part of the operator rather than as a term to be estimated as an error. The arguments used here make it difficult to bring to bear the finer results of [1] in the periodic setting, as the incorporation of the damping term into the operator one is trying to estimate makes the proof of our Proposition 4 fail badly. To obtain a stronger estimate, one would therefore have to follow the second-microlocal arguments of [1] directly rather than simply using the resulting estimate on the torus. As refined estimates linked to the regularity of the damping term remain a difficult subject of current research, we will not pursue such an approach in this note.

In what follows, the constant C will change from line to line, but will always be independent of the spectral parameter. As noted above, $\Delta \geq 0$ denotes the nonnegative Laplacian, and we also denote $D_{x_j} = i^{-1}\partial_{x_j}$, so that $\Delta = \sum D_{x_j}^2$, or, abusing notation slightly, $\Delta = D_x^2$. With no subscript, the notation $\|\bullet\|$ denotes L^2 norm. We use the notation for the standard torus $\mathbb{T}^n \equiv \mathbb{R}^n/2\pi\mathbb{Z}^n$.

2. Proofs of Main results

2.1. **Twisted Laplacian.** We begin by establishing observability estimates on a bundle Laplacian on the flat torus (cf. Lemma 2.4 of [9] for a related estimate).

Let $\alpha \in \mathbb{R}^n$. Set

$$(3) \qquad H_\alpha = (D_x - \alpha)^2.$$

Note that these operators are all self-adjoint with the same domain independent of α.

Proposition 4. *Let $\Upsilon \subset \mathbb{T}^n$ be open and nonempty. For all $\alpha \in [0,1)^n$,*

$$(4) \qquad (H_\alpha - \lambda)u = f \text{ on } \mathbb{T}^n \Longrightarrow \|u\|_{L^2(\mathbb{T}^n)} \le C\|f\|_{L^2(\mathbb{T}^n)} + C\|u\|_{L^2(\Upsilon)},$$

with constants independent of α and $\lambda \in \mathbb{R}$.

Proof. With $\alpha = 0$ the result is known, from the estimates of Jaffard [10] in dimension $n = 2$ and Komornik [11] in higher dimension. We will use these results to generalize to variable α.

We recall that one approach to proving estimates of the form (4) is based on an observability estimate for the the Schrödinger propagator (see Theorem 4 of [2]): we say that *Schrödinger observability* holds for H_α if for every open, nonempty $\omega \subset \mathbb{T}^n$ and every $T > 0$ there exists $C = C(T,\omega)$ such that

$$(5) \qquad \|f\|^2 \le C \int_0^T \left\| e^{itH_\alpha} f \right\|_{L^2(\omega)}^2 dt.$$

That Schrödinger observability (5) for H_α is equivalent to the resolvent estimate (4) for H_α follows from Theorem 5.1 of Miller [13].

Thus we will prove the proposition by proving Schrödinger observability for any $\alpha \in \mathbb{R}^n$. Given an open $\Upsilon \subset \mathbb{T}^n$, fix a nonempty open $\omega \subset \mathbb{T}^n$ such that $\overline{\omega} \subset \Upsilon$, hence there exists $T > 0$ such that $x \in \omega$ and $d(x,y) < 4\sqrt{n}T$ implies $y \in \Upsilon$. (Here $d(x,y)$ denotes the distance function between points on $\mathbb{T}^n$.)

Now we note that the propagators

$$U_\alpha(t) \equiv e^{itH_\alpha}$$

all commute with one another, and indeed we may factor

$$(6) \qquad U_\alpha(t) = U_0(t) \exp(-2it\alpha \cdot D + it|\alpha|^2)$$

$$(7) \qquad\qquad = e^{it|\alpha|^2} \tau_{-2t\alpha} U_0(t),$$

where for $\theta \in \mathbb{R}^n$, τ_θ denotes the translation operator $\tau_\theta f(x) = f(x + \theta)$. Thus, by H_0-observability and the choice of $T \ll 1$ so that $\tau_{2t\alpha}(\omega) \subset \Upsilon$ for

$t \in [0, T]$, we obtain

$$
(8) \qquad \|f\|^2 \leq C \int_0^T \|U_0(t)f\|_{L^2(\omega)}^2 \, dt
$$

$$
(9) \qquad \leq C \int_0^T \left\| e^{it|\alpha|^2} \tau_{-2\alpha t} U_0(t)f \right\|_{L^2(\tau_{2\alpha t}(\omega))}^2 \, dt
$$

$$
(10) \qquad \leq C \int_0^T \left\| e^{it|\alpha|^2} \tau_{-2\alpha t} U_0(t)f \right\|_{L^2(\Upsilon)}^2 \, dt
$$

$$
(11) \qquad \leq C \int_0^T \|U_\alpha(t)f\|_{L^2(\Upsilon)}^2 \, dt.
$$

As noted above, Theorem 5.1 of [13] now shows that this Schrödinger observability estimate implies our resolvent estimate. $\qquad \square$

2.2. Observability estimate.
The proof of Theorem 2 now proceeds as follows. For $g \in \langle x \rangle^{-s} H^{-\infty}(\mathbb{R}^n)$ with $s > n/2$, define its periodization $\Pi g \in \mathcal{D}'(\mathbb{T}^n)$ by

$$
\Pi g(x) = \sum_{\ell \in \mathbb{Z}^n} g(x + 2\pi\ell).
$$

More generally, for $\alpha \in \mathbb{R}^n$ we set

$$
(\Pi_\alpha g) = \Pi(e^{i\alpha x} g).
$$

Note that this quantity is quasi-$\mathbb{Z}^n$-periodic in $\alpha \in \mathbb{R}^n$: we have for $k \in \mathbb{Z}^n$,

$$
(\Pi_{\alpha+k} g)(x) = e^{ikx}(\Pi_\alpha g)(x).
$$

Lemma 5. *We have the equality of L^2 norms*

$$
(12) \qquad \|g\|_{L^2(\mathbb{R}^n)}^2 = \int_{[0,1)^n} \|\Pi_\alpha g\|_{L^2(\mathbb{T}^n)}^2 \, d\alpha.
$$

More generally, if $\Omega \subset \mathbb{R}^n$ is $2\pi\mathbb{Z}^n$-invariant and Ω_0 denotes its projection to $\mathbb{T}^n$,

$$
\|g\|_{L^2(\Omega)}^2 = \int_{[0,1)^n} \|\Pi_\alpha g\|_{L^2(\Omega_0)}^2 \, d\alpha.
$$

Proof. We use Fubini to compute the Fourier coefficients of the periodic functions $\Pi_\alpha g$ on $\mathbb{T}^n$:

$$
\widehat{\Pi_\alpha g}(\ell) = (2\pi)^{-n/2} \int_{[0,2\pi]^n} \sum_{m \in \mathbb{Z}^n} g(x + 2\pi m) e^{i\alpha(x+2\pi m)} e^{-i\ell x} \, dx
$$

$$
= (2\pi)^{-n/2} \int_{[0,2\pi]^n} \sum_{m \in \mathbb{Z}^n} g(x + 2\pi m) e^{i\alpha(x+2\pi m)} e^{-i\ell(x+2\pi m)} \, dx
$$

$$
= (2\pi)^{-n/2} \int_{\mathbb{R}^n} g(y) e^{i\alpha y} e^{-i\ell y} \, dy
$$

$$
= \mathcal{F}(g)(\ell - \alpha).
$$

Integrating the sum of squares of the RHS over the unit cube gives $\|g\|^2_{L^2(\mathbb{R}^n)}$ by Fubini and Plancherel on $\mathbb{R}^n$, while on the LHS, we get

$$\int_{[0,1)^n} \sum \left|\widehat{\Pi_\alpha g}(\ell)\right|^2 d\alpha = \int_{[0,1)^n} \|\Pi_\alpha g\|^2_{L^2(\mathbb{T}^n)} d\alpha$$

by Plancherel on the torus.

The generalization to taking the norm over Ω is proved simply by applying (12) to the function $\mathbb{1}_\Omega g$. $\qquad \square$

Now we note that

$$H_\alpha - \lambda \equiv (D_x - \alpha)^2 - \lambda$$
$$= e^{i\alpha x}(\Delta - \lambda)e^{-i\alpha x}.$$

Thus $(\Delta - \lambda)u = f$ yields

$$(H_\alpha - \lambda)e^{i\alpha x}u = e^{i\alpha x}f \quad \text{on } \mathbb{R}^n.$$

Applying Π to both sides and using translation-invariance of H_α, we get an equation on the torus:

$$(H_\alpha - \lambda)(\Pi_\alpha u) = \Pi_\alpha f \quad \text{on } \mathbb{T}^n.$$

Applying Proposition 4, we obtain for every α in a fundamental domain (and with constants independent of α)

$$\|\Pi_\alpha u\|^2 \leq C\|\Pi_\alpha f\|^2 + C\|\Pi_\alpha u\|^2_{L^2(\Omega_0)},$$

Now by Lemma 5 we may integrate both sides in $\alpha \in [0,1)^n$ to obtain

$$\|u\|^2 \leq C\|f\|^2 + C\|u\|^2_{L^2(\Omega)}.$$

This concludes the proof of Theorem 2. $\qquad \square$

2.3. **Resolvent estimate.** We now prove Theorem 3. We will be brief, as this is ground well-trodden by other authors.

We start by noting that if we pair the equation (2) with u and take the real part, we obtain (using Cauchy-Schwarz) for $|s| \leq s_0 \equiv \sqrt{m}/2$

$$\|u\|^2_{H^1(\mathbb{R}^n)} \leq C\|f\|^2_{L^2(\mathbb{R}^n)}.$$

This proves the estimate near $s = 0$, so we will take $|s| > s_0$ for fixed s_0 below.

Again pairing (2) with u and this time taking the imaginary part yields the usual estimate

$$\|\sqrt{\gamma}u\|^2 \leq \frac{C}{|s|}\|f\|\|u\|.$$

On the other hand, applying Theorem 2 to (2) with the damping term on the right-hand side (and $\lambda = s^2 - m$) yields

(13)
$$\|u\| \leq C\|f\| + C|s|\|\gamma u\| + C\|u\|_{L^2(\Omega)}$$
$$\leq C\|f\| + C|s|\|\gamma u\|,$$

where we chose Ω contained in the set where $\gamma \geq \epsilon$ a.e. for some $\epsilon > 0$ (and used $s \geq s_0$). Combining these estimates and observing that $\gamma \leq C\sqrt{\gamma}$ a.e. yields for $|s| \geq s_0$

$$\|\sqrt{\gamma}u\|^2 \leq \frac{C}{|s|}\|f\|^2 + C\|f\|\|\sqrt{\gamma}u\|.$$

Applying Cauchy-Schwarz we obtain

$$\|\sqrt{\gamma}u\|^2 \leq C\|f\|^2, \quad |s| > s_0.$$

Finally returning to (13) gives

$$\|u\| \leq C\|f\| + C|s|\|f\|. \quad \square$$

2.4. **Proof of energy decay.** In this section, we apply the resolvent estimate, Theorem 3, to prove our result on energy decay for the damped Klein–Gordon equation, Theorem 1. To do this we follow the strategy used by Anantharaman–Léautaud [1], albeit in the much simpler framework of [8], in which low energy issues are rendered moot by the positive Klein–Gordon mass. (We cannot simply quote Proposition 2.4 of [1] verbatim, however, as its hypotheses include a compact resolvent assumption that fails here.)

The strategy consists of employing the following theorem of Borichev–Tomilov [6] (this is in fact just one part of Theorem 2.4 of [6]):

Theorem (Borichev–Tomilov). Let e^{tA} be a bounded $\mathcal{C}^0$ semigroup on a Hilbert space with generator A with $\mathrm{spec}(A) \cap i\mathbb{R} = \emptyset$. Then

$$\left\|(A - is\,\mathrm{Id})^{-1}\right\| = O(|s|^\alpha),\ |s| \to \infty \Longleftrightarrow \left\|e^{tA}A^{-1}\right\| = O(t^{-1/\alpha}),\ t \to \infty.$$

(This represents a slight strengthening of a prior result of Batty-Duyckaerts [5] in Banach spaces.)

We will apply this theorem to the semigroup generated by

$$A = \begin{pmatrix} 0 & \mathrm{Id} \\ -\Delta - m\,\mathrm{Id} & -\gamma \end{pmatrix}$$

acting on the energy space $\mathcal{H} \equiv H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

The resolvent estimate from Theorem 3 implies the condition on non-imaginary spectrum on A as well as the resolvent estimate on A as follows: if $\mathbf{u} = (u_0, u_1)^t$ and $\mathbf{f} = (f_0, f_1)^t$ then

$$(A - is\,\mathrm{Id})\mathbf{u} = \mathbf{f}$$

is equivalent to

$$(14) \qquad \begin{aligned} (\Delta + m + is\gamma - s^2)u_0 &= f_1 + (\gamma + is)f_0, \\ u_1 &= f_0 + isu_0, \end{aligned}$$

i.e., if we let $R(is)$ denote the inverse of $(\Delta + m + is\gamma - s^2)$, we have

$$\mathbf{u} = (A - is\,\mathrm{Id})^{-1}\mathbf{f} = \begin{pmatrix} R(is)(\gamma + is) & R(is) \\ \mathrm{Id} + R(is)(is)(\gamma + is) & isR(is) \end{pmatrix}\mathbf{f}.$$

Existence of $R(is)$ on L^2 (with norm $O(\langle s \rangle)$) is Theorem 3, and pairing (2) with u as usual and taking real parts easily establishes that $R(is) : L^2 \to H^1$ with norm $O(\langle s \rangle^2)$. Thus $(A - is\,\mathrm{Id})$ is invertible and we have verified the spectral condition. We can further use these methods to estimate

$$\left\| (A - is\,\mathrm{Id})^{-1} \right\|_{\mathcal{H} \to \mathcal{H}} = O(\langle s \rangle^2);$$

details of the argument can be found in, e.g., Lemma 4.6 of [1] (cf. also [12] and [7]). This yields the decay rate $\langle t \rangle^{-1/2}$ for the damped Klein Gordon equation by the theorem of Borichev–Tomilov. $\qquad\square$

References

[1] Nalini Anantharaman and Matthieu Léautaud, *Sharp polynomial decay rates for the damped wave equation on the torus*, Analysis & PDE **7** (2014), no. 1, 159–214.

[2] Nalini Anantharaman and Fabricio Macià, *Semiclassical measures for the Schrödinger equation on the torus*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 6, 1253–1288. MR3226742

[3] C. Bardos, G. Lebeau, and J. Rauch, *Un exemple d'utilisation des notions de propagation pour le contrôle et la stabilisation de problèmes hyperboliques*, Rend. Sem. Mat. Univ. Politec. Torino **1988**, no. Special Issue, 11–31 (1989). Nonlinear hyperbolic equations in applied sciences.

[4] _____, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM Journal on Control and Optimization **30** (1992), no. 5, 1024–1065.

[5] Charles J. K. Batty and Thomas Duyckaerts, *Non-uniform stability for bounded semi-groups on Banach spaces*, J. Evol. Equ. **8** (2008), no. 4, 765–780. MR2460938 (2009m:47104)

[6] Alexander Borichev and Yuri Tomilov, *Optimal polynomial decay of functions and operator semigroups*, Math. Ann. **347** (2010), no. 2, 455–478. MR2606945 (2011c:47091)

[7] Nicolas Burq and Michael Hitrik, *Energy decay for damped wave equations on partially rectangular domains*, Math. Res. Lett. **14** (2007), no. 1, 35–47. MR2289618 (2008a:35180)

[8] Nicolas Burq and Romain Joly, *Exponential decay for the damped wave equation in unbounded domains*, 2014.

[9] Nicolas Burq and Maciej Zworski, *Control for Schrödinger operators on tori*, Math. Res. Lett. **19** (2012), no. 2, 309–324. MR2955763

[10] S. Jaffard, *Contrôle interne exact des vibrations d'une plaque rectangulaire*, Portugal. Math. **47** (1990), no. 4, 423–429. MR1090480 (91j:93051)

[11] V. Komornik, *On the exact internal controllability of a Petrowsky system*, J. Math. Pures Appl. (9) **71** (1992), no. 4, 331–342. MR1176015 (93j:93019)

[12] G. Lebeau, *Équations des ondes amorties*, Séminaire sur les équations aux dérivées partielles, 1993–1994, 1994, pp. Exp. No. XV, 16.

[13] Luc Miller, *Controllability cost of conservative systems: resolvent condition and transmutation*, J. Funct. Anal. **218** (2005), no. 2, 425–444. MR2108119 (2005i:93015)

[14] Jeffrey Rauch and Michael Taylor, *Exponential decay of solutions to hyperbolic equations in bounded domains*, Indiana Univ. Math. J. **24** (1974), 79–86. MR0361461 (50 #13906)

Department of Mathematics, Northwestern University, 2033 Sheridan Rd., Evanston IL 60208, USA

E-mail address: jwunsch@math.northwestern.edu