

NON-CONCENTRATION OF QUASIMODES FOR INTEGRABLE SYSTEMS

JARED WUNSCH

ABSTRACT. We consider the possible concentration in phase space of a sequence of eigenfunctions (or, more generally, a quasimode) of an operator whose principal symbol has completely integrable Hamilton flow. The semiclassical wavefront set WF_h of such a sequence is invariant under the Hamilton flow. In principle this may allow concentration of WF_h along a single closed orbit if all frequencies of the flow are rationally related. We show that, subject to non-degeneracy hypotheses, this concentration may not in fact occur. Indeed, in the two-dimensional case, we show that WF_h must fill out an entire Lagrangian torus. The main tools are the spreading of Lagrangian regularity previously shown by the author, and an analysis of higher order transport equations satisfied by the principal symbol of a Lagrangian quasimode. These yield a unique continuation theorem for the principal symbol of Lagrangian quasimode, which is the principal new result of the paper.

1. INTRODUCTION

A central question in spectral geometry is how a sequence of eigenfunctions of the Laplace-Beltrami operator on an n -dimensional manifold X may concentrate in phase space. The *semiclassical wavefront set* or *frequency set*, here denoted WF_h , is a closed set measuring the locations in phase space (i.e., T^*X) where a sequence of functions is non-negligible; for a sequence of eigenfunctions of the Laplacian, WF_h is known to be invariant under the geodesic flow. In the case when the geodesic flow is completely integrable, this leaves open the possibility that WF_h may concentrate on a single closed orbit or some other small set invariant under the geodesic flow. The results of this paper put limitations on this possible concentration. We show that in many circumstances, concentration may not occur on the smallest possible set allowed by standard propagation arguments, which is to say, a single closed orbit of the bicharacteristic flow. These results can be viewed as analogues of results of Bourgain, Jakobson, Macià and Anantharaman-Macià in

Date: September 25, 2011.

The author is grateful to Michael Koehn and Steve Zelditch for raising the question answered here. He thanks Zelditch for many instructive conversations on Lagrangian quasimodes and semiclassical analysis. He is also grateful to Nalini Anantharaman for helpful discussions and to András Vasy for comments on the manuscript, as well as to an anonymous referee. This work was partially supported by NSF grants DMS-0700318 and DMS-1001463.

the case when X is a flat torus (and of more recent work of Burq-Zworski on Schrödinger operators on 2-tori), extended to the broader context of (nondegenerate) completely integrable systems.

The results of this paper apply somewhat more generally than to sequences of eigenfunctions: they are equally applicable to approximate eigenfunctions or *quasimodes* (see [2], [6]). We write such approximate solutions in the formalism of semiclassical analysis: for instance, instead of having a sequence of approximate eigenfunctions of the (non-negative) Laplace-Beltrami operator:

$$(\Delta - \lambda_k^2)u_k = O(\lambda_k^{-\infty}),$$

we may set $h = 1/\lambda_k$, suppress the index, and write

$$(h^2\Delta - 1)u = O(h^\infty).$$

This is the notation we employ below.

Our first results show how the results obtained by the author [20] (and refined by Vasy and the author [19, 18]) on the spreading of Lagrangian regularity of quasimodes for quantizations of classically integrable systems can be easily applied to obtain results on the nonconcentration of semiclassical wavefront set in the same setting. Consider a semiclassical pseudodifferential operator P with semiclassical principal symbol p , acting on half-densities.¹ We assume that

- (A) $p = \sigma_h(P)$ is real.
- (B) The subprincipal symbol of P (which is well defined for an operator on half-densities) is a real constant on² T^*X .
- (C) The bicharacteristic flow of the Hamilton vector field H_p is *completely integrable*.

The integrability hypothesis means that there exist action-angle variables $(I_1, \dots, I_n, \theta_1, \dots, \theta_n)$, i.e. symplectic coordinates $I \in \mathbb{R}^n$, $\theta \in \mathbb{R}^n/\mathbb{Z}^n$ such that $p = p(I)$ is independent of θ . In fact *we only need to assume non-degeneracy of these coordinates in some region of interest in phase space* (in a neighborhood of the Lagrangian torus introduced below) rather than globally, where it is unlikely to hold in general. The assumption on subprincipal symbols is certainly satisfied for geometric Schrödinger operators $h^2\Delta_g + V(x)$ with g a Riemannian metric and V a real potential, as the subprincipal symbol vanishes identically; for more details on this hypothesis, we refer the reader to §5 of [19].

The Arnol'd-Liouville tori of the system are the sets where the I variables are held constant. We consider one such torus \mathcal{L} . Let

$$\omega_i = \partial p / \partial I_i$$

¹For a discussion of semiclassical pseudodifferential operators, we refer the reader to [7] or [9].

²It will in fact suffice for the subprincipal symbol to be real everywhere on T^*X and constant on the Lagrangian \mathcal{L} discussed below. In what follows we will indicate in footnotes where the proof varies in the case of this weaker hypothesis.

and

$$\omega_{ij} = \partial^2 p / \partial I_i \partial I_j.$$

Let $\bar{\omega}_i$ and $\bar{\omega}_{ij}$ denote the corresponding quantities restricted to \mathcal{L} (where they are constant). We make the assumption, standard in KAM theory, that

- (D) The system is (in a neighborhood of \mathcal{L}) *isoenergetically nondegenerate*;

this means that the matrix

$$(1) \quad \Omega = \begin{pmatrix} \bar{\omega}_{11} & \dots & \bar{\omega}_{1n} & \bar{\omega}_1 \\ \vdots & \ddots & \vdots & \vdots \\ \bar{\omega}_{n1} & \dots & \bar{\omega}_{nn} & \bar{\omega}_n \\ \bar{\omega}_1 & \dots & \bar{\omega}_n & 0 \end{pmatrix}$$

is nonsingular, or equivalently, that the map from the energy surface to the projectivization of the frequencies

$$\{p = 0\} \ni I \mapsto [\omega_1(I) : \dots : \omega_n(I)] \in \mathbb{RP}^n$$

is a local diffeomorphism.

Now we consider a normalized quasimode of P , concentrated on \mathcal{L} , i.e. a family of distributions $u = u(x; h)$ satisfying

- (E) $Pu = O(h^\infty)$, $\|u\|_{L^2} = 1$, $\text{WF}_h u \subset \mathcal{L}$.

For example $u = u(x; h)$ may be a family of exact, normalized Schrödinger eigenfunctions in the nullspace of $P = h^2\Delta + V - E(h)$, with $E(h) \sim E_0 + hE_1 + \dots$ as $h \downarrow 0$; more generally, it may be a family of approximate eigenfunctions. We recall that the *semiclassical wavefront set* or *frequency set* of u is defined as the closed set

$$\text{WF}_h u = \{\rho \in T^*X : \exists A \in \Psi_h(X), \sigma_h(A)(\rho) \neq 0, Au = O(h^\infty)\}^{\text{cl}}.$$

This set is well known, by the semiclassical analogue of the Duistermaat-Hörmander theorem on propagation of singularities, to be invariant under \mathbf{H}_p . Thus if there exist no rational linear relations among the $\bar{\omega}_i$, then as a closed invariant set, $\text{WF}_h u$ must fill out the whole of the torus \mathcal{L} . (It cannot be empty, as that would contradict L^2 -normalization.) The results of Bourgain and Jakobson [13] in the case of the Laplacian on flat tori show that even on completely rational tori, however, the semiclassical limit measure of a sequence of eigenfunctions must project to the base to be absolutely continuous; Macià [14] Anantharaman-Macià [15] show analogous results for more general limit measures arising from Schrödinger flow. We prove similar results in the setting of WF_h , in the more general setting described above. Estimates roughly of the form that we employ here were previously used by Burq-Zworski [4] in showing that a sequence of eigenfunctions on the Bunimovich stadium cannot concentrate along a single bouncing-ball orbit, and generalized to the general setting of integrable systems by the author [20] and further by the author and Vasy [19, 18]. Very recent results

of Burq-Zworski [5] also yield stronger, control-theoretic, estimates in the special case of a Schrödinger operator $\Delta + V$ on a two-dimensional torus.

To begin, we recall a key result from [20].³ This result describes how Lagrangian regularity with respect to \mathcal{L} is forced to spread on \mathcal{L} . (The definition of Lagrangian regularity is discussed below.)

Theorem 1 ([20]). *Let P and u satisfy hypotheses (A)–(E).*

- (1) *If the dimension $n = 2$ and u enjoys Lagrangian regularity at some point on \mathcal{L} , then u is Lagrangian with respect to \mathcal{L} .*
- (2) *If the $\bar{\omega}_i$ are all rationally related, and u enjoys Lagrangian regularity outside a single closed bicharacteristic $\gamma \subset \mathcal{L}$, then u is Lagrangian with respect to \mathcal{L} .*

Semiclassical Lagrangian distributions are a special class of distributions with WF_h lying only on \mathcal{L} . There are two principal ways to describe this class. One characterization is simply that u has (locally near each point) an oscillatory integral representation of a well-understood form, which we discuss below. The second characterization is that u has iterated regularity with respect to pseudodifferential operators characteristic on \mathcal{L} . We describe these definitions below in §2; for further details, we refer the reader to the paper of Alexandrova [1], where the results of the Hörmander-Melrose theory [12] are adapted to the semiclassical setting. The simplest and most instructive example of a semiclassical Lagrangian distribution is just a family of the form

$$(2) \quad h^{-s} a(x; h) e^{i\phi(x)/h}$$

where $s \in \mathbb{R}$, a is in \mathcal{C}^∞ , uniformly in h and $\phi \in \mathcal{C}^\infty$. This is a Lagrangian distribution with respect to

$$\mathcal{L} = \text{Graph}(d\phi) \subset T^*X.$$

Indeed, if the projection of \mathcal{L} to X is a diffeomorphism, every Lagrangian distribution with respect to \mathcal{L} is locally of this form. When the projection is not a diffeomorphism, we need instead to employ oscillatory integral representations, i.e., integral superpositions of expressions of the form 2—see (3) below.

As a simple consequence of Theorem 1, we are able to prove:

Theorem 2. *Let P and u satisfy hypotheses (A)–(E).*

- (1) *Let $n = 2$. Then $\text{WF}_h u$ has nonempty interior (in the relative topology of \mathcal{L}).*
- (2) *If the $\bar{\omega}_i$ are all rationally related, then $\text{WF}_h u$ cannot be a subset of a single closed bicharacteristic.*

³The hypothesis used in [20] was that of vanishing subprincipal symbol; the use of weaker hypotheses is discussed in [19, 18].

The upshot of these results is that a normalized quasimode may not concentrate on too small a subset of a Lagrangian torus. We now state an additional result, with stronger hypotheses, that implies that indeed the quasimode must be supported on the *whole* invariant torus.

The necessary stronger hypothesis is:

- (F) The system is (in a neighborhood of \mathcal{L}) *quasi-convex* in the sense of Nekhoroshev.

This means that the Hessian $\partial^2 p / \partial I_i \partial I_j$ is strictly positive definite on the fixed energy surface $p = 0$ (in a neighborhood of \mathcal{L})—see [16] for details of the use of this hypothesis in proving exponential stability of orbits for perturbed integrable systems. *Note that the hypothesis of quasi-convexity implies isoenergetic nondegeneracy (hypothesis (D)).* The main new result in this paper is then the following.

Theorem 3. *Let P and u satisfy (A)–(C) and (E)–(F). Assume that u is Lagrangian with respect to \mathcal{L} . Then the support of the principal symbol of u is all of \mathcal{L} . The same conclusion still holds if (E) is weakened to merely $Pu = O(h^{2+\delta})$ for some $\delta > 0$.*

Remark 4. The principal symbol of a Lagrangian distribution is locally a half-density on \mathcal{L} which characterizes its leading order behavior; in the example (2), it can be taken to be the amplitude a , modulo $O(h)$. The global description of the principal symbol is more involved, as one must take into account both Maslov factors and the cohomology class of the canonical one-form restricted to \mathcal{L} . For accounts of this construction, we refer the reader to work of Duistermaat [8] and Bates and Weinstein [3].

A priori, we have not assumed that the Lagrangian distribution is *classical*, i.e. enjoys a power series expansion in h : $a \sim a_0 + ha_1 + \dots$. Thus, the support theorem as stated merely tells us that there is no open set $\mathcal{O} \subset \mathcal{L}$ on which the localization of a is $O(h)$. In the special case of classical Lagrangians, though, this implies that the support of a_0 is all of \mathcal{L} .

In the general, non-classical case, we in fact show more than nonvanishing modulo $O(h)$: to wit, we obtain the stronger statement that there do not exist an open set $\mathcal{O} \subset \mathcal{L}$ and a sequence $h_j \downarrow 0$ with $\sigma_h(a)(x; h_j) \rightarrow 0$ pointwise a.e. on \mathcal{O} .

By Theorem 3, we immediately obtain a very strong non-concentration result in two dimensions, subject to the quasi-convexity hypothesis:

Corollary 5. *Let P and u satisfy (A)–(C) and (E)–(F). Let $n = 2$. Then $\text{WF}_h u = \mathcal{L}$.*

Remark 6. More generally, we may relax the assumption $\text{WF}_h u \subset \mathcal{L}$: if we merely have $\text{WF}_h u \cap \mathcal{L} \neq \emptyset$, then it follows that $\mathcal{L} \subset \text{WF}_h u$.

2. PROOF OF QUASIMODE NONCONCENTRATION

We begin by recalling in more detail the notion of Lagrangian regularity. Throughout this paper, we will let $\Psi_h(X)$ denote the algebra of semiclassical

pseudodifferential operators on the manifold X (acting on half-densities) obtained locally by quantization of Kohn-Nirenberg symbols, as described for instance in §9 of [9].

We say that u is Lagrangian with respect to $h^s L^2$ at $\rho \in \mathcal{L}$ if there exists a neighborhood U of ρ in T^*X such that for all $k \in \mathbb{N}$ and all $A_1, \dots, A_k \in \Psi_h(X)$ with $\sigma_h(A_j) = 0$ on \mathcal{L} , and $\text{WF}'(A_j) \subset U$, we have

$$h^{-k} A_1 \cdots A_k u \in h^s L^2.$$

In other words, u enjoys “iterated regularity” under the application of operators of the form $h^{-1}A$ with A characteristic on \mathcal{L} and microsupported near ρ .

We now restate Theorem 1 slightly more precisely: In [20] (see also [19, 18]) it was shown that, subject to the above hypotheses, if u is in L^2 and is Lagrangian⁴ in an annular region (i.e., a hollow tube) surrounding a single closed bicharacteristic on a rational Lagrangian torus, then for every $\epsilon > 0$ it is Lagrangian with respect to $h^{-\epsilon} L^2$ on γ as well. Likewise, in the special case $n = 2$, the same argument proved that if u is in L^2 and Lagrangian at a single point on \mathcal{L} , then u is Lagrangian with respect to $h^{-\epsilon} L^2$ globally on \mathcal{L} .

We now prove the first part of Theorem 2. Either all of \mathcal{L} lies in $\text{WF}_h u$ or there exists $\rho \in \mathcal{L}$ such that $\rho \notin \text{WF}_h u$. In the latter case, then *a fortiori* u is Lagrangian at ρ . Thus, by the results of [19], u is Lagrangian on all of \mathcal{L} with respect to $h^{-\epsilon} L^2$ for all $\epsilon > 0$. Now by the semiclassical analog of the Hörmander-Melrose theory of Lagrangian distributions (see Alexandrova [1]) this means that microlocally near \mathcal{L} , u can be written as an oscillatory integral

$$(3) \quad u = h^{-\epsilon + N/2} \int_{\mathbb{R}^N} a(x, \xi; h) e^{i\phi(x, \xi)/h} d\xi$$

for some $N \in \mathbb{N}$ determined by the geometry of \mathcal{L} (in particular by the local form of its projection to the base) and with an amplitude $a \in C_c^\infty$, uniformly in h . Here ϕ parametrizes \mathcal{L} in the sense that

$$(4) \quad \mathcal{L} = \{(x, d_x \phi) : (x, \xi) \in C\} \equiv \Phi(C).$$

where

$$(5) \quad C = \{(x, \xi) : d_\xi \phi = 0\}.$$

By stationary phase, for $x, \xi \in C$,

$$\Phi(x, \xi) \in \text{WF}_h u \iff a(\bullet, h) \neq O(h^\infty) \text{ in a neighborhood of } (x, \xi).$$

By smoothness of a , the set of such points is the closure of an open set. (Recall that the semiclassical wavefront set cannot be empty, as that would

⁴It suffices to assume Lagrangian regularity at this point with respect to $h^{-t} L^2$ for any $t \in \mathbb{R}$, as by interpolation with $u \in L^2$ we automatically obtain Lagrangian regularity with respect to $h^{-\delta} L^2$ for all $\delta > 0$.

contradict the L^2 normalization.) This concludes the proof of the first part of the theorem.

The proof of the second part is analogous. By the hypotheses, we have Lagrangian regularity everywhere but along γ . Theorem 1 then allows us to conclude global Lagrangian regularity, which, by the argument above, shows that $\text{WF}_h u$ has nonempty interior, and in particular, cannot have been a subset of γ after all. \square

3. LAGRANGIAN QUASIMODES

In this section, we prove Theorem 3.

We recall that the principal symbol $\sigma_h(u)$ of a semiclassical Lagrangian distribution $u \in h^{-s}L^2$ given locally by

$$(6) \quad u = h^{-s+N/2} \int_{\mathbb{R}^N} a(x, \xi; h) e^{i\phi(x, \xi)/h} d\xi$$

can be *locally* identified with the half-density a , restricted to the manifold C given by (5) (which is, in turn, diffeomorphically identified with \mathcal{L} via Φ defined in (4)), modulo $O(h)$. By standard results in the calculus of Lagrangians, we know that $\sigma_h(u)$ is necessarily invariant under $\mathcal{L}_{H_p} + ip'$ with p' denoting (the operation of multiplication by) a subprincipal symbol of P . (Indeed, this same invariance property holds globally, where we interpret the symbol as a section of the tensor product of $\Omega^{1/2}$ with a flat complex line bundle.)

3.1. The model case. We begin our finer analysis of $\sigma_h(u)$ by considering the model case $X = \mathbb{T}_x^n = \mathbb{R}_x^n / \mathbb{Z}^n$, with the action-angle variables given by

$$I_j = \xi_j, \quad \theta_j = x_j$$

and the Lagrangian torus given by the zero section:

$$\mathcal{L}_0 = \{\xi = 0\} \subset T^*(\mathbb{T}_x^n).$$

We will use $|dx_1 \cdots dx_n|^{1/2}$ to trivialize the half-density bundle; note that this has vanishing Lie derivative along constant coefficient vector fields in x , hence we will identify symbols with functions and the action of \mathcal{L}_{H_p} with that of H_p .

Since the principal symbol of P is assumed to be a function only of ξ , vanishing on \mathcal{L}_0 , its Taylor expansion in the ξ variables reads

$$p = \sum \bar{\omega}_j \xi_j + \bar{\omega}_{ij} \xi_i \xi_j + O(|\xi|^3)$$

and hence, since the subprincipal symbol of P is assumed to be a real constant⁵ c ,

$$P = h \sum \bar{\omega}_j D_j + hc + h^2 \sum \bar{\omega}_{ij} D_i D_j + \sum h^3 D_i D_j D_k Q_{ijk} + h^2 R$$

⁵If the subprincipal symbol is only constant on \mathcal{L} , there is an extra term in this expression of the form $h^2 \sum \kappa_i(x) D_i$.

where $Q_{ijk}, R \in \Psi_h(X)$. The hypothesis of quasi-convexity of p near \mathcal{L}_0 means that the matrix $\bar{\omega}_{ij}$ is positive definite on the orthocomplement of the span of the vector $\sum \bar{\omega}_j \partial_j$. Note that the term $\sum \bar{\omega}_j D_j$ is $i^{-1}H_p$ in this setting, and we will use this latter notation as well.

We may further Taylor expand the principal symbol of R into

$$r(x) + \sum \xi_i r_i(x, \xi),$$

hence we may express

$$h^2 R = h^2 r(x) + h^3 \sum R_i D_i + h^3 E$$

with $E, R_i \in \Psi_h(X)$ and $r(x)$ denoting the multiplication operator by the function of the same name. Plugging this into our expression for P we now obtain

$$(7) \quad P = h \sum \bar{\omega}_j D_j + hc + h^2 \sum \bar{\omega}_{ij} D_i D_j + h^2 r(x) \\ + h^3 \sum D_i D_j D_k Q_{ijk} + h^3 \sum D_i R_i + h^3 \tilde{R}$$

with $\tilde{R} \in \Psi_h(X)$. We will collectively write the last three terms as $O(h^3)$ below.

We remark⁶ that in this model situation, the iterated regularity definition of Lagrangian distributions and the definition by oscillatory integrals actually coincide: a parametrization of \mathcal{L}_0 is by the phase function $\phi = 0$, and u is Lagrangian with respect to $h^{-\epsilon} L^2$ if and only if

$$u = u(x; h) \in \mathcal{C}^\infty(\mathbb{T}^n), \text{ with } \|\partial_x^\alpha u\| \lesssim h^{-\epsilon} \text{ for all } \alpha,$$

i.e. u is $h^{-\epsilon}$ times a function that is smooth in X , uniformly in h . In this situation, the distribution u and its total symbol coincide. The principal symbol of u as a Lagrangian distribution with respect to $h^{-\epsilon} L^2$ is simply the equivalence class.

$$\sigma_h(u) = u \text{ mod } O(h^{1-\epsilon}).$$

We now prove a unique continuation theorem for the principal symbol of a Lagrangian quasimode of P in the model setting. This will constitute the main step in the proof of Theorem 3, with the remainder of the proof being a conjugation to this model problem.

Proposition 7. *Let \mathcal{L}_0 be the zero section of $T^*(\mathbb{T}^n)$ and P as described above. If $Pu = O(h^{2+\delta})$ with $\delta > 0$ and u is Lagrangian with respect to \mathcal{L}_0 and L^2 normalized, then there do not exist $\mathcal{O} \subset \mathcal{L}_0$ open and $h_j \downarrow 0$ with $\sigma_h(u)(x; h_j) \rightarrow 0$ pointwise a.e. on \mathcal{O} .*

Proof. As discussed above, we have

$$(8) \quad P = h \sum \bar{\omega}_j D_j + hc + h^2 \sum \bar{\omega}_{ij} D_i D_j + h^2 r(x) + O(h^3) \\ = h(-iH_p + c) + h^2 \sum \bar{\omega}_{ij} D_i D_j + h^2 r(x) + O(h^3)$$

⁶Indeed, this remark will justify our $O(h^3)$ notation just introduced.

Certainly then the principal symbol, $\sigma_h(u) = u \bmod O(h^{1-\epsilon})$, is annihilated by

$$-i\mathbf{H}_p + c,$$

hence if it is nonvanishing at a point $\rho \in \mathcal{L}_0$, it is also nonvanishing along the whole orbit γ_ρ of ρ under the flow along \mathbf{H}_p . We let T denote the closure of the orbit through one such ρ ; T is necessarily a sub-torus of dimension $k \geq 1$. Identifying \mathbb{T}^n with $\mathbb{R}^n/\mathbb{Z}^n$ we may lift T to \mathbb{R}^n and translate coordinates to obtain a vector subspace $V \subset \mathbb{R}^n$ of dimension k . Letting $L = V \cap \mathbb{Z}^n$ we obtain a sub-lattice of \mathbb{Z}^n , with \mathbb{Z}^n/L torsion-free. Hence by taking a basis of the quotient module we may extend a basis $\mathbf{e}_1, \dots, \mathbf{e}_k$ of L to a basis $\mathbf{e}_1, \dots, \mathbf{e}_k, \mathbf{f}_1, \dots, \mathbf{f}_{n-k}$ of \mathbb{Z}^n . Now identifying

$$\mathbb{T}^n = \mathbb{R}^n / (\mathbb{Z}\mathbf{e}_1 + \dots + \mathbb{Z}\mathbf{e}_k + \mathbb{Z}\mathbf{f}_1 + \dots + \mathbb{Z}\mathbf{f}_{n-k}),$$

we may split

$$\mathbb{T}^n \cong T \times T'$$

with

$$\begin{aligned} T &= (\mathbb{R}\mathbf{e}_1 + \dots + \mathbb{R}\mathbf{e}_k) / (\mathbb{Z}\mathbf{e}_1 + \dots + \mathbb{Z}\mathbf{e}_k), \\ T' &= (\mathbb{R}\mathbf{f}_1 + \dots + \mathbb{R}\mathbf{f}_{n-k}) / (\mathbb{Z}\mathbf{f}_1 + \dots + \mathbb{Z}\mathbf{f}_{n-k}). \end{aligned}$$

We may thus introduce new coordinates $x = (y_1, \dots, y_k, z_1, \dots, z_{n-k})$ on \mathbb{T}^n with T defined by $z = 0$, $y \in \mathbb{R}^k/\mathbb{Z}^k$ and T' defined by $z = 0$ with $y \in \mathbb{R}^{n-k}/\mathbb{Z}^{n-k}$. On the torus T (and its translates $\{z = z_0\}$, for $z_0 \in T'$), the Hamilton flow has dense orbits, hence the space of solutions v of functions on T satisfying

$$(9) \quad (-i\mathbf{H}_p + c)v = 0$$

has dimension at most 1, since specifying v on a single point then determines v on a dense set in T . Equivalently, Fourier analyzing in $y \in T$, we may write

$$v(y) = \sum_{\alpha \in \mathbb{Z}^k} \hat{v}(\alpha) e_\alpha(y)$$

with $e_\alpha(y) = e^{2\pi i \alpha \cdot y}$. Since $\mathbf{H}_p = \sum \bar{\omega}_i \partial_{x_i}$ is tangent to T , and since its orbit closure is dense in T , we may write it in our new coordinates as

$$\sum \bar{\omega}_i \partial_{x_i} = \sum \tilde{\omega}_j \partial_{y_j}$$

for some $\tilde{\omega}_i$, $i = 1, \dots, k$, with no rational relation holding among the $\tilde{\omega}_i$. Then

$$(-i\mathbf{H}_p + c)v(y) = 0 \iff \sum_{\alpha \in \mathbb{Z}^k} (\tilde{\omega} \cdot \alpha + c) \hat{v}(\alpha) e_\alpha(y) = 0.$$

By the irrationality assumption, there can be at most one value of α such that $\tilde{\omega} \cdot \alpha + c = 0$, hence v can only have at most this one Fourier mode.

If the dimension of the solution space of (9) is 0, then since the principal symbol must satisfy (9), there cannot be an L^2 -normalized quasimode. Thus, we assume that the dimension of smooth solutions to (9) is 1, hence that there is exactly one frequency vector α_0 satisfying $\tilde{\omega} \cdot \alpha_0 + c = 0$.

Now we decompose both sides of (8) in Fourier modes along T , i.e. in the y variables. We let

$$u = \sum e_\alpha(y) \hat{u}_\alpha(z; h)$$

denote the Fourier series of u . We also note that the second order constant coefficient operator

$$Q = \sum \bar{\omega}_{ij} D_{x_i} D_{x_j}$$

becomes, under the change to y, z variables, a new operator of the same form, which is *elliptic in the z variables* as a consequence of the quasi-convexity hypothesis (which tells us that $\sum \bar{\omega}_{ij} D_i D_j$ is elliptic on the orthocomplement of H_p). We may rewrite this operator as

$$Q = \sum \rho_{ij}^1 D_{y_i} D_{y_j} + \sum \rho_{ij}^2 D_{y_i} D_{z_j} + \sum_{i,j=1}^{n-k} \Omega_{ij} D_{z_i} D_{z_j}$$

with the matrix Ω positive definite. Thus, it acts on

$$u = \sum e_\alpha(y) \hat{u}_\alpha(z; h),$$

as

$$(10) \quad \begin{aligned} Qu &= \sum_{\alpha \in \mathbb{Z}^k} e_\alpha(y) \left[\sum_{i,j=1}^{n-k} \Omega_{ij} D_{z_i} D_{z_j} + \sum \gamma_i(\alpha) D_{z_i} + \rho(\alpha) \right] (\hat{u}_\alpha) \\ &\equiv \sum_{\alpha} e_\alpha(y) Q_\alpha (\hat{u}_\alpha) \end{aligned}$$

with γ_i, ρ depending linearly resp. quadratically on α . Each operator Q_α is an elliptic constant coefficient operator in z . We also decompose

$$r(x) = r(y, z) = \sum e_\alpha(y) \hat{r}_\alpha(z).$$

Thus, our Fourier analysis of (8) (for an $O(h^{2+\delta})$ quasimode) finally yields

$$(11) \quad \sum e_\alpha(y) (h(\tilde{\omega} \cdot \alpha + c) \hat{u}_\alpha + h^2 Q_\alpha \hat{u}_\alpha) + h^2 \sum_{\alpha, \beta} e_{\alpha+\beta}(y) \hat{r}_\beta \hat{u}_\alpha + O(h^{3-\epsilon}) = O(h^{2+\delta}).$$

(where we have used the fact that overall, $u = O(h^{-\epsilon})$). Examining this equation modulo $O(h^{2-\epsilon})$ reveals that all terms \hat{u}_α with $\alpha \neq \alpha_0$ are $O(h^{1-\epsilon})$. Thus, we focus attention on the coefficient of e_{α_0} . The term $\tilde{\omega} \cdot \alpha + c$ in (11) then vanishes, and the terms in the discrete convolution are $O(h^{3-\epsilon})$ except when $\beta = 0, \alpha = \alpha_0$; thus, we obtain, by examining the coefficient of e_{α_0} in (11) modulo $O(h^{2+\delta})$,

$$h^2 (Q_{\alpha_0} + \hat{r}_0) \hat{u}_{\alpha_0} = O(h^{2+\delta}).$$

Hence for some $g(z; h)$, uniformly bounded as $h \downarrow 0$,

$$(Q_{\alpha_0} + \hat{r}_0) \hat{u}_{\alpha_0}(z; h) = h^\delta g(z; h).$$

Let L denote the elliptic operator⁷

$$L = Q_{\alpha_0} + \hat{r}_0$$

on $T' \cong \mathbb{R}^{n-k}/\mathbb{Z}^{n-k}$. Since it is elliptic on a compact manifold, L has finite-dimensional nullspace on $L^2(T')$, with a partial inverse G satisfying

$$LG = \pi_{\text{Ran}(L)}.$$

hence for each h ,

$$L(h^\delta G(g)) = h^\delta g,$$

and we conclude that

$$\hat{u}_{\alpha_0}(z; h) = (h^\delta G(g)(z; h) + v(z; h)),$$

with

$$Lv(z; h) = 0 \text{ on } T'$$

for each h . Note that there exists h_0 such that for $h < h_0$, $\|v\|_{L^2(T')} > 1/2$, since u was L^2 -normalized, and Fourier modes other than the α_0 mode have decaying mass.

Now suppose that there exists $\mathcal{O}' \subset T'$ with $\hat{u}_{\alpha_0}(z; h) \rightarrow 0$ pointwise a.e. for $z \in \mathcal{O}'$ as $h = h_j \downarrow 0$. Then we must have

$$(12) \quad v(z; h_j) \rightarrow 0$$

for all $z \in \mathcal{O}'$. As the nullspace of L is finite-dimensional, and its elements enjoy the property of unique continuation (see, e.g., Theorem 17.2.6 of [11]), there exists $c > 0$ such that

$$(13) \quad Lf(z) = 0, \|f\|_{L^2(T')} \geq 1/2 \implies \int_{\mathcal{O}'} |f|^2 dz \geq c.$$

By Dominated Convergence (which we may apply since elements of the nullspace of L are uniformly bounded above), this is a contradiction with (12). Thus, such a set \mathcal{O}' cannot exist.

Now since all Fourier coefficients with $\alpha \neq \alpha_0$ vanish as $O(h^{1-\epsilon})$, we may take $u_{\alpha_0}(z; h)e_{\alpha_0}(y)$ to be a representative of $\sigma_h(u)$; by the above considerations, we may take

$$\sigma_h(u) = v(z; h)e_{\alpha_0}(y) + O(h^\delta).$$

Thus, there cannot exist an open set in \mathcal{L}_0 on which $\sigma_h(u) \rightarrow 0$ pointwise a.e. along any sequence $h_j \downarrow 0$, as this would entail the existence of $\mathcal{O}' \subset T'$ on which $v \rightarrow 0$. \square

⁷In the case of the weaker assumption on the subprincipal symbol, this operator would have first order terms in it as well; the second order part would be unchanged, however.

3.2. The general case (proof of Theorem 3). Finally, we turn to the general case of the theorem, in which X is arbitrary and \mathcal{L} is any Liouville torus in T^*X . The proof is by conjugating to the normal form studied in Proposition 7, with the difficulty that there do exist obstructions to the global existence of semiclassical Fourier integral operators quantizing a given symplectomorphism (microlocally, there is no obstruction). In particular, the Bohr-Sommerfeld-Maslov quantization conditions are known to obstruct this process. Fortunately, in the situation at hand, we know a priori that there exists a nontrivial Lagrangian distribution supported along one of our orbit closures, and this ensures that the cohomological obstruction vanishes on the homology classes represented in the orbit closure.

As u is normalized, there is some point at which it has nonvanishing principal symbol, hence by invariance of the symbol under $-i\mathcal{L}_{H_p} + c$, there is some orbit closure T along which the principal symbol of u as a Lagrangian distribution with respect to $h^{-\epsilon}L^2$ is nonvanishing. This principal symbol takes values in $L \otimes \Omega^{1/2} \otimes \mathcal{E}$ where L is the Maslov bundle over \mathcal{L} , $\Omega^{1/2}$ is the half-density bundle, and \mathcal{E} is the *pre-quantum* line bundle (see [3] §4.1, 4.4, where this object is denoted $\iota^*\mathcal{E}_{M,h}$). As observed by Maslov [17] and further amplified, for instance, in [8], [3], the existence of a nonvanishing section of this bundle has a topological implication: if it exists globally on \mathcal{L} , we obtain a periodicity:⁸

$$(14) \quad \frac{\lambda}{2\pi h} \equiv \frac{1}{4}\alpha \pmod{H^1(\mathcal{L}; \mathbb{Z})}.$$

Here λ denotes the cohomology class of the canonical one-form ξdx restricted to \mathcal{L} (the ‘‘Liouville class’’), and α denotes the Maslov class in $H^1(\mathcal{L}; \mathbb{Z})$ (cf. equation (1.5.3) of [8]). By contrast, in the case at hand, the section only exists locally along T (and so along its translates as well), hence (14) holds but only *when paired with cycles in $H_1(T) \subset H_1(\mathcal{L})$* , i.e., if ι denotes the inclusion $T \rightarrow \mathcal{L}$,

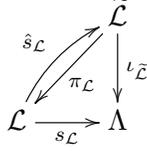
$$(15) \quad \frac{\iota^*\lambda}{2\pi h} \equiv \frac{1}{4}\iota^*\alpha \pmod{H^1(T; \mathbb{Z})}.$$

By our (local) integrability hypothesis there exists a (locally defined) symplectomorphism κ from T^*X to $T^*(\mathbb{T}^n)$ mapping \mathcal{L} to \mathcal{L}_0 , the zero section of $T^*(\mathbb{T}^n)$: we simply set

$$x \circ \kappa = \theta, \quad \xi \circ \kappa = I.$$

In general, we may or may not be able to quantize such a symplectomorphism to a unitary FIO. In the case at hand, however, it turns out that we may

⁸We note that there exists a distinction, at this point, between the case when $h \downarrow 0$ is taken to be a continuous parameter versus a discrete family; in the former case, (14) then entails that λ and $(1/4)\alpha$ must separately vanish, while in the latter there is the possibility that for a given sequence of $h \downarrow 0$, (14) is satisfied with a nontrivial left-hand side.

FIGURE 1. The maps among \mathcal{L} , $\tilde{\mathcal{L}}$, Λ .

do so microlocally along T . To prove this, we proceed as follows. Let

$$\Lambda = \text{Graph}(\kappa)' \subset T^*(X \times \mathbb{T}^n).$$

(This is a Lagrangian manifold; the prime denotes inversion of dual variables in the second factor.) Let $\tilde{\mathcal{L}}$ denote the restriction of the twisted graph to \mathcal{L} :

$$\tilde{\mathcal{L}} = \{(\rho, \kappa(\rho)') : \rho \in \mathcal{L}\} \subset \Lambda;$$

let \tilde{T} denote the further restriction to the torus T :

$$\tilde{T} = \{(\rho, \kappa(\rho)') : \rho \in T\} \subset \tilde{\mathcal{L}}.$$

and let \mathcal{U} be a tubular neighborhood of \tilde{T} in Λ .

Now let $\tilde{\lambda}$ and $\tilde{\alpha}$ denote the Liouville class and Maslov class of Λ . By Example 5.26 of [3], if $s_{\mathcal{L}}$ denotes the embedding $\mathcal{L} \hookrightarrow \tilde{\mathcal{L}} \subset \Lambda$, then $s_{\mathcal{L}}$ induces an isomorphism between $H^*(\mathcal{L})$ and $H^*(\Lambda)$, and this isomorphism preserves Maslov classes:

$$\alpha = s_{\mathcal{L}}^*(\tilde{\alpha}).$$

Also, if $\iota_{\tilde{\mathcal{L}}}$ denotes the inclusion of $\tilde{\mathcal{L}}$ into Λ and π the projection from $T^*X \times T^*(\mathbb{T}^n)$ to the left factor, since the right projection of $\tilde{\mathcal{L}}$ has range in the zero section, we have

$$\iota_{\tilde{\mathcal{L}}}^*(\tilde{\lambda}) = \iota_{\tilde{\mathcal{L}}}^* \circ \pi^*(\xi \cdot dx) = \pi_{\mathcal{L}}^*(\lambda) = (\hat{s}_{\mathcal{L}}^*)^{-1} \lambda$$

where $\pi_{\mathcal{L}}$ is the projection from $\tilde{\mathcal{L}} \subset \Lambda$ to \mathcal{L} and $\hat{s}_{\mathcal{L}}$ is the diffeomorphism from \mathcal{L} to $\tilde{\mathcal{L}}$ (hence $\pi_{\mathcal{L}} \circ \hat{s}_{\mathcal{L}} = \text{Id}$). As a result, we obtain

$$\lambda = s_{\mathcal{L}}^*(\tilde{\lambda}).$$

Thus, since $\pi_{\mathcal{L}}$ and $s_{\mathcal{L}}$ induce isomorphisms on cohomology, (15) shows that for any cycle γ on $\tilde{\mathcal{L}}$ lying in \tilde{T} ,

$$(16) \quad \left(\frac{\tilde{\lambda}}{2\pi\hbar} - \frac{1}{4}\tilde{\alpha}, \gamma\right) = \left(\frac{s_{\mathcal{L}}^*(\tilde{\lambda})}{2\pi\hbar} - \frac{1}{4}s_{\mathcal{L}}^*(\tilde{\alpha}), (\pi_{\mathcal{L}})_*(\gamma)\right) = \left(\frac{\lambda}{2\pi\hbar} - \frac{1}{4}\alpha, (\pi_{\mathcal{L}})_*(\gamma)\right) \in \mathbb{Z}.$$

Since any cycle $\gamma \in H_1(\Lambda; \mathbb{Z})$ lying in \mathcal{U} is homologous to a cycle in $\tilde{T} \subset \tilde{\mathcal{L}}$ we in fact obtain

$$(17) \quad \frac{\iota_{\mathcal{U}}^* \tilde{\lambda}}{2\pi\hbar} \equiv \frac{1}{4} \iota_{\mathcal{U}}^* \tilde{\alpha} \pmod{H^1(\mathcal{U}; \mathbb{Z})},$$

with $\iota_{\mathcal{U}}$ denoting the inclusion of \mathcal{U} into Λ .

As a consequence of the quantization condition (17), there exists a Maslov canonical operator microlocally defined over a neighborhood of \tilde{T} i.e., we can find a Lagrangian distribution U_0 on $X \times \mathbb{T}^n$, microsupported in a neighborhood of \tilde{T} and Lagrangian with respect to Λ , with principal symbol of norm 1 on a sub-neighborhood of \tilde{T} . Viewing U_0 as the Schwartz kernel of an operator $\mathcal{D}'(X) \rightarrow \mathcal{D}'(\mathbb{T}^n)$, there exists $R \in \Psi_h(X)$ such that

$$U_0^* U_0 = \text{Id} + hR$$

when *acting on the space of distributions microsupported near T* . Consequently, on such distributions, the h-FIO

$$U = U_0 (U_0^* U_0)^{-1/2}$$

acts unitarily (i.e., is a partial isometry in a microlocal sense).

In fact, it turns out that we can refine the above argument to demand a little more of U than mere microlocal unitarity. In [10, Theorem 2.4], Hitrik-Sjöstrand show that we may construct U so as to obey an improved Egorov theorem: if a is a semiclassical symbol on T^*X and Op_W denotes the Weyl quantization, then we may achieve

$$(18) \quad U \text{Op}_W(a) U^* = \text{Op}_W(a \circ \kappa^{-1} + O(h^2)),$$

i.e. the Egorov theorem holds to one order better than is usual. In general, such a U cannot be taken to be single valued—it is, rather, Floquet periodic; however as described above, the obstruction to its global single-valued construction is the condition (17), hence we may in fact find a single-valued U microlocally unitary along a neighborhood of T such that (18) holds for a with essential support in a neighborhood of T .

We may of course employ the same construction over a neighborhood of any translate of T inside \mathcal{L} . However, the resulting microlocally defined operators may not fit together to be globally defined U over \mathcal{L} : in general, the construction of [10] now yields U that is Floquet-periodic with respect to cycles in T' under the splitting $\mathcal{L} \cong T \times T'$ as described in the model setting. This is the conjugating operator that we shall employ.

Now we finally turn to the proof of the theorem. With U as constructed above, we obtain

$$(UPU^*)Uu = O(h^{2+\delta})$$

microlocally near $\kappa(T)$; moreover, UPU^* satisfies the hypotheses of Proposition 7 in this neighborhood, with the condition on the subprincipal symbol being guaranteed by the improved Egorov property (18), and with the slight variation of taking values in a flat bundle over \mathcal{L}_0 with trivial holonomy

along cycles lying in T .⁹ Thus, the principal symbol of $U(u)$ must satisfy the unique continuation property. \square

REFERENCES

- [1] Ivana Alexandrova, *Semi-classical wavefront set and Fourier integral operators*, Canad. J. Math. **60** (2008), no. 2, 241–263. MR MR2398747 (2009m:35565)
- [2] V. I. Arnol'd, *Modes and quasimodes*, Funkcional. Anal. i Priložen. **6** (1972), no. 2, 12–20. MR MR0297274 (45 #6331)
- [3] Sean Bates and Alan Weinstein, *Lectures on the geometry of quantization*, Berkeley Mathematics Lecture Notes, vol. 8, American Mathematical Society, Providence, RI, 1997. MR 1806388 (2002f:53151)
- [4] Nicolas Burq and Maciej Zworski, *Control for schrödinger operators on tori*, arXiv:1106.1412.
- [5] ———, *Bouncing ball modes and quantum chaos*, SIAM Rev. **47** (2005), no. 1, 43–49 (electronic). MR MR2149100 (2006d:81111)
- [6] Yves Colin de Verdière, *Quasi-modes sur les variétés Riemanniennes*, Invent. Math. **43** (1977), no. 1, 15–52. MR MR0501196 (58 #18615)
- [7] Mouez Dimassi and Johannes Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Mathematical Society Lecture Note Series, vol. 268, Cambridge University Press, Cambridge, 1999. MR MR1735654 (2001b:35237)
- [8] J. J. Duistermaat, *Oscillatory integrals, Lagrange immersions and unfolding of singularities*, Comm. Pure Appl. Math. **27** (1974), 207–281. MR MR0405513 (53 #9306)
- [9] Lawrence Evans and Maciej Zworski, *Lectures on semiclassical analysis, version 0.95*, available at <http://math.berkeley.edu/~zworski/semiclassical.pdf>.
- [10] Michael Hitrik and Johannes Sjöstrand, *Non-selfadjoint perturbations of selfadjoint operators in 2 dimensions. I*, Ann. Henri Poincaré **5** (2004), no. 1, 1–73. MR MR2036816 (2004m:47110)
- [11] Lars Hörmander, *The analysis of linear partial differential operators. III*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 274, Springer-Verlag, Berlin, 1985, Pseudodifferential operators. MR MR781536 (87d:35002a)
- [12] ———, *The analysis of linear partial differential operators. IV*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 275, Springer-Verlag, Berlin, 1994, Fourier integral operators, Corrected reprint of the 1985 original. MR MR1481433 (98f:35002)
- [13] Dmitry Jakobson, *Quantum limits on flat tori*, Ann. of Math. (2) **145** (1997), no. 2, 235–266. MR 1441877 (99e:58194)
- [14] Fabricio Macià, *High-frequency propagation for the Schrödinger equation on the torus*, J. Funct. Anal. **258** (2010), no. 3, 933–955. MR 2558183 (2011b:81143)
- [15] Fabricio Macià and Nalini Anantharaman, *Semiclassical measures for the schrödinger equation on the torus*, arXiv:1005.0296, 2010.
- [16] Jean-Pierre Marco and David Sauzin, *Stability and instability for Gevrey quasi-convex near-integrable Hamiltonian systems*, Publ. Math. Inst. Hautes Études Sci. (2002), no. 96, 199–275 (2003). MR MR1986314 (2004m:37112)

⁹This makes virtually no difference in the proof of Proposition 7: the only change is that the elliptic operator to which we apply the unique continuation theorem acts on sections of a flat (h -dependent) complex line bundle. This does not affect our application of Theorem 17.2.6 of [11], which is a merely local statement. The constant c in our quantitative statement of unique continuation (13) can be taken uniform with respect to the compact set of possible cycles defining such a bundle, hence the same argument applies as in the scalar case.

- [17] V. P. Maslov, *Théorie des perturbations et méthodes asymptotiques*, Dunod, Paris, 1972.
- [18] András Vasy and Jared Wunsch, *Erratum to “Semiclassical second microlocal propagation of regularity and integrable systems”*, to appear.
- [19] ———, *Semiclassical second microlocal propagation of regularity and integrable systems*, *J. Anal. Math.* **108** (2009), 119–157. MR MR2544756
- [20] Jared Wunsch, *Spreading of lagrangian regularity on rational invariant tori*, *Comm. Math. Phys.* **279** (2008), 487–496.