# LOCAL SMOOTHING FOR THE SCHRÖDINGER EQUATION WITH A PRESCRIBED LOSS

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Abstract. We consider a family of surfaces of revolution, each with a single periodic geodesic which is degenerately unstable. We prove a local smoothing estimate for solutions to the linear Schrödinger equation with a loss that depends on the degeneracy, and we construct explicit examples to show our estimate is saturated on a weak semiclassical time scale. As a byproduct of our proof, we obtain a cutoff resolvent estimate with a sharp polynomial loss.

## 1. Introduction

Local smoothing for the linear Schrödinger equation has a long and rich history. First observed by Kato [15] for the KdV equation and later studied by Constantin-Saut [8], Sjölin [19], Vega [24], and Kato-Yajima [16] for the Schrödinger equation, the local smoothing estimate expresses that, on average in time and locally in space, solutions to a linear homogeneous dispersive equation gain some regularity compared to the initial data. Since dispersive equations are time-reversible, the propagator at time $t$ preserves the initial energy, but local smoothing shows there is greater regularity if we also integrate in time.

For the linear Schrödinger equation in $\mathbb{R}^n$, it is well known (see, for example, [23]) that for any $\epsilon > 0$, there exists $C > 0$ such that one has the estimate

$$\int_0^T \| \langle x \rangle^{-1/2-\epsilon} e^{it\Delta} u_0 \|_{H^{1/2}}^2 dt \leqslant C \|u_0\|_{L^2}^2 .$$

There are several ways to prove this estimate, the simplest of which is a positive commutator method (see below), although estimates on the cutoff free resolvent also imply this estimate (this argument seems to have its origin in the work of Kato [14]; see also, for example, [2, 5, 6]). However, for the Schrödinger equation on a non-compact manifold, the situation is not so simple. A remarkable result of Doi [11] states that one has the sharp $H^{1/2}$ local smoothing effect on an asymptotically Euclidean manifold if and only if the geodesic flow is non-trapping. That is, the presence of geodesics which do not "escape to infinity" cause a loss in how much regularity the solution can gain. This has been generalized to boundary value problems in [2]. In the case of sufficiently "thin" hyperbolic trapped sets it has been demonstrated in [2, 5, 6, 9] that one has only a "trivial" loss of $\epsilon > 0$ derivatives. In fact, with some care in definitions, the loss is only logarithmic. These examples include Ikawa's examples [2, 12, 13], a single periodic hyperbolic geodesic (with or without boundary reflections) [6], very general fractal trapped sets without boundary [5, 9, 17], and normally hyperbolic trapped sets [26]. That is, in all of these cases, the authors prove that for any $\epsilon > 0$, there exists a constant $C > 0$ such that

$$\int_0^T \| \langle x \rangle^{-1/2-\epsilon} e^{it\Delta} u_0 \|_{H^{1/2-\epsilon}}^2 dt \leqslant C \| u_0 \|_{L^2}^2 .$$

In this case, we call the loss due to trapping "trivial".

To contrast, if a manifold admits an elliptic trapped set, the existence of resonances converging exponentially to the real axis and the existence of infinite order quasimodes prevents polynomial gain in regularity.

The purpose of this note is to exhibit a class of manifolds with only one periodic geodesic which is weakly hyperbolic, and prove a (sharp) local smoothing effect with loss that lies somewhere between the complete loss of an elliptic trapped set and the trivial loss of a strictly hyperbolic trapped set.

We consider the manifold $X = \mathbb{R}_x \times \mathbb{R}_\theta / 2\pi\mathbb{Z}$, equipped with a metric of the form

$$ds^2 = dx^2 + A^2(x) d\theta^2,$$

where $A \in \mathcal{C}^\infty$ is a smooth function, $A \geqslant \epsilon > 0$. From this metric, we get the volume form

$$d\text{Vol} = A(x) dx d\theta,$$

and the Laplace-Beltrami operator acting on 0-forms

$$\Delta f = (\partial_x^2 + A^{-2} \partial_\theta^2 + A^{-1} A' \partial_x) f.$$

We observe that we can conjugate $\Delta$ by an isometry of metric spaces and separate variables so that spectral analysis of $\Delta$ is equivalent to a one-variable semiclassical problem with potential. That is, let $T : L^2(X, d\text{Vol}) \to L^2(X, dx d\theta)$ be the isometry given by

$$Tu(x, \theta) = A^{1/2}(x) u(x, \theta).$$

Then $\widetilde{\Delta} = T \Delta T^{-1}$ is essentially self-adjoint on $L^2(X, dx d\theta)$ with mild assumptions on $A$ (for example in this paper $X$ has two ends which are short range perturbations of $\mathbb{R}^2$). A simple calculation gives

$$-\widetilde{\Delta} f = (-\partial_x^2 - A^{-2}(x) \partial_\theta^2 + V_1(x)) f,$$

where the potential

$$V_1(x) = \frac{1}{2} A'' A^{-1} - \frac{1}{4} (A')^2 A^{-2}.$$

If we now separate variables and write $\psi(x, \theta) = \sum_k \varphi_k(x) e^{ik\theta}$, we see that

$$(-\widetilde{\Delta} - \lambda^2) \psi = \sum_k e^{ik\theta} (P_k - \lambda^2) \varphi_k(x),$$

where

$$(P_k - \lambda^2) \varphi_k(x) = (-\frac{d^2}{dx^2} + k^2 A^{-2}(x) + V_1(x) - \lambda^2) \varphi_k(x).$$

Setting $h = k^{-1}$, we have the semiclassical operator

$$P(z, h) \varphi(x) = (-h^2 \frac{d^2}{dx^2} + V(x) - z) \varphi(x),$$

where the potential is

$$V(x) = A^{-2}(x) + h^2 V_1(x)$$

and the spectral parameter is $z = h^2 \lambda^2$.

In this paper, we are primarily interested in the case $A(x) = (1 + x^{2m})^{1/2m}$, $m \in \mathbb{Z}_+$. If $m \geqslant 2$, then $X$ is asymptotically Euclidean (with two ends), and the

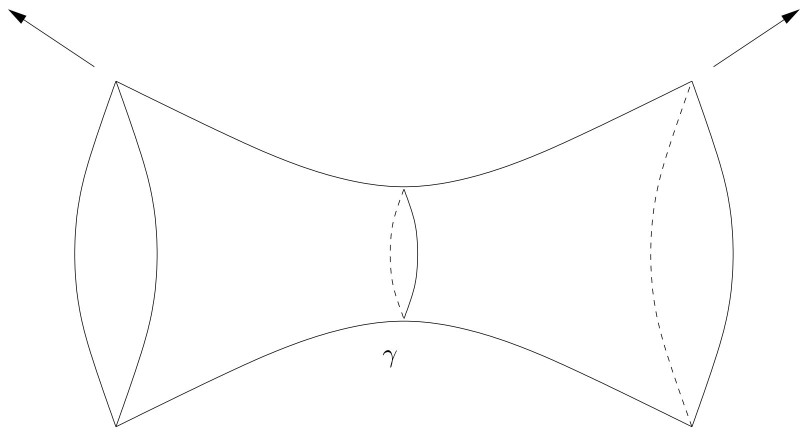


FIGURE 1. A piece of the manifold $X$ and the periodic geodesic $\gamma$. The Gaussian curvature $K = -A''/A = -(2m-1)x^{2m-2}(1+x^{2m})^{-2}$ vanishes to order $2m-2$ at $x = 0$ and is asymptotically flat as $|x| \to \infty$

subpotential $h^2V_1(x)$ is seen to be lower order in both the semiclassical and the scattering sense. If $m = 1$, a trivial modification must be made to make the metric a short-range perturbation, but we completely ignore this issue here. The point is that for $m \geqslant 2$, the principal part of the potential $V(x)$ is $A^{-2}(x)$ which has a degenerate maximum at $x = 0$. The corresponding periodic geodesic $\gamma \subset X$ is *weakly* hyperbolic in the sense that it is unstable and isolated, but degenerate (see Figure 1).

Our main result is the following theorem, which says that for every $m \geqslant 2$, there is still some local smoothing, but with a polynomial loss depending on $m$.

**Theorem 1** (Local Smoothing)**.** *Suppose $X$ is as above for $m \geqslant 2$, and assume $u$ solves*

$$\begin{cases} (D_t - \Delta)u = 0 \text{ in } \mathbb{R} \times X, \\ u|_{t=0} = u_0 \in H^s \end{cases}$$

*for some $s \geqslant m/(m+1)$. Then for any $T < \infty$, there exists a constant $C > 0$ such that*

$$\int_0^T \| \langle x \rangle^{-3/2} u\|_{H^1(X)}^2 \, dt \leqslant C(\| \langle D_\theta \rangle^{m/(m+1)} u_0\|_{L^2}^2 + \| \langle D_x \rangle^{1/2} u_0\|_{L^2}^2).$$

**Remark 1.1.** Observe that there is no polynomial local smoothing effect in the limit $m \to \infty$. In Theorem 3 below, we show Theorem 1 is sharp, and that in fact the estimate is saturated on a weak semiclassical time scale.

We are also able to prove, using the same techniques, a polynomial bound on the resolvent of the Laplacian in the same geometric setting. We now assume for simplicity that our surface of revolution is Euclidean at infinity, i.e. that $A(x) = x$ for $|x| \gg 0$. (More generally we could just require merely dilation analyticity at infinity; this would allow us to include asymptotically conic spaces as treated in [25].)

We denote by $R(\lambda)$ the resolvent on $X$

$$R(\lambda) = (-\Delta_g - \lambda^2)^{-1},$$

where it exists. If we take $\operatorname{Im}\lambda < 0$ as our physical sheet, then, since $X$ is Euclidean near infinity, there is a meromorphic continuation of $\chi R(\lambda)\chi$ to the logarithmic covering space, for any $\chi \in \mathcal{C}_c^\infty(X)$ (see, e.g., [20]). In particular, by choosing an appropriate branch cut, $\chi R(\lambda)\chi$ continues meromorphically to $\{\lambda \in \mathbb{R}, \lambda \gg 0\}$.

**Theorem 2.** *Fix $m \geqslant 2$. For any $\chi \in \mathcal{C}_c^\infty(X)$, there exists a constant $C = C_{m,\chi} > 0$ such that for $\lambda \gg 0$,*

$$\|\chi R(\lambda - i0)\chi\|_{L^2 \to L^2} \leqslant C\lambda^{-2/(m+1)}.$$

*Moreover, this estimate is sharp, in the sense that no better polynomial rate of decay holds true.*

**Remark 1.2.** The estimate in this theorem represents a loss over the non-trapping case, when generally $\lambda^{-1}$ order bounds are known. In the non-degenerate hyperbolic trapping case ($m = 1$), most known estimates are of the order $\lambda^{-1}\log(\lambda)$, and in the elliptic trapping cases, generally one expects at best exponential bounds. Hence Theorem 2 represents a family of estimates with a sharp polynomial loss. To the best of our knowledge, no other such examples are known, although an analogous estimate with a logarithmic loss is stated in [3] in the context of semiclassical scattering.

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## 2. Positive commutators

### 2.1. **The smoothing estimate on Euclidean space.** 

In this section we write out the standard positive commutator proof of local smoothing for the Schrödinger equation in polar coordinates. We then try to mimic the proof in the case of degenerate hyperbolic orbits ($m \geqslant 2$ above) to see where the proof fails.

In polar coordinates, the homogeneous Schrödinger equation on $\mathbb{R}_t \times \mathbb{R}^2$ is

$$\begin{cases} (D_t - \partial_r^2 - r^{-1}\partial_r - r^{-2}\partial_\theta^2)u = 0, \\ u|_{t=0} = u_0; \end{cases}$$

we will of course write

$$\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2.$$

We recall that in polar coordinates the radial, or scaling, vector field is $x \cdot \partial_x = r\partial_r$. By scaling, we immediately compute

$$[r\partial_r, \Delta] = 2\Delta;$$

however, as $r\partial_r$ is not a bounded map between Sobolev spaces, we change the weight and employ the commutant $B = r\langle r\rangle^{-1}\partial_r$. The function $a(r) = r\langle r\rangle^{-1}$ is non-negative and bounded, and satisfies $a'(r) = \langle r\rangle^{-3}$. Thus, we compute

$$[B,\Delta] = 2a'\partial_r^2 + (a'' + a'r^{-1} + ar^{-2})\partial_r + 2ar^{-3}\partial_\theta^2$$
$$= 2\langle r\rangle^{-3}\partial_r^2 + 2\langle r\rangle^{-1} r^{-2}\partial_\theta^2 + O(r^{-1}\langle r\rangle^{-1})\partial_r. \tag{2.1}$$

Using the Schrödinger equation, we write

$$\begin{aligned}0 =& 2i\,\mathrm{Im}\int_0^T \langle B(D_t-\Delta)u,u\rangle\,dt\\ =& \int_0^T \left\langle B(D_t-\partial_r^2 - r^{-1}\partial_r - r^{-2}\partial_\theta^2)u,u\right\rangle\\ &- \left\langle u, B(D_t-\partial_r^2 - r^{-1}\partial_r - r^{-2}\partial_\theta^2)u\right\rangle dt\\ =& \int_0^T \left\langle [B,(-\partial_r^2 - r^{-1}\partial_r - r^{-2}\partial_\theta^2)]u,u\right\rangle dt + i\,\langle Bu,u\rangle|_0^T\,.\end{aligned}$$

The last term is bounded using energy estimates by

$$\left|\langle Bu,u\rangle|_0^T\right| \leqslant \|u_0\|_{H^{1/2}}^2.$$

Rearranging, we thus obtain

$$\int_0^T \langle [B,\Delta]u,u\rangle\,dt \leqslant C_T\|u_0\|_{H^{1/2}}^2.$$

Employing (2.1) and integrating by parts thus yields

$$\int_0^T \left\|\langle r\rangle^{-3/2}\partial_r u\right\|^2 + \left\|\langle r\rangle^{-1/2} r^{-1}\partial_\theta u\right\|^2 dt \leqslant C_T\|u_0\|_{H^{1/2}}^2.$$

where we have absorbed on the right the term involving $\int_0^T \langle\partial_r u,u\rangle\,dt$ as well as the similar error terms from commuting $\partial_r$ with a multiplier. This is the local smoothing estimate on the manifold $\mathbb{R}^2$.

2.2. **Degenerate hyperbolic trapping.** In this section, we prove our main local smoothing estimate.

Let us begin by reproducing the positive commutator computation in the previous section for the degenerate case. Let $A(x) = (1+x^{2m})^{1/2m}$, the metric $ds^2 = dx^2 + A^2 d\theta^2$ as before, and conjugate the Laplacian to Euclidean space:

$$-\widetilde{\Delta} f = (-\partial_x^2 - A^{-2}(x)\partial_\theta^2 + V_1(x))f,$$

where the potential

$$V_1(x) = \frac{1}{2}A''A^{-1} - \frac{1}{4}(A')^2A^{-2}.$$

The following proposition is the statement of local smoothing for the conjugated equation, and evidently implies Theorem 1 by conjugating back.

**Proposition 2.1.** *Suppose $m \geqslant 2$ and $u$ solves*

$$\begin{cases}(D_t - \widetilde{\Delta})u = 0,\\ u(0,x,\theta) = u_0.\end{cases} \tag{2.2}$$

*Then for any $T < \infty$ there exists a constant $C > 0$ such that*

$$\int_0^T (\| \langle x \rangle^{-1} \partial_x u\|_{L^2}^2 + \| \langle x \rangle^{-3/2} \partial_\theta u\|_{L^2}^2)\, dt$$
$$\leqslant C(\| \langle D_\theta \rangle^{m/(m+1)} u_0\|_{L^2}^2 + \| \langle D_x \rangle^{1/2} u_0\|_{L^2}^2).$$

2.3. **Proof of Proposition 2.1.** Let us summarize briefly the strategy of the proof. Using a positive commutator argument similar to the previous section, we prove local smoothing except at the periodic orbit $\gamma = \{x = 0\}$. Moreover, solutions to (2.2) exhibit perfect local smoothing in the $x$ direction and only lose smoothing in the directions tangential to $\gamma$ (that is, only in the $\theta$ direction). Thus it suffices to prove local smoothing with a loss for $\theta$ derivatives, in a neighbourhood of $x = 0$. We separate variables in the $\theta$ direction (Fourier series decomposition) and prove estimates uniform in each Fourier mode. To do this, we further decompose, say, the $k$th Fourier mode into a low-frequency part where $|k| \leqslant |D_x|$ and a high-frequency part where $|D_x| \leqslant |k|$. The low frequency part is estimated using the positive commutator technique modulo a term which is localized to high-frequencies, so it suffices to estimate a solution cut off to high frequencies. For this, we introduce a semiclassical rescaling, and reduce the estimate to a cutoff semiclassical resolvent estimate, which implies local smoothing via [5, Theorem 1].

**Step 1: Positive commutators and the estimate away from $x = 0$.**

If $B = \arctan(x)\partial_x$, we have

$$[\widetilde{\Delta}, B] = 2 \langle x \rangle^{-2} \partial_x^2 - 2x \langle x \rangle^{-4} \partial_x + 2A'A^{-3} \arctan(x)\partial_\theta^2 + V_1' \arctan(x).$$

Now

$$iB - (iB)^* = i[\arctan(x), \partial_x]$$

is $L^2$ bounded, so

$$\begin{aligned} 0 =& \int_0^T \int u\overline{(\arctan(x)D_x(D_t - \widetilde{\Delta})u)}dxd\theta\, dt \\ =& \int_0^T \int \arctan(x)D_x u\overline{((D_t - \widetilde{\Delta})u)}dxd\theta\, dt \\ &+ \int_0^T \int (iB - (iB)^*)u\overline{((D_t - \widetilde{\Delta})u)}dxd\theta\, dt \\ =& i \langle \arctan(x)D_x u, u\rangle |_0^T + \int_0^T \left\langle (D_t - \widetilde{\Delta})i^{-1}Bu, u \right\rangle dt. \end{aligned}$$

Hence, using the notation $P = D_t - \widetilde{\Delta}$,

$$\begin{aligned} 0 =& 2i \operatorname{Im} \int_0^T \left\langle i^{-1}BPu, u \right\rangle dt \\ =& \int_0^T \left\langle i^{-1}BPu, u \right\rangle dt - \int_0^T \left\langle u, i^{-1}BPu \right\rangle dt \\ =& \int_0^T \left\langle [i^{-1}B, P]u, u \right\rangle dt - i \langle \arctan(x)D_x u, u\rangle |_0^T, \end{aligned}$$

or

$$\int_0^T \left\langle [B, -\widetilde{\Delta}]u, u \right\rangle dt = - \langle \arctan(x)D_x u, u\rangle |_0^T,$$

since $B$ does not depend on $t$. By writing $\partial_x = \langle D_x \rangle^{1/2} \langle D_x \rangle^{-1/2} \partial_x$, and using energy estimates, we can control the right hand side by $\|u_0\|^2_{H^{1/2}}$. The left hand side is computed as above:

$$\begin{aligned}\int_0^T & \left\langle [B, -\widetilde{\Delta}]u, u \right\rangle dt \\ &= \int_0^T \Big\langle (2 \langle x \rangle^{-2} \partial_x^2 - 2x \langle x \rangle^{-4} \partial_x + 2A' A^{-3} \arctan(x) \partial_\theta^2 \\ &\quad + V_1' \arctan(x))u, u \Big\rangle \, dt.\end{aligned}$$

Using the energy estimates,

$$\begin{aligned}\Big| \int_0^T \Big\langle (-2x \langle x \rangle^{-4} \partial_x + V_1' \arctan(x))u, u \Big\rangle \, dt \Big| &\leqslant CT \sup_{0 \leqslant t \leqslant T} \|u(t)\|^2_{H^{1/2}} \\ &\leqslant C_T \|u_0\|^2_{H^{1/2}}. \end{aligned} \tag{2.3}$$

Integrating by parts in $x$ and $\theta$ and adding the lower order terms into the right hand side as in (2.3) yields the estimate

$$\int_0^T (\| \langle x \rangle^{-1} \partial_x u\|^2_{L^2} + \| \sqrt{A' A^{-3} \arctan(x)} \partial_\theta u\|^2_{L^2}) \, dt \leqslant C \|u_0\|^2_{H^{1/2}}.$$

We observe that

$$A' A^{-3} \arctan(x) = \arctan(x) x^{2m-1} (1 + x^{2m})^{-1/m-1}$$

is even, non-negative, bounded below by $C|x|^{2m}$ for $|x| \leqslant 1$ and $C'|x|^{-3}$ for $|x| \geqslant 1$. Hence

$$|x|^{2m} \langle x \rangle^{-2m-3} \leqslant C A' A^{-3} \arctan(x),$$

and hence,

$$\Big\langle |x|^{2m} \langle x \rangle^{-2m-3} \partial_\theta u, \partial_\theta u \Big\rangle \leqslant C \left\langle A' A^{-3} \arctan(x) \partial_\theta u, \partial_\theta u \right\rangle$$

plus terms which can be absorbed into the energy, so up to lower order terms,

$$\| |x|^m \langle x \rangle^{-m-3/2} \partial_\theta u \| \leqslant C \| \sqrt{A' A^{-3} \arctan(x)} \partial_\theta u \|.$$

Hence we have the estimate

$$\int_0^T (\| \langle x \rangle^{-1} \partial_x u\|^2_{L^2} + \| |x|^m \langle x \rangle^{-m-3/2} \partial_\theta u\|^2_{L^2}) \, dt \leqslant C \|u_0\|^2_{H^{1/2}}. \tag{2.4}$$

**Step 2: Low frequency estimate.**

The estimate (2.4) shows we have perfect local smoothing away from the periodic geodesic at $x = 0$, and moreover shows we have perfect local smoothing in the $x$ direction. That is, the only loss is in the direction tangential to the periodic geodesic, which is expected since a point in $T^*X$ which is transversal to the periodic geodesic will flow out to infinity, and only the tangential directions stay localized for long times. In this subsection we show how to get an estimate in the tangential directions with a loss.

Let us decompose

$$u(t, x, \theta) = \sum_k e^{ik\theta} u_k(t, x),$$

and

$$u_0(x,\theta) = \sum_k e^{ik\theta} u_{0,k}(x).$$

By orthogonality it suffices to prove Proposition 2.1 for each mode. Observe the zero mode $u_0(t,x)$ satisfies

$$\begin{cases} (D_t - \widetilde{\Delta}) u_0(t,x) = 0, \\ u_0(0,x) = u_{0,0}(x), \end{cases}$$

and $\partial_\theta u_0(t,x) = 0$, so that, from Step 1, we have

$$\begin{aligned} \int_0^T (\| \langle x \rangle^{-1} \partial_x u_0 \|_{L^2}^2 + \| \langle x \rangle^{-3/2} \partial_\theta u_0 \|_{L^2}^2 ) \, dt &= \int_0^T \| \langle x \rangle^{-1} \partial_x u_0 \|_{L^2}^2 \, dt \\ \leqslant \int_0^T (\| \langle x \rangle^{-1} \partial_x u_0 \|_{L^2}^2 + \| |x|^m \langle x \rangle^{-m-3/2} \partial_\theta u_0 \|_{L^2}^2 ) \, dt \\ \leqslant C \| u_{0,0} \|_{H^{1/2}}^2 \\ \leqslant C ( \| \langle D_\theta \rangle^{m/(m+1)} u_0 \|_{L^2}^2 + \| \langle D_x \rangle^{1/2} u_0 \|_{L^2}^2 ). \end{aligned}$$

To prove Proposition 2.1 for the nonzero modes, we show

$$\int_0^T \| \chi(x) k u_k \|_{L^2(\mathbb{R})}^2 \, dt \leqslant C ( \| \langle k \rangle^{m/(m+1)} u_{0,k} \|_{L^2}^2 + \| u_{0,k} \|_{H^{1/2}}^2 )$$

for some $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\chi(x) \equiv 1$ near $x = 0$, for $|k| \geqslant 1$.

For simplicity in exposition, let us drop the $k$ notation for $u$ and $u_0$, and just observe that now the time-dependent Schrödinger operator depends on $k$:

$$D_t + P_k = D_t - \widetilde{\Delta} = D_t + D_x^2 + A^{-2}(x) k^2 + V_1(x).$$

The idea is to use $k^{-1}$ as a semiclassical parameter, and decompose $u$ into a part where $|k| \leqslant |D_x|$ and a part where $k$ is not controlled by $D_x$. It turns out that then the part not controlled by $D_x$ can be handled with a second commutator argument plus a $TT^*$ argument (see Step 3 below and §2.4).

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ be an even function satisfying $\psi(r) \equiv 1$ for $|r| \leqslant 1$ and $\psi(r) \equiv 0$ for $|r| \geqslant 2$. Let

$$u = u_{\text{hi}} + u_{\text{lo}},$$

where

$$u_{\text{hi}} = \psi(D_x/k) u, \ \ u_{\text{lo}} = (1 - \psi(D_x/k)) u.$$

Observe $u_{\text{lo}}$ satisfies the equation

$$(D_t + P_k) u_{\text{lo}} = -[P_k, \psi(D_x/k)] u = k \langle x \rangle^{-3} L \tilde{\psi}(D_x/k) u,$$

where $L$ is $L^2$ bounded and $\tilde{\psi} \in \mathcal{C}_c^\infty$ is 1 on $\operatorname{supp}\psi$. If we try to apply the positive commutator argument from the previous step to $u_{\rm lo}$, we now have

$$\begin{aligned}
&2i \operatorname{Im} \int_0^T \left\langle i^{-1} B k \langle x \rangle^{-3} L\tilde{\psi}(D_x/k)u, u_{\rm lo} \right\rangle dt \\
&= 2i \operatorname{Im} \int_0^T \left\langle i^{-1} B(D_t + P_k)u_{\rm lo}, u_{\rm lo} \right\rangle dt \\
&= \int_0^T \left\langle i^{-1} B(D_t + P_k)u_{\rm lo}, u_{\rm lo} \right\rangle dt - \int_0^T \left\langle u_{\rm lo}, i^{-1} B(D_t + P_k)u_{\rm lo} \right\rangle dt \\
&\quad - \int_0^T \int (iB - (iB)^*)u_{\rm lo}\overline{(D_t + P_k)u_{\rm lo}} dx d\theta\, dt \\
&= \int_0^T \left\langle [i^{-1}B, P_k]u, u \right\rangle dt - i \left\langle \arctan(x) D_x u_{\rm lo}, u_{\rm lo} \right\rangle |_0^T \\
&\quad - i \int_0^T \left\langle \langle x \rangle^{-2} u_{\rm lo}, k \langle x \rangle^{-3} L\tilde{\psi}(D_x/k)u \right\rangle dt,
\end{aligned}$$

or

$$\begin{aligned}
&\left| \int_0^T \left\langle [i^{-1}B, P_k]u, u \right\rangle dt \right| \\
&\leqslant C \Bigg( \left| \int_0^T \left\langle i^{-1} B k \langle x \rangle^{-3} L\tilde{\psi}(D_x/k)u, u_{\rm lo} \right\rangle dt \right| \\
&\quad + \left| \left\langle \arctan(x) D_x u_{\rm lo}, u_{\rm lo} \right\rangle |_0^T \right| + \left| \int_0^T \left\langle \langle x \rangle^{-2} u_{\rm lo}, k \langle x \rangle^{-3} L\tilde{\psi}(D_x/k)u \right\rangle dt \right| \Bigg) \\
&\leqslant C \Big( \| k \langle x \rangle^{-3/2} \tilde{\psi}(D_x/k)u \|^2 + \| \langle x \rangle^{-3/2} D_x u \|^2 + \|u_0\|_{H^{1/2}}^2 \Big),
\end{aligned}$$

where we have again used the energy estimate where appropriate. The right hand side is now controlled by (2.4) except near $x = 0$. What we have gained is the cutoff in frequency $\tilde{\psi}(D_x/k)$. Clearly if we can show for any $\chi \in \mathcal{C}_c^\infty$, $\chi \equiv 1$ near $x = 0$,

$$\int_0^T \|\chi k \tilde{\psi}(D_x/k)u\|_{L^2}^2\, dt \leqslant C \|k^{m/(m+1)} u_0\|_{L^2}^2$$

we can control the remaining term from the estimate on $u_{\rm lo}$ as well as the estimate of $u_{\rm hi}$.

**Step 3: The high frequency estimate.**

Let us now try to estimate $u_{\rm hi}$ near $x = 0$, or more generally a solution to $(D_t + P_k)u = 0$ microlocalized near $(0, 0)$. For some $0 \leqslant r \leqslant 1/2$ to be determined, let $F(t)$ be defined by

$$F(t)g = \chi(x)\psi(D_x/k)k^r e^{-itP_k} g,$$

where $e^{-itP_k}$ is the free propagator. Our goal is to determine for what values of $r$ we have a mapping $F : L_x^2 \to L^2([0,T])L_x^2$, since then

$$\|k^{1-r} F(t) u_0\|_{L^2([0,T]);L^2} \leqslant C \|k^{1-r} u_0\|_{L^2} \tag{2.5}$$

is the desired local smoothing estimate. We have such a mapping if and only if $FF^* : L^2L^2 \to L^2L^2$. We compute

$$FF^*f(x,t) = \psi(D_x/k)\chi(x)k^{2r}\int_0^T e^{i(t-s)P_k}\chi(x)\psi(D_x/k)f(x,s)ds,$$

and it suffices to estimate $\|FF^*f\|_{L^2L^2} \leqslant C\|f\|_{L^2L^2}$. We write $FF^*f(x,t) = \psi\chi(v_1+v_2)$, where

$$v_1 = k^{2r}\int_0^t e^{i(t-s)P_k}\chi(x)\psi(D_x/k)f(x,s)ds,$$

and

$$v_2 = k^{2r}\int_t^T e^{i(t-s)P_k}\chi(x)\psi(D_x/k)f(x,s)ds,$$

so that

$$(D_t+P_k)v_j = \pm ik^{2r}\chi\psi f,$$

and it suffices to estimate

$$\|\psi\chi v_j\|_{L^2L^2} \leqslant C\|f\|_{L^2L^2}.$$

Since the Fourier transform in time is an $L^2$ isometry, it suffices to estimate

$$\|\psi\chi\hat{v}_j\|_{L^2L^2} \leqslant C\|\hat{f}\|_{L^2L^2},$$

but this is the same as estimating

$$\|\psi\chi k^{2r}(\tau \pm i0 + P_k)^{-1}\chi\psi\|_{L^2_x\to L^2_x} \leqslant C.$$

Let us factor out the $k^2$ in $P_k$ to get the operator

$$k^{-2r}(\tau \pm i0 + P_k) = k^{2(1-r)}(-z \pm i0 + k^{-2}D_x^2 + A^{-2}(x) + k^{-2}V_1(x))$$

for $-z = \tau k^{-2}$, and if we let $h = k^{-1}$, we are left with the task of finding $r$ so that

$$\|\psi(hD_x)\chi(x)(-z \pm i0 + (hD_x)^2 + V)^{-1}\chi(x)\psi(hD_x)\|_{L^2\to L^2} \leqslant Ch^{-2(1-r)},$$

where $V = A^{-2}(x) + h^2V_1(x)$. Let

$$\widetilde{Q} = (hD_x)^2 + V - z.$$

We observe that the cutoff $\psi(hD_x)\chi(x)$ shows we only need to estimate this for $z$ in a bounded interval near $z=1$. Indeed, $\psi\chi$ cuts off to a neighbourhood of $(0,0)$, and $V(0)=1$, so for $|z-1|$ sufficiently large, we have elliptic regularity. The cutoff estimate on $\widetilde{Q}$ is the content of the following Proposition, which is proved in the next subsection.

**Proposition 2.2.** *Let $\varphi \in \Phi^0$ have wavefront set sufficiently close to $(0,0)$. Then for each $\epsilon > 0$ sufficiently small, there exists a constant $C > 0$ such that*

$$\|\varphi(\widetilde{Q} \pm i0)^{-1}\varphi\|_{L^2\to L^2} \leqslant Ch^{-2m/m+1}, \;\; z \in [1-\epsilon, 1+\epsilon].$$

With Proposition 2.2 in hand, we observe

$$\|\psi\chi k^{2r}(\tau \pm i0 + P_k)^{-1}\chi\psi\|_{L^2_x\to L^2_x} \leqslant C$$

holds if

$$k^{2(r-1)} = k^{-2m/(m+1)},$$

or

$$r = \frac{1}{m+1}.$$

From (2.5), this implies Proposition 2.1 (see also [5, Theorem 1]).

2.4. **Proof of Proposition 2.2.** The technique of proof is to prove an invertibility estimate microlocally near $(0,0)$ in Lemma 2.3 below. From this, one easily obtains a resolvent estimate with complex absorbing potential, and then the gluing techniques of [6, Proposition 2.2] imply the Proposition (see also the recent paper of Datchev-Vasy [10]).

The proof of the microlocal invertibility estimate proceeds through several steps. First, we rescale the principal symbol of $\widetilde{Q}$ to introduce a calculus of two parameters. We then quantize in the second parameter which eventually will be fixed as a constant in the problem. This technique has been used in [4, 7, 21, 22].

Our central result to achieve microlocal invertibility is a lower bound for the resolvent of the semiclassical operator $\widetilde{Q}$, whose potential has a degenerate barrier top. This result is of independent interest, and is used to prove the sharp resolvent estimate in Theorem 2.

**Lemma 2.3.** *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x,\xi)| \leqslant \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$\|\widetilde{Q}\varphi^w u\| \geqslant C_\epsilon h^{2m/(m+1)} \|\varphi^w u\|, \;\; z \in [1-\epsilon, 1+\epsilon]. \tag{2.6}$$

2.5. **The two-parameter calculus.** Following Sjöstrand-Zworski [22, §3.3], we introduce a calculus with two parameters, designed to enable symbolic computations in the $h^{-1/2}$ calculus which would otherwise involve global considerations rather than a local Moyal product of symbols. We use a somewhat more general version of this calculus than in [22], involving inhomogenous powers of $h$.

For $\alpha \in [0,1]$ and $\beta \leqslant 1-\alpha$, we let

$$\begin{aligned} &\mathcal{S}^{k,m,\widetilde{m}}_{\alpha,\beta}(T^*(\mathbb{R}^n)) := \\ &= \Bigg\{ a \in \mathcal{C}^\infty\left(\mathbb{R}^n \times (\mathbb{R}^n)^* \times (0,1]^2\right) : \\ &\qquad \left|\partial_x^\rho \partial_\xi^\gamma a(x,\xi;h,\tilde{h})\right| \leqslant C_{\rho\gamma} h^{-m} \tilde{h}^{-\widetilde{m}} \left(\frac{\tilde{h}}{h}\right)^{\alpha|\rho|+\beta|\gamma|} \langle \xi \rangle^{k-|\gamma|} \Bigg\}. \end{aligned}$$

Throughout this work we will always assume $\tilde{h} \geqslant h$. We let $\Psi^{k,m,\widetilde{m}}_{\alpha,\beta}$ denote the corresponding spaces of semiclassical pseudodifferential operators obtained by Weyl quantization of these symbols. We will sometimes add a subscript of $h$ or $\tilde{h}$ to indicate which parameter is used in the quantization; in the absence of such a parameter, the quantization is assumed to be in $h$. The class $\mathcal{S}_{\alpha,\beta}$ (with no superscripts) will denote $\mathcal{S}^{0,0,0}_{\alpha,\beta}$ for brevity.

In [22], it is observed that in the special case $\alpha = \beta = 1/2$, the composition in the calculus can be computed in terms of a symbol product that converges in the sense that terms improve in $\tilde{h}$ and $\xi$ orders, but not in $h$ orders (owing to the marginality of the $h^{-1/2}$ calculus, which is what the introduction of the second parameter $\tilde{h}$ mitigates). We will restrict our attention in what follows to a generalization of this marginal case:

$$\alpha + \beta = 1.$$

By the same arguments employed in [22], we may easily verify that the calculus $\Psi_{\alpha,\beta}$ is closed under composition: if $a \in \mathcal{S}^{k,m,\widetilde{m}}_{\alpha,\beta}$ and $b \in \mathcal{S}^{k',m',\widetilde{m}''}_{\alpha,\beta}$ then

$$\operatorname{Op}_h^w(a) \circ \operatorname{Op}_h^w(b) = \operatorname{Op}_h^w(c) \;\; \text{with} \;\; c \in \mathcal{S}^{k+k',m+m',\widetilde{m}+\widetilde{m}'}_{\alpha,\beta}.$$

The presence of the additional parameter $\tilde{h}$ allows us to conclude that

$$c \equiv \sum_{|\rho|<M} \frac{1}{\rho!} \partial_\xi^\rho a D_x^\rho b \mod \mathcal{S}_{\alpha,\beta}^{k+k'-M, m+m', \widetilde{m}+\widetilde{m}'-M},$$

that is, we have a symbolic expansion in powers of $\tilde{h}$.

We also require a more general version of [22, Lemma 3.6], which gives error estimates on remainders:

**Lemma 2.4.** *Suppose that $a, b \in \mathcal{S}_{\alpha,\beta}$, and that $c^w = a^w \circ b^w$. Then*

$$c(x,\xi) = \sum_{k=0}^{N} \frac{1}{k!} \left( \frac{ih}{2} \sigma(D_x, D_\xi; D_y, D_\eta) \right)^k a(x,\xi) b(y,\eta)|_{x=y,\xi=\eta} + e_N(x,\xi), \tag{2.7}$$

*where for some $M$*

$$\begin{aligned} &|\partial^\gamma e_N| \leqslant C_N h^{N+1} \\ &\quad \times \sum_{\gamma_1+\gamma_2=\gamma} \sup_{\substack{(x,\xi)\in T^*\mathbb{R}^n \\ (y,\eta)\in T^*\mathbb{R}^n}} \sup_{|\rho|\leqslant M,\ \rho\in\mathbb{N}^{4n}} \left| \Gamma_{\alpha,\beta,\rho,\gamma}(D)(\sigma(D))^{N+1} a(x,\xi) b(y,\eta) \right|, \end{aligned} \tag{2.8}$$

*where $\sigma(D) = \sigma(D_x, D_\xi; D_y, D_\eta)$ as usual, and*

$$\Gamma_{\alpha,\beta,\rho,\gamma}(D) = (h^\alpha \partial_{(x,y)}, h^\beta \partial_{(\xi,\eta)}))^\rho \partial^{\gamma_1} \partial^{\gamma_2}.$$

*Proof.* Following [22, Lemma 3.6] we recall that

$$c(x,\xi) = \exp(ih\sigma(D)/2) a(x,\xi) b(y,\eta)|_{x=y,\eta=\xi}$$

and hence by Taylor's theorem the remainder may be expressed as

$$e_N(x,\xi) = \frac{1}{N!} \int_0^1 (1-t)^N \exp(ith\sigma(D)/2)(ih\sigma(D)/2)^{N+1} \big(a(x,\xi)b(y,\eta)\big)\, dt \big|_{x=y,\eta=\xi}. \tag{2.9}$$

Likewise $\partial^\gamma e_N$ is a sum of terms of the form

$$\begin{aligned} &(\text{const})\times \\ &\int_0^1 (1-t)^N \partial_{(x,\xi)}^{\gamma_1} \partial_{(y,\eta)}^{\gamma_2} \exp(ith\sigma(D)/2)(ih\sigma(D)/2)^{N+1} \big(a(x,\xi)b(y,\eta)\big)\, dt \big|_{x=y,\eta=\xi} \end{aligned} \tag{2.10}$$

where $\gamma_1 + \gamma_2 = \gamma$. We further recall that for any non-degenerate real quadratic form $A$ there exists $M$ such that for all $f$,

$$|\partial^\gamma \exp(iA(D)/2) f| \leqslant C \sum_{|\rho|<M} \sup \left| \partial^{\gamma+\rho} f \right|$$

(where the sup is over all phase and base variables—in our case, $(x,\xi,y,\eta)$). Now we take $A(D) = th\sigma(D)$ and note that

$$h\sigma(D) = \sigma(h^\alpha D_x, h^\beta D_\xi; h^\alpha D_y, h^\beta D_\eta),$$

Rescaling the $x, y$ variables by $h^\alpha$ and $\xi, \eta$ by $h^\beta$ shows that we may estimate

$$|\partial^{\gamma_1} \partial^{\gamma_2} \exp(ith\sigma(D)/2) f| \leqslant C \sum_{|\rho|<M} \sup \left| \partial^{\gamma_1} \partial^{\gamma_2} (h^\alpha \partial_{(x,y)}, h^\beta \partial_{(\xi,\eta)})^\rho f \right|$$

Thus we may estimate the integrand of of (2.10) uniformly by a constant times

$$h^{N+1} \sup \sum_{|\rho|<M} \left|\partial^{\gamma_1}\partial^{\gamma_2}(h^\alpha \partial_{x,y}, h^\beta \partial_{\xi,\eta})^\rho \sigma(D)^{N+1} a(x,\xi) b(y,\eta)\right|,$$

and the result follows. □

As a particular consequence we notice that if $a \in \mathcal{S}_{\alpha,\beta}(T^*\mathbb{R}^n)$ and $b \in \mathcal{S}(T^*\mathbb{R}^n)$ then

$$c(x,\xi) = \sum_{k=0}^{N} \frac{1}{k!} \left(ih\sigma(D_x, D_\xi; D_y, D_\eta)\right)^k a(x,\xi) b(y,\eta)|_{x=y,\xi=\eta} + \mathcal{O}_{\mathcal{S}_{\alpha,\beta}}(h^{N+1} \max\{(\tilde{h}/h)^{(N+1)\alpha}, (\tilde{h}/h)^{(N+1)\beta}\}. \tag{2.11}$$

We will let $\mathcal{B}$ denote the "blowdown map"

$$(x,\xi) = \mathcal{B}(X,\Xi) = ((h/\tilde{h})^\alpha X, (h/\tilde{h})^\beta \Xi). \tag{2.12}$$

Observe that introducing the coordinates $(X,\Xi)$ is tantamount to performing an anisotropic blowup centered at $x = \xi = h = 0$. The spaces of operators $\Psi_h$ and $\Psi_{\tilde{h}}$ are related via a unitary rescaling in the following fashion. Let $a \in \mathcal{S}^{k,m,\tilde{m}}_{\alpha,\beta}$, and consider the rescaled symbol

$$a\left(\left(h/\tilde{h}\right)^\alpha X, \left(h/\tilde{h}\right)^\beta \Xi\right) = a \circ \mathcal{B} \in \mathcal{S}^{k,m,\tilde{m}}_{0,0}.$$

Define the unitary operator $T_{h,\tilde{h}} u(X) = \left(h/\tilde{h}\right)^{\frac{n\alpha}{2}} u\left(\left(h/\tilde{h}\right)^\alpha X\right)$, so that

$$\mathrm{Op}_{\tilde{h}}^w(a \circ B) T_{h,\tilde{h}} u = T_{h,\tilde{h}} \mathrm{Op}_h^w(a) u.$$

2.6. **Proof of Lemma 2.3.** By virtue of the cutoff $\varphi^w$, to begin we are working microlocally in $\{|(x,\xi)| \leqslant \epsilon\}$. We observe that since $2m/(m+1) < 2$, if we can show the estimate (2.6) for $Q_1 = \widetilde{Q} - h^2 V_1$, the estimate follows also for $\widetilde{Q}$. Let

$$q_1 = \xi^2 + A^{-2} - z$$

be the principal symbol of $Q_1$. The function $A^{-2} = (1 + x^{2m})^{-1/m}$ is analytic near $x = 0$, and since $|x| \leqslant \epsilon$ is small, we expand $A^{-2}$ in a Taylor series about $x = 0$ and write

$$q_1 = \xi^2 - \frac{1}{m} x^{2m}(1 + a(x)) - z_1,$$

where $z_1 = z - 1 \in [-\epsilon, \epsilon]$, and $a(x) = \mathcal{O}(x^{2m})$.

The Hamilton vector field $\mathsf{H}$ associated to the symbol $q_1$ is given by

$$\mathsf{H} = 2\xi\partial_x + (2x^{2m-1} + \mathcal{O}(x^{4m-1}))\partial_\xi.$$

We will consider a commutant localizing in this region and singular at the origin in a controlled way: as above we introduce new variables

$$\Xi = \frac{\xi}{(h/\tilde{h})^{m\alpha}}, \quad X = \frac{x}{(h/\tilde{h})^\alpha},$$

with

$$\alpha = \frac{1}{m+1}.$$

(When we wish to be more precise below, we will explicitly use the map $(x,\xi) = \mathcal{B}(X,\Xi)$ in this coordinate change; for the moment, we simply abuse notation.) As $m\alpha + \alpha = 1$, we note that quantizations of symbolic functions of $X, \Xi$ lie in the pseudodifferential calculus, hence the symbol of the composition of two such operators depends *globally* on the symbols of the two operators. It is in order to cope with this issue that we employ the two parameter calculus.

We remark that in the new "blown-up" coordinates $\Xi, X$,

$$\mathsf{H} = (h/\tilde{h})^{\frac{m-1}{m+1}} \left( \Xi \partial_X + X^{2m-1} \partial_\Xi + \mathcal{O}((h/\tilde{h})^{2m\alpha} X^{2m}) \partial_\Xi \right) \tag{2.13}$$

Now fix a small $\epsilon_0 > 0$ and set

$$\Lambda(s) = \int_0^s \langle s' \rangle^{-1-\epsilon_0} \, ds';$$

$\Lambda$ is of course a symbol of order 0, with $\Lambda(s) \sim s$ near $s = 0$.

We introduce the singular symbol

$$a(x, \xi; h) = \Lambda(\Xi)\Lambda(X)\chi(x)\chi(\xi) = \Lambda(\xi/(h/\tilde{h})^{m\alpha})\Lambda(x/(h/\tilde{h})^{\alpha})\chi(x)\chi(\xi),$$

where $\chi(s)$ is a cutoff function equal to 1 for $|s| < \delta_1$ and 0 for $s > 2\delta_1$ ($\delta_1$ will be chosen shortly). Then $a$ is bounded, and a 0 symbol in $X, \Xi$ :

$$\left| \partial_X^\alpha \partial_\Xi^\beta a \right| \leqslant C_{\alpha,\beta}.$$

(Recall that $x = (h/\tilde{h})^\alpha X$ and $\xi = (h/\tilde{h})^{m\alpha} \Xi$.) Using (2.13), it is simple to compute (2.14)

$$\begin{aligned} \mathsf{H}(a) =& (h/\tilde{h})^{\frac{m-1}{m+1}} \chi(x)\chi(\xi) \big( \Lambda(\Xi) \langle X \rangle^{-1-\epsilon_0} \Xi \\ &+ X^{2m-1} \langle \Xi \rangle^{-1-\epsilon_0} \Lambda(X)(1 + \mathcal{O}(x^{2m})) \big) + r \\ =& (h/\tilde{h})^{\frac{m-1}{m+1}} \chi(x)\chi(\xi) \bigg( (h/\tilde{h})^{-m\alpha} \xi \Lambda(\xi/(h/\tilde{h})^{m\alpha}) \Big\langle x/(h/\tilde{h})^\alpha \Big\rangle^{-1-\epsilon_0} \\ &+ (h/\tilde{h})^{-(2m+1)\alpha} x^{2m-1} \Lambda(x/(h/\tilde{h})^\alpha) \Big\langle \xi/(h/\tilde{h})^{m\alpha} \Big\rangle^{-1-\epsilon_0} (1 + \mathcal{O}(x^{2m})) \bigg) + r \\ \equiv& (h/\tilde{h})^{\frac{m-1}{m+1}} g + r \end{aligned}$$

with

$$\operatorname{supp} r \subset \{|x| > \delta_1\} \cup \{|\xi| > \delta_1\}$$

($r$ comes from terms involving derivatives of $\chi(x)\chi(\xi)$). Note that near $X = \Xi = 0$, since $\Lambda(s) \sim s$ for $s \sim 0$, the term

$$g = \Lambda(\Xi)\langle X \rangle^{-1-\epsilon_0} \Xi + \langle \Xi \rangle^{-1-\epsilon_0} \Lambda(X) X^{2m-1} (1 + \mathcal{O}(x^{2m})) \tag{2.15}$$

in $\mathsf{H}(a)$ is bounded below by a multiple of $\Xi^2 + X^{2m}$. Provided $\delta_1$ is chosen small enough (so we can absorb the $\mathcal{O}(x^{2m})$ error term), $g$ is in fact strictly positive away from $X = \Xi = 0$, while in the region $|(X, \Xi)| \geqslant 1$, we find that since $\operatorname{sgn} \Lambda(s) = \operatorname{sgn}(s)$, when $|\Xi| \geqslant \max(|X|^{1+\epsilon_0}, 1)$ then

$$g \geqslant \Lambda(\Xi)\langle X \rangle^{-1-\epsilon_0} \Xi \gtrsim \frac{|\Xi|}{\langle \Xi \rangle} \geqslant C > 0,$$

while for $|X|^{1+\epsilon_0} \geqslant \max(|\Xi|, 1)$, we have (providing $\delta_1 \ll 1$)

$$g \geqslant (1/2)\langle \Xi \rangle^{-1-\epsilon_0} \Lambda(X) X^{2m-1} \gtrsim |X|^{-2(1+\epsilon_0)} |X|^{2m-1} \geqslant C > 0,$$

provided $2(1 + \epsilon_0) < 2m - 1$. Thus, since the larger of $|\Xi|$ and $|X|^{1+\epsilon_0}$ is assuredly greater than 1 in the region of interest, we have in fact shown that

$$g \geqslant C > 0 \quad \text{in } \{\Xi^2 + X^2 > 1\}.$$

Thus, we find

$$\mathsf{H}(a) = (h/\tilde{h})^{\frac{m-1}{m+1}} g + r$$

with

$$r = \mathcal{O}_{\mathcal{S}_{\alpha,\beta}}((h/\tilde{h})^{(m-1)/(m+1)}((h/\tilde{h})^{\alpha}|\Xi| + (h/\tilde{h})^{\beta}|X^{2m-1}|)$$

supported as above and

$$g(X,\Xi;h) = \begin{cases} c(\Xi^2 + X^{2m})(1+r_2), & \Xi^2 + X^2 \leqslant 1 \\ b, & \Xi^2 + X^2 \geqslant 1, \end{cases}$$

where $c > 0$ is a constant,$r_2 = \mathcal{O}_{\mathcal{S}_{\alpha,\beta}}(\delta_1)$, and $b > 0$ is elliptic.

We will require a positivity result dealing with operators satisfying estimates of this type.

**Lemma 2.5.** *Let a real-valued symbol $\tilde{g}(x,\xi;h)$ satisfy*

$$\tilde{g}(x,\xi;h) = \begin{cases} c(\xi^2 + x^{2m})(1+r_2), & \xi^2 + x^2 \leqslant 1 \\ b, & \xi^2 + x^2 \geqslant 1, \end{cases}$$

*where $c > 0$ is constant, $r_2 = \mathcal{O}_{\mathcal{S}_{\alpha,\beta}}(\delta_1)$, and $b > 0$ is elliptic. Then there exists $c_0 > 0$ such that*

$$\langle \mathrm{Op}_h^w(\tilde{g})u, u\rangle \geqslant c_0 h^{2m/(m+1)}\|u\|^2$$

*for $h$ sufficiently small.*

*Proof.* Since $b > 0$ is elliptic, there exists $\sigma > 0$ sufficiently small and independent of $h > 0$ so that if $\langle \mathrm{Op}_h^w(\tilde{g})u, u\rangle \leqslant \sigma\|u\|^2$, then $u$ has semiclassical wavefront set contained in the set $\{|x|^2 + |\xi|^2 \leqslant 1/2\}$. On this set, we may write

$$\tilde{g} = (\xi^2 + x^{2m})K^2$$

with $K$ a strictly positive symbol. The Weyl quantization has the convenient feature that we thus have

$$\mathrm{Op}_h^w(\tilde{g}) = \mathrm{Op}_h^w(K)^*(h^2 D_x^2 + x^{2m})\mathrm{Op}_h^w(K) + \mathcal{O}(h^2),$$

and $\mathrm{Op}_h^w(K) \geqslant \epsilon_1 > 0$. Then for $u$ microsupported in $\{|x|^2 + |\xi|^2 \leqslant 1/2\}$ we thus compute

$$\langle \mathrm{Op}_h^w(\tilde{g})u, u\rangle \geqslant \epsilon_1 \left\langle \mathrm{Op}_h^w(\xi^2 + x^{2m})u, u\right\rangle - \mathcal{O}(h^2)\|u\|^2.$$

The lower bound follows, for $h > 0$ sufficiently small, from Lemma A.2.

□

We now employ this result to estimate $\mathrm{Op}_h^w(\mathsf{H}(a))$.

**Lemma 2.6.** *For $\tilde{h} > 0$ sufficiently small, there exists $c > 0$ such that $\mathrm{Op}_h^w(g) > c\tilde{h}^{2m/(m+1)}$, uniformly as $h \downarrow 0$, where $g$ is given by* (2.15).

*Proof.* Note that we have written $g$ as a function of $X, \Xi$, so in changing variables to $x, \xi$ we are tacitly employing the blowdown map $\mathcal{B}$. In particular, we are interested in estimating $\mathrm{Op}_h^w(g \circ \mathcal{B}^{-1})$ from below. By (2.13),

$$\mathrm{Op}_{\tilde{h}}^w(g)T_{h,\tilde{h}}u = T_{h,\tilde{h}}\mathrm{Op}_h^w(g \circ \mathcal{B}^{-1})u,$$

hence

$$\left\langle \mathrm{Op}_h^w(g \circ \mathcal{B}^{-1})u, u\right\rangle = \left\langle T_{h,\tilde{h}}\mathrm{Op}_{\tilde{h}}^w(g)T_{h,\tilde{h}}u, u\right\rangle \geqslant c\tilde{h}^{2m/(m+1)}\|u\|^2$$

for $\tilde{h}$ sufficiently small, by unitarity of $T_{h,\tilde{h}}$ and Lemma 2.5, with $\tilde{h}$ replacing $h$. This establishes the Lemma. □

Before completing the proof of Lemma 2.3, we need the following lemma about the lower order terms in the expansion of the commutator of $Q_1$ and $a^w$.

**Lemma 2.7.** *The symbol expansion of* $[Q_1, a^w]$ *in the* $h$*-Weyl calculus is of the form*

$$[Q_1, a^w] = \mathrm{Op}_h^w\Bigg(\Big(\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\Big)(q_1(x,\xi)a(y,\eta) - q_1(y,\eta)a(x,\xi))|_{x=y,\xi=\eta} + e(x,\xi) + r_3(x,\xi)\Bigg),$$

*where* $e$ *satisfies*

$$\mathrm{Op}_h^w(e) \leqslant C\tilde{h}^{-(m-3)/(m+1)}h^{2m/(m+1)}\mathrm{Op}_h^w(g),$$

*with* $g$ *given by* (2.15) *and* $r_3$ *supported in* $\{|(x,\xi)| \geqslant \delta_1\}$.

*Proof.* Since everything is in the Weyl calculus, only the odd terms in the exponential composition expansion are non-zero. Hence the $h^2$ term is zero in the Weyl expansion. Now according to Lemma 2.4 and the standard $L^2$ continuity theorem for $h$-pseudodifferential operators, we need to estimate a finite number of derivatives of the error:

$$|\partial^\gamma e_2| \leqslant Ch^3 \sum_{\gamma_1+\gamma_2=\gamma} \sup_{\substack{(x,\xi)\in T^*\mathbb{R} \\ (y,\eta)\in T^*\mathbb{R}}} \sup_{|\rho|\leqslant M,\ \rho\in\mathbb{N}^4} \left|\Gamma_{\alpha,\beta,\rho,\gamma}(D)(\sigma(D))^3 q_1(x,\xi)a(y,\eta)\right|.$$

However, since $q_1(x,\xi) = \xi^2 - x^{2m}(1 + a(x))$, we have

$$D_x D_\xi q_1 = D_\xi^3 q_1 = 0,$$

so that

$$\begin{aligned}
&\sigma(D)^3 q_1(x,\xi)a(y,\eta)|_{x=y,\xi=\eta} \\
&\quad = D_x^3 q_1 D_\eta^3 a|_{x=y,\xi=\eta} \\
&\quad = cx^{2m-3}(1 + \mathcal{O}(x^{2m}))(\tilde{h}/h)^{3m/(m+1)}\Lambda'''((\tilde{h}/h)^{m/(m+1)}\eta) \\
&\qquad \times \Lambda((\tilde{h}/h)^{1/(m+1)}y)\chi(y)\chi(\eta) + r_3,
\end{aligned}$$

where $r_3$ is supported in $\{|(x,\xi)| \geqslant \delta_1\}$. Owing to the cutoffs $\chi(y)\chi(\eta)$ in the definition of $a$ (and the corresponding implicit cutoffs in $q_1$), we only need to estimate this error in compact sets. The derivatives $h^\beta\partial_\eta$ and $h^\alpha\partial_y$ preserve the order of $e_2$ in $h$ and increase the order in $\tilde{h}$, while the other derivatives lead to higher powers in $h/\tilde{h}$ in the symbol expansion. Hence we need only estimate $e_2$, as the derivatives satisfy similar estimates.

In order to estimate $e_2$, we again use conjugation to the 2-parameter calculus. We have

$$\|\mathrm{Op}_h^w(e_2)u\| = \|T_{h,\tilde{h}}\mathrm{Op}_h^w(e_2)T_{h,\tilde{h}}^{-1}T_{h,\tilde{h}}u\| \leqslant \|T_{h,\tilde{h}}\mathrm{Op}_h^w(e_2)T_{h,\tilde{h}}^{-1}\|_{L^2\to L^2}\|u\|,$$

by unitarity of $T_{h,\tilde{h}}$. But $T_{h,\tilde{h}}\mathrm{Op}_h^w(e_2)T_{h,\tilde{h}}^{-1} = \mathrm{Op}_{\tilde{h}}^w(e_2 \circ \mathcal{B})$ and

$$\begin{aligned}
e_2 \circ \mathcal{B} = {} & h^3(h/\tilde{h})^{(2m-3)\alpha}X^{2m-3}(1 + \mathcal{O}(x^{2m}))(\tilde{h}/h)^{3m/(m+1)}\Lambda'''(\Xi) \\
& \times \Lambda(X)\chi(x)\chi(\xi) + r_3 \circ \mathcal{B},
\end{aligned}$$

and we may estimate the first term above by

$$Ch^{2m/(m+1)}\tilde{h}^{(m+3)/(m+1)}X^{2m-3}\Lambda'''(\Xi)\chi(x)\chi(\xi),$$

which in turn is bounded above by

$$\begin{cases} Ch^{2m/(m+1)}\tilde{h}^{(m+3)/(m+1)}, & |X|\leqslant 1,\\ Ch^{2m/(m+1)}\tilde{h}^{(m+3)/(m+1)}g, & |X|\geqslant 1.\end{cases} \tag{2.16}$$

It now suffices to verify that for

$$k = X^{2m-3}\Lambda'''(\Xi)\chi(x)\chi(\xi),$$
$$\operatorname{Op}_h^w(k\circ\mathcal{B}^{-1}) \leqslant C\tilde{h}^{-2m/(m+1)}\operatorname{Op}_h^w(g\circ\mathcal{B}^{-1}),$$

i.e., that for all $u(X)$,

$$\big\langle \operatorname{Op}_h^w(k\circ\mathcal{B}^{-1})u,u\big\rangle \leqslant C\tilde{h}^{-2m/(m+1)}\big\langle\operatorname{Op}_h^w(g\circ\mathcal{B}^{-1})u,u\big\rangle.$$

We now rescale and return to performing $\tilde{h}$ quantization in the $X$ variable. For $u(X)$ microsupported away from the origin in $(X,\Xi)$, the desired estimate follows from the second inequality in (2.16) (indeed the $\tilde{h}^{-2m/(m+1)}$ factor on the RHS may be omitted), while for $u$ microsupported near the origin, it follows from the lower bound of Lemma 2.5. □

We are now able to prove the resolvent estimate Lemma 2.3. Let $v=\varphi^w u$, with $\varphi$ chosen to have support inside the set where $\chi(x)\chi(\xi)=1$; thus the terms $r$ and $r_3$ above are supported away from the support of $\varphi$. Then Lemmas 2.6 and 2.7 yield

$$\begin{aligned} i\langle[Q_1-z,a^w]v,v\rangle &= h\langle\operatorname{Op}_h^w(\mathsf{H}(a))v,v\rangle + \langle\operatorname{Op}_h^w(e_2)u,u\rangle\\ &= h(h/\tilde{h})^{(m-1)/(m+1)}\langle\operatorname{Op}_h^w(g)v,v\rangle + \langle\operatorname{Op}_h^w(e_2)u,u\rangle\\ &= h^{2m/(m+1)}\big(\tilde{h}^{-(m-1)/(m+1)}+\mathcal{O}(\tilde{h}^{-(m-3)/(m+1)})\big)\langle\operatorname{Op}_h^w(g)v,v\rangle\\ &\geqslant Ch^{2m/(m+1)}\tilde{h}\|v\|^2, \end{aligned}$$

for $\tilde{h}$ sufficiently small. On the other hand, we certainly have

$$\big|\langle[Q_1-z,a^w]v,v\rangle\big| \leqslant C\|(Q_1-z)v\|\|v\|,$$

hence the desired bound follows once we fix $\tilde{h}>0$. □

## 3. Resonances and Quasimodes

In this section, we construct quasimodes for the model operator near $(0,0)$ in phase space. Let

$$\widetilde{P} = -h^2\partial_x^2 - m^{-1}x^{2m}$$

locally near $x=0$. We will construct quasimodes which are localized very close to $x=0$, so this should be a decent approximation.

Complex scaling $(x,\xi)\mapsto(e^{i\pi/(2m+2)}x, e^{-i\pi/(2m+2)}\xi)$ sends $\widetilde{P}$ to a multiple of the quantum anharmonic oscillator. As in the appendix, we find there is a Schwartz class function $v(x)=v_0(xh^{-1/(m+1)})$ which is an un-normalized ground state for the equation

$$(-h^2\partial_x^2 + m^{-1}x^{2m})v = h^{2m/(m+1)}\lambda_0 v.$$

This suggests there are resonances for the operator $\widetilde{P}$ with imaginary part to leading order $c_0h^{2m/(m+1)}$, although this is only a heuristic. We use a complex WKB approximation to get an explicit formula for a localized approximate resonant state.

Let $E = (\alpha + i\beta)h^{2m/(m+1)}$, $\alpha, \beta > 0$ independent of $h$. Let the phase function

$$\varpi(x) = \int_0^x (E + m^{-1}y^{2m})^{1/2} dy,$$

where the branch of the square root is chosen to have positive imaginary part. Let

$$u(x) = (\varpi')^{-1/2} e^{i\varpi/h},$$

so that

$$(hD)^2 u = (\varpi')^2 u + fu,$$

where

$$\begin{aligned} f &= (\varpi')^{1/2}(hD)^2(\varpi')^{-1/2} \\ &= -h^2 \left( \frac{3}{4}(\varpi')^{-2}(\varpi'')^2 - \frac{1}{2}(\varpi')^{-1}\varpi''' \right). \end{aligned}$$

**Lemma 3.1.** *The phase function $\varpi$ satisfies the following properties:*

**(i):** *There exists $C > 0$ independent of $h$ such that*

$$|\operatorname{Im}\varpi| \leqslant C \begin{cases} h(1 + \log(x/h^{1/2})), & m = 1, \\ h, & m \geqslant 2. \end{cases}$$

*In particular, if $|x| \leqslant Ch^{1/(m+1)}$, $|\operatorname{Im}\varpi| \leqslant C'$ for some $C' > 0$ independent of $h$.*

**(ii):** *There exists $C > 0$ independent of $h$ such that*

$$C^{-1}\sqrt{h^{2m/(m+1)} + m^{-1}x^{2m}} \leqslant |\varpi'(x)| \leqslant C\sqrt{h^{2m/(m+1)} + m^{-1}x^{2m}}$$

**(iii):**

$$\begin{cases} \varpi' = (E + m^{-1}x^{2m})^{1/2}, \\ \varpi'' = x^{2m-1}(\varpi')^{-1}, \\ \varpi''' = \left((1 - 1/m)x^{4m-2} + E(2m-1)x^{2m-2}\right)(\varpi')^{-3}, \end{cases}$$

*In particular,*

$$f = -h^2 x^{2m-2} \left( \left( \frac{1}{4} + \frac{1}{2m} \right) x^{2m} - \left( m - \frac{1}{2} \right) E \right) (\varpi')^{-4}.$$

*Proof.* For (i) we write $\varpi' = s + it$ for $s$ and $t$ real valued, and then

$$E + m^{-1}x^{2m} = s^2 - t^2 + 2ist.$$

Hence

$$s^2 \geqslant s^2 - t^2 = \alpha h^{2m/(m+1)} + m^{-1}x^{2m},$$

so that

$$t = \frac{\beta h^{2m/(m+1)}}{2s} \leqslant \frac{\beta h^{2m/(m+1)}}{2\sqrt{h^{2m/(m+1)}\alpha + m^{-1}x^{2m}}}.$$

Then

$$\begin{aligned} |\operatorname{Im} \varpi(x)| &\leqslant \int_0^{|x|} |\operatorname{Im} \varpi'(y)| dy \\ &\leqslant C \int_0^{h^{1/(m+1)}} h^{m/(m+1)} dy + C \int_{h^{1/(m+1)}}^{x} h^{2m/(m+1)} y^{-m} dy \\ &= \begin{cases} \mathcal{O}(h(1+\log(x/h^{1/2}))), & m = 1, \\ \mathcal{O}(h), & m > 1. \end{cases} \end{aligned}$$

Parts (ii) and (iii) are simple computations.

□

In light of this lemma, $|u(x)|$ is comparable to $|\varpi'|^{-1/2}$ for all $x$ for $m \geqslant 2$, and provided $|x| \leqslant Ch^{1/2}$ when $m = 1$. We are only interested in sharply localized quasimodes, so let

$$\gamma = h^{1/(m+1)},$$

choose $\chi(s) \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\chi \equiv 1$ for $|s| \leqslant 1$ and $\operatorname{supp} \chi \subset [-2, 2]$. Let

$$\tilde{u}(x) = \chi(x/\gamma) u(x),$$

and, since $|\varpi'(x)| \sim h^{m/(m+1)}$ for $|x| \leqslant 2h^{1/(m+1)}$, we compute:

$$\begin{aligned} \|\tilde{u}\|_{L^2}^2 &= \int_{|x| \leqslant 2\gamma} \chi(x/\gamma)^2 |u|^2 dx \\ &\sim \int_{|x| \leqslant 2\gamma} \chi(x/\gamma)^2 |\varpi'|^{-1} dx \\ &\sim h^{1/(m+1)} h^{-m/(m+1)} \\ &\sim h^{(1-m)/(1+m)}. \end{aligned}$$

Further, $\tilde{u}$ satisfies the following equation:

$$\begin{aligned} (hD)^2 \tilde{u} &= \chi(x/\gamma)(hD)^2 u + [(hD)^2, \chi(x/\gamma)] u \\ &= (\varpi')^2 \tilde{u} + f \tilde{u} + [(hD)^2, \chi(x/\gamma)] u \\ &= (\varpi')^2 \tilde{u} + R, \end{aligned}$$

where

$$R = f \tilde{u} + [(hD)^2, \chi(x/\gamma)] u.$$

**Lemma 3.2.** *The remainder $R$ satisfies*

$$\|R\|_{L^2} = \mathcal{O}(h^{2m/(m+1)}) \|\tilde{u}\|_{L^2}. \tag{3.1}$$

*Proof.* We have already computed the function $f$, which is readily seen to satisfy

$$\|f\|_{L^\infty(\operatorname{supp}(\tilde{u}))} = \mathcal{O}(h^{2m/(m+1)}),$$

since $\operatorname{supp}(\tilde{u}) \subset \{|x| \leqslant 2h^{1/(m+1)}$.

On the other hand, since $\|\tilde{u}\|_{L^2} \sim h^{(1-m)/2(1+m)}$, we need only show that

$$\|[(hD)^2, \chi(x/\gamma)] u\|_{L^2} \leqslant Ch^{(3m+1)/2(m+1)}.$$

We compute:

$$\begin{aligned}[(hD)^2, \chi(x/\gamma)]u &= -h^2\gamma^{-2}\chi'' u + 2\frac{h}{i}\gamma^{-1}\chi' hDu \\ &= -h^2\gamma^{-2}\chi'' u + 2\frac{h}{i}\gamma^{-1}\chi'\left(-\frac{h}{2i}\frac{\varpi''}{\varpi'} + \varpi'\right)u \\ &= -h^2\gamma^{-2}\chi'' u + 2\frac{h}{i}\gamma^{-1}\chi'\left(-\frac{h}{2i}\frac{x^{2m-1}}{(\varpi')^2} + \varpi'\right)u.\end{aligned}$$

The first term is estimated:

$$\|h^2\gamma^{-2}\chi'' u\|_{L^2} = \mathcal{O}(h^{2m/(m+1)})\|u\|_{L^2(\operatorname{supp}(\tilde{u}))} = \mathcal{O}(h^{(3m+1)/2(m+1)}).$$

Similarly, the remaining two terms are estimated:

$$\begin{aligned}&\left\| 2\frac{h}{i}\gamma^{-1}\chi'\left(-\frac{h}{2i}\frac{x^{2m-1}}{(\varpi')^2} + \varpi'\right)u\right\|_{L^2} \\ &\quad = \mathcal{O}(h^{m/(m+1)}h^1 h^{(2m-1)/(m+1)}h^{-2m/(m+1)})\|u\|_{L^2(\operatorname{supp}(\tilde{u}))} \\ &\qquad + \mathcal{O}(h^{m/(m+1)}h^{2m/(m+1)})\|u\|_{L^2(\operatorname{supp}(\tilde{u}))} \\ &\quad = \mathcal{O}(h^{(3m+1)/2(m+1)}).\end{aligned}$$

□

3.1. **Sharp local smoothing of quasimodes.** In this subsection, we show that the quasimode $\tilde{u}$ constructed above can be used to saturate the local smoothing estimate of Theorem 1. We observe that due to the estimate (2.4), we have perfect local smoothing in the "radial" direction $x$. Hence we only consider $\theta$ regularity.

**Theorem 3.** *Let $\varphi_0(x,\theta) = e^{ik\theta}\tilde{u}(x)$, where $\tilde{u} \in \mathcal{C}_c^\infty(\mathbb{R})$ was constructed in the previous section, with $h = |k|^{-1}$, where $|k|$ is taken sufficiently large, and the parameter $m \geqslant 2$ as usual. Suppose $\psi$ solves*

$$\begin{cases}(D_t + \widetilde{\Delta})\psi = 0, \\ \psi|_{t=0} = \varphi_0.\end{cases}$$

*Then for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on $\operatorname{supp}\tilde{u}$, and $A > 0$ sufficiently large, independent of $k$, there exists a constant $C_0 > 0$ independent of $k$ such that*

$$\int_0^{|k|^{-2/(m+1)}/A} \| \langle D_\theta\rangle \chi\psi\|_{L^2}^2 dt \geqslant C_0^{-1} \| \langle D_\theta\rangle^{m/(m+1)} \varphi_0\|_{L^2}^2. \tag{3.2}$$

**Remark 3.3.** The theorem states that on the *weak semiclassical* time scale $|t| \lesssim |k|^{-2/(m+1)}$, the local smoothing estimate in this paper is sharp. Evidently this implies that on any fixed finite time scale the theorem is sharp; the theorem stated above gives more information. That is, it demonstrates that even on a semiclassical time scale, the local smoothing estimate really cannot be improved.

In addition, as the proof will indicate, no weight $\chi$ is necessary, because on the semiclassical time scale we have essentially finite propagation speed.

**Remark 3.4.** The analogue of Theorem 3 when the parameter $m = 1$, where a log loss is expected, is contained in the work of Bony-Burq-Ramond [1].

*Proof.* The technique of proof is to simply evolve the stationary quasimode as if the equation were separated. The advantage is that this separated stationary "solution" remains compactly supported for all time. Of course this is not an exact solution, and generates an inhomogeneous error which must be estimated using energy estimates. It is here that we use the semiclassical time scale. Combining the two estimates yields the theorem.

Let $\tilde{u}$ be as above for $h = |k|^{-1}$ (after a suitable $L^2$ normalization), and let

$$\varphi_0(x, \theta) = \tilde{u} e^{ik\theta},$$

as in the statement of the theorem. Let

$$\varphi(t, x, \theta) = e^{it\tau} \varphi_0$$

for some $\tau \in \mathbb{C}$ to be determined. Since the support of $\tilde{u}$ is very small, contained in $\{|x| \leqslant h^{1/(m+1)}/\gamma\}$, we have

$$A^{-2} = (1 + x^{2m})^{-1/m} = 1 - \frac{1}{m} x^{2m} + \mathcal{O}(h^{4m/(m+1)})$$

on $\operatorname{supp} \tilde{u}$. Then

$$\begin{aligned}
(D_t + \widetilde{\Delta})\varphi &= (D_t + P_k)\varphi \\
&= (\tau - D_x^2 - A^{-2}k^2 - V_1(x))\varphi \\
&= k^2 e^{it\tau} e^{ik\theta} \left[ \left( \tau k^{-2} - (k^{-2} D_x^2 + 1 - \frac{1}{m} x^{2m}) \right) \tilde{u} + \mathcal{O}(k^{-2}) \tilde{u} \right] \\
&= k^2 e^{it\tau} e^{ik\theta} \left[ \left( \tau k^{-2} - 1 - E_0 \right) \tilde{u} + R + \mathcal{O}(k^{-2}) \tilde{u} \right],
\end{aligned}$$

where $R$ satisfies the remainder estimate (3.1), i.e.,

$$\|R\|_{L^2} = \mathcal{O}(h^{2m/(m+1)}).$$

Set

$$\tau = k^2 + k^2 E_0 = k^2(1 + \alpha k^{-2m/(m+1)}) + i\beta k^{2/(m+1)}, \ \ \alpha, \beta > 0$$

so that we have

$$\begin{cases} (D_t + \widetilde{\Delta})\varphi = \tilde{R}, \\ \varphi(0, x, \theta) = \varphi_0 \end{cases}$$

with

$$\tilde{R} = k^2 e^{it\tau} e^{ik\theta} (R(x, k) + \mathcal{O}(k^{-2}) \tilde{u}). \tag{3.3}$$

Thus, we obtain

$$\|\tilde{R}\| \leqslant C k^2 |e^{it\tau}| k^{-2m/(m+1)} \|\tilde{u}\| \tag{3.4}$$

and since on every function in question, $\langle D_\theta \rangle = \langle k \rangle$, we furthermore have

$$\| \langle D_\theta \rangle \tilde{R} \| \leqslant C k^2 |e^{it\tau}| k^{-2m/(m+1)} \| \langle D_\theta \rangle \varphi_0 \| \tag{3.5}$$

We can readily verify that $\varphi$ saturates the local smoothing estimate of Theorem 1 *on any time scale*:

$$\begin{aligned}
\| \langle D_\theta \rangle \varphi\|^2_{L^2([0,T])L^2} &= \int_0^T \|e^{it\tau} \langle D_\theta \rangle \varphi_0\|^2_{L^2} dt \\
&= \int_0^T e^{-2t\beta|k|^{2/(m+1)}} \| \langle D_\theta \rangle \varphi_0\|^2_{L^2} dt \\
&= \frac{1-e^{-2T\beta|k|^{2/(m+1)}}}{2\beta|k|^{2/(m+1)}} \| \langle D_\theta \rangle \varphi_0\|^2_{L^2} \\
&= \frac{1-e^{-2T\beta|k|^{2/(m+1)}}}{2\beta} \| |D_\theta|^{-1/(m+1)} \langle D_\theta \rangle \varphi_0\|^2_{L^2} \\
&= B \| |D_\theta|^{-1/(m+1)} \langle D_\theta \rangle \varphi_0\|^2_{L^2}, \qquad (3.6)
\end{aligned}$$

where we let

$$B = \frac{1-e^{-2T\beta|k|^{2/(m+1)}}}{2\beta}. \qquad (3.7)$$

What we aim to do, then, is show that $\varphi$ is close enough to a solution of the Schrödinger equation that the same form of estimate holds for the nearby solution with initial data $\varphi_0$.

Thus, let $L(t)$ be the unitary Schrödinger propagator:

$$\begin{cases} (D_t + \widetilde{\Delta})L = 0, \\ L(0) = \operatorname{id}, \end{cases}$$

and write using Duhamel's formula:

$$\varphi(t) = L(t)\varphi_0 + i \int_0^t L(t)L^*(s)\tilde{R}(s)ds =: \varphi_{\mathrm{h}} + \varphi_{\mathrm{ih}},$$

where $\varphi_{\mathrm{h}}$ and $\varphi_{\mathrm{ih}}$ are the homogeneous and inhomogeneous parts respectively. We want to show the homogeneous smoothing effect is saturated, for which we need to show the inhomogeneous term is small and can be absorbed into the homogeneous term.

For this, we use an energy estimate, and localize in time to a scale depending on $k$. That is, if

$$E(t) = \| \langle D_\theta \rangle \varphi_{\mathrm{ih}}\|^2_{L^2},$$

we have

$$\begin{aligned}
E' &= 2\,\mathrm{Re}\, \langle \langle D_\theta \rangle \partial_t \varphi_{\mathrm{ih}}, \langle D_\theta \rangle \varphi_{\mathrm{ih}} \rangle_{L^2} \\
&= 2\,\mathrm{Re} \left\langle \langle D_\theta \rangle (-i\widetilde{\Delta}\varphi_{\mathrm{ih}} + i\tilde{R}), \langle D_\theta \rangle \varphi_{\mathrm{ih}} \right\rangle_{L^2} \\
&= 2\,\mathrm{Re} \left\langle i \langle D_\theta \rangle \tilde{R}, \langle D_\theta \rangle \varphi_{\mathrm{ih}} \right\rangle_{L^2} \\
&\leqslant \nu \| \langle D_\theta \rangle \tilde{R}\|^2_{L^2} + \nu^{-1} E,
\end{aligned}$$

for $\nu > 0$ to be determined, so that

$$\begin{aligned}
E(t) &\leqslant e^{\nu^{-1}t}\int_0^t e^{-\nu^{-1}s}\left(\nu\| \langle D_\theta\rangle \tilde{R}(s)\|_{L^2}^2\right) ds \\
&\leqslant e^{\nu^{-1}t}\left(\nu\| \langle D_\theta\rangle \tilde{R}\|_{L^2([0,T])L^2}^2\right) \\
&\leqslant C\nu e^{\nu^{-1}t}\left(\int_0^T |e^{2is\tau}|ds\right)(k^2k^{-2m/(m+1)})^2\| \langle D_\theta\rangle \varphi_0\|_{L^2}^2 \\
&\leqslant C(\nu e^{\nu^{-1}t})\left(\frac{1-e^{-2\beta|k|^{2/(m+1)}T}}{2\beta}\right) \\
&\quad\times |k|^{-2/(m+1)}(k^2k^{-2m/(m+1)})^2\| \langle D_\theta\rangle \varphi_0\|_{L^2}^2.
\end{aligned}$$

Here we have used the estimate (3.5) in the middle inequality. Integrating in $0 \leqslant t \leqslant T$, we get

$$\begin{aligned}
\| \langle D_\theta\rangle \varphi_{\mathrm{ih}}\|_{L^2([0,T])L^2}^2 &\leqslant C\nu\left(\int_0^T e^{\nu^{-1}t}dt\right) B|k|^{2/(m+1)}\| \langle D_\theta\rangle \varphi_0\|_{L^2}^2 \\
&\leqslant C\nu^2(e^{\nu^{-1}T}-1)B|k|^{2/(m+1)}\| \langle D_\theta\rangle \varphi_0\|_{L^2}^2,
\end{aligned}$$

where $B$ is given by (3.7).

If $\nu = |k|^{-2/(m+1)}/A$ for some large $A > 0$ independent of $k$, and if we take $T = \nu$, we have

$$\begin{aligned}
\| \langle D_\theta\rangle \varphi_{\mathrm{ih}}\|_{L^2([0,T])L^2}^2 &\leqslant \frac{C}{A^2}(e-1)B|k|^{-2/(m+1)}\| \langle D_\theta\rangle \varphi_0\|_{L^2}^2 \\
&= c_0B\||D_\theta|^{-1/(m+1)}\langle D_\theta\rangle \varphi_0\|_{L^2}^2, \qquad (3.8)
\end{aligned}$$

where

$$c_0 = \frac{C}{A^2}(e-1) \leqslant \frac{1}{4}$$

if we take $A$ sufficiently large.

Now, since $T$ depends on $k$, so does $B$ in principle, but

$$B = \left(\frac{1-e^{-2\beta|k|^{2/(m+1)}T}}{2\beta}\right) = \left(\frac{1-e^{-2\beta/A}}{2\beta}\right) > 0$$

independent of $k$. Hence, combining (3.6) with (3.8), we have for this choice of $T$ and $A$, and $\chi$ as in the statement of the theorem,

$$\begin{aligned}
\| \langle D_\theta\rangle \chi L(t)\varphi_0\|_{L^2([0,T])L^2} &\geqslant \| \langle D_\theta\rangle \chi\varphi\|_{L^2([0,T])L^2} - \| \langle D_\theta\rangle \chi\varphi_{\mathrm{ih}}\|_{L^2([0,T])L^2} \\
&\geqslant \| \langle D_\theta\rangle \varphi\|_{L^2([0,T])L^2} - \| \langle D_\theta\rangle \varphi_{\mathrm{ih}}\|_{L^2([0,T])L^2} \\
&\geqslant \frac{1}{2}B^{1/2}\||D_\theta|^{-1/(m+1)}\langle D_\theta\rangle \varphi_0\|_{L^2}
\end{aligned}$$

(using $\operatorname{supp}\varphi \subset \{\chi = 1\}$), with $B > 0$ independent of $k$ as above.

Thus, $\psi = \varphi_{\mathrm{h}}$ satisfies (3.2), since $\varphi$ satisfies the estimate, while $\varphi_{\mathrm{ih}}$ is small enough to be absorbed. □

## 4. Sharp resolvent estimates with loss

We now prove Theorem 2.

We begin with the sharpness. By the $TT^*$ argument (see, for example [5],if a better resolvent estimate held true, then a better local smoothing estimate would also hold true. But Theorem 3 shows the local smoothing estimate in Theorem 1 is sharp. Hence no better polynomial rate of decay in the resolvent estimate can hold true.

In order to show the estimate is true in the first place, we conjugate the Laplacian on $X$ and decompose in Fourier modes as usual

$$(-\widetilde{\Delta} - \lambda^2) = \bigoplus_{k=-\infty}^{\infty} (L_k - \lambda^2).$$

We break this sum into two pieces where either $k^2 \leqslant \lambda^2/2$ or not. If $k^2 \leqslant \lambda^2/2$, then we use $h = \lambda^{-1}$ as our semiclassical parameter, while if $k^2 > \lambda^2/2$, we use $h = |k|^{-1}$. We get

$$(-\widetilde{\Delta} - \lambda^2) = \bigoplus_{k^2 \leqslant \lambda^2/2} \lambda^2(\tilde{L}_k - z_k) \oplus \bigoplus_{k^2 > \lambda^2/2} k^2(\tilde{L}_k - z_k).$$

Here

$$\tilde{L}_k = \begin{cases} -h^2\partial_x^2 + \frac{k^2}{\lambda^2} A^{-2}(x) + h^2 V_1(x), \text{ if } k^2 \leqslant \lambda^2/2, \\ -h^2\partial_x^2 + A^{-2}(x) + h^2 V_1(x), \text{ if } k^2 > \lambda^2/2, \end{cases}$$

and

$$z_k = \begin{cases} 1, \text{ if } k^2 \leqslant \lambda^2/2, \\ k^{-2}\lambda^2, \text{ if } k^2 > \lambda^2/2. \end{cases}$$

In the case $k^2 \leqslant \lambda^2/2$, the energy level $z_k$ is non-trapping, so by Proposition 2.2 the operator $\tilde{L}_k$ satisfies the estimate

$$\|\chi(\tilde{L}_k - z_k)^{-1}\chi\|_{L^2 \to L^2} \leqslant Ch^{-1} = C\lambda,$$

with constants uniform as $|k|$ and $\lambda$ both go to infinity. In the case $k^2 > \lambda^2/2$, the operator $\tilde{L}_k$ satisfies the estimate

$$\|\chi(\tilde{L}_k - z_k)^{-1}\chi\|_{L^2 \to L^2} \leqslant Ch^{-2m/(m+1)} = C|k|^{2m/(m+1)},$$

with constants independent of $\lambda$ as $|k| \to \infty$. Hence by orthogonality of the Fourier eigenspaces,

$$\begin{aligned} \|\chi R(\lambda)\chi\|_{L^2 \to L^2} &\leqslant \max\{ \max_{|k|^2 \leqslant \lambda^2/2} \lambda^{-2} \|\chi(\tilde{L}_k - z_k)^{-1}\chi\|, \\ &\qquad \sup_{|k| > \lambda^2/2} |k|^{-2} \|\chi(\tilde{L}_k - z_k)^{-1}\chi\|\} \\ &\leqslant \max\{C\lambda^{-1}, C \sup_{|k|^2 > \lambda^2/2} |k|^{-2/(m+1)}\} \\ &\leqslant C\lambda^{-2m/(m+1)}. \quad \square \end{aligned}$$

## Appendix A. Eigenfunction properties

In this section we recall some standard results on eigenfunctions.

We consider the classical eigenfunction problem

$$Pu = \lambda u,$$

where

$$P = -\partial_x^2 + x^{2m},$$

with $m \in \mathbb{Z}$, $m \geqslant 2$ and $\lambda \geqslant 0$.

The following is a standard result in spectral theory (see, for example, [18]).

**Lemma A.1.** *As an operator on $L^2$ with domain $\mathcal{S}(\mathbb{R})$, $P$ is essentially self-adjoint. It has pure point spectrum $\lambda_j \to \infty$, $\lambda_0 > 0$, and every eigenfunction is of Schwartz class.*

We now remark the semiclassical anharmonic oscillator estimate, which follows from Lemma A.1 via the rescaling $X = h^{-1/(m+1)}x$.

**Lemma A.2.** *Let $Q = -h^2\partial_x^2 + x^{2m}$ for some $m \in \mathbb{Z}$, $m \geqslant 1$. Then there exists $c = c_m > 0$ such that for any $u \in L^2(\mathbb{R})$,*

$$\|Qu\| \geqslant ch^{2m/(m+1)}\|u\|.$$

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