SHARP STRICHARTZ ESTIMATES ON NON-TRAPPING ASYMPTOTICALLY CONIC MANIFOLDS

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ABSTRACT. We obtain the Strichartz inequalities

$$\|u\|_{L_t^q L_x^r([0,1] \times M)} \leq C\|u(0)\|_{L^2(M)}$$

for any smooth n-dimensional Riemannian manifold M which is asymptotically conic at infinity (with either short-range or long-range metric perturbation) and non-trapping, where u is a solution to the Schrödinger equation $iu_t + \frac{1}{2}\Delta_M u = 0$, and $2 < q, r \leq \infty$ are admissible Strichartz exponents ($\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$). This corresponds with the estimates available for Euclidean space (except for the endpoint $(q,r) = (2, \frac{2n}{n-2})$ when $n > 2$). These estimates imply existence theorems for semi-linear Schrödinger equations on M, by adapting arguments from Cazenave and Weissler [6] and Kato [16].

This result improves on our previous result in [12], which was an $L_{t,x}^4$ Strichartz estimate in three dimensions. It is closely related to the results in [24], [1], [28], [21], which consider the case of asymptotically flat manifolds.

CONTENTS

1. INTRODUCTION

The purpose of this paper is to establish the full range (except for the endpoint) of (local-in-time) Strichartz inequalities on a class of non-Euclidean spaces,

namely smooth asymptotically conic Riemannian manifolds (M, g) which obey a non-trapping condition. As in our earlier paper [12], by an *asymptotically conic manifold* we mean a smooth complete noncompact Riemannian manifold $M = M^n$ of dimension $\dim M = n \geq 2$, with metric[1] $g := g_{jk}(z)dz^j dz^k$ such that there is a compact set $K_0 \subset M$ and an $(n-1)$-dimensional smooth compact Riemannian manifold $(\partial M, h)$, with metric $h := h_{jk}(y)dy^j dy^k$ such that the *scattering region* or *asymptotic region* $M \backslash K_0$ can be parametrized as the collar neighbourhood

$$M \backslash K_0 \equiv (0, \epsilon_0) \times \partial M := \{(x, y) : 0 < x < \epsilon_0, y \in \partial M\}$$

for some $\epsilon_0 > 0$, with metric which can be brought to the form

$$(1.1) \qquad g = \frac{dx^2}{x^4} + \frac{h_{jk}(x, y)dy^j dy^k}{x^2} = dr^2 + r^2 h_{jk}(\frac{1}{r}, y)dy^j dy^k.$$

Here $r := \frac{1}{x} \in (\frac{1}{\epsilon_0}, +\infty)$ is the "radial" variable, y^j are the $n-1$ "angular" variables, the "scattering co-ordinate" $x = \frac{1}{r}$ is the defining function for ∂M, and $h_{jk}(x, y)$ is a smooth function on $[0, \epsilon_0) \times \partial M$ such that $h_{jk}(0, y) = h_{jk}(y)$. If h_{ij} is independent of x (so that $h_{ij}(x, y) = h_{ij}(y)$) we say that M is *perfectly conic near infinity* and write g_{conic} instead of g. It is then natural to compactify M to $\overline{M} := M \cup \partial M$ by identifying ∂M with $\{0\} \times \partial M$ in this co-ordinate chart. In the terminology of Melrose [19], the metric g on $\overline{M}$ is thus a *short-range scattering metric*. As is customary, we say that M is *non-trapping* if every geodesic $\gamma : \mathbf{R} \to M$ reaches ∂M at $\pm\infty$.

We shall also consider long range metrics in this paper. We say that g is a *long range scattering metric* on M if it can be brought to the form (1.1) in the scattering region, but where the function $h_{jk}(x, y)$ is no longer smoothup to the boundary $x = 0$, but instead obeys the expansion

$$(1.2) \qquad h_{jk}(x, y) = h_{jk}(y) + x^\delta e_{jk}(x, y)$$

for some $0 < \delta \leq 1$, where e_{jk} are symbols of order 0 (see Definition 2.4).

Remark 1.1. By a result of Joshi and Sá Barreto, every short range scattering metric can be brought to the form (1.1). In section 10.6 we prove a generalization (closely related to results of [2], [4]): any metric of the form

$$(1.3) \qquad \frac{dx^2}{x^4} + \frac{h(y, dy)}{x^2} + x^\delta \left(k_{00} \frac{dx^2}{x^4} + k_{0j} \frac{dx}{x^2} \frac{dy^j}{x} + k_{ij} \frac{dy^i}{x} \frac{dy^j}{x} \right)$$

where the k_{ij} are symbols of order zero, can be brought to the form (1.1), with h_{jk} as in (1.2). If $M = \mathbf{R}^n$ and h is the standard metric on S^{n-1}, then (1.3) may be expressed as follows: the metric is of the form

$$g_{ij} = \delta_{ij} + g'_{ij},$$

where the g'_{ij} are symbols of order $-\delta$. Thus our class of metrics include the long range metrics considered by Burq in [1], and is also somewhat more general than the definition given in [29].

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[1] We use $z \in \mathbf{R}^n$ as the spatial co-ordinate on M, in order to reserve the letter x for the scattering co-ordinate $x := 1/r$. We neglect the case $n = 1$ as such manifolds are isometrically equivalent to a finite number of copies of the Euclidean real line $\mathbf{R}$.

It will be convenient to select a somewhat artificial, but globally defined radial positive weight $\langle z \rangle$ on M, chosen so that $\langle z \rangle$ is equal to r in the scattering region $(0, \epsilon_0) \times \partial M$, is comparable to $1/\epsilon_0$ in the compact interior region K_0, and is smooth throughout.

Example 1.2. The most important example of an asympotically conic manifold is Euclidean space $M = \mathbf{R}^n$ itself, which is perfectly conic and nontrapping (with ∂M being the unit sphere S^{n-1} with the standard metric, and (r, y) being the usual polar co-ordinates). More generally, any compact perturbation of Euclidean space is also asymptotically conic (and perfectly conic at infinity), although it may or may not be nontrapping. Any compactly supported perturbation of Euclidean space which is sufficiently small in C^2 will be nontrapping.

Let (M, g) be an asymptotically conic manifold. We let $dg(z) := \sqrt{g} \, dz$ be the measure induced by the metric g, and let $L^2(M)$ be the complex Hilbert space given by the inner product

$$\langle f_1, f_2 \rangle_{L^2(M)} := \int_M f_1(z) \overline{f_2(z)} \, dg(z).$$

Let $H = \frac{1}{2} \nabla^* \nabla$ be minus one-half the Laplace-Beltrami operator on M. As is well known, H is self-adjoint and positive-definite on $L^2(M)$ and has a functional calculus on its spectrum $[0, \infty)$, and one can define Sobolev spaces $H^s(M)$ for all $s \in \mathbf{R}$ by taking the closure of test functions under the norm $\|u\|_{H^s(M)} := \|(1 + H)^{s/2} u\|_{L^2(M)}$.

We now consider Schwartz[2] solutions $u : \mathbf{R} \times M \to \mathbf{C}$ to the (time-dependent) Schrödinger equation

$$(1.4) \qquad\qquad u_t = -iHu;$$

this is the natural quantization of the geodesic flow equation. It is well known (see e.g. [8]) that if $u(0)$ is Schwartz then there is a unique global Schwartz solution to (1.4).

For fixed n, let us call a pair of exponents (q, r) *admissible* if we have the conditions

$$(1.5) \qquad \frac{2}{q} + \frac{n}{r} = \frac{n}{2}; \quad 2 \leq q, r \leq \infty; \quad (q, r, n) \neq (2, \infty, 2).$$

In Euclidean space $(M, g) = (\mathbf{R}^n, \delta)$ we have the global-in-time *Strichartz estimate* (see e.g. [17] and the references therein)

$$(1.6) \qquad\qquad \|u\|_{L^q(\mathbf{R}; L^r(\mathbf{R}^n))} \leq C_{n,q,r} \|u(0)\|_{L^2(\mathbf{R}^n)}$$

for all admissible exponents (q, r). The admissibility conditions are best possible (see [17] for a discussion, and [20], [27] for the failure of the estimates at the endpoint $(q, r, n) = (2, \infty, 2)$). One can also generalize these estimates to other Sobolev spaces via Sobolev embedding (exploiting the fact that the Schrödinger flow commutes with fractional differentiation operators $(1 + H)^{s/2}$). The purpose of this paper is to show that these Strichartz estimates extend to more general non-trapping manifolds, at least locally in time and excluding the endpoint $q = 2$.

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[2]This is in order to avoid needless technicalities, but our estimates will not depend on any of the Schwartz semi-norms of the solution and so can be extended to rougher solutions.

Theorem 1.3 (Strichartz estimate). *For any n-dimensional asymptotically conic non-trapping manifold M, with either short range or long range metric g, and any (Schwartz) solution u to (1.4), we have*

$$(1.7) \qquad \|u\|_{L^q([0,1];L^r(M))} \leq C_{n,q,r,M} \|u(0)\|_{L^2(M)}$$

for any admissible pair (q,r) with $q > 2$.

Remark 1.4. We have recently learnt that similar estimates have been independently announced by Tataru [28] and by Robbiano-Zuily [21]. These authors treat asymptotically Euclidean, rather than asymptotically conic metrics, but in other respects their results may be more general than ours (for example, Tataru treats C^2 metrics).

Remark 1.5. There are some standard extensions of this estimate, as well as applications to low regularity local existence theory of semilinear wave equations, in the spirit of [6]; we discuss these in Section 7. The extension to long range metrics is routine but somewhat technical, requiring a re-proof of a key lemma (Lemma 2.5) which we defer to Appendix §10.

We contrast Theorem 1.3 with previously known results. In the rest of the paper we allow all our constants C to depend on n, q, r, M and will not mention this dependence explicitly again. Without the non-trapping condition, one does not expect to obtain the estimate (1.6) without conceding some derivatives, because of quasimode-type solutions to (1.4) (possibly allowing for a small forcing term on the right-hand side). However, it is still possible to obtain a (local in time) Strichartz estimate with a loss of derivatives. For instance, for admissible (q,r), Burq, Gerard and Tzvetkov [3] showed that the estimate

$$(1.8) \qquad \|u\|_{L^q([0,1];L^r(M))} \leq C \|u(0)\|_{H^{\frac{1}{q}}(M)}$$

holds for arbitrary smooth manifolds (trapping or non-trapping) in general dimension; a key tool here is the use of parametrices for the propagator e^{-itH} for frequency-localized solutions and for extremely short time intervals. This should be compared to what one can obtain purely from Sobolev embedding, which loses $2/q$ derivatives. This loss of $1/q$ derivatives, as well as the localization in time, was shown in [3] to be sharp in the case of the sphere (which is in some sense maximally trapping).

In the non-trapping case one expects better estimates, as local smoothing estimates become available. For our purposes we need a somewhat refined "half-angular local smoothing estimate" introduced in [12] (see also [26] for a Euclidean space variant). We need a rather technical Banach space X, defined formally in Section 4, but which is heuristically of the form

$$\|f\|_X \approx \|f\|_{L^2(M)} + \|\langle z \rangle^{-1/2-\varepsilon}|\nabla|^{1/2}f\|_{L^2(M)} + \|\langle z \rangle^{-1/2}|\slashed{\nabla}|^{1/2}f\|_{L^2(M)}$$

where $\slashed{\nabla} := x\nabla_y = \frac{1}{r}\nabla_y$ is the angular portion of the derivative[3].

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[3]We have to insert a smooth cutoff to the scattering region $r \gg 1$ in order to ensure that $\slashed{\nabla}$ is well-defined and has smooth coefficients, and to justify taking the square root of $|\slashed{\nabla}|$ we will need the scattering pseudo-differential calculus. We will do all this rigorously in Section 4.

Lemma 1.6 (Half-angular local smoothing effect [12]). *Let M be a smooth non-trapping asymptotically conic manifold, and let u be a Schwartz solution to (1.4). Then we have*

$$(1.9) \qquad \|u\|_{L^2([0,1];X)} := \left(\int_0^1 \|u(t)\|_X^2 \, dt \right)^{1/2} \le C \|u(0)\|_{L^2(M)}.$$

Remarks 1.7. This Lemma shall follow immediately from [12, Lemmas 10.4-10.5] once we define the space X properly in Section 4. The point is that if one wants to gain a half-derivative smoothing effect in all directions then one must lose an additional factor of $\langle z \rangle^{-\varepsilon}$, but if one only wishes for a half-derivative smoothing effect in just the *angular* directions then one does not lose this ε of decay. This is crucial in order not to lose even an epsilon of derivatives in (1.7). The well known Morawetz estimate (based on commuting the flow with the vector field ∂_r) allows one to obtain a slightly weaker "full-angular local smoothing effect" in which involves one full angular derivative and $-1/2$ of a general derivative, but this estimate turns out to be too weak for our arguments here. See [12] for some further comparisons among these various local smoothing estimates. For our argument we shall also need to prove (1.9) not just for the actual solution $u = e^{-itH}u(0)$ of the Schrödinger equation, but also for various localized parametrices to this solution, but it turns out that this will also follow by a modification of the positive commutator method used in [12].

The non-trapping hypothesis was exploited by Staffilani and Tataru [24], who were able to remove the $1/q$ loss of derivatives in (1.8) completely (i.e. to prove (1.7)) for metrics on $\mathbf{R}^n$ which are non-trapping and Euclidean outside a compact set (requiring only $C^{1,1}$ regularity of the metric); a key tool was the use of local smoothing estimates such as (1.9) to control the error term arising from localization to a compact region of space. They were also able to attain the $q = 2$ endpoint with some additional arguments. After the work of Staffilani and Tataru, Burq [1] gave an alternative proof of the same result (using the local parametrices from [3]), and also considered metrics which were asymptotically flat rather than flat outside of a compact set. In this more general setting Burq was able to recover almost the same Strichartz estimate as Staffilani and Tataru (for $q > 2$ at least), but with an epsilon loss of derivatives:

$$\|u\|_{L_t^q L_x^r([0,1] \times M)} \le C \|u(0)\|_{H^\varepsilon(M)}.$$

The idea was to divide M into dyadic shells $R \le |z| \le 2R$ and apply a variant of the Staffilani-Tataru argument on each shell, relying on the non-angular component of the local smoothing estimate (1.9) to control the error terms. The presence of the ε loss of decay in the non-angular local smoothing estimate eventually contributes to the epsilon loss of derivatives in Burq's result[4].

In [12], we were able to remove this ε loss, but only in the special context of the specific Strichartz estimate

$$\int_0^1 \int_M |u(t,z)|^4 \, dg(z) dt \le C_M \|u(0)\|_{H^{1/4}(M)}^4$$

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[4]We have learnt that this epsilon loss has very recently been removed by independent work of Tataru [28] and Robbiano-Zuily [21].

in the three-dimensional case $n = 3$. The technique here relied on an interaction Morawetz (or positive commutator) method rather than a parametrix-based approach, and as such was restricted to the $L_{t,x}^4$ set of exponents. However, the proof relied geometrically on some refined estimates on the metric function $d_M(z, z')$—in particular, that it is smooth and obeys product symbol estimates on thin cones—and we will use these estimates again to prove Theorem 1.3. Another key feature of the argument in [12] was a crucial reliance on the half-angular component of the local smoothing estimate (1.9). This estimate avoids the epsilon loss problem, but on the other hand only controls angular derivatives. This forced us in [12] to rely primarily on cutoff functions which could vary in the angular variable y but not in the radial variable x, thus localizing our solutions to thin cones rather than dyadic annuli. For similar reasons[5] we shall adopt a similar localization to cones in this paper.

Our argument here extends to more general exponents than the L^4 estimate in [12] (but not the endpoint estimates). We sketch the proof as follows. We cover the compact manifold $\overline{M}$ by a finite number of small open sets; back on the original manifold M, this corresponds to a covering by a finite number of small balls and a finite number of "thin cones." We then apply cutoff functions to localize to each of these sets, and then construct an approximate "local" parametrix for the propagator e^{-itH}. Indeed, our parametrix is rather simple, and is based on the parametrix $U(t)$ from [11], whose kernel $U(t, z, z')$ had the form

$$U(t, z, z') := (2\pi i t)^{-n/2} e^{i d_M(z, z')^2/2t} a(z, z', t)$$

when (z, z') are sufficiently close, so that there is a unique minimizing geodesic between them. For flat Euclidean space, of course, the kernel has this form globally, with $a \equiv 1$; in [11] a was smooth in t with the coefficients of its Taylor series at $t = 0$ given by a solution of certain transport equations. Here, we simply take this as an ansatz, restrict $a(z, z, t)$ to $t = 0$, and insert a cutoff function which is supported in a region, containing the diagonal, where z' and z are joined by unique minimizing geodesics. More precisely, and to fix some notation, we take our local parametrix $U(t)$ for the propagator e^{-itH} to be

$$(1.10) \qquad U(t)f(z) := \frac{1}{(2\pi i t)^{n/2}} \int_M e^{i\Phi(z, z')/t} \chi(z, z') a(z, z') f(z') \, dg(z')$$

where Φ is the phase function

$$(1.11) \qquad \Phi(z, z') := d_M(z, z')^2/2,$$

and d_M is the metric function on M, and $a(z, z') dg(z)^{1/2} dg(z')^{1/2}$ is the half-density defined (for z and z' sufficiently close) as[6]

$$(1.12) \qquad a(z, z') dg(z)^{1/2} dg(z')^{1/2} := \det(-\nabla_z \nabla_{z'} \Phi(z, z'))^{1/2},$$

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[5]Semi-classically, the reason why the localization to dyadic annuli costs an epsilon of derivatives while the localization to cones do not is that each geodesic in M can (and will) pass through an infinite number of dyadic annuli, but can only cross a finite number of cones (and cannot stay in the compact region indefinitely, by the non-trapping hypothesis).

[6]Note that $-\nabla_z \nabla_{z'}\Phi$ is most naturally a section of the tensor bundle $T_z M \otimes T_{z'} M$, and so its determinant naturally is a section of the tensor bundle $(\bigwedge^n T_z M) \otimes (\bigwedge^n T_{z'} M)$, and is thus a scalar multiple of $dg(z) dg(z')$. Thus the equation (1.12) is tensorial, or equivalently is invariant under changes of co-ordinates. Indeed, the choices of Φ and a are not arbitrary, but are dictated by the geodesic flow on phase space.

and χ is a sum of cutoff functions, each of which localizes z, z' to either a small ball or a thin cone. When z and z' are widely separated, this parametrix cannot be expected to work since the metric function can develop singularities[7]. However, when z, z' are restricted to a small ball or to a thin cone then this parametrix is reasonably accurate, essentially because Φ and a solve the eikonal and Hamilton-Jacobi equations associated to the Schrödinger flow (1.4). The parametrix here has two key advantages over that in [11]. First it is specified when both variables z and z' approach infinity (provided the angular separation between them remains small), while in [11] the first variable z was restricted to lie in a compact set in M, although it was completely global in the other variable z'. Second, it satisfies the $O(|t|^{-n/2})$ $L^1 \to L^\infty$ estimate, while the parametrix of [11] in general does not satisfy such an estimate. In fact, it follows from [11] that *the propagator itself fails to satisfy such an $L^1 \to L^\infty$ estimate* at any point (z, z') such that there is a geodesic emanating from z with a conjugate point at z', so the cutoff in (1.10) is essential for the method used here.

The fact that the parametrix obeys $L^1(M) \to L^\infty(M)$ type dispersive estimates with an $O(|t|^{-n/2})$ bound allows us[8] to apply the abstract Strichartz estimate in [17] to obtain the required estimates for each of the localized parametrices. The proof of the Strichartz estimates for the actual propagators e^{-itH} then hinges on dealing with the error terms caused by the inaccuracies of the parametrices and by the cutoff functions. For the terms localized to small balls these terms can be controlled by the standard local smoothing estimates (as in the arguments of Staffilani-Tataru [24] and Burq [1]). For the terms localized to thin cones one uses half-angular local smoothing estimates such as those in Lemma 1.6 to control the error terms; a key point here is that because the cutoff depends mainly on the angular variable y and not on the radial variable x, the terms arising from differentiating the cutoff will mainly involve *angular* derivatives and are thus compatible with the half-angular X spaces in Lemma 1.6 (a precise formulation of this is in Lemma 4.4). The non-endpoint hypothesis $q \neq 2$ is exploited (as in [24], [1]) via the Christ-Kiselev lemma [7], which allows one to decouple the Strichartz and the local smoothing aspects of our estimates.

As in [12], a key component of the argument is a detailed analysis of the metric function $d_M(z, z')$ and its derivatives for various values of z, z', including cases in which z and z' both lie in a thin cone but where $\langle z \rangle / \langle z' \rangle$ or $\langle z' \rangle / \langle z \rangle$ could be arbitrarily large. However, the geometric information we need on the metric is slightly different from that in [12]. There, the main objective was to obtain a certain geodesic (weak) convexity of the metric function $d_M(z, z')$, modulo lower order errors. Here, by constrast, it is more important to obtain non-degeneracy estimates on the various Hessians of this metric function (in order to apply stationary phase).

This paper is organized as follows. In Section 2 we state the necessary geometric estimates we need on asymptotically conic manifolds; the proof of these estimates are technical and will be deferred to Appendix §9, with one particular long-range

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[7]Microlocally, one can think of this parametrix as an approximation to the initial portion of the flow, before the geodesic flow develops bad behaviour such as caustics or conjugate points. One can obtain a more accurate parametrix for the full evolution by composing this local parametrix with itself a finite number of times using the Duhamel formula, but we will not do so in this paper.

[8]More precisely, we also need dispersive inequalities on the TT^* version of this parametrix, but this can be obtained by stationary phase arguments. Also we need to prove energy and local smoothing estimates for the parametrix.

estimate deferred to Appendix §10. Then in Section 3 we introduce the concept of a *local Schrödinger integral operator* (LSIO), which are basically Fourier integral operators associated to the Schrödinger flow, which are localized with respect to the compactified manifold $\overline{M}$ but not the original manifold M. We give the basic theory for these operators in that section, but defer the proofs (which are mostly standard but technical applications of the TT^* method and the principle of stationary phase) to Appendix §11. Then, in Section 4 we review from [12] and [19] the calculus of scattering pseudo-differential operators, which we need to rigorously define the space X alluded to earlier, and to establish local smoothing estimates for both the Schrödinger propagator and the localized parametrix (which we construct in Section 5). Finally, in Section 6 we assemble all these ingredients together, using the Duhamel formula and the Christ-Kiselev lemma (which we review in Appendix §8), to quickly prove Theorem 1.3. In Section 7 we discuss some extensions and applications of Theorem 1.3.

We thank Ben Andrews and Nicolas Burq for illuminating conversations.

2. The geometry of asymptotically conic manifolds

In this section we set out our notation and describe some of the geometry of asymptotically conic manifolds. While the statements here are intuitively plausible, some of the proofs are a little technical and will be deferred to an Appendix §9. For this discussion we shall consider both short-range and long-range metrics.

The Riemannian metric g induces a distance function d_M on M, and similarly $h(0, \cdot)$ induces a distance function $d_{\partial M}$ on ∂M. It will be convenient to also select a smooth Riemannian metric $\overline{g}$ on compactified manifold $\overline{M}$; the exact choice of this metric is not particularly important since $\overline{M}$ is compact (and hence all Riemannian metrics are equivalent), however for sake of concreteness we require $\overline{g}$ to be given in the asymptotic region $0 \leq x < \epsilon_0$ by the formula[9]

$$(2.1) \qquad \overline{g} = \frac{dx^2}{(1+x^2)^2} + \frac{h_{ij}(x,y)y^i y^j}{1+x^2}$$

(compare with (1.1)); the choice of metric for $\overline{g}$ in the near region K_0 is not relevant for our argument so long as it is smooth. The advantage of using (2.1) for the compactified metric $\overline{g}$ is that in the perfectly conic case $h(x, y) = h(0, y)$, the geodesics corresponding to g and to $\overline{g}$ (near infinity at least); see Lemma 2.1.

Let $d_{\overline{M}}$ be the distance function on $\overline{M}$ associated to $\overline{g}$. Observe that $d_{\overline{M}}(z, z')$ will be comparable to $d_M(z, z')$ when z, z' are restricted to a fixed compact set (with the comparability constants depending of course on this set), and when $z = (x, y), z' = (x', y')$ are in the asymptotic region then $d_{\overline{M}}(z, z')$ will be comparable to $|x - x'| + d_{\partial M}(y, y')$. Thus in particular, small balls in the $d_{\overline{M}}$ metric will either correspond to small balls in the near region of M, or thin cones in the asymptotic region of M. For any $\epsilon > 0$, we define the neighbourhoods of the diagonal $\Delta_\epsilon := \{(z, z') \in M^2 : d_{\overline{M}}(z, z') \leq \epsilon\}$, together with their compactifications $\overline{\Delta_\epsilon} := \{(z, z') \in \overline{M}^2 : d_{\overline{M}}(z, z') \leq \epsilon\}$.

The basic philosophy of this section is that for sufficiently small $\epsilon > 0$, the geodesic structure of the original metric d_M (as well as the structure of the closely

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[9]One can interpret this metric as the natural metric obtained after using a "radial compactification" to identify M with a "hemisphere" with boundary ∂M; see the proof of Lemma 2.1 in Appendix §9.

related functions Φ and a) on Δ_ϵ behaves very much like its Euclidean counterpart. The crucial point (which was also key in [12]) is that we will still control all the geometry in the case when the radial variables r, r' are widely separated, so long as we have the angular closeness condition $d_{\partial M}(y, y') \ll 1$; this is essential in order for us not to lose an epsilon of derivatives or decay in our final Strichartz estimates, essentially because the Schrödinger flow tends to propagate along geodesics, and (thanks to the non-trapping condition) any given geodesic eventually ends up on a thin cone where the angular variable y is in a small set, but the radial variable r goes to infinity. When $d_{\overline{M}}(z, z')$ is large, we do not attempt to control the geometry at all, and our parametrix will be highly inaccurate in this region; however we can still recover estimates for the genuine propagator e^{-itH} from that of the parametrix by means of Duhamel's formula and local smoothing estimates, which are genuinely global estimates relying on the non-trapping property of the manifold.

We treat $\epsilon > 0$ as a small parameter, and adopt the following convenient notation: we use $o(1)$ to denote any (possibly tensor-valued) quantity which is bounded in magnitude by $c(\epsilon)$ for some quantity $c(\epsilon)$ depending only on ϵ (and on the manifold M) which goes to zero as $\epsilon \to 0$. This quantity $c(\epsilon)$ will vary from line to line. More generally we abbreviate $o(1)X$ as $o(X)$ for any non-negative quantity X. In what follows we shall always assume that ϵ is sufficiently small (depending only on M). We also adopt the convention that if a statement involved a point $z \in M$ in the asymptotic region, then $r = r(z), y = y(z), x = x(z)$ are assumed to be the radial, angular, and scattering co-ordinates of z respectively; similarly for z' and r', y', x', etc.

We begin with a basic lemma on the geodesic structure of M near the diagonal.

Lemma 2.1 (Geodesic local connectedness of $\overline{M}$). *If $z, z' \in M$ is such that $d_{\overline{M}}(z, z') = o(1)$, there is a unique length-minimizing geodesic[10] $\gamma_{z \to z'} : [0, 1] \to M$ which goes from z to z', and furthermore we have $\mathrm{diam}_{\overline{M}}(\gamma_{z \to z'}) = o(1)$ (i.e. $\gamma_{z \to z'}$ lies inside a ball of radius $o(1)$ in the $d_{\overline{M}}$ metric). In particular, if z and z' are in the asymptotic region $x, x' = o(1)$, then so is the geodesic $\gamma_{z \to z'}$. Furthermore, in this case we have the additional estimate*

$$(2.2) \qquad r(\gamma_{z \to z'}(\theta)) = (1 + o(1))((1 - \theta)r + \theta r')$$

for all $0 \le \theta \le 1$, where $r(\gamma_{z \to z'}(\theta))$ is the radial co-ordinate of $\gamma_{z \to z'}(\theta)$. If $g = g_{\mathrm{conic}}$ is a perfectly conic metric in this asymptotic region, then we also have the cosine rule

$$(2.3) \qquad 2\Phi_{\mathrm{conic}}(z, z') = d_{\mathrm{conic}}(z, z')^2 = r^2 + (r')^2 - 2rr' \cos d_{\partial M}(y, y')$$

and furthermore the geodesics associated to g_{conic} and to the compactified metric $\overline{g}$ coincide in this region.

Proof. For short-range metrics, most of these claims are essentially in [12, Lemma 9.1, Proposition 9.2], but we shall give a complete proof (covering the long-range case also) in Appendix §9. □

Remark 2.2. From this lemma we can uniquely define the geodesic $\gamma_{z \to z'}$ for any pair of points z, z' which are sufficiently close in the compactified metric $d_{\overline{M}}$. One can

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[10]All our geodesics will be parameterized in the usual constant-speed (or energy-minimizing) manner, i.e. they will be projections to physical space of the geodesic flow on the cotangent bundle T^*M.

think of the map $z \mapsto \gamma'_{z_0 \to z}(0)$ as normal co-ordinates around z_0, thus Lemma 2.1 implies that these normal co-ordinates are smooth as long as $d_{\overline{M}}(z_0, z)$ is sufficiently small.

Example 2.3. In Euclidean space $M = \mathbf{R}^n$, we can define $\gamma_{z \to z'}$ for all pairs z, z', indeed we have $\gamma_{z \to z'}(\theta) = (1 - \theta)z + \theta z'$. Also we have $\Phi(z, z') = |z - z'|^2/2$ and $a(z, z') = 1$ in this case.

Now we study the phase function Φ defined in (1.11). This function is clearly real-valued and symmetric. From normal co-ordinates around z and Lemma 2.1 we see that Φ is smooth on Δ_ϵ for sufficiently small ϵ. Also, we recall *Gauss's Lemma*, which in our notation asserts that

(2.4)
$$\nabla^{z'}\Phi(z, z') := \gamma'_{z \to z'}(1) = -\gamma'_{z' \to z}(0); \quad \nabla^z\Phi(z, z') := -\gamma'_{z \to z'}(0) = \gamma'_{z' \to z}(1),$$

valid when $d_{\overline{M}}(z, z')$ is sufficiently small. Also, since geodesics have constant speed, we have

(2.5)
$$|\gamma'_{z \to z'}(\theta)|_{g(\gamma_{z \to z'}(\theta))} = d_M(z, z')$$

for all $0 \leq \theta \leq 1$ and all z, z' sufficiently close in $d_{\overline{M}}$. Combining these two equations we obtain the *eikonal equation*

(2.6)
$$\Phi(z, z') = \frac{1}{2}\big|\nabla^z\Phi(z, z')\big|^2_{g(z)} = \frac{1}{2}\big|\nabla^{z'}\Phi(z, z')\big|^2_{g(z')}$$

again valid when $d_{\overline{M}}(z, z')$ is sufficiently small. The first equation in (2.6) can be rewritten as

(2.7)
$$(it^2\partial_t - t^2 H_z)e^{i\Phi(z, z')/t} = O_{z, z'}(t)$$

where we use $O_{z, z'}(t)$ to denote a quantity which vanishes to first order at $t = 0$ for any fixed z, z', and $H_z = \frac{1}{2}\nabla^*_z\nabla_z$ is the operator H applied to the z variable. From (2.7) we naturally expect the phase $e^{i\Phi(z, z')/t}$ to arise any parametrix for the Schrödinger operators e^{-itH}, at least in the region where z, z' are geometrically close; much of this paper is devoted to the justification of such a heuristic.

Another consequence of (2.4) is the identities

$$\nabla^{z'}\Phi(\gamma_{z_0 \to z_1}(\theta), \gamma_{z_0 \to z_1}(\theta')) = (\theta' - \theta)\gamma'_{z_0 \to z_1}(\theta')$$
$$\nabla^z\Phi(\gamma_{z_0 \to z_1}(\theta), \gamma_{z_0 \to z_1}(\theta')) = -(\theta' - \theta)\gamma'_{z_0 \to z_1}(\theta)$$

for any $0 \leq \theta, \theta' \leq 1$ and z_0, z_1 sufficiently close in $d_{\overline{M}}$. In particular we have

(2.8)
$$\nabla_{z''}\big((1 - \theta)\Phi(z'', z) + \theta\Phi(z'', z')\big)\big|_{z'' = \gamma_{z \to z'}(\theta)} = 0,$$

whenever $d_{\overline{M}}(z, z')$ is sufficiently small and $0 \leq \theta \leq 1$. In fact this identity can be used to define $\gamma_{z \to z'}(\theta)$, locally at least; see (2.16) below.

Now we show that Φ is well-behaved whenever the compactified distance $d_{\overline{M}}(z, z')$ is small. First we give

Definition 2.4. We say that a function $b : M \to \mathbf{C}$ is a *symbol of order j on M* if we have the bounds

$$\big|\nabla^\alpha_y \nabla^j_r b\big| \leq C_\alpha \langle z \rangle^j$$

for all $\alpha, j \geq 0$ and $z \in M$. This is equivalent to requiring that

$$\big|(\langle z \rangle \nabla_z)^\alpha b(z)\big| \leq C_\alpha \langle z \rangle^j$$

for all $\alpha \geq 0$ and $z \in M$, where we use the metric to measure the length of the tensor $\nabla^\alpha b$.

Similarly, a smooth function $b : \Delta_{o(1)} \to \mathbf{C}$ defined on a small diagonal neighbourhood $\Delta_{o(1)} \subset M^2$ of M^2 is said to be a *local product symbol* if we have the bounds

$$|(\langle z \rangle \nabla_z)^\alpha (\langle z' \rangle \nabla_{z'})^\beta b(z, z')| \leq C_{\alpha,\beta} \langle z \rangle^j \langle z' \rangle^{j'}$$

for all $\alpha, \beta \geq 0$ and all $(z, z') \in \Delta_{o(1)}$. Thus for instance any function which extends smoothly to $\overline{M}$ is a symbol of order 0 on M, while $\langle z \rangle^j$ is a symbol of order j, and if $b_j(z)$ and $b_{j'}(z)$ are symbols of order j and j' respectively then $b_j(z)b_{j'}(z')$ is a local product symbol of order (j, j').

For a conic metric, the cosine rule (2.3) implies in particular $\Phi_{\mathrm{conic}}(z, z') - (\langle z \rangle^2 + \langle z' \rangle^2)/2$ is a product symbol of order $(1, 1)$. For asymptotically conic metrics, there is no exact cosine rule, nevertheless one can approximate Φ by Φ_{conic} up to a lower order error:

Lemma 2.5 (Phase regularity estimates [12])**.** *Let g_{conic} be the perfectly conic metric associated to g, and let Φ_{conic} be the associated phase function. Then for $d_{\overline{M}}(z, z') = o(1)$, we can write $\Phi(z, z') = \Phi_{\mathrm{conic}}(z, z') + e(z, z')$, where we have the error estimates*

$$(2.9) \qquad e(z, z') = o(\Phi(z, z'))$$

$$(2.10) \qquad \nabla_z e(z, z'), \nabla_{z'} e(z, z') = o(\langle z \rangle + \langle z' \rangle)$$

$$(2.11) \qquad \nabla_z \nabla_{z'} e(z, z') = o(1)$$

$$(2.12) \qquad \nabla_z \nabla_z e(z, z') = o(1 + \frac{\langle z' \rangle}{\langle z \rangle})$$

$$(2.13) \qquad \nabla_{z'} \nabla_{z'} e(z, z') = o(1 + \frac{\langle z \rangle}{\langle z' \rangle}).$$

Furthermore, the function $\Phi(z, z') - (\langle z \rangle^2 + \langle z' \rangle^2)/2$ is a local product symbol of order $(1, 1)$.

Proof. For short-range metrics, the claims follow from [12, Proposition 9.4] (in fact, a more precise estimates are proven there). The arguments extend to long-range metrics also but are a little more technical; we give the details in Appendix 10. $\square$

Next, we describe certain non-degeneracy and regularity estimates on the phase $\Phi(z, z')$.

Lemma 2.6 (Phase nondegeneracy estimates)**.** *If $z, z', z'' \in M$ are within $o(1)$ of each other in the $d_{\overline{M}}$ metric, then for any $0 \leq \theta \leq 1$ we have the estimates*

$$(2.14) \qquad \det(-\nabla_{z''} \nabla_z \Phi(z, z'')) = 1 + o(1)$$

$$(2.15) \qquad \frac{\left| \nabla^{z''} \big(\Phi(z'', z) - \Phi(z'', z') \big) \right|_{g(z'')}}{d_M(z, z')} = 1 + o(1)$$

$$(2.16) \qquad \frac{\left| \nabla^{z''} \big((1 - \theta)\Phi(z'', z) + \theta\Phi(z'', z') \big) \right|_{g(z'')}}{d_M(z'', \gamma_{z \to z'}(\theta))} \geq 1 - o(1).$$

We emphasize that the error terms $o(1)$ go to zero as $\epsilon \to 0$ uniformly in the choice of z, z', z'', θ.

As an easy corollary of the above estimates we can obtain some further regularity bounds on $\gamma_{z\to z'}(\theta)$:

Lemma 2.7 (Geodesic regularity estimates). *We have the estimate*

$$(2.17) \qquad C^{-1} \leq \frac{\langle \gamma_{z\to z'}(\theta)\rangle}{\theta\langle z'\rangle + (1-\theta)\langle z\rangle} \leq C$$

whenever $0 \leq \theta \leq 1$ and $d_{\overline{M}}(z, z')$ is sufficiently small. In fact in this region we have the additional symbol-type estimates

$$(2.18) \qquad |(\langle z\rangle\nabla_z)^\alpha (\langle z'\rangle\nabla_{z'})^\beta \gamma_{z\to z'}(\theta)| \leq C_{\alpha,\beta}(\theta\langle z'\rangle + (1-\theta)\langle z\rangle)$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$. Furthermore we have the estimates

$$(2.19) \qquad C^{-1}d_M(z', z'') \leq |\gamma'_{z_0\to z'}(0) - \gamma'_{z_0\to z''}(0)|_{g(z_0)} \leq Cd_M(z', z'')$$

whenever $z_0, z', z'' \in M$ are sufficiently close in the compactified metric $d_{\overline{M}}$.

The proof of these results will be deferred to Appendix §9.

Remarks 2.8. The estimate (2.14) can be interpreted as the statement as the "mixed" Hessian matrix $-\nabla_{z''}\nabla_z\Phi(z, z'')$, when measured in an orthonormal frame around $T_z M$ and $T_{z''}M$, is approxiamtely unimodular. Note that for Euclidean space, the quantities in the middle of (2.14), (2.15), and (2.16) are precisely equal to 1. These estimates are important for establishing (via stationary phase arguments) basic Fourier integral operator type estimates for oscillatory integrals whose phase is given by Φ. The estimate (2.19) asserts that normal co-ordinates are bilipschitz with respect to the metric d_M as long as one works on a small ball in $d_{\overline{M}}$.

We now turn to the amplitude $a(z, z')$, defined in (1.12). From (2.14) we see that a is well-defined and positive on Δ_ϵ for ϵ sufficiently small. We in fact have a number of additional estimates:

Lemma 2.9 (Amplitude estimates and identities). *The functions a, a^{-1} are symmetric and are local product symbols of order $(0,0)$. We also have the diagonal identity*

$$(2.20) \qquad a(z, z) = 1 \text{ for all } z \in M$$

and the transport equations

$$(2.21) \quad (\nabla^{z'}\Phi \cdot \nabla_{z'} + \frac{1}{2}\Delta_{z'}\Phi - \frac{n}{2})a(z, z') = (\nabla^z\Phi \cdot \nabla_z + \frac{1}{2}\Delta_z\Phi - \frac{n}{2})a(z, z') = 0$$

for $d_{\overline{M}}(z, z')$ sufficiently small. Here Δ_z and $\Delta_{z'}$ denotes the Laplace-Beltrami operator in the z and z' variables, and we use the metric g to raise the derivatives ∇_z, $\nabla_{z'}$ in the usual manner.

Remark 2.10. The equations (2.21) are the natural Hamilton-Jacobi equations associated to the Schrödinger flow (1.4) and the phase Φ, which justifies the use of $a(z, z')$ as an amplitude for a parametrix (1.10) for e^{-itH}; one can also use (2.20), (2.21) instead of (1.12) as one's definition for a. The choice (1.12) can also be justified on the grounds that it will make our parametrix (1.10) "locally unitary". Namely, if we regard (1.10) as a Fourier integral operator, then choosing a as in (1.12) means that the symbol of the operator is equal to 1, which will ensure that the parametrix is an approximate semigroup (see Theorem 3.9).

Proof. The claim (2.20) follows by working in normal co-ordinates around z, and observing that Φ agrees with the Euclidean counterpart $\frac{1}{2}|z - z'|^2$ in those co-ordinates up to errors of cubic or higher order. The smoothness follows from Lemma 2.5 and (2.14), since the mixed Hessian of $(\langle z \rangle^2 + \langle z' \rangle^2)/2$ vanishes. The symmetry of a is clear from definition. Now we prove (2.21). By symmetry, it suffices to establish the second of the equalities in (2.21). We rewrite this identity in terms of the half-density $a(z, z')dg(z)^{1/2} dg(z')^{1/2}$ as

$$(2.22) \qquad \mathcal{L}_{\nabla^z \Phi \cdot \nabla_z}(a(z, z')dg(z)^{1/2} dg(z')^{1/2}) - \frac{n}{2}a(z, z')dg(z)^{1/2} dg(z')^{1/2} = 0,$$

where $\mathcal{L}$ denotes the Lie derivative.

To prove (2.22), we shall work on the product manifold

$$N := \mathbf{R}_+ \times \Delta_{o(1)} = \{(t, z, z') : t > 0, d_{\overline{M}}(z, z') < o(1)\}.$$

We parameterize the cotangent bundle T^*N using "scattering co-ordinates" by $(t, z, z', \tau, \zeta, \zeta')$, where τ, ζ, ζ' are the dual variables to $-\partial_t/t^2$, ∇_z/t, and $\nabla_{z'}/t$; in other words, the canonical one-form is

$$-\tau \frac{dt}{t^2} + \zeta \cdot \frac{dz}{t} + \zeta' \cdot \frac{dz'}{t}.$$

We now view rewrite the eikonal equation (2.7) semi-classically, noting that the symbol of $it^2\partial_t - t^2 H$ is $\sigma := \tau - \frac{1}{2}|\zeta|^2_{g(z)}$, and that the graph $L \subset T^*N$ of the derivative of the phase function Φ/t is given by the $2n + 1$-dimensional manifold

$$L := \operatorname{graph} d\left(\frac{\Phi}{t}\right)$$
$$(2.23) \qquad = \{(t, z, z', \Phi(z, z'), \nabla_z \Phi(z, z'), \nabla_{z'} \Phi(z, z')) : (t, z, z') \in N\};$$

note that our use of scattering co-ordinates allows us to extend this manifold smoothly to the boundary $t = 0$. The eikonal equation (2.6) (or (2.7)) shows that the symbol σ vanishes on L. On the other hand, since L is the graph of an exact form, it is a Lagrangian submanifold of T^*N. Thus if we let X denote the Hamilton vector field associated to σ, then X preserves L. This vector field X can also be computed explicitly (cf. [11, (3.1)]) as

$$(2.24) \qquad X = t(t\partial_t + \zeta \cdot \partial_\zeta + \zeta' \cdot \partial_{\zeta'} - |\zeta|^2\partial_\tau + H_{z,\zeta})$$

where $H_{z,\zeta}$ generates geodesic flow on M. Thus $t^{-1}X$ extends smoothly (in scattering co-ordinates) to the boundary $\{t = 0\}$ of T^*N, and preserves that boundary, and thus also preserves the n-dimensional manifold $L|_{t=0}$.

We now restrict to the boundary $t = 0$ of T^*N, and consider the canonical half-density $\alpha := (dzd\zeta)^{1/2}$. Since $H_{z,\zeta}$ preserves the symplectic form $d\zeta \wedge dz$, we easily compute

$$\mathcal{L}_{t^{-1}X}(\alpha) - \frac{n}{2}\alpha = 0.$$

We then restrict this identity to $L|_{t=0}$ (which as observed before, was preserved by $t^{-1}X$) and then push forward from $L|_{t=0}$ down to $\Delta_{o(1)}$ by the projection map

$$(0, z, z', \Phi(z, z'), \nabla_z \Phi(z, z'), \nabla_{z'} \Phi(z, z')) \mapsto (z, z').$$

When one does so, the vector field $t^{-1}X$ becomes $\zeta \cdot \nabla^z = \nabla^z \Phi \cdot \nabla_z$ by (2.24), (2.23) and the half-density α_L becomes $a(z, z')dg(z)^{1/2} dg(z')^{1/2}$ by (2.23), (1.12). Thus we obtain (2.22) as desired. $\qquad \square$

3. Local Schrödinger integral operators (LSIOs)

In our arguments we shall frequently be dealing with a number of oscillatory integral operators of the form (1.10), as approximate parametrices for the Schrödinger propagators e^{-itH} and associated operators. These operators are almost of the standard oscillatory integral type, except that the support of the amplitude function is on a thin cone rather than a compact set, which unfortunately means that we cannot quite cite a reference for the (standard) results for these operators we will need here. Nevertheless, the control on the geometry of thin cones obtained in the previous section is good enough that it turns out that the basic calculus of these operators is almost identical to those of standard oscillatory integral operators on compact sets.

We now define the family of oscillatory integral operators we shall use.

Definition 3.1. A *local Schrödinger integral operator (or LSIO)* is a family of operators $S(t) = S_b(t)$ parametrized by $t \in [-1, 1]$ given by the formula

$$(3.1) \qquad S_b(t)f(z) := \frac{1}{(2\pi it)^{n/2}} \int_M e^{i\Phi(z,z')/t} a(z, z') b(z, z', t) f(z') \, dg(z')$$

when $t \neq 0$ (compare with (1.10)) and

$$(3.2) \qquad S_b(0)f(z) := b(z, z, 0)f(z)$$

when $t = 0$, where $b(\cdot, \cdot, t)$ is an $L^\infty(t)$ family of local product symbols of order $(0,0)$ (which we refer to as the *symbol of $S(t)$*). Here we take the standard branch of $z^{1/2}$ (with branch cut at the negative real axis) to define $(2\pi it)^{n/2}$.

Remarks 3.2. One can heuristically think of an LSIO $S_b(t)$ as a Fourier integral operator (FIO) of order 0, whose canonical relation is given by geodesic flow for time t; such operators can track the Schrödinger flow (and operators related to that flow) accurately as long as one does not travel too far in the compactified metric $d_{\overline{M}}$, but become useless once the distances in $d_{\overline{M}}$ become large. Unfortunately as our spatial parameters can extend to the boundary ∂M of our compactified manifold, it is not easy to use the standard theory of FIOs directly to handle LSIOs, and so we must redevelop much of this theory in our setting. From a geometrical viewpoint it turns out that it is more natural to view the amplitude function $a(z, z')$ as a half-density $a(z, z') dg(z)^{1/2} dg(z')^{1/2}$ rather than as a scalar function $a(z, z')$ (cf. (1.12)), and similarly to view $S_b(t)$ as acting on half-densities $f(z') dg(z')^{1/2}$ rather than on scalar functions $f(z')$ (which is of course extremely natural for the L^2 theory, although less so for the L^p theory); we already saw this in the proof of (2.21). Thus for instance

(3.3)
$$\int_M f dg^{1/2} S_b(t)(f' dg^{1/2}) =$$
$$\frac{1}{(2\pi it)^{n/2}} \int_{M^2} e^{i\Phi(z,z')/t} b(z, z', t) (a(z, z') dg(z)^{\frac{1}{2}} dg(z')^{\frac{1}{2}}) f(z) \, dg(z)^{\frac{1}{2}} f'(z') \, dg(z')^{\frac{1}{2}}.$$

In this formulation $S_b(t)$ becomes manifestly invariant under co-ordinate changes (with b being a scalar function). We shall thus adopt this invariant viewpoint when convenient (e.g. if we want to work in normal co-ordinates).

Remark 3.3. Naively, it would natural to approximate the Schrödinger flow e^{-itH} by the global parametrix $S_1(t)$. Unfortunately, neither the phase Φ from (1.11) nor the amplitude a from (1.12) will be smooth away from Δ_ϵ in general[11]. Thus we must apply an additional cutoff χ, which on the one hand makes our parametrix into a "local" operator, with phase supported in $\overline{\Delta_\epsilon}$ with $\epsilon \ll 1$, but which creates additional error terms that must be dealt with. It turns out that the resulting errors can be controlled by local smoothing estimates, at least for the purposes of proving non-endpoint Strichartz estimates.

Remark 3.4. While our parametrix will have a time-independent symbol, error terms will crop up involving time-dependent symbols. It is for this reason that we allow time-dependence in b. The more accurate parametrix constructed in [11], which satisfies the Schrödinger equation up to $O(t^\infty)$, also has a time-dependent symbol.

We now give the basic properties of LSIOs.

Lemma 3.5. *Let b be a local product symbol of order (0,0) and let $|t| \leq 1$. If $\tilde{b}$ is the local product symbol of order (0,0) defined by $\tilde{b}(z, z', t) := \overline{b(z', z, -t)}$ then $S_b(t)^* = S_{\tilde{b}}(-t)$. In particular if b is real and symmetric in z, z' and constant in t then $S_b(-t) = S_b(t)^*$. Also, if $b(t)$ is continuous in t at $t = 0$, we have the continuity property*

$$(3.4) \qquad \lim_{t \to 0} S_b(t)f(z) = S_b(0)f(z)$$

for any test function f, and for both the $t \to 0^+$ and $t \to 0^-$ approach, where the limit can be taken in either the pointwise or the distributional sense[12].

Proof. The first set of claims follow from (3.3) since Φ and a is real and symmetric. The claim (3.4) can be proven pointwise for a single z by working in normal co-ordinates around z and using stationary phase (see e.g. [25]), observing from Gauss's Lemma (2.4) that $\Phi(z, z')$ is only stationary at $z' = z$, and in normal co-ordinates is equal to $\frac{1}{2}|z - z'|^2$ plus terms of cubic or higher order. The distributional convergence then follows by using (2.20) and the hypothesis that f is a test function and quantitative stationary phase estimates to place uniform bounds on $S_b(t)f$; we omit the details. $\qquad \square$

The basic L^2 estimate (analogous to the Hörmander-Eskin theorem [13], [10] for FIOs) is given by the following theorem, which we prove in Appendix §11.

Theorem 3.6 (L^2 estimate for LSIOs)**.** *If $S_b(t)$ is a LSIO then we have the estimate*

$$\|S_b(t)f\|_{L^2(M)} \leq C_b \|f\|_{L^2(M)}$$

for all test functions f and all times $|t| \leq 1$. The bound C_b depends only on M and on finitely many of the constants in the product symbol bounds for the symbol b.

From (3.1) it is clear that we have the $L^1 \to L^\infty$ estimate

$$(3.5) \qquad \|S_b(t)f\|_{L^\infty} \leq C_b |t|^{-n/2} \|f\|_{L^1}$$

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[11]In the case when M has non-positive curvature everywhere, then the phase function Φ is globally smooth, and it may be possible to use $S_1(t)$ as a global parametrix without recourse to any cutoff. We have not pursued any simplifications in this case however.

[12]In fact one has convergence in much smoother topologies, such as the Schwartz topology, but we will not need to use this here.

for all test functions f and all $0 < |t| < 1$. We can generalize this estimate as follows.

Theorem 3.7 (Dispersive estimate for SFIOs). *If b, b' are local product symbols of order (0,0), then we have the estimate*

$$(3.6) \qquad \|S_b(t)^* S_{b'}(s) f\|_{L^\infty(M)} \le C_{b,b'} |t - s|^{-n/2} \|f\|_{L^1(M)}$$

for all test functions f and all times $-1 \le s, t \le 1$ with $s \ne t$. The bound $C_{b,b'}$ depends only on finitely many of the constants in the product symbol bounds for b, b'.

We prove this theorem in Appendix §11. From these two theorems and the abstract Strichartz estimate in [17, Theorem 1.2] we thus have the Strichartz estimate

$$(3.7) \qquad \|S_b(t) f\|_{L_t^q L_x^r([-1,1] \times M)} \le C_b \|f\|_{L^2(M)}$$

for all admissible exponents (q, r) and all test functions f, where for each t, $b(t)$ is a local product symbol of order (0,0), with bounds uniform in t. Indeed we even obtain the endpoint case $q = 2$ this way (as long as $n \ge 3$). Unfortunately we will not be able to exploit this endpoint as we also use the Christ-Kiselev lemma (see Lemma 8.1), which requires $q > 2$. Note that we do not require any regularity of $b(t)$ in time in order to establish (3.7), only that each $b(t)$ is a local product symbol of order (0,0) uniformly in t.

As discussed in the introduction, our parametrix (1.10) can now be interpreted as an LSIO of the form $S_{\chi a}(t)$ for some suitable cutoff χ, where a is the amplitude (1.12). The motivation for this parametrix is justified by the following identity, which shows that $S_{\chi a}(t)$ locally solves the Schrödinger equation (1.4) in the top two orders in t^{-1}:

Lemma 3.8 (LSIOs as parametrices). *Let $\psi_1, \psi_2 : M \to \mathbf{C}$ be two local symbols of order 0, and let $\chi : M^2 \to \mathbf{C}$ be a local product symbol of order (0,0) such that $\chi = 1$ on $\operatorname{supp}(\psi_1) \times \operatorname{supp}(\psi_2)$. Then for all $|t| \le 1$, $\psi_1(\frac{d}{dt} S_\chi(t) + i S_\chi(t) H)\psi_2$ is an LSIO; in fact we have the explicit identities*

$$\psi_1\Big(\frac{d}{dt} S_\chi(t) + i S_\chi(t) H\Big)\psi_2 = S_{\psi_1(z)\psi_2(z')(H_{z'}a)/a}(t)$$

$$\psi_1\Big(\frac{d}{dt} S_{\chi a}(t) + i H S_{\chi a}(t)\Big)\psi_2 = S_{\psi_1(z)\psi_2(z')(H_z a)/a}(t).$$

Proof. We prove just the former claim, as the latter is similar (and can also be deduced from the first by duality and Lemma 3.5). From (3.1) and hypothesis on χ, we have

$$\psi_1\Big(\frac{d}{dt} S_\chi(t) + i S_\chi(t) H\Big)\psi_2 f(z) = \int_M (\partial_t + i H_{z'})\Big(\frac{e^{i\Phi/t}a}{(2\pi i t)^{n/2}}\Big)\psi_1(z)\psi_2(z')f(z')\, dg(z')$$

By the Leibniz rule and chain rule, we can expand

$$(3.8) \quad \begin{aligned} (\partial_t + i H_{z'})\Big(\frac{e^{i\Phi/t}a}{(2\pi i t)^{n/2}}\Big) = \frac{e^{i\Phi/t}}{(2\pi i t)^{n/2}}\Big[& t^{-2}(-i\Phi + \frac{i}{2}|\nabla_{z'}\Phi|^2)a \\ & + t^{-1}(-\frac{n}{2} + \frac{1}{2}(\Delta_{z'}\Phi) + \nabla^{z'}\Phi \cdot \nabla_{z'})a - i H_z a \Big]. \end{aligned}$$

But the t^{-2} terms vanish thanks to the eikonal equation (2.6) (see also (2.7)), and the t^{-1} terms vanish thanks to the transport equation (2.21). $\qquad \square$

Finally, we require a composition law for LSIOs. It turns out that one has a good law as long as one composes two operators moving in the same direction in time.

Theorem 3.9 (Composition law for LSIOs). *Let $0 < t, t'$ be such that $t + t' \leq 1$, and let b, b' be local product symbols of order (0,0). Then we have*[13] *the composition law*

$$S_b(t)S_{b'}(t') = S_c(t + t') + S_{\tilde{e}_{t,t'}}(t + t')$$

where c is the local product symbol of order (0,0)

(3.9) $$c(z, z') := b(z, w, t)b'(w, z', t')$$

where $w := \gamma_{z \to z'}(\frac{t}{t+t'})$ and $\tilde{e}_{t,t'} = \frac{t+t'}{\langle z \rangle + \langle z' \rangle}e_{t,t'}$ where $e_{t,t'}$ is another local product symbol of order (0,0) (uniformly in t, t').

Remarks 3.10. Note that the point w arises naturally from geometric optics heuristics: if a particle is at z' at time zero and at z at time $t + t'$, we expect it to be at $w = \gamma_{z \to z'}(\frac{t}{t+t'}) = \gamma_{z' \to z}(\frac{t'}{t+t})$ at time t'. The presence of the amplitude factors a in (3.9) reflects the fact that the operators e^{-itH} form a semigroup, and that the operators $U(t)$ with amplitude a behave like e^{-itH}. We prove this theorem in Appendix §11. We remark that by using Lemma 3.5 we also have an analogue of this theorem for negative times t, t', but we will not need to use that analogue here. However, the lemma fails when t, t' have opposite sign as one loses control of the geometry in that case[14]. The factor $\frac{t+t'}{\langle z \rangle + \langle z' \rangle}$ is not best possible (indeed one can obtain a more accurate asymptotic expansion for the error term $\tilde{e}_{t,t'}$), but will suffice for our arguments. There are also additional regularity estimates for $\tilde{e}_{t,t'}$ in the time variables t, t', but again we will not need these estimates (because (3.7) does not require any regularity of the symbol in time).

It seems of interest to develop a more systematic calculus of "scattering Fourier integral operators" which would include these LSIOs as a special case, but can also extend to more non-local operators in which the canonical relation contains folds (and would therefore not be representable in an oscillatory integral form such as (3.1)). We will however not pursue such a theory here.

4. SCATTERING SYMBOL CALCULUS AND THE SPACE X

In this section we set up some standard notation for scattering pseudo-differential operators, as described by Melrose [19]; we will need this to properly define the X space used in Lemma 1.6, and also to prove local smoothing estimates for a certain parametrix for the Schrödinger equation which we will encounter later in this paper. Our notation is repeated from [12], and so we shall be somewhat brief in our descriptions, referring to [19] and [12] for more detail.

The first step is to choose co-ordinates for the cotangent bundle ${}^{sc}T^*\overline{M}$ in the scattering region $r > r_0$. We shall use the co-ordinates (x, y^j, ν, μ_j), where $0 <$

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[13]Strictly speaking, for this theorem to work, b and b' have to be supported on $\Delta_{o(1)/2}$ rather than $\Delta_{o(1)}$. This will however cause no difficulties as we shall only use this theorem a finite number of times and can take $o(1)$ to be as small as we wish.

[14]The point is that if a geodesic is at z at time 0 and at z' at time $t + t'$, then when t, t' have opposite sign, we do not necessarily have any good control on where the geodesic is at time t, even if $d_{\overline{M}}(z, z') = o(1)$.

$x < \epsilon_0$, and $y \in \partial M$, while the co-ordinates ν and μ_j are dual to the vector fields $-x^2 \partial_x = \partial_r$ and $x \partial_{y^j}$ respectively, and the canonical one-form is thus $-\nu \, dx/x^2 + \mu^j \, dy_j/x$. Following Melrose [19], we define the *scattering cotangent bundle* over the compact manifold $\overline{M}$ as the bundle ${}^{sc}T^*\overline{M}$ whose sections are locally spanned over $\mathcal{C}^\infty(\overline{M})$ by dx/x^2 and dy/x (hence can be paired with the *scattering vector fields*, spanned by $-x^2 \partial_x$ and $x \partial_y$). On any compact set away from the boundary ∂M, we shall parameterize the cotangent bundle ${}^{sc}T^*\overline{M}$ in the usual manner by a spatial variable z and a frequency variable ζ.

Definition 4.1 (Scattering symbol classes). Let m, l be real numbers and $0 \le \rho < 1$. A smooth function $a : {}^{sc}T^*\overline{M} \to \mathbf{C}$ is said to be in the *scattering symbol class* $S^{m,l}_{1,\rho}(\overline{M})$ provided that one has the usual Kohn-Nirenberg symbol estimates

$$|\nabla_z^\alpha \nabla_\zeta^\delta a(z, \zeta)| \le C_{\alpha,\delta,K}(1 + |\zeta|)^{m-|\delta|}$$

for z ranging in any compact subset K of M, and all $\alpha, \beta \ge 0$, as well as the scattering region estimates

$$\left| \partial_x^\alpha \nabla_y^\beta \partial_\nu^\gamma \nabla_\mu^\delta a(x, y, \nu, \mu) \right| \le C_{\alpha,\beta,\gamma,\delta} \left(1 + |\nu| + |\mu| \right)^{m-\gamma-|\delta|} x^{l-\alpha-\rho(\gamma+|\delta|)}$$

in the scattering region $(x, y) \in (0, \epsilon_0) \times \partial M$. Every symbol a in $S^{m,l}_{1,\rho}(\overline{M})$ can be quantized to give a *scattering pseudo-differential operator* $\mathrm{Op}(a)$ in the class $\Psi^{m,l;\rho}_{sc}(M)$; the exact means of quantization is not particularly important for our purposes, but we can for instance use the Kohn-Nirenberg quantization (on any n-dimensional asymptotically conic manifold)

$$(4.1) \qquad \mathrm{Op}(a)u(z') := (2\pi)^{-n} \int_{T^*M} \chi(z', z'') e^{-i\langle \gamma'_{z' \to z''}(0), \zeta \rangle} a(z', \zeta) u(z'') \, d\zeta dz'',$$

to define the kernel of $\mathrm{Op}(a)$ near ∂M; here, χ is a cutoff to Δ_ε for some suitably small $\varepsilon > 0$, and $d\zeta dz''$ is the canonical Liouville measure on the cotangent bundle T^*M.

We now briefly review the scattering calculus for these symbols (see [12] and especially [19] for more details). If A is an operator in $\Psi^{m,l;\rho}_{sc}(\overline{M})$, then its (principal) symbol $\sigma(A)$ is well defined in $S^{m,l}_{1,\rho}(\overline{M})/S^{m-1,l+1-\rho}_{1,\rho}(\overline{M})$, and its adjoint A^* (with respect to the $L^2(M)$ inner product) is also in this class with symbol

$$(4.2) \qquad\qquad\qquad \sigma(A^*) = \overline{\sigma(A)}$$

If $A \in \Psi^{m,l;\rho}_{sc}(\overline{M})$ and $B \in \Psi^{m',l';\rho}_{sc}(\overline{M})$ then

$$(4.3) \qquad AB \in \Psi^{m+m',l+l';\rho}_{sc}(\overline{M}); \quad i[A, B] \in \Psi^{m+m'-1,l+l'+1-\rho;\rho}_{sc}(\overline{M}).$$

Moreover

$$(4.4) \qquad \begin{aligned} \sigma(AB) &= \sigma(A)\sigma(B) \\ \sigma(i[A,B]) &= \{\sigma(A), \sigma(B)\} \end{aligned}$$

where $\{,\}$ is the Poisson bracket on the scattering cotangent bundle ${}^{sc}T^*\overline{M}$. In particular $\{\sigma(H), \cdot\}$ is the Hamilton vector field in the direction of geodesic flow.

We introduce the weighted Sobolev spaces $H^{m,l}(M)$ as

$$H^{m,l}(M) := \{u : \langle z \rangle^l u \in H^m(M)\}.$$

From (4.3), (4.4) and L^2-boundedness of $\Psi^{0,0;\rho}_{sc}(\overline{M})$ it is easy to verify that operators in $\Psi^{m,l;\rho}_{sc}(\overline{M})$ map $H^{m',l'}(M)$ to $H^{m'-m,l'+l}(M)$.

We now define a half-angular operator B.

Definition 4.2 (Half-angular derivative). Let $\phi : \mathbf{R} \to \mathbf{R}$ be a smooth non-decreasing function such that $\phi = 0$ on $(-\infty, 1]$ and $\phi = 1$ on $[2, +\infty)$. For any $\rho \in (0, 1)$, we let $B \in \Psi_{\mathrm{sc}}^{1/2, 1/2; \rho}(\overline{M})$ be a quantization of the $S_{1,\rho}^{1/2, 1/2}(\overline{M})$ symbol

$$(4.5) \qquad \sigma(B) := \frac{1}{2} \phi(\frac{\epsilon_0}{x}) \phi(|\mu|^2 + |\nu|^2) \phi(\frac{|\mu|}{|\nu| x^\rho}) x^{1/2} |\mu|^{1/2}$$

where $|\mu|^2 = |\mu|^2_{h(x)} := h^{jk}(x, y) \mu_j \mu_k$. By adding a lower order term to B if necessary we may assume that B is self-adjoint. We then define the X norm on test functions f as

$$(4.6) \qquad \|f\|_X := \|f\|_{L^2(M)} + \|f\|_{H^{1/2, -1/2 - \rho/2}(M)} + \|Bf\|_{L^2(M)}.$$

Remark 4.3. Ignoring all the cutoffs, the symbol $\sigma(B)$ is essentially $\frac{1}{2} x^{1/2} |\mu|^{1/2}$; thus B should be thought of heuristically as $B \approx \langle z \rangle^{-1/2} |\overline{\nabla}|^{1/2}$. The cutoffs are designed to keep B close to ∂_M, away from the frequency origin $(\mu, \nu) = 0$, and away from the radial directions $\mu = 0$, in order to avoid a number of singularities in the symbol.

The local smoothing estimate in Lemma 1.6 then follows directly from [12, Lemmas 10.4-10.5]. Later on we will also need to modify that argument so that it gives a local smoothing estimate for our parametrix.

A key fact about the X norm is its ability to absorb "half" of a commutator $i[H, \psi]$; the main point is that that commutator consists primarily of an angular derivative, and each X norm can absorb roughly half of such an angular derivative. A more precise statement is the following.

Lemma 4.4. *Let $\psi : \overline{M} \to \mathbf{R}$ be a smooth function. Then we have the estimate*

$$|\langle f, i[H, \psi] g \rangle_{L^2(M)}| \le C_\psi \|f\|_X \|g\|_X$$

for all test functions f, g.

Proof. If f or g is supported on the near region $r \le 2r_0$ then the claim follows simply by observing from (4.6) that the X norm controls the $H^{1/2}(M)$ norm in this region, and that $i[H, \psi]$ is a first-order operator in this region with bounded coefficients and thus maps $H^{1/2}(M)$ to $H^{-1/2}(M)$. Now suppose that f, g are supported in the scattering region $r > r_0$. Ignoring the zeroth order component of $i[H, \psi]$ (which can be dealt with by the L^2 component of the X norm in (4.6)), we see from the Liebniz rule that we can write $i[H, \psi] = a x^4 D_x + b \cdot x^2 D_y$ for some symbols $a(z)$, $b(z)$ of order 0; the point is that the angular derivatives of ψ decay like $O(\langle z \rangle^{-1})$ but the radial derivatives of ψ will decay like $O(\langle z \rangle^{-2})$ since ψ is smooth on the boundary $\overline{M}$. Using the scattering calculus (4.4) and the fact that μx is bounded by a multiple of $\nu x^{1+\rho}$ on $\mathrm{supp}(1 - \phi(\frac{|\mu|}{|\nu| x^\rho}))$, we can thus write

$$i[H, \psi] \in B^* \Psi_{\mathrm{sc}}^{0,0;\rho}(M) B + \Psi_{\mathrm{sc}}^{1,1+\rho;\rho}(M).$$

Since elements of $\Psi_{\mathrm{sc}}^{1,1+\rho;\rho}(M)$ map $H^{1/2, -1/2 - \rho/2}(M)$ to $H^{-1/2, 1/2 + \rho/2}(M)$ and those of $\Psi_{\mathrm{sc}}^{0,0;\rho}(M)$ map $L^2(M)$ to $L^2(M)$, the corresponding terms are controlled by the last two terms in (4.6). $\qquad \square$

Remark 4.5. The commutators $i[H, \psi]$ arise naturally when trying to approximate the flow e^{-itH} by certain approximate flows localized to the support of ψ—see §6. Lemma 4.4, combined with the local smoothing estimate in Lemma 1.6 (and its analogue for the parametrix in Proposition 5.3) will be essential in allowing us to pass from Strichartz control of the localized flows to Strichartz control on the global flow.

5. CONSTRUCTION OF THE PARAMETRIX

In this section we define precisely our parametrix for the Schrödinger propagator e^{-itH}, and prove a smoothing estimate for the parametrix which is the exact analogue of Lemma 1.6. We start by constructing a family of partitions of unity and associated cutoff functions on M.

Lemma 5.1. *Let $\epsilon > 0$ be a small number. Then there exist local symbols $\psi_\alpha^{(j)}$: $M \to \mathbf{R}$ for $j = 0, 1, 2, 3, 4$ of order 0, each supported on a ball of radius ϵ in the $d_{\overline{M}}$ metric, which enjoy the partition of unity property*

$$(5.1) \qquad\qquad 1 = \sum_{\alpha \in A} \psi_\alpha^{(0)},$$

the nesting property

$$(5.2) \qquad \psi_\alpha^{(j)} \psi_\alpha^{(j')} = \psi_\alpha^{(j)} \text{ for all } 0 \le j < j' \le 4, \ \alpha \in A,$$

the square root property

$$(5.3) \qquad \psi_\alpha^{(j)}, \ |\nabla \psi_\alpha^{(j)}|_{g(z)} \text{ are squares of smooth functions}$$

and the extension property

$$(5.4) \quad \nabla \psi_\alpha^{(j)} / |\nabla \psi_\alpha^{(j)}|_{g(z)} \text{ extends to a smooth vector field on } \overline{M}$$

$$\text{in a neighbourhood of supp } \nabla \psi_\alpha^{(j)}, \ j = 0 \dots 3.$$

Also, we can strengthen the nesting property (5.2) in two different ways:

    (i) *For any $0 \le j \le 3$, $\alpha \in A$ and any two points $z, z' \in \mathrm{supp}(\psi_\alpha^{(j)})$, the geodesic $\gamma_{z \to z'}([0, 1])$ is contained in the region where $\psi_\alpha^{(j+1)} = 1$.*

    (ii) *For any $0 \le j \le 2$, $\alpha \in A$ there exists $c > 0$ such that, if $z' \in \mathrm{supp}\, \psi_\alpha^{(j)}$ and $z \in \mathrm{supp}\, \nabla \psi_\alpha^{(j+1)}$, then*

$$(5.5) \qquad \nabla_z \psi_\alpha^{(j+1)} \cdot \gamma'_{z \to z'}(0) \ge c |\gamma'_{z \to z'}(0)|_{g(z)} |\nabla_z \psi_\alpha^{(j+1)}|_{g(z)}.$$

Proof. Let z_α be an arbitrary point in M, and let $\psi_\alpha^{(4)}$ be a local symbol of order 0 which equals 1 on a small ball $B_{\overline{M}}(z_\alpha, r_4)$ in the compactified metric $d_{\overline{M}}$ centered at z_α. By Lemma 2.1 we can then find an even smaller ball $B_{\overline{M}}(z_\alpha, r_3)$ centered at z_α with the property that any minimal length geodesic connecting two points in that neighbourhood lies in the region where $\psi_\alpha^{(4)} = 1$. We now claim that there is a local symbol $\psi_\alpha^{(3)}$ of order 0 supported in $B_{\overline{M}}(z_\alpha, r_3)$ which equals one on a smaller ball centered at z_α, and with the property that $\psi_\alpha^{(3)}$ is monotone decreasing on any outward ray from z_α. This is clear for $z_\alpha \in K_0$, since we can take $\psi_\alpha^{(3)}$ to be a function of the distance $d_M(z_\alpha, \cdot)$, in which case (5.5) holds with $c = 1$. Moreover, by continuity we can find a smaller ball $B_{\overline{M}}(z_\alpha, r_2)$ such that (5.5) holds, with $c = 1/2$, say, for all $z' \in B_{\overline{M}}(z_\alpha, r_2)$ and $z' \in \mathrm{supp}\, \nabla \psi_\alpha^{(3)}$. Now consider z_α in the

scattering region $M \setminus K_0$. Suppose first that M is perfectly conic. Then we use the fact from Lemma 2.1 that the geodesic curves given by g_{conic} are also the geodesic curves given by the compactified metric $\overline{g}$. Then we can take $\psi_\alpha^{(3)} = h(d_{\overline{M}}(z_\alpha, z))$ where h is a smooth non-increasing function with $h(s) = 1$ for s small and $h(s) = 0$ for $s \geq s_0 > 0$; this is non-increasing along every geodesic emanating from $z' = z_\alpha$, and hence, by continuity, non-increasing along every geodesic emanating from z' in a ball $B_{\overline{M}}(z_\alpha, r_2)$, where r_2 may be taken uniform over the scattering region. Therefore (5.5) holds, with the positivity of c following from compactness. In the general case, the metric g is very close to the perfectly conic metric g_{conic} for x sufficiently small,

$$\left(1 - \frac{\delta}{2}\right) g_{\mathrm{conic}} \leq g \leq \left(1 + \frac{\delta}{2}\right) g_{\mathrm{conic}} \text{ for } x \text{ small,}$$

and $\gamma'_{z \to z'}(0)$ is very close to the conic $\gamma'_{z \to z'}(0)$ then (this follows from (2.4) and the estimate (2.10)). Hence (5.5) holds also for g (with constant $c/2$, say) for x sufficiently small. This concludes the proof of (5.5). The extension property (5.4) holds because $\psi_\alpha^{(3)}$ is a smooth function of distance in the compactified metric, hence (5.4) is equal to $\nabla d_{\overline{M}}(z_\alpha, \cdot) / |\nabla d_{\overline{M}}(z_\alpha, \cdot)|$ which is smooth in a neighbourhood of the support of $\nabla \psi_\alpha^{(3)}$. The square root property (5.3) holds for a suitable choice of function h.

Now $\psi_\alpha^{(3)}$ is also a local symbol of order 0 which equals 1 on a small ball in the compactified metric $d_{\overline{M}}$ centered at z_α. We repeat the construction three more times, constructing local symbols $\psi_\alpha^{(j)}$ which equal 1 on a small ball $B_{\overline{M}}(z_\alpha, r_j)$ in the compactified metric $d_{\overline{M}}$ centered at z_α, $j = 2, 1, 0$. Note that none of these radii r_j of balls used in this construction need to depend on z_α.

As z_α ranges over M, the balls $B_{\overline{M}}(z_\alpha, r_0)$ cover $\overline{M}$. By compactness and a standard partition of unity construction we can thus find a finite index set A and a local symbol $\psi_\alpha^{(0)}$ of order 0 for each $\alpha \in A$ supported on $B_{\overline{M}}(z_\alpha, r_0)$ obeying (5.1). The claims of the lemma are then easily verified. $\square$

We now formally define the local parametrices that we shall use:

Definition 5.2. For $\alpha \in A$, let $U_\alpha(t)$ be the LSIO with symbol $b_\alpha(z, z') := \psi_\alpha^{(2)}(z) \psi_\alpha^{(2)}(z')$; this is a product symbol of order $(0,0)$ by Lemma 2.9.

From Lemma 3.5 we have the symmetry condition

$$(5.6) \qquad\qquad U_\alpha^*(t) = U_\alpha(-t)$$

and the condition

$$(5.7) \qquad\qquad \text{w-}\lim_{t \to 0} U_\alpha(t) = U_\alpha(0) = \psi^{(2)}(z)^2,$$

where the limit is meant in the distribution sense and the right hand side is regarded as a multiplication operator.

Recall from Section 3 that $U_\alpha(t)$ is bounded on $L^2(M)$ uniformly in time (Theorem 3.6), satisfies a composition law (Theorem 3.9), and an $L^1 \to L^\infty$ estimate (Theorem 3.7), and thus also obeys a Strichartz estimate (3.7). We shall now round out this collection of estimates by also demonstrating that $U_\alpha(t)$ satisfies a smoothing estimate which is the analogue of Lemma 1.6.

Proposition 5.3 (Half-angular local smoothing effect for parametrix)**.** *The parametrix* $U_\alpha(t)$ *satisfies the smoothing estimate*

$$\|U_\alpha(t)\tilde{f}\|_{L^2([-1,0];X)} \le C\|\tilde{f}\|_{L^2(M)}$$

(cf. Lemma 1.6).

Proof. The $L^2(M)$ portion of (4.6) of this estimate follows from Theorem 3.6, so it will suffice to prove the estimates

(5.8) $\|U_\alpha(t)\tilde{f}\|_{L^2_t H^{1/2,-1/2-\rho/2}([-1,0]\times M)} + \|BU_\alpha(t)\tilde{f}\|_{L^2_t L^2([-1,0]\times M)} \le C\|\tilde{f}\|_{L^2(M)}.$

For technical reasons it will be convenient to replace U_α by the slightly larger operator $\psi_\alpha^{(3)}\tilde{U}_\alpha$, where $\tilde{U}_\alpha(t)$ is defined the same as $U_\alpha(t)$ but with amplitude $\psi_\alpha^{(4)}(z)a(z,z')\psi_\alpha^{(2)}(z')$ instead of $\psi_\alpha^{(2)}(z)a(z,z')\psi_\alpha^{(2)}(z')$. This replacement is justified since the space $H^{1/2,-1/2-\rho/2}$ is stable under multiplication by cutoff functions such as $\psi_\alpha^{(j)}$ (which are special cases of operators in $\Psi_{sc}^{0,0;\rho}(\overline{M})$), and because the commutator of B with $\psi_\alpha^{(3)}$ is of zeroth order and hence gives an acceptable contribution to (5.8) by Theorem 3.6.

We therefore aim to prove

(5.9)
$$\|\psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}\|_{L^2_t H^{1/2,-1/2-\rho/2}([-1,0]\times M)} + \|B\psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}\|_{L^2_t L^2([-1,0]\times M)} \le C\|\tilde{f}\|_{L^2(M)}.$$

We use the positive commutator method. Let $A \in \Psi_{sc}^{0,0;\rho}(\overline{M})$ be a self-adjoint scattering pseudo-differential operator to be chosen later; the constants below are allowed to depend on A. Since A is bounded on $L^2(M)$, we see from Theorem 3.6 that

$$\left|\langle A\psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}, \psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}\rangle\right| \le C\|\tilde{f}\|_{L^2(M)}^2$$

for all $-1 \le t \le 1$. Similarly, from Lemma 3.8 we see that $\psi_\alpha^{(3)}(\partial_t + iH)\tilde{U}_\alpha(t)$ is an LSIO, and hence by Lemma 3.6 again

$$\left|\langle A\psi_\alpha^{(3)}(\partial_t + iH)\tilde{U}_\alpha(t)\tilde{f}, \psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}\rangle\right| \le C\|\tilde{f}\|_{L^2(M)}^2$$

Thus from the identity

$$\frac{d}{dt}\langle A\psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}, \psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}\rangle = \text{Re}\langle i[H,A]\psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}, \psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}\rangle$$
$$+ 2\text{Re}\langle Ai[H,\psi_\alpha^{(3)}]\tilde{U}_\alpha(t)\tilde{f}, \psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}\rangle$$
$$+ 2\text{Re}\langle A\psi_\alpha^{(3)}(\partial_t + iH)\tilde{U}_\alpha(t)\tilde{f}, \psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}\rangle$$

and the fundamental theorem of calculus we see that

(5.10) $\left| \int_{-1}^0 \text{Re}\langle i[H,A]\psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}, \psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}\rangle \right.$
$$\left. + 2\text{Re}\langle Ai[H,\psi_\alpha^{(3)}]\tilde{U}_\alpha(t)\tilde{f}, \psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}\rangle \, dt \right| \le C\|\tilde{f}\|_{L^2(M)}^2.$$

To exploit this we will choose A so that both operators on the left-hand side are mostly positive. Note that in the second term, semiclassical intuition says that the operator $\tilde{U}_\alpha(t)^* i[H,\psi_\alpha^{(3)}]\tilde{U}_\alpha(t)$ is mostly *negative*, since the derivative of $\psi_\alpha^{(3)}$ along

geodesics starting on the support of $\psi_\alpha^{(2)}$ is negative thanks to (5.5)[15]. Thus we should choose A so that A has a negative symbol and $i[H, A]$ has a positive symbol. We now interrupt the proof of Proposition 5.3 to give two lemmas formalizing this intuition.

Lemma 5.4 (Positive commutator inequality for parametrix)**.** *Suppose that* $A \in \Psi_{sc}^{0,0;\rho}(\overline{M})$ *and* $\sigma(A)$ *is strictly negative. Then we have*

$$(5.11) \qquad Re\langle Ai[H, \psi_\alpha^{(3)}]\tilde{U}_\alpha(t)\tilde{f}, \psi_\alpha^{(3)}\tilde{U}_\alpha(t)\tilde{f}\rangle \geq -C_A\|\tilde{f}\|_{L^2(M)}^2,$$

uniformly for $-1 \leq t < 0$ *and all test functions* $\tilde{f}$.

Proof. We begin by noting that if $A \in \Psi_{sc}^{-\infty,0}(M)$, then $Ai[H, \psi^{(3)}]$ is zeroth order, hence bounded on $L^2(M)$, and the result follows from Theorem 3.6. Thus we may without loss of generality take the kernel of A to have support in Δ_ϵ with any desired $\epsilon > 0$.

We now write the identity operator as the sum of two (approximate) pseudodifferential projections. Recall from (5.4) that the vector field $\widehat{\nabla}\psi_\alpha^{(3)} = \nabla\psi_\alpha^{(3)}/|\nabla\psi_\alpha^{(3)}|$ extends to a smooth vector field (which we shall also denote $\widehat{\nabla}\psi_\alpha^{(3)}$) in a neighbourhood N_1 of the support of $\nabla\psi_\alpha^{(3)}$ (in the topology of $\overline{M}$). Let N_2 be an open set in $\overline{M}$ with $N_1 \cup N_2 = \overline{M}$ and such that the closure of N_2 is disjoint from $\operatorname{supp}\nabla\psi_\alpha^{(3)}$. Writing $\hat{\zeta} = \zeta/|\zeta|$, we note that the sets

$$(5.12) \qquad \{(z, \hat{\zeta}) \mid z \in N_1, \ \hat{\zeta} \cdot \widehat{\nabla}\psi_\alpha^{(3)}(z) < -\frac{c}{2}\} \cup \{(z, \hat{\zeta}) \mid z \in N_2\}$$

and

$$(5.13) \qquad \{(z, \hat{\zeta}) \mid z \in N_1, \ \hat{\zeta} \cdot \widehat{\nabla}\psi_\alpha^{(3)}(z) > -\frac{3c}{4}\}$$

(where c is as in (5.5)) are open sets whose union is the entire bundle of unit vectors in the scattering cotangent bundle over $\overline{M}$. Thus by a partition of unity relative to these open sets one can construct an operator $P = P^\alpha \in \Psi_{sc}^{0,0}(\overline{M})$ such that $\operatorname{WF}'(P^*P)$, the essential support of the symbol of P^*P, is contained in (5.12), while the essential support of the symbol of $\operatorname{Id} - P^*P$ is contained in (5.13). Recall now that by (5.5), we have[16]

$$(5.14) \qquad \begin{aligned} &\zeta = \nabla_z d(z, z')^2, \ (z, z') \in \operatorname{supp}\nabla\psi_\alpha^{(3)} \times \operatorname{supp}\psi_\alpha^{(2)} \\ &\implies \sigma(i[H, \psi_\alpha^{(3)}])(z, \zeta) \equiv \zeta \cdot \nabla\psi_\alpha^{(3)}(z) \leq -c|\zeta||\nabla\psi_\alpha^{(3)}|. \end{aligned}$$

Indeed this implies that $\widehat{\nabla}\psi_\alpha^{(3)}(z) \cdot \zeta \leq -c|\zeta|$ where $\zeta = \nabla_z d(z, z')^2$, z' in the support of $\psi_\alpha^{(2)}$, z in the support of $\nabla\psi_\alpha^{(3)}$, which by continuity means that $\widehat{\nabla}\psi_\alpha^{(3)}(z) \cdot \zeta < -3c|\zeta|/4$ for z on all of N_1, provided N_1 is chosen small enough. It follows that
$$(5.15)$$
$$\operatorname{WF}'(\operatorname{Id} - P^*P) \cap \{(z, \zeta) \mid \zeta = \nabla_z d(z, z')^2, (z, z') \in \operatorname{supp}\nabla\psi_\alpha^{(3)} \times \operatorname{supp}\psi_\alpha^{(2)}\} = \emptyset;$$

we may further assume that P^*P and $\operatorname{Id} - P^*P$ have kernels supported in Δ_ϵ for any desired $\epsilon > 0$.

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[15]Note that in (5.5), the vector field $\gamma'_{z \to z'}(0)$ points from z in $\operatorname{supp}\nabla\psi_\alpha^{(j+1)}$ to z' in $\operatorname{supp}\psi_\alpha^{(j)}$ which accounts for the sign reversal here.

[16]Here, as in the previous footnote, there is a sign reversal because ζ is the negative of $\gamma'_{z \to z'}(0)$ from (5.5).

Now we write

$$(5.16) \quad \langle Ai[H, \psi_\alpha^{(3)}] \tilde{U}_\alpha(t) \tilde{f}, \tilde{U}_\alpha(t) \tilde{f} \rangle = \langle \psi_\alpha^{(3)} Ai[H, \psi_\alpha^{(3)}] P^* P \tilde{U}_\alpha(t) \tilde{f}, \tilde{U}_\alpha(t) \tilde{f} \rangle$$
$$+ \langle Ai[H, \psi_\alpha^{(3)}] (\mathrm{Id} - P^* P) \tilde{U}_\alpha(t) \tilde{f}, \tilde{U}_\alpha(t) \tilde{f} \rangle.$$

We claim that the first of the operators on the right hand side, $\psi_\alpha^{(3)} Ai[H, \psi_\alpha^{(3)}] P^* P$, has symbol which is the square of a symbol of order $(1/2, 1/2)$. We may first factor out the quantity $\psi_\alpha^{(3)} |\nabla \psi_\alpha^{(3)}|$, which is a square of a symbol by (5.3). Next, $-A$ has a strictly positive symbol by assumption, so this is a square of a symbol. Finally, consider the symbol of $-i[H, \psi_\alpha^{(3)}] |\nabla \psi_\alpha^{(3)}|^{-1}$. We only have to consider $(z, \hat{\zeta})$ in (5.12) which is a neighbourhood of $\mathrm{WF}'(P^* P)$, and with z in some neighbourhood N_1' of $\mathrm{supp} \nabla \psi_\alpha^{(3)}$ disjoint from N_2. By (5.12), its symbol is at least as big as $c/2$ there, so this is also the square of a symbol on a neighbourhood of the essential support of $\psi_\alpha^{(3)} Ai[H, \psi_\alpha^{(3)}] P^* P$. Thus, up to an operator of order $(0, 1)$ this operator may be written in the form $G^* G$, where G is an order $(1/2, 1/2)$ pseudodifferential operator. The contribution of $G^* G$ to (5.11) is non-negative and can thus be discarded, while the order $(0, 1)$ term is L^2 bounded and hence is acceptable.

We now consider the $\mathrm{Id} - P^* P$ term. Heuristically, the contribution of this term should be extremely small because $i[H, \psi_\alpha^{(3)}] \tilde{U}_\alpha(t)$ should be microlocally supported in 'outgoing' directions (starting from the support of $\psi_\alpha^{(2)}$ and emanating towards the support of $\nabla \psi_\alpha^{(3)}$), whereas by (5.15) $\mathrm{Id} - P^* P$ is microlocally supported away from the outgoing directions. To make this intuition precise we use stationary phase arguments[17]. We begin by observing that we may replace $Ai[H, \psi_\alpha^{(3)}]$ by $i[H, \psi_\alpha^{(3)}] A$, as the difference is in $\Psi_{sc}^{0,2}(M)$, hence its composition with $\tilde{U}_\alpha(t)$ is L^2-bounded by Theorem 3.6. Now we may suppose that the symbol q of $A(\mathrm{Id} - P^* P)$ is written in right-reduced form. Then the kernel of $i[H, \psi_\alpha^{(3)}] A(\mathrm{Id} - P^* P) \tilde{U}_\alpha(t)$ is
$$(5.17)$$
$$t^{-n/2} \int -i\chi(z, z'') (\nabla \psi_\alpha^{(3)}(z) \cdot \zeta) e^{i(z - z'') \cdot \zeta} q(z'', \zeta) e^{i\Phi(z'', z')/t} a(z'', z') \psi_\alpha^{(2)}(z') \, d\zeta \, dg_{z''};$$

here we have explicitly inserted a cutoff χ, supported in $\Delta_{2\epsilon}$, as permitted by our support assumptions on the kernels of A and $\mathrm{Id} - P^* P$. By construction, $q(z'', \zeta)$ vanishes when $\zeta = \nabla_z \Phi(z'', z')/t$, $(z'', z') \in \mathrm{supp} \nabla \psi_\alpha^{(3)} \times \mathrm{supp} \psi_\alpha^{(2)}$. Thus we may integrate by parts in z''. By Lemma 5.1, the angle between ζ and $\nabla_{z''} \Phi(z'', z')/t$ is bounded away from zero, when $(z'', z') \in \mathrm{supp} \nabla \psi_\alpha^{(3)} \times \mathrm{supp} \psi_\alpha^{(2)}$ and (z'', ζ) is in the cone support of q. Therefore, there is an $\epsilon' > 0$ such that

$$(5.18) \quad |t\zeta - \nabla_{z''} \Phi(z'', z')| \geq \epsilon' \Big(t|\zeta| + |\nabla_{z''} \Phi(z'', z')| \Big) = \epsilon' \Big(t|\zeta| + d_M(z'', z') \Big) \geq C,$$

when z'', z', ζ are as above, since $d_{\overline{M}}(z'', z')$ (and hence $d_M(z'', z')$) is bounded away from zero. Let

$$v_j(z'', z', \zeta, t) = \frac{t\zeta_j - \nabla_{z_j''} \Phi(z'', z')}{\left| t\zeta - \nabla_{z''} \Phi(z'', z') \right|^2_{g(z'')}}.$$

---

[17] A slightly different alternative to the arguments given here would be to dyadically decompose the region of integration and apply rescaled versions of the localization principle (see e.g. [25]) to each dyadic component.

Then for all k, the operator $(itv^j \nabla_{z_j''})^k$ leaves $e^{i(z-z'')\cdot\zeta}e^{i\Phi(z'',z')/t}$ invariant. Integrating by parts gives the kernel

$$(5.19) \quad t^{-n/2} \int -i\chi(z,z'')(\nabla\psi_\alpha^{(3)}(z)\cdot\zeta)e^{i(z-z'')\cdot\zeta}e^{i\Phi(z'',z')/t}$$
$$\times (-it\nabla_{z_j''}v^j)^k \Big(q(z'',\zeta)a(z'',z')\Big)\psi_\alpha^{(0)}(z')\,d\zeta\,dg_{z''}.$$

Note that for (z',z'') in the support of the symbol, we have $d_{\overline{M}}(z',z'') > \delta > 0$ for some δ, hence

$$d(z',z'') \geq C\min(\langle z'\rangle, \langle z''\rangle)$$

for some C. Now this implies that

$$\left|tv^j\right|^{-1} \leq C\min(|\zeta|^{-1}, t(\langle z'\rangle + \langle z''\rangle)^{-1}),$$

hence, for any N, taking k sufficiently large (and using symbol estimates on q and a) we may estimate those terms in (5.19) in which no derivative hits a v^j by a multiple of

$$t^N \int \chi(z,z'')\Big(\langle z'\rangle + \langle z''\rangle\Big)^{-N} dg_{z''};$$

the ζ integral has disappeared as we have taken k sufficiently large that the integrand is L^1 in ζ with bounds in z', z'' as above. This term is L^2-bounded, uniformly in t, as it is Hilbert-Schmidt for N large.

To estimate terms in which derivatives fall upon v_i's, note that by Lemma 2.5, we may estimate, for $k \geq 1$,

$$|\nabla_{z''}^k(t\zeta - \Phi(z'',z'))| \leq C((r'')^{2-k} + r'(r'')^{1-k});$$

on the support of a, this is bounded by $C((r')^{1-k} + (r'')^{1-k})$. This allows us to conclude that

$$\left|(t\nabla_{z''})^k v_j\right| \leq C\min(|\zeta|^{-k}, t^k(r' + r'')^{-k}),$$

and the estimates on terms involving derivatives follow as above. $\qquad\square$

Lemma 5.5 (Construction of commutant). *There exists a scattering pseudo-differential operator $A \in \Psi_{sc}^{0,0;\rho}(\overline{M})$ such that $\sigma(A) < 0$ and*

$$(5.20) \qquad i[H,A] \geq (1+H)^{1/4}\langle z\rangle^{-1+\rho}(1+H)^{1/4} + B^*B - O(\Psi_{sc}^{0,0;\rho}(\overline{M})),$$

where we use $A \geq B$ to denote the assertion that $A - B$ is positive semi-definite and $O(\Psi_{sc}^{0,0;\rho}(\overline{M}))$ denotes some term in $\Psi_{sc}^{0,0;\rho}(\overline{M})$.

Proof. This argument is essentially contained in [12, Lemmas 10.4, 10.5], and we give only a sketch here. We first observe that the condition $\sigma(A) < 0$ is automatic, since if $\sigma(A)$ is not negative we may simply subtract a large constant from A (taking advantage of the symbol bounds $A \in \Psi_{sc}^{0,0;\rho}(\overline{M})$); note that this does not affect the commutator $i[H,A]$). Thus we can ignore the constraint $\sigma(A) < 0$.

We still have to find an operator $A \in \Psi_{sc}^{0,0;\rho}(\overline{M})$ which obeys the inequality (5.20). In practice we accomplish this in several separate pieces, which we will add together at the end.

To start with, we exploit the non-trapping hypothesis on the manifold M in the usual manner (see [8], [9]) to construct, for any given compactly supported bump function φ, an operator $A_0 \in \Psi_{sc}^{-1,0;\rho}(\overline{M})$ such that $i[H,A_0] \geq \varphi^2 + O(\Psi_{sc}^{-1,0;\rho}(\overline{M}))$. Multiplying this on the left and right by $(1+H)^{1/4}$ (which is self-adjoint and

commutes with H) we obtain a symbol $(1+H)^{1/4}A_0(1+H)^{1/4} \in \Psi_{sc}^{0,0;\rho}(\overline{M})$ such that

$$i[H,(1+H)^{1/4}A_0(1+H)^{1/4}] \geq (1+H)^{1/4}\varphi^2(1+H)^{1/4} + O(\Psi_{sc}^{0,0;\rho}(\overline{M})).$$

This gives a commutant with the desired properties over any compact subset of $\overline{M}$ (note that B is of order $1/2$). Thus it will suffice to obtain the inequality (5.20) in the asymptotic region $\langle z \rangle \gg 1$, since the claim will then follow by adding a sufficiently large multiple of $(1+H)^{1/4}A_0(1+H)^{1/4}$ to A.

We now test the radial vector field $A_1 := \eta^2 \partial_r \in \Psi_{sc}^{1,0;\rho}(\overline{M})$, where r is the radial co-ordinate in the scattering region and η is a smooth cutoff to the asymptotic region $\langle z \rangle \gg 1$. A computation (see [12, Lemma 10.4]) shows that

$$i[H,A_1] \geq c(\langle z \rangle^{-1/2} \slashed{\nabla}_j \eta)^* h^{jk} (\langle z \rangle^{-1/2} \slashed{\nabla}_k \eta) - \varphi O(\Psi_{sc}^{2,0;\rho}(\overline{M}))\varphi - O(\Psi_{sc}^{1,0;\rho}(\overline{M}))$$

for some compactly supported bump function φ and some $c > 0$. A similar computation (again see [12, Lemma 10.4]) shows that if we let $A_2 := -\eta^2 \langle z \rangle^{-2\varepsilon} \partial_r \in \Psi_{sc}^{1,0;\rho}(\overline{M})$ for some small $\varepsilon > 0$ then

$$i[H,A_2] \geq C_\varepsilon (\langle z \rangle^{-1/2-\varepsilon}\nabla)^* (\langle z \rangle^{-1/2-\varepsilon}\nabla) - (x^{1/2}\slashed{\nabla}_j \eta)^* O(\Psi_{sc}^{0,0;\rho}(\overline{M}))(x^{1/2}\slashed{\nabla}_k \eta)$$
$$- \varphi O(\Psi_{sc}^{2,0;\rho}(\overline{M}))\varphi - O(\Psi_{sc}^{1,0;\rho}(\overline{M})$$

for some $C_\rho > 0$. Adding a suitable combination of the two, we can find $A_3 \in \Psi_{sc}^{1,0;\rho}(\overline{M})$ such that

$$i[H,A_3] \geq (\langle z \rangle^{-1/2-\varepsilon}\nabla)^* (\langle z \rangle^{-1/2-\varepsilon}\nabla) - \varphi O(\Psi_{sc}^{2,0;\rho}(\overline{M}))\varphi - O(\Psi_{sc}^{1,0;\rho}(\overline{M})).$$

Multiplying on left and right by $(1+H)^{-1/4}$, and then adding a large multiple of $(1+H)^{1/4}A_0(1+H)^{1/4}$, we can find $A_4 \in \Psi_{sc}^{0,0;\rho}(\overline{M})$ such that

$$i[H,A_4] \geq (1+H)^{1/4}\langle z \rangle^{-1-2\varepsilon}(1+H)^{1/4} - O(\Psi_{sc}^{0,0;\rho}(\overline{M}))$$

which gives part of (5.20) (with $\varepsilon := \rho/2$). Finally, from [12, Lemma 10.5] we can obtain $A_5 \in \Psi_{sc}^{0,0;\rho}(\overline{M})$ such that

$$i[H,A_5] \geq B^*B - O(\Psi_{sc}^{1,1+2\varepsilon;\rho}(\overline{M})) - O(\Psi_{sc}^{0,0;\rho}(\overline{M})))$$

for some $\varepsilon > 0$. By adding a large multiple of A_4 to A_5 we can eliminate the $O(\Psi_{sc}^{1,1+2\varepsilon;\rho}(\overline{M}))$ error and obtain (5.20) as desired. $\square$

Proof of Proposition 5.3 (completion). We choose A as in Lemma 5.5 and use this in (5.10). Using (5.20) and (5.11), we see that (5.9) holds, which completes the proof. $\square$

6. Proof of Theorem 1.3

We can now deduce Theorem 1.3 from the machinery developed in the previous sections (the proofs of some of which are deferred to Appendices). We fix M, q, r, u, and allow all constants C to depend on M, n, q, r. Write $f := u(0)$, thus $u := e^{-itH}f$.

It suffices by (5.1) and the triangle inequality to establish the estimate

$$\|\psi_\alpha^{(0)}u(t)\|_{L^q([0,1];L^r(M))} \leq C_{n,q,r,M}\|f\|_{L^2(M)}$$

for each $\alpha \in A$. By (5.7), (1.4) and the fundamental theorem of calculus we have the Duhamel identity

$$\psi_\alpha^{(0)} u(t) = U_\alpha(t) \psi_\alpha^{(0)} u(0) + \int_0^t \frac{d}{ds}(U_\alpha(t-s)\psi_\alpha^{(0)} u(s)) \ ds$$

$$= U_\alpha(t) \psi_\alpha^{(0)} f + \int_0^t -U_\alpha'(t-s)\psi_\alpha^{(0)} u(s) - U_\alpha(t-s)\psi_\alpha^{(0)} iHu(s) \ ds$$

$$= U_\alpha(t) \psi_\alpha^{(0)} f + \int_0^t -(U_\alpha' + iU_\alpha H)(t-s)\psi_\alpha^{(0)} u(s) + U_\alpha(t-s)[iH, \psi_\alpha^{(0)}]u(s) \ ds.$$

We can multiply both sides by $\psi_\alpha^{(1)}$ and use (5.2) to obtain

$$\psi_\alpha^{(0)} u(t) = \psi_\alpha^{(1)} U_\alpha(t) \psi_\alpha^{(0)} f +$$

$$\int_0^t -\psi_\alpha^{(1)}(U_\alpha' + iU_\alpha H)(t-s)\psi_\alpha^{(0)} u(s) + \psi_\alpha^{(1)} U_\alpha(t-s)[iH, \psi_\alpha^{(0)}]u(s) \ ds.$$

Now we take $L_t^q L_x^r$ norms of both sides and use the triangle inequality. The first term will be acceptable from (3.7). From Lemma 3.8 we see that $\psi_\alpha^{(1)}(U_\alpha' + iU_\alpha H)(t-s)\psi_\alpha^{(0)}$ is an LSIO, and so the second term will also be acceptable from (3.7) and Minkowski's inequality (noting that $u(s)$ has L^2 bounded by a multiple of that of $u(0) = f$). It thus will suffice to show that

$$\| \int_0^t \psi_\alpha^{(1)} U_\alpha(t-s)[iH, \psi_\alpha^{(0)}]u(s) \ ds\|_{L_t^q L_x^r([0,1]\times M)} \leq C\|f\|_{L^2(M)}.$$

From Lemma 1.6 it will thus suffice to show that

$$\| \int_0^t \psi_\alpha^{(1)} U_\alpha(t-s)[iH, \psi_\alpha^{(0)}]v(s) \ ds\|_{L_t^q L_x^r([0,1]\times M)} \leq C\|v\|_{L_t^2([0,1];X)}$$

for all Schwartz functions v. But by the Christ-Kiselev lemma (see Lemma 8.1 in Appendix §8) and the non-endpoint hypothesis $q > 2$, it will suffice to show that

$$(6.1) \qquad \| \int_0^{t_0} \psi_\alpha^{(1)} U_\alpha(t-s) i[H, \psi_\alpha^{(0)}]v(s) \ ds\|_{L^q([t_0,1];L^r(M))} \leq C\|v\|_{L_t^2([0,1];X)}$$

for all $0 \leq t_0 \leq 1$.

Fix t_0. We shall take advantage of the approximate semigroup properties of U_α to approximately factorize the expression in (6.1), thus decoupling the hybrid Strichartz/local smoothing estimate (6.1) into a Strichartz estimate component for the parametrix (which we can deal with by (3.7)) and a local smoothing component for the parametrix (which we can deal with by Lemma 5.4). More precisely, we decompose

$$U_\alpha(t-s) = U_\alpha(t-t_0)U_\alpha(t_0-s) + (U_\alpha(t-s) - U_\alpha(t-t_0)U_\alpha(t_0-s)).$$

Let us first consider the contribution of the error term $U_\alpha(t-s) - U_\alpha(t-t_0)U_\alpha(t_0-s)$. From the definition of U_α and Theorem 3.9 we see that this error term is an LSIO of the form $S_c(t-s)$, where $c = c_{t-t_0, t_0-s}$ is defined by

$$c(z, z') := \psi_\alpha^{(2)}(z)\psi_\alpha^{(2)}(z')(1 - \psi_\alpha^{(2)}(w)^2) - \frac{t-s}{\langle z \rangle + \langle z' \rangle} e(z, z') =: c_1 + c_2$$

for some product symbol $e = e_{t-t_0, t_0-s}$ of order $(0,0)$, and $w := \gamma_{z \to z'}(\frac{t-t_0}{t-s})$. From Lemma 5.1 we know that $\psi_\alpha^{(1)} c_1 i[H, \psi_\alpha^{(0)}] = 0$. By (3.1) we thus have

$$\psi_\alpha^{(1)} S_c(t-s) i[H, \psi_\alpha^{(0)}] f(z)$$
$$:= \frac{1}{(2\pi i(t-s))^{n/2}} \int_M e^{i\frac{\Phi(z,z')}{t-s}} \frac{t-s}{\langle z \rangle + \langle z' \rangle} \tilde{e}(z,z') i[H, \psi_\alpha^{(0)}] f(z') \, dg(z').$$

If we now integrate the first-order operator $i[H, \psi_\alpha^{(0)}]$ by parts, we thus see that the operator $\psi_\alpha^{(1)} S_c(t-s) i[H, \psi_\alpha^{(0)}] = S_d(t-s)$ for some product symbol $d = d_{t-s}$ of order $(0,0)$ (the damping factor $\frac{t}{\langle z \rangle + \langle z' \rangle}$ can absorb any term that arises when the z' derivative hits $\Phi(z,z')/t$, thanks for instance to Lemma 2.5). Furthermore the symbol bounds of d are uniform in s, t in the range $0 \le s \le t \le 1$. In particular, we see from (3.7) that for any $0 < s < 1$ we have

$$\|\psi_\alpha^{(1)} S_{c_{t-t_0, t_0-s}}(t-s) i[H, \psi_\alpha^{(0)}] v(s)\|_{L_t^q L_x^r([0,1] \times M)} \le C \|v(s)\|_{L^2(M)}$$

The contribution of the error term to (6.1) is thus acceptable from Minkowski's inequality since the X norm controls the $L^2(M)$ norm by (4.6).

It remains to consider the contribution of the main term $U_\alpha(t-t_0) U_\alpha(t_0-s)$ to (6.1). We can factorize this as

$$\|\psi_\alpha^{(1)} U_\alpha(t-t_0) \int_0^{t_0} U_\alpha(t_0-s) i[H, \psi_\alpha^{(0)}] v(s) \, ds\|_{L^q([t_0,1]; L^r(M))} \le C \|v\|_{L_t^2([0,1]; X)}$$

for all $0 \le t_0 \le 1$. From (3.7) it will suffice to show that

$$\| \int_0^{t_0} U_\alpha(t_0-s) i[H, \psi_\alpha^{(0)}] v(s) \, ds\|_{L^2(M)} \le C \|v\|_{L_t^2([0,1]; X)}$$

We test this inequality against an arbitrary test function $\tilde{f}$, and see that it suffices by duality and (5.6) to show that

$$\left| \int_0^{t_0} \langle U_\alpha(s-t_0) \tilde{f}, i[H, \psi_\alpha^{(0)}] v(s) \rangle \, ds \right| \le C \|v\|_{L_t^2([0,1]; X)} \|\tilde{f}\|_{L^2(M)}.$$

This follows from Proposition 5.3 and Lemma 4.4. This completes the proof of Theorem 1.3.   □

7. Remarks

We comment briefly on some standard and automatic extensions of the estimate in Theorem 1.3. The time interval $[0,1]$ can easily be replaced with any other finite interval by time translation invariance and concatenating several time intervals together (and exploiting the fact that free propagators e^{-itH} preserve the $L^2(M)$ norm), although at present we are unable to prevent the constants in the estimates from growing with the length of the interval and so cannot prove a global-in-time Strichartz estimate (in analogy with (1.6)) though we conjecture that such estimates are true. Also one can add a bounded lower order potential term V to H without affecting this estimate by using Duhamel's formula and exploiting the time localization. Also one can extend the Strichartz estimate to allow for the presence of a forcing term by use of TT^* arguments and the Christ-Kiselev lemma (see e.g. [27]). The condition that u and F are Schwartz can be removed by the usual limiting arguments. Again, we can use Sobolev embedding and commute the equation with fractional powers of H to extend these results to other Sobolev spaces.

Somewhat trickier would be to obtain the endpoint $q = 2$ (in dimensions $n \geq 3$, of course). The Christ-Kiselev lemma (Lemma 8.1) is no longer applicable, so one would have to prove the hybrid local smoothing/Strichartz estimate (6.1) by other means. We will pursue this matter elsewhere.

As is well known, Strichartz estimates can be used to obtain local well-posedness results for semilinear Schrödinger equations such as

$$u_t = -iHu + \lambda|u|^{p-1}u; \quad u(0) = f$$

where $p > 1$ and $\lambda = \pm 1$ is fixed. Indeed, the Strichartz estimates given above imply that this equation is locally well-posed in $H^s(M)$ whenever $s \geq \max(0, \frac{n}{2} - \frac{2}{p-1})$, and (in order to ensure $|u|^{p-1}u$ is sufficiently regular) the exponent p is either odd, or greater than $\lfloor s \rfloor + 1$; this can be achieved by applying the arguments of Cazenave and Weissler [6] with Euclidean space replaced by the manifold M. (For p not an odd integer, one needs to develop a fractional Liebniz rule for Sobolev spaces on manifolds, but this can be done by working in local co-ordinates, in which the Sobolev norms are comparable to their Euclidean counterparts). We omit the details. For similar reasons one can adapt the arguments in [16] to obtain global well-posedness for this equation in the energy norm $\|u(t)\|_{H^1}$ when $p < 1 + 4/(n-2)$ (in the defocusing case $\lambda = -1$) or when $p < 1 + 4/n$ (in the focusing case $\lambda = +1$), and to obtain the respective endpoint values of p when the H^1 norm is sufficiently small. Note that none of these results require the endpoint Strichartz estimate (which was only obtained later, in [17]). On the other hand, these Strichartz estimates are not strong enough to obtain a scattering theory for any of these equations (as is done for instance in [5]), because they are only local in time. To obtain global-in-time Strichartz estimates would require one to extend local smoothing estimates such as those in Lemma 1.6 to also be global in time, and would also require one to treat lower order terms in the above arguments more carefully.

8. Appendix: Christ-Kiselev lemma

In this section we prove a minor variant of the Christ-Kiselev lemma [7].

Lemma 8.1 (Christ-Kiselev Lemma [7]). *Let X, Y be Banach spaces, and for all $s, t \in \mathbf{R}$ let $K(s,t) : X \to Y$ be an operator-valued kernel from X to Y. Suppose we have the estimate*

$$(8.1) \qquad \| \int_{s<t_0} K(s,t)f(s) \, ds \|_{L^q([t_0,\infty);Y)} \leq A\|f\|_{L^p(\mathbf{R};X)}$$

for some $A > 0$ and $1 \leq p < q \leq \infty$, and all $t_0 \in \mathbf{R}$ and $f \in L^p((-\infty, t_0); X)$ and $t_0 \in \mathbf{R}$. Then we have

$$\| \int_{s<t} K(s,t)f(s) \, ds \|_{L^q(\mathbf{R};Y)} \leq C_{p,q}A\|f\|_{L^p(\mathbf{R};X)}.$$

Remarks 8.2. Note that the hypothesis (8.1) is in particular satisfied if we have the global estimate

$$(8.2) \qquad \| \int_{\mathbf{R}} K(s,t)f(s) \, ds \|_{L^q(\mathbf{R};Y)} \leq A\|f\|_{L^p(\mathbf{R};X)},$$

since one can simply restrict f to $(-\infty, t_0)$ and restrict the quantity on the left-hand side to the region $t \geq t_0$. For sake of completeness we shall prove the above

variant of the Christ-Kiselev lemma in Appendix §8. This Lemma is very convenient in the non-endpoint setting for eliminating the causality constraint $s < t$ in the operators that one wishes to bound, and has been utilized in previous work on the Schrödinger equation, see [27], [24], [1]. Clearly we may replace the time domain **R** by any smaller interval such as $[0, 1]$ and obtain a similar result in which all time variables s, t, t_0 are restricted to this interval. The estimate unfortunately fails at the endpoint $p = q$; see [27] for an instance of this failure in a context of obtaining Strichartz estimates for the Schrödinger equation.

Proof of Lemma 8.1. This is a minor modification of the proof of [27], Lemma 3.1 or [23], Lemma 3.1, but for sake of completeness we give the full proof (as our hypotheses are slightly different). By a limiting argument we may assume that the cumulative distribution function $F(t_0) := \int_{-\infty}^{t_0} \|f(t)\|_X^p \, dt$ is continuous and strictly increasing in t_0. We may also normalize $\|f\|_{L^p(\mathbf{R};X)} = 1$, hence F is a bijection from **R** to $[0, 1]$.

We impose the usual dyadic grid on $[0, 1]$. Given two dyadic intervals[18] I, J strictly contained in $[0, 1]$, we define the relation $I \sim J$ if I and J have the same length, I is to the left of J, and that I and J are not adjacent but have adjacent parents; let us write S_n for the set of pairs (I, J) with $|I| = |J| = 2^{-n}$ and $I \sim J$. Also, for almost every $0 < x < y < 1$ there is exactly one pair of intervals I, J such that $x \in I$, $y \in J$, and $I \sim J$. Applying F^{-1} to this fact, we thus have (cf. [27], [23])

$$\int_{s<t} K(s,t)f(s) \, ds = \sum_{I,J:I\sim J} 1_{F^{-1}(J)} \int_{F^{-1}(I)} K(s,t)f(s) \, ds.$$

We group the intervals based on their dyadic length, take $L^q(\mathbf{R};Y)$ norms, and apply the triangle inequality to obtain

$$\|\int_{s<t} K(s,t)f(s) \, ds\|_{L^q(\mathbf{R};Y)} \leq \sum_{n=1}^{\infty} \| \sum_{(I,J)\in S_n} 1_{F^{-1}(J)} \int_{F^{-1}(I)} K(s,t)f(s) \, ds\|_{L^q(\mathbf{R};Y)}.$$

Observe that for fixed n and J, there are at most two intervals I for which $I \sim J$. Thus the inner summands have finitely overlapping supports, and we can estimate

$$\|\int_{s<t} K(s,t)f(s) \, ds\|_{L^q(\mathbf{R};Y)} \leq C_q \sum_{n=1}^{\infty} \Big(\sum_{(I,J)\in S_n} \big\|1_{F^{-1}(J)} \int_{F^{-1}(I)} K(s,t)f(s) \, ds\big\|_{L^q(\mathbf{R};Y)}^q \Big)^{1/q}.$$

Applying the hypothesis (8.1) (with t_0 chosen to be some intermediate time between $F^{-1}(I)$ and $F^{-1}(J)$) we obtain

$$\|\int_{s<t} K(s,t)f(s) \, ds\|_{L^q(\mathbf{R};Y)} \leq C_q A \sum_{n=1}^{\infty} \Big(\sum_{(I,J)\in S_n} \|f(s)\|_{L^p(F^{-1}(J);X)}^q \Big)^{1/q}.$$

From the definition of F we have $\|f(s)\|_{L^p(F^{-1}(J);X)} = |J|^{1/p} = 2^{-n/p}$, thus we have

$$\|\int_{s<t} K(s,t)f(s) \, ds\|_{L^q(\mathbf{R};Y)} \leq CA \sum_{n=1}^{\infty} \Big(\sum_{(I,J)\in S_n} 2^{-nq/p} \Big)^{1/q} = C_q A \sum_{n=1}^{\infty} (2^n 2^{-nq/p})^{1/q}$$

which is less than infinity, so the claim follows.   □

---

[18]We will ignore sets of measure zero and so we will not care whether these intervals are open or closed.

9. Appendix: Phase nondegeneracy estimates

In this section we provide the proof of Lemmas 2.1, 2.6 and 2.7.

9.1. **Proof of Lemma 2.1.** We begin with the proof of Lemma 2.1. If we restrict attention to any fixed compact subset K_0 of M, then the claim about the existence of a unique geodesic $\gamma_{z \to z'}$ follows from local Riemannian geometry by taking ϵ (and hence all the $o(1)$ factors) sufficiently small. Thus by letting K_0 grow very slowly in ϵ, it suffices to work in the asymptotic region $\{x \le o(1)\}$. Now let us consider what happens in this region in the perfectly conic case $g = g_{\mathrm{conic}}$.

Consider any metric of the form $f(r)dr^2 + g(r)h(y, dy)$ on M, where h is a metric on a cross-section manifold N. Consider a geodesic arc $\theta \mapsto \gamma(\theta)$ in the cross-section (N, h). It is not difficult to check that the two-dimensional cone $\mathcal{C}_\gamma = \{(r, \gamma(\theta))\} \subset M$ over γ is totally geodesic in M. Let us write $d\theta^2$ for the metric on the geodesic γ. For the conic metric, $dr^2 + r^2 h$, the induced metric on $\mathcal{C}_\gamma$ is exactly equal to the Euclidean metric $dr^2 + r^2 d\theta^2$ on the plane $\mathbf{R}^2$ in polar co-ordinates. Thus the geometry of geodesics on the cone $\mathcal{C}_\gamma$ is identical to that on the Euclidean plane $\mathbf{R}^2$. Conversely, every geodesic λ in M is contained in a cone of the form $\mathcal{C}_\gamma$, since one can easily locate such a cone for which λ is tangent to at least second order, and then the claim follows from uniqueness of the second-order geodesic flow ODE. Note that we can now conclude uniqueness of geodesics and the cosine rule (2.3) for perfectly conic manifolds, since this rule is known to already hold in the Euclidean plane $\mathbf{R}^2$. For similar reasons we also obtain (2.2) in the perfectly conic case.

Similarly, the cone $\mathcal{C}_\gamma$ is totally geodesic for the incomplete metric $\overline{g}$, and on $\mathcal{C}_\gamma$ the metric induced by $\overline{g}$ takes the form

$$\overline{g} = \frac{dx^2}{(1+x^2)^2} + \frac{d\theta^2}{1+x^2} = \frac{dr^2}{(1+r^2)^2} + \frac{r^2}{1+r^2}d\theta^2.$$

Following [19], we let S_+^2 denote the upper hemisphere of the standard unit sphere in $\mathbf{R}^3$ and RC $: \mathbf{R}^2 \to S_+^2$ denote the "radial compactification" (or "inverse gnomonic projection")

$$z \mapsto \Big(\frac{z}{\sqrt{1+|z|^2}}, \frac{1}{\sqrt{1+|z|^2}}\Big)$$

(see figure). Letting $g_{S_+^2}$ denote the standard metric on the hemisphere, we then compute

$$\mathrm{RC}^*(g_{S_+^2}) = \overline{g}$$

The map RC identifies lines on the plane with great semi-circles on the hemisphere, and thus maps geodesics to geodesics. From this we conclude that g_{conic} and $\overline{g}$ do indeed give the same geodesics as claimed.

It remains to prove the claims of Lemma in the asymptotically conic case in the asymptotic region where $x, x' \le o(1)$ and $d_{\overline{M}}(z, z') = o(1)$. We begin with the claim that the geodesic $\gamma_{z \to z'}$ is unique and has diameter $o(1)$ in the $d_{\overline{M}}$ metric. We repeat the arguments from [12, Lemma 9.1]. We observe the crude estimate

$$(9.1) \qquad\qquad d_M(z, z') = |r - r'| + o(\min(r, r'))$$

for (z, z') in the above region; the lower bound follows from the trivial comparison $g \ge dr^2$, and the upper bound follows (when $r \le r'$, say) by considering the
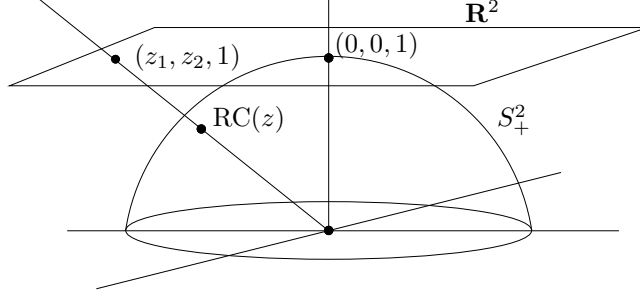
FIGURE 1. The radial compactification map

path from z' to z formed by taking the radial ray from $z' = (r', y')$ to (r, y') and then the angular arc connecting (r, y') to (r, y). This inequality, combined with the triangle inequality, already shows that any length-minimizing geodesic in M connecting z to z' must stay within the asymptotic region $\{x \leq o(1)\}$, as a path which strays too close to the compact region K_0 will necessarily have longer length than $|r - r'| + o(\min(r, r'))$. In this asymptotic region we have the comparision $g \geq \tilde{g}_{\text{conic}}$, where $\tilde{g}_{\text{conic}}$ is the same perfectly conic metric as g_{conic} but with h replaced by $(1 - o(1))h$. Thus $d_M(z, z') \geq \tilde{d}_{\text{conic}}(z, z')$ in this region. From this we also see that the d_M-length-minimizing geodesic must have diameter at most $o(1)$ in the $d_{\overline{M}}$ metric, since again a path which wanders further than $o(1)$ away from z and z' in this metric will require too much length in $\tilde{d}_{\text{conic}}$ (as can be seen from (9.1) and (2.3), and a quantitative version of the triangle inequality in the Euclidean plane $\mathbf{R}^2$; see [12, Lemma 9.1]) and hence in d_M.

To prove the uniqueness of length-minimizing geodesics, we must show that given two points z, z' whose distance is $o(1)$ in the compactified metric $d_{\overline{M}}$, there is only one geodesic between them which remains at distance $o(1)$ from z. For short-range metrics this is part of [12, Proposition 9.2]; for long-range metrics, we prove this in Corollary 10.5.

Now we prove (2.2) for asymptotically conic metrics. For this we use the comparison $g = (1 + o(1))g_{\text{conic}}$ in the asymptotic region to obtain the estimate[19]

$$(9.2) \qquad\qquad d_M(z, z') = (1 + o(1))d_{\text{conic}}(z, z').$$

Now set $Z'' := \gamma_{z \to z'}(\theta)$, and $Z''_{\text{conic}} := \gamma_{z \to z', \text{conic}}(\theta)$, where $\gamma_{z \to z', \text{conic}}$ is the geodesic from z to z' in the perfectly conic metric g_{conic}. From (9.2) we see that

$$d_{\text{conic}}(z, Z'') = (1+o(1))\theta d_{\text{conic}}(z, z''); \quad d_{\text{conic}}(z', Z'') = (1+o(1))(1-\theta)d_{\text{conic}}(z, z''),$$

while of course

$$d_{\text{conic}}(z, Z''_{\text{conic}}) = \theta d_{\text{conic}}(z, z''); d_{\text{conic}}(z', Z''_{\text{conic}}) = (1 - \theta)d_{\text{conic}}(z, z'').$$

From the triangle inequality we then see that

$$d_{\text{conic}}(Z'', Z''_{\text{conic}}) = o(1)\min(\theta, 1 - \theta)d_{\text{conic}}(z, z'') = o(\langle Z''_{\text{conic}}\rangle)$$

(by (2.2) in the perfectly conic case), and we can now extend (2.2) to the asymptotically conic case by the triangle inequality. $\qquad\square$

---

[19]This is still quite crude. For a more precise estimate in the short-range case, see [12, Proposition 9.4].

9.2. **Proof of Lemma 2.6.** Now we prove Lemma 2.6. We begin by assuming that z, z', z'' all lie in a fixed compact set $K_0 \subset M$. Then the estimates (2.14), (2.15) are immediate from normal coordinates around z'' (with constants depending on K_0 of course). Indeed, for z, z' close enough to z'', $z, z', z'' \in K_0$, the function $\Phi(z, z'')$ agrees with the Euclidean function $|z - z''|^2/2$ up to terms of third order in $z - z''$; similarly if we replace z by z'. Also in these normal co-ordinates $|z - z'|$ is comparable to $d_M(z, z')$. Since the claims (2.14), (2.15) are easily verified in the Euclidean case, the claim follows if z, z', z'' are close enough. The estimate (2.16) is proven similarly, but using Fermi normal co-ordinates around the geodesic $\gamma_{z \to z'}$ instead of normal co-ordinates around z''; we omit the details.

In light of the above statements, we see that in order to prove Lemma 2.6, we can now fix attention to the asymptotic region $x = o(1)$ (by choosing K_0 to be growing slowly as $\epsilon \to 0$. As in the proof of Lemma 2.1, we begin this task by considering the case of perfectly conic metrics $g = g_{\mathrm{conic}}$; the general case turns out to be treatable by an easy perturbation argument (using Lemma 2.5) which we give at the end of this argument.

First we prove (2.14) for perfectly conic metrics. From (2.3) (and Gauss's Lemma for ∂M) we see that the matrix $-\nabla_{z''} \nabla_z \Phi(z, z'')$ of partial derivatives in (2.14) (with respect to a standard orthonormal frame associated to the co-ordinates x, y) takes the form

(9.3)

$$
-\nabla_{z''} \nabla_z \Phi(z, z'') = \begin{pmatrix} \cos d_{\partial M}(y, y'') & \cdots & \sin d_{\partial M}(y, y'') \hat{\mu} & \cdots \\ \vdots & & & \\ \sin d_{\partial M}(y, y'') \hat{\mu}'' & & \nabla_y \nabla_{y''} \cos d_{\partial M}(y, y'') & \\ \vdots & & & \end{pmatrix}
$$

where $\hat{\mu}$ is the unit vector at y'' parallel to the geodesic from y'' to y, and similarly for $\hat{\mu}'$. The $(n-1) \times (n-1)$ lower right submatrix is close to the identity since $\cos d_{\partial M} = 1 - d_{\partial M}^2/2 + O(d_{\partial M}^4)$, and $d_{\partial M} = d_{\partial M}(y, y'')$ is assumed to be small, namely $o(1)$. Thus this matrix is the identity plus $o(1)$, and its determinant is $1 + o(1)$, and hence we have (2.14) as desired.

Next, we prove (2.15) for perfectly conic metrics. We first do this assuming that z and z' are relatively close in the uncompactified metric d_M, say $d_M(z, z') < o(\langle z \rangle)$ where the $o()$ decay is as slow as we please. In this case, we consider a curve $s \mapsto \gamma(s)$ between $z = (x, y)$ and $z' = (x', y')$ to be chosen shortly, and compute

$$
(9.4) \quad \nabla_{z''}\big(\Phi_{\mathrm{conic}}(z'', z) - \Phi_{\mathrm{conic}}(z'', z')\big) = \int_0^1 \gamma'(s) \cdot \nabla^z \nabla_{z''} \Phi_{\mathrm{conic}}(z'', \gamma(s)) \, ds.
$$

The vector $\gamma'(s)$ can be taken to be $ax^2 \partial_x + bx \partial_{y_1}$ in a suitable local coordinate system by choosing γ appropriately. A short calculation using the exact formula (2.3) shows that the length of this curve is comparable to $d_{\mathrm{conic}}(z, z')$, or more precisely that $a^2 + b^2 = (1 + o(1)) d_{\mathrm{conic}}(z, z')^2$. Since the matrix of partial derivatives in (9.3) is equal $I + o(1)$, we thus have

$$
\gamma' \cdot \nabla_z \nabla_{z''} \Phi_{\mathrm{conic}}(z'', \gamma(s)) = a(x'')^2 \partial_{x''} + bx'' \partial_{y_1} + o(1) \nabla_{x', y'}
$$

Thus the length of this vector is equal to $\sqrt{a^2 + b^2} + o(1)$, the integral (9.4) is equal to $(1 + o(1)) d_{\mathrm{conic}}(z, z')$.

To complete the proof of (2.15) for perfectly conic metrics we have to consider the case when $d_{\text{conic}}(z, z') > o(\langle z \rangle)$, we may choose the decay in the $o()$ factor to be as slow as we please. By (9.1) we thus have $|r - r'| \geq o(\langle z \rangle)$. By (2.3) we have

$$|\nabla^{z''} (\Phi_{\text{conic}}(z'', z) - \Phi_{\text{conic}}(z'', z'))|^2_{g(z'')}$$

$$= \left(\frac{\cos d_{\partial M}(y, y'')}{x} - \frac{\cos d_{\partial M}(y', y'')}{x'} \right)^2 + \left| \frac{\sin d_{\partial M}(y, y'')\hat{\mu}}{x} - \frac{\sin d_{\partial M}(y', y'')\hat{\mu}'}{x'} \right|^2_{h(y'')}$$

$$= d_{\text{conic}}(z, z')^2 + 2rr'T,$$

where T is the quantity

$$T := \cos d_{\partial M}(y, y') - \cos d_{\partial M}(y, y'') \cos d_{\partial M}(y', y'') - \sin d_{\partial M}(y, y'') \sin d_{\partial M}(y', y'')\hat{\mu} \cdot \hat{\mu}'.$$

But since z, z', z'' are within $o(1)$ of each other in the compactified metric $d_{\overline{M}}$, we have $T = o(1)$. Since $|r - r'| \geq o(\langle z \rangle)$ for some slowly decaying $o()$, we thus have $rr'T = o(d_{\text{conic}}(z, z')^2)$, and (2.15) follows.

We now we verify (2.16) for perfectly conic metrics. Fix z, z', θ, and let $\Psi(z'')$ denote the function

$$\Psi(z'') := (1 - \theta)\Phi_{\text{conic}}(z'', z) + \theta\Phi_{\text{conic}}(z'', z')$$

appearing in (2.16), and set $Z'' := \gamma_{z \to z'}(\theta)$; our task is to show that

(9.5)
$$|\nabla_{z''} \Psi(z'')| \geq (1 + o(1))d(z'', Z'').$$

Note that this is already true in the case $z'' = Z''$ thanks to (2.8).

Next, let us study the Hessian $\nabla_{z''} \nabla_{z''} \Phi_{\text{conic}}(z, z'')$, or equivalently let us study the second derivatives $\frac{d^2}{d\tau^2} \Phi_{\text{conic}}(z, z''(\tau))$ when $z'' = z''(\tau)$ moves along a geodesic for g_{conic} at unit speed. We first observe that

$$\frac{d^2}{d\tau^2} \frac{1}{2} r''(\tau)^2 = 1$$

since the geodesic $z''()$ can be contained inside a flat totally geodesic sector $\mathcal{C}\gamma$, at which point the claim follows from the corresponding claim for the Euclidean plane $\mathbf{R}^2$. From the cosine rule (2.3) we thus see that

$$\frac{d^2}{d\tau^2} \Phi_{\text{conic}}(z, z''(\tau)) = 1 - r\frac{d^2}{d\tau^2}(r''(\tau) \cos d_{\partial M}(y, y''(\tau))).$$

In flat space, $(r''(\tau) \cos d_{\partial M}(y, y''(\tau)))$ is a linear function of τ (it is the y component of $z''(\tau)$) and so the second derivative vanishes. For perfectly conic manifolds, this quantity is homogeneous of order 1 in $z''(\tau)$ and so by working in normal coordinates in ∂M around y we obtain the bound

$$\frac{d^2}{d\tau^2} \Phi_{\text{conic}}(z, z''(\tau)) = 1 + o(\frac{r}{r''(\tau)}).$$

and hence we can control the Hessian as

(9.6)
$$\nabla_{z''} \nabla_{z''} \Phi_{\text{conic}}(z, z'') = I + o(\frac{r}{r''})$$

where I is the bilinear form $I := g(z'')^{-1}$ on $T^*_{z''}M$ (i.e. the identity matrix, if measured in an orthonormal frame). Hence

$$\nabla_{z''} \nabla_{z''} \Psi(z'') = I + o(\frac{\theta r + (1 - \theta)r'}{r''}) = I + o(\frac{R''}{r''})$$

where the latter estimate follows from (2.2). This allows us to establish (9.5) in the region $r'' \geq o(R'')$ (where the $o()$ factor decays sufficiently slowly in ϵ), by integrating this estimate along the geodesic $\gamma_{Z'' \to z'}$ and using the fact that (9.5) is already true at Z''. Note that we can control the radial variable of this geodesic without difficulty since this geodesic lies in a flat totally geodesic sector $\mathcal{C}\gamma$, and thus behaves like a Euclidean geodesic.

Now consider what happens if $r'' = o(R'')$, so that $d(z'', Z'') = (1 + o(1))R''$ by (9.1). To prove (9.5) it will suffice to bound the radial gradient, i.e. to prove

$$(9.7) \qquad |\partial_{r''} \Psi(z'')| \geq (1 + o(1))R''.$$

Using the cosine rule (2.3) again, we have

$$(9.8) \qquad \partial_{r''} \Phi_{\mathrm{conic}}(z, z'') = r'' - r \cos d_{\partial M}(y, y'') = o(R'') - (1 + o(1))r$$

and hence

$$\partial_{r''} \Psi(z'') = o(R'') - (1 - \theta)(1 + o(1))r - \theta(1 + o(1))r'.$$

But by (2.2) we have

$$(1 - \theta)(1 + o(1))r + \theta(1 + o(1))r' = (1 + o(1))R'',$$

and (9.7) follows. This completes the proof of (2.16), and hence Lemma 2.6 has been established for perfectly conic metrics.

Finally, we show (by perturbative arguments) that the estimates (2.14), (2.15), (2.16) for the perfectly conic metric will imply the corresponding estimates for the original metric in the asymptotic region $x = o(1)$. We recall the error function $e := \Phi - \Phi_{\mathrm{conic}}$ used in Lemma 2.5. Since (2.15) was already proven for the perfectly conic function Φ_{conic}, it suffices by the triangle inequality to prove that

$$\left| \nabla_{z''} e(z'', z) - \nabla_{z''} e(z'', z') \right| = o(d_M(z, z'))$$

whenever $d_{\overline{M}}(z, z'), d_{\overline{M}}(z, z''), d_{\overline{M}}(z', z'') = o(1)$ and $x, x', x'' = o(1)$. But this follows by integrating (2.11) on $\gamma_{z \to z'}$. Note that a similar (and simpler) argument also gives (2.14) in the asymptotically conic case.

We next prove (2.16) for asymptotically conic manifolds in the asymptotic region $x = o(1)$. We repeat the argument in the perfectly conic case, except now with some additional error terms that can be controlled with Lemma 2.5. As before, it suffices to prove (9.5), which is already true when $z'' = Z''$ thanks to (2.8). By Lemma 2.5, the replacement of Φ_{conic} with Φ introduces errors of $o(\frac{r}{r''}) + o(1)$ to (9.6), and errors of $o(r) + o(r'')$ to (9.8), and none of these errors significantly affect the argument. This completes the proof of (2.16), and thus of Lemma 2.6. $\qquad \square$

9.3. **Proof of Lemma 2.7.** We now prove Lemma 2.7. This lemma is trivially true by compactness arguments if z, z', z'', z_0 are restricted to any given compact subset K_0 of M, so we shall assume instead that we are in the asymptotic region $x = o(1)$.

The claim (2.17) already follows from (2.2), so we turn to the symbol estimates (2.18). From (2.8) and (2.16) we already know that $Z'' := \gamma_{z \to z'}(\theta)$ can be defined implicitly by the equation

$$(9.9) \qquad \nabla_{z''} \big((1 - \theta)\Phi(Z'', z) + \theta\Phi(Z'', z') \big) = 0.$$

Then (2.18) follows from (2.17) and the implicit function theorem, or more precisely by applying several powers of $\langle z \rangle \nabla_z$ and $\langle z' \rangle \nabla_{z'}$ to (9.9) and solving recursively for $(\langle z \rangle \nabla_z)^\alpha (\langle z' \rangle \nabla_{z'})^\beta \gamma_{z \to z'}(\theta)$; we omit the details.

Finally, the bound (2.19) follows directly from (2.15) and Gauss's Lemma (2.4). This completes the proof of Lemma 2.7.   □

10. Appendix: The geometry of long range metrics

In this appendix we prove Lemma 2.5 for long-range metrics. In the course of doing so, we shall also establish uniqueness of length-minimizing geodesics between points which are close in the compactified metric $d_{\overline{M}}$. We may work entirely in the scattering region $\{x = o(1)\}$ since in the near region K_0 there is no distinction between long-range and short-range metrics (and Lemma 2.5 is easy to prove from normal co-ordinates and a compactness argument in that case anyway).

Recall that we call g a long range scattering metric if it takes the form in some smooth coordinates (x, y) near infinity, with $r = x^{-1}$,

$$(10.1) \qquad g = dr^2 + r^2(h(y, dy) + r^{-\delta}k_{ij}dy^i dy^j),$$

where the k_{00}, k_{0j}, k_{ij} are symbols of order zero on $\overline{M}$, and $\delta > 0$. We can assume that δ is small, and allow all constants (for instance in the $o(1)$ notation) to depend on δ. In particular, the principal symbol $\sigma(H)$ of the Hamiltonian H in the scattering co-ordinates (x, y, ν, μ) of the scattering cotangent bundle introduced in Section 4 becomes

$$(10.2) \qquad \sigma(H) = \frac{1}{2}\Big(\nu^2 + (h^{ij} + x^\delta k^{ij})\mu_i \mu_j\Big).$$

Before we prove Lemma 2.5, we need a number of geometric preliminaries.

10.1. The blown up product manifold $\overline{M}_b^2$.
It will be convenient to work on the compactified product manifold-with-boundary $\overline{M} \times \overline{M} := \{(z, z') : z, z' \in \overline{M}\}$, which contains in particular the compactified diagonal $\overline{\Delta_0} := \{(z, z) : z \in \overline{M}\}$. We also need to introduce the blown-up double space M_b^2 given by
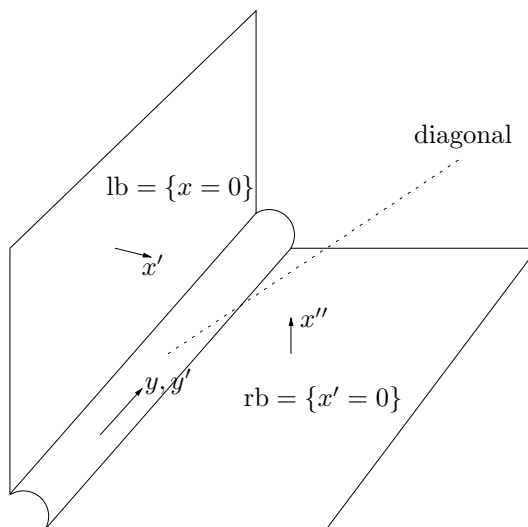
$$M_b^2 := [M^2; (\partial M)^2],$$

i.e. it is the double space M^2 with the corner blown up. The corner is thus removed and replaced with a new hypersurface consisting of the inward-pointing spherical normal bundle of $(\partial M)^2$. This new hypersurface will be denoted bf and is a quarter-circle bundle over $(\partial M)^2$. Coordinates on bf are y, y' and $x/(x + x')$, which varies between 0 at the left boundary lb of M_b^2 and 1 at the right boundary rb of M_b^2 (see figure).

10.2. Algebras of conormal functions.
Since long-range metrics do not smoothly extend to the boundary $\overline{M}$, we need the algebra $\mathcal{C}^\infty(\overline{M})$ by the larger algebra of *conormal functions* $C_\delta^\infty(\overline{M})$.

In the case of a manifold with boundary, $C_\delta^\infty(\overline{M})$ is defined as a set to be the sum $\mathcal{C}^\infty(\overline{M}) + \mathcal{A}_\delta(\overline{M})$, where $\mathcal{A}_\delta(\overline{M})$ is the Fréchet space of conormal functions of order δ [18]:

$$(10.3) \qquad \mathcal{A}_\delta(\overline{M}) = \{g \in L^\infty(\overline{M}) \mid (x\partial_x)^k \partial_y^\alpha g \in x^\delta L^\infty(\overline{M}) \text{ for all } k, \alpha\}.$$

Given a continuous extension operator $E : \mathcal{C}^\infty(\partial \overline{M}) \to \mathcal{C}^\infty(\overline{M})$, we may equivalently say that $f \in C_\delta^\infty(\overline{M})$ if $f|\partial \overline{M} \in \mathcal{C}^\infty(\partial \overline{M})$ and $f - E(f|\partial \overline{M}) \in \mathcal{A}_\delta(\overline{M})$. This naturally gives rise to a countable family of seminorms on $C_\delta^\infty(\overline{M})$ which makes $C_\delta^\infty(\overline{M})$ into a Fréchet space.

FIGURE 2. The blown up space $\overline{M}_b^2$

To define the space of conormal functions on the b-double space $\overline{M}_b^2$, we must we consider a smooth manifold X with corners of codimension two. To define $C_\delta^\infty(X)$ we require first that $f \in C_\delta^\infty(X)$ restricts to each boundary hypersurface H to be in $C_\delta^\infty(H)$. As a consequence the restriction of f to the codimension two corner is smooth. We then use smooth extension operators E_K, E_{H_i} from each hypersurface H and the corner K, and require that

$$f - E_K(f|_K) - \sum_i E_{H_i}\big((f - E_k(F|_K))\big) \in \mathcal{A}_{(\delta,\delta)}(X),$$

where $\mathcal{A}_{(\delta,\delta)}$ is defined by modifying (10.3), with iterated regularity under $x_1\partial_{x_1}$, $x_2\partial_{x_2}$, and ∂_y leaving the distribution in $x_1^\delta x_2^\delta L^\infty$, where x_i are the defining functions for the boundary hypersurfaces, and y are tangential coordinates. We give $C_\delta^\infty(X)$ the natural Fréchet topology.

Let X be a manifold with corners of codimension at most two. It is not hard to check that $C_\delta^\infty(X)$ is an algebra, and that if $f \in C_\delta^\infty(X)$ is nowhere zero, then also $1/f \in C_\delta^\infty(X)$. When $X = \overline{M}_b^2$, it is straightforward to check that

(10.4) $\qquad f \in C_\delta^\infty(\overline{M}_b^2) \implies f|_{M^2}$ is a local product symbol of order $(0,0)$.

From now on in this section, by a "conormal" function we mean one in in space $C_\delta^\infty(X)$ where X is the manifold under discussion. By a conormal change of variables on a manifold with boundary, we mean a change of variables of the form $\tilde{x}(x,y), \tilde{y}(x,y)$ where $\tilde{x}(x,y) = xf(x,y)$ and f and $\tilde{y}_i$ are in $C_\delta^\infty(\overline{M})$; this definition extends in an obvious way to manifolds with corners.

10.3. **Proof of Lemma 2.5 in the long-range case.** We can now prove Lemma 2.5 in the long-range case, by checking that the proof of [12, Proposition 9.2] extends to the Hamilton vector field associated to Hamiltonian symbols of the form (10.19). We assume some familiarity with our arguments in our previous paper.

As in [12, Proposition 9.4], we parameterize the cotangent bundle of $\overline{M}_b^2$ by $(\rho, y', x'', y''; \nu', \mu', \nu'', \mu'')$, where (x', y', ν', μ') and (x'', y'', ν'', μ'') parameterize the two factor cotangent bundles ${}^{sc}T^*\overline{M}$, and $\rho := x'/x''$. We divide the Hamilton vector field of $\sigma(H)$ by x' (i.e. we reparameterize the time variable s by $ds' := x'ds$, and observe that the vector field now takes the form (compare with equations (9.10) and (9.12) of [12])

(10.5)

$$\frac{d}{ds'}\rho = -\rho\nu - \rho^3(x'')^2 c(\mu, \nu)$$

$$\frac{d}{ds'}(y')^i = (h^{ij} + (\rho x'')^\delta k^{ij})\mu'_j + \rho^2(x'')^2 c(\nu)$$

$$\frac{d}{ds'}\nu' = (h^{ij} + (\rho x'')^\delta k^{ij})\mu'_i\mu'_j + \frac{\rho}{2}\frac{\partial(h^{ij} + (\rho x'')^\delta k^{ij})}{\partial\rho}\mu'_i\mu'_j + \rho^2(x'')^2 c(\mu\nu, \nu^2)$$

$$\frac{d}{ds'}\mu'_k = -\mu'_k\nu' - \frac{1}{2}\frac{\partial(h^{ij} + (\rho x'')^\delta k^{ij})}{\partial(y')^k}\mu'_i\mu'_j + \rho^2(x'')^2 c(\nu^2, \nu\mu, \mu^2)$$

$$\frac{d}{ds'}x'' = \frac{d}{ds'}y'' = \frac{d}{ds'}\nu'' = \frac{d}{ds'}\mu'' = 0$$

Here we write $c(\mu)$, $c(\mu\nu)$, etc, for any function which is linear in μ, $\mu\nu$, etc, with coefficients in $\mathcal{A}_0(\overline{M})$. The evolution of the function $\kappa := x'\Psi$, where $\Psi(z', z'') := d_M(z', z'')$ is the distance function, is given by

$$\frac{d}{ds'}\kappa = 1 - \kappa\nu' + \kappa\rho^2(x'')^2 c(\nu, \mu).$$

Following [12], we introduce $M := \mu/\rho$ and $K := (\kappa - 1)/\rho$ and divide the flow by ρ (i.e. reparameterize the time variable as $ds'' = \rho ds'$) to obtain

(10.6)
$$\frac{d}{ds''}\rho = -\nu + \rho(x'')^2 c(M, \nu)$$

$$\frac{d}{ds''}y^i = (h^{ij} + (\rho x'')^\delta k^{ij})M_j + \rho(x'')^2 c(\nu)$$

$$\frac{d}{ds''}\nu = \rho h^{ij}M_iM_j + \frac{\rho^3}{2}\frac{\partial(h^{ij} + (\rho x'')^\delta k^{ij})}{\partial\rho}M_iM_j + \rho(x'')^2 c(M\nu, \nu^2)$$

$$\frac{d}{ds''}M_k = -\frac{1}{2}\frac{\partial(h^{ij} + (\rho x'')^\delta k^{ij})}{\partial y^k}M_iM_j + x''c(M^2, M\nu)$$

$$\frac{d}{ds''}K = \frac{(h^{ij} + (\rho x'')^\delta k^{ij})M_iM_j}{1 + \nu} + (x'')^2 c(\mu, \nu).$$

This is a vector field with conormal coefficients, which is tangent to bf and transverse to $\{\rho = 0\}$. We need to prove that the flowout from the set $\{y' = y'', \rho = 1, \mu' = -\mu'', \nu' = -\nu''\}$ has conormal regularity. This will follow from

Lemma 10.4. *Suppose that V is a conormal, nonvanishing vector field on a manifold with codimension two corners.*

(i) Suppose that near the boundary hypersurface H, with local coordinates (x, y), V takes the form

$$V = \sum_i f_i\frac{\partial}{\partial y^i}, \quad f_1 = 1.$$

Then there is a conormal change of coordinates in which V takes the form

$$V = \frac{\partial}{\partial y^1}.$$

(ii) Suppose that, near a codimension two corner, with local coordinates (x^1, x^2, y), V takes the form

$$V = \frac{\partial}{\partial x^1} + \sum_i f_i \frac{\partial}{\partial y^i}.$$

Then there is a conormal change of coordinates in which V takes the form

$$V = \frac{\partial}{\partial x^1}.$$

Proof. (i) The flow of this vector field, starting say from the hypersurface $\{y^1 = 0\}$, is given by

$$(10.7) \qquad (t, x, y) \mapsto (x, Y_1(t), \dots, Y_m(t)), \quad Y_1(t) = t$$

where $Y' = (Y_j)$, $j \geq 2$ satisfies the ODE

$$(10.8) \qquad \frac{dY'}{dy_1}(x, t, y') = f_k(x, t, Y'(x, t, y'))) \quad Y'(x, 0, y') = y'.$$

Here we have written $y' = (y_2, \dots, y_m)$. Consequently Y' solves the integral equation

$$(10.9) \qquad Y'(x, y_1, y') = y' + \int_0^{y_1} f_k(x, t, Y'(x, t, y'))) \, dt.$$

It follows from this equation that Y'_j are conormal functions. Indeed, by iteratively differentiating the ODE with respect to $x\partial_x$ or ∂_y, we see that $Y' \in \mathcal{A}_0(\overline{M})$ locally. To show membership in $C_\delta^\infty(X)$ we derive a linear ODE for the function $x^{-\delta}(Y'(x, y_1, y') - Y'(0, y_1, y'))$ (whose coefficients depend on Y') and then interatively differentiate the ODE with respect to $x\partial_x$ or ∂_y, as above; we omit the details of this argument which follows standard lines.

Then we change to coordinates (x, y_1, z'), where

$$(10.10) \qquad z'(x, y_1, Y'(x, y_1, y')) = y'.$$

Thus the value of z' at (x, y_1, Y') is the value of y' at $y_1 = 0$ along the integral curve passing through (x, y_1, Y'). It follows that the Jacobian determinant of z' as a function of Y' is conormal (using the algebra and inverse properties of $C_\delta^\infty(X)$). Using a similar argument as above, one checks that z' is also conormal. Finally z' is constant along the flow, so V takes the required form in the coordinates (x, y_1, z').

The proof of (ii) is similar. $\qquad \square$

Continuing the proof of Lemma 2.5, we consider the flowout G from the set $G_{\text{init}} := \{y' = y'', \rho = 1, \mu' = -\mu'', \nu' = -\nu'', g' = g'' = 1\}$ (with g', g'' denoting the metric function $2\sigma(H)$ in each factor); G is the graph of $d\Psi$, where Ψ is the distance function $d(z', z'')$, and the value of Ψ is given by $(\rho x'')^{-1}\kappa = (\rho x'')^{-1} + (x'')^{-1}K$ on G. By symmetry of the distance function, we may restrict to the set where $x' < 2x''$, that is $\rho < 2$. (In the complementary region, $x'' < 2x'$, we would look at the flow in the doubly-primed variables instead.) Then the integral curve joining such a point to G_{init} lies wholly within the set $x' < Cx''$ for some $C < \infty$ by Lemma 2.1, so we restrict attention to this region. This is away from the boundary

hypersurface rb $= \{x''/x' = 0\}$, so we may use boundary defining functions x'' for bf and ρ for lb throughout the discussion.

Near any given point we will have either $(d/ds'')\rho \neq 0$ or $(d/ds'')(y')^k \neq 0$ for some k; for definiteness let us assume that $(d/ds'')\rho \neq 0$. Then by dividing the flow equations by the value of $(d/ds'')\rho$ (which does not change the flowout from G_{init}), we may assume that $\dot{\rho} = \pm 1$ locally. Applying the lemma above, we find that y' and κ are conormal functions of y'', μ'', x'', ρ on G, provided that $\rho > 0$. Similar reasoning shows that y' and K are conormal functions of these variables near $\rho = 0$. Moreover, the flow restricted to bf is the same as the exact conic flow, and since the distance function has the explicit form (2.3), we see that (y', y'', ρ) are coordinates on G restricted to $\{x'' = 0\}$, provided that y' and y'' are close in the h-metric. Also, we have $dx'' \neq 0$ at the intersection of $G_{\text{init}} \cap \{x'' = 0\}$, and the Lie derivative of dx'' vanishes under the flow, so $dx'' \neq 0$ on $G \cap \{x'' = 0\}$ in the region under consideration. Thus (y', y'', ρ, x'') are coordinates on G on a small neighbourhood of bf, when y' and y'' are close in the h-metric, which is equivalent to saying that

$$(10.11) \quad (y', y'', \rho, x'') \text{ are coordinates on } G \text{ on } \overline{\Delta}_\epsilon \subset \overline{M}_b^2 \text{ provided } \epsilon \text{ is small.}$$

It follows that κ (or K when ρ is small) is a conormal function of these variables. Hence,

$$\Phi = \frac{1}{2}\Psi^2 = \frac{1}{2}\Big(\frac{1 + \rho K}{x'}\Big)^2 = \frac{1 + 2\rho K + \rho^2 K^2}{(x')^2}$$

near $\rho = 0$, where K is conormal. Therefore

$$(10.12) \quad \frac{1}{\langle z'\rangle\langle z''\rangle}\Big(\Phi - \frac{\langle z'\rangle^2 + \langle z''\rangle^2}{2}\Big) = \frac{1}{2}\Big(2K + \rho K^2 - \rho\Big)$$

is a conormal function on $\overline{\Delta}_\epsilon \subset \overline{M}_b^2$. By (10.4), the expression in (10.12) is thus a local product symbol of order (0,0). Now we prove the error estimates. We have

$$e = \Phi - \Phi_{\text{conic}} = \frac{\langle z'\rangle\langle z''\rangle}{2}\Big(2(K - K_{\text{conic}}) + \rho(K^2 - K_{\text{conic}}^2)\Big).$$

Since K and K_{conic} agree at bf, we see from this that $e/\langle z\rangle\langle z'\rangle$ extends smoothly to $\overline{\Delta}_\epsilon$ and vanishes to order δ at bf. This implies all the claims (2.10), (2.11), (2.12), (2.13), while (2.9) follows from (9.2). This proves Lemma 2.5 in the long-range case. $\square$

Corollary 10.5. *Given two points z, z' whose distance is $o(1)$ in the compactified metric $d_{\overline{M}}$, there is only one geodesic between them which remains at distance $o(1)$ from z.*

Proof. Any geodesic from z to z' that stays in a $o(1)$ neighbourhood of z in the compactified metric $\overline{g}$ is represented by an integral curve of the flow (10.6) which is contained in G and remains uniformly close to bf. If there were more than one, this would contradict (10.11). $\square$

10.6. **Long-range metrics not in normal form.** In this section we consider more general long range metrics, as in Remark 1.1, and justify the statement made there that such metrics can be brought to the normal form (1.1) where $h_{ij}(y)$ has the form (1.2).

Consider a more general long range metric which in the asymptotic region takes the form

$$g = dr^2 + r^2 h_{ij}(y) dy^i dy^j + r^{-\delta}\Big(k_{00} dr^2 + r k_{0j} dr dy^j + r^2 k_{ij} dy^i dy^j \Big),$$

for some symbols k_{00}, k_{0j}, k_{ij} of order 0. In particular, the principal symbol of the Hamiltonian H takes the form

$$(10.13) \qquad \sigma(H) = \frac{1}{2}\Big((1 + x^\delta k^{00})\nu^2 + 2x^\delta k^{0j}\nu\mu_j + (h^{ij} + x^\delta k^{ij})\mu_i\mu_j \Big).$$

Let us sketch a proof that there is a conormal change of variables bringing such a metric to normal form. This was shown in the asymptotically flat case, and for analytic perturbations, by Burq in Appendix A of [2], and his argument adapts to the conormal setting as observed by Cardoso-Vodev [4]. Our argument follows Joshi-Sá Barreto [15]; their proof is only written for short range metrics, but we now sketch why the argument extends to the long range setting. By decreasing δ if necessary we may assume that $1/\delta$ is not an integer. We first establish a preliminary result, which is that given a symbol of the form

$$(10.14) \qquad \frac{1}{2}\Big((1 + x^{n\delta} k^{00})\nu^2 + 2x^{n\delta} k^{0j}\nu\mu_j + (h^{ij} + x^\delta k^{ij})\mu_i\mu_j \Big)$$

for some integer $n \geq 0$, that there is a conormal change of coordinates such that in the new coordinates, the symbol becomes

$$(10.15) \qquad \frac{1}{2}\Big((1 + x^{(n+1)\delta} k^{00})\nu^2 + 2x^{(n+1)\delta} k^{0j}\nu\mu_j + (h^{ij} + x^\delta k^{ij})\mu_i\mu_j \Big)$$

where the functions $k^{00}, k^{0j}, k^{ij} \in \mathcal{A}_0(\overline{M})$ may of course change from (10.14) to (10.15).

Given a change of coordinates

$$\tilde{x} = x\alpha(y, x), \quad \tilde{y} = \beta(y, x),$$

the variables ν and μ change according to

$$(10.16) \qquad \nu = \frac{\tilde{\nu}}{\alpha} + \frac{\tilde{\nu}}{\alpha^2} x\partial_x\alpha + \tilde{\mu}_i x\partial_x\beta^i, \quad \mu_i = -\frac{\tilde{\nu}}{\alpha^2}\partial_{y^i}\alpha + \tilde{\mu}_j\partial_{y^i}\beta^j.$$

We try to choose α and β in the form

$$(10.17) \qquad \alpha = 1 + x^{n\delta}a, \quad \beta^i = y^i + x^{n\delta}b^i, \quad a, b^i \in \mathcal{A}_0(\overline{M}),$$

hence $\alpha, \beta \in C^\infty_\delta(\overline{M})$. Substituting these expressions into (10.14), we find that we can arrange (10.15) provided that

$$(10.18) \qquad \begin{aligned} 2\big(x\partial_x a + (-1 + n\delta)a\big) &= k^{00}, \\ x\partial_x b^i + n\delta b^i &= k^{0i} - h^{ij}\partial_{y^i}a. \end{aligned}$$

These have solutions $a, b^i \in \mathcal{A}_0(\overline{M})$. A finite number of applications of this result shows that there is a conormal change of co-ordinates under which the symbol $\sigma(H)$ takes the form

$$(10.19) \qquad \sigma(H) = \frac{1}{2}\Big((1 + x^2 k^{00})\nu^2 + 2x^2 k^{0j}\nu\mu_j + (h^{ij} + x^\delta k^{ij})\mu_i\mu_j \Big).$$

(for instance). Now one applies the arguments of Joshi-Sá Barreto involves starting at the submanifold $S_\epsilon = \{x = \epsilon\}$ and showing that Fermi normal coordinates with respect to S_ϵ extend all the way to infinity. This argument goes through in our setting; effectively we use a single space version of the argument in Lemma 2.5,

introducing the blowup coordinate $M = \mu/x$ and dividing by a power of x to make the flow transverse to the boundary. It is then a simple matter to show that the new y coordinates generated by the flow (i.e. the Y coordinates of the proof of Lemma 10.4) remain nonsingular to $x = 0$, provided that ϵ is sufficiently small. We omit the details.

11. Appendix: Basic estimates for LSIOs

In this section we establish the basic estimates for LSIOs that were needed in the paper, or more specifically Theorem 3.6, Theorem 3.9, and Theorem 3.7. In our arguments we will not bother to explicitly establish the fact that our constants depends only on finitely many of the product symbol constants for the symbols b or b', but such an estimate can be extracted from the argument we give. Henceforth we allow all our constants to depend on b and b'.

11.1. L^2 boundedness.
We begin with the proof of Theorem 3.6.

Proof. We first observe that it suffices to prove this result assuming that $b(z, z', t)$ is supported in the region where $\langle z \rangle > \langle z' \rangle/2$ (using the symmetry of the distance function). Under this assumption, we use the T^*T method and show that $V(t)^*V(t)$ is bounded on L^2. (In the region where $\langle z' \rangle > \langle z \rangle/2$, we would simply look at $V(t)V(t)^*$ instead.) Note further that, given that we will show L^2 bounds depends only upon a finite number of symbol seminorms, we may in fact drop the time-dependence of the symbol.

Consider the kernel of $V(t)^*V(t)$. This is given by an integral[20]

$$(11.1) \qquad t^{-n} \int e^{-i\Phi(z'', z)/t} e^{i\Phi(z'', z')/t} \overline{b(z, z'')} b(z'', z') a(z, z'') a(z'', z) \, dg_{z''}.$$

Because of the χ cutoff in the kernel, we may suppose (by Lemma 2.1 and a partition of unity) that z, z' and z'' lie in a small ball B in $\overline{M}$ of some small radius ϵ in the compactified metric $d_{\overline{M}}$. If B is contained compactly within the interior of M, then the result follows by the standard Hörmander-Eskin L^2 bound [13], [10] for non-degenerate oscillatory integrals (Fourier integral operators) (see e.g. [25] for a proof). Hence we may assume that B is a "thin cone"; i.e., y, y' and y'' lie in a ball of radius $o(1)$ in $\partial \overline{M}$, and $\langle z \rangle$, $\langle z' \rangle$ and $\langle z'' \rangle$ are larger than $1/o(1)$. Under this assumption, we shall show that $V^*(t)V(t)$ is a semiclassical pseudodifferential operator of order zero (with t playing the role of small parameter, usually denoted h), which proves the lemma since it is a standard result that such operators are uniformly bounded on L^2 [19], [30].

From Lemma 2.5 we see that $\nabla_{z,z''}\Phi(z, z'')$ is a local product symbol of order $(0,0)$. In particular on the support of b, where $\langle z'' \rangle > \langle z \rangle/2$, we have

$$(11.2) \qquad \left| \nabla_z \nabla_{z''}^\beta \nabla_{z,z''} \Phi(z, z'') \right| \leq C_\beta \langle z'' \rangle^{1-\beta}.$$

Let us fix a co-ordinate chart on B (e.g. by using x and y^j), recalling that the measure $a(z, z'') a(z'', z) \, dg_{z''}$ is invariant under changes of co-ordinates. With $z(s)$

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[20]This is an oscillatory integral, not a convergent integral, and is given meaning by inserting a cutoff $\chi(\langle z'' \rangle/R)$ into the integral, and showing that the limit as $R \to \infty$ exists as a distribution. We omit the standard details.

the point given in coordinates by $z(s) = (1-s)z + sz'$, we define a function $\zeta = \zeta(z, z', z'')$ implicitly by setting

$$\Phi(z'', z) - \Phi(z'', z') = (z - z') \cdot \zeta, \qquad \zeta = \int_0^1 \frac{\partial}{\partial z_j} \Phi(z'', z(s)) \, ds, .$$

This allows us to write the phase in pseudodifferential form, $(z - z') \cdot \zeta / t$. From (11.2) and (2.14) we have

$$(11.3) \qquad \frac{\partial \zeta_i}{\partial z_j''} = I + o(1)$$

where I denotes the identity operator on $\mathbf{R}^n$. Thus we may invert and express z'' as a function of (z, z', ζ), $z'' = Z''(z, z', \zeta)$.

We must now show that the function $s(z, z', \zeta)$ defined by

$$s(z, z', \zeta) := \overline{ab(z, Z''(z, z', \zeta))} ab(z', Z''(z, z', \zeta)) \chi(z, Z''(z, z', \zeta)) \chi(z', Z''(z, z', \zeta))$$

is a classical pseudodifferential symbol of order zero in ζ; explicitly, we must show that

$$(11.4) \qquad \left| \left(\langle z \rangle \frac{\partial}{\partial z}\right)^\alpha \left(\langle z' \rangle \frac{\partial}{\partial z'}\right)^\beta \left(\langle \zeta \rangle \frac{\partial}{\partial \zeta}\right)^\gamma s(z, z', \zeta) \right| \le C_{\alpha, \beta, \gamma}.$$

Since $Z''(z, z', \zeta(z, z', z'')) = z''$, we see that $\partial_z Z'' = \partial_{z'} Z'' = 0$. Thus z or z' derivatives only hit the first argument of ab and symbolic estimates for these derivatives follow directly from the symbolic estimates on ab.

As for the ζ partial derivatives, we first need good bounds on partial ζ-derivatives of Z''. It is straightforward to show by induction that

$$(11.5) \qquad \frac{\partial^{k+1} Z''}{\partial \zeta^{k+1}} = \sum_{(\alpha_1, \ldots, \alpha_l)}^{\alpha_1 + \ldots \alpha_l = k} \prod_{j=1}^l \left(\frac{\partial Z''}{\partial \zeta}\right)^{b_j} \frac{\partial^{\alpha_j + 1} \zeta}{(\partial z'')^{\alpha_j + 1}}.$$

Here the product on the right hand side is shorthand for a sum of terms each of which is a monomial of degree b_j in the components of $\partial Z'' / \partial \zeta$ and linear in the components of $\partial^{\alpha_j + 1} \zeta / (\partial z'')^{\alpha_j + 1}$. By (11.3), the matrix $\partial Z'' / \partial \zeta$ is bounded between $1/2$ and 2, while from the definition of ζ and (11.2), $\partial \zeta / \partial z''$ is a symbol of order zero in z'', uniformly in $z, z' \in U$. It follows from these facts and (11.5) that

$$\left| \frac{\partial^{k+1} Z''}{\partial \zeta^{k+1}} \right| \le C_k \langle z'' \rangle^{-k}.$$

Recall now that the integrand in (11.1) is supported where $\langle z'' \rangle > \max(\langle z \rangle / 2, \langle z' \rangle / 2)$.

We now note that from (1.1) (and (1.2)) that the geodesic evolution of the radial co-ordinate r is

$$\ddot{r} = r^{-1} \left(h^{ij} + o(1) \right) \mu_i \mu_j;$$

in particular, since h^{ij} is positive definite, we see that $r(s)$ is a convex function of s in the asymptotic region. Using this convexity, and by comparing the metric with a larger conic metric and using the exact formula (2.3) for conic distance, we have

$$|\zeta|^2 \le \sup_s d^2(z'', z(s)) \le \langle z'' \rangle^2 + \left(\langle z \rangle + \langle z' \rangle\right)^2 + \eta \langle z'' \rangle \left(\langle z \rangle + \langle z' \rangle\right) \le C \langle z'' \rangle^2.$$

Hence we find that

$$\left| \frac{\partial^{k+1} Z''}{\partial \zeta^{k+1}} \right| \le C_k \langle \zeta \rangle^{-k},$$

and using this it is simple to check that s is symbolic of order zero in ζ. This completes the proof of the lemma. $\qquad\square$

11.2. **The composition law.** We now prove Theorem 3.9. Fix b, b'. From (3.1) we have $S_b(t)S_{b'}(t') = S_{\tilde{c}_{t,t'}}(t+t')$, where $\tilde{c}_{t,t'}$ is the function

$$\tilde{c}_{t,t'}(z, z'') :=$$
$$\frac{(2\pi i(t+t'))^{n/2}}{a(z,z'')(2\pi it)^{n/2}(2\pi it')^{n/2}} \int_M e^{i(\frac{\Phi(z,z')}{t} + \frac{\Phi(z',z'')}{t'} - \frac{\Phi(z,z'')}{t+t'})} ab(z,z',t)ab'(z',z'',t')\, dg(z').$$

Since b, b' are localized to a region of the form Δ_ϵ for some small ϵ, so is $\tilde{c}_{t,t'}$ (with a slightly larger value of ε. It thus suffices to show that $\tilde{c}_{t,t'}$ has the expansion $\tilde{c}_{t,t'} = c + \frac{t+t'}{\langle z \rangle + \langle z' \rangle}e_{t,t'}$ for some local product symbol $e_{t,t'}$ of order $(0,0)$, and c given by (3.9); observe from (2.18) that this would in fact ensure that $\tilde{c}_{t,t'}$ is a product local symbol of order $(0,0)$ as claimed.

Let us make the "semi-classical" change of variables $\hbar := \frac{tt'}{t+t'}$ and $\theta := \frac{t}{t+t'}$, and set

$$f_{\hbar,\theta} = a(z,z'')\tilde{c}_{t,t'}$$

so that we have

$$f_{\hbar,\theta}(z,z'') = \frac{1}{(2\pi i\hbar)^{n/2}} \int_M e^{\frac{i}{\hbar}\Psi_\theta(z,z',z'')} ab(z,z',t)ab'(z',z'',t')\, dg(z')$$

where Ψ_θ is the phase

$$\Psi_\theta(z, z', z'') := (1-\theta)\Phi(z,z') + \theta\Phi(z',z'') - \theta(1-\theta)\Phi(z,z'').$$

Here we have elected not to apply the change of variables to the t, t' in the symbols (one may think of these as auxiliary parameters) so as not to require any time-regularity of the symbols later on.

Our task is to show that

$$(11.6) \quad f_{\hbar,\theta}(z,z'') = a(z,z'')b(z,z'_0,t)b'(z'_0,z'',t') + \frac{\hbar}{\theta(1-\theta)(\langle z \rangle + \langle z'' \rangle)}e_{\hbar,\theta}(z,z'')$$

where $e_{\hbar,\theta}$ is a local product symbol of order $(0,0)$ which depends on θ and $\hbar$ but obeys local product symbol bounds which are uniform with respect to these parameters, and $z'_0 := \gamma_{z \to z''}(\theta)$. (We have absorbed the a factor into $e_{\hbar,\theta}$ since a and a^{-1} are both local product symbols of order $(0,0)$ on the support of $f_{\hbar,\theta}$, thanks to Lemma 2.9.)

We first verify the claim in the classical limit $\hbar = 0$, or more precisely that

$$(11.7) \quad \lim_{\hbar \to 0} \frac{1}{(2\pi i\hbar)^{n/2}} \int_M e^{\frac{i}{\hbar}\Psi_\theta(z,z',z'')} ab(z,z',t)ab'(z',z'',t')\, dg(z')$$
$$= a(z,z'')b(z,z'_0,t)b'(z'_0,z'',t').$$

From Lemma 2.6 we have the phase oscillation estimate

$$(11.8) \quad |\nabla_{z'} \frac{\Psi_\theta(z,z',z'')}{\hbar}| \geq c\frac{1}{\hbar}d(z',z'_0).$$

In particular the only stationary point of the integral occurs at $z' = z'_0$ (cf. (2.8)). At this point, the phase is equal to
$$(11.9)$$
$$\Psi_\theta(z,z'_0,z'') = (1-\theta)(\theta d(z,z''))^2/2 + \theta((1-\theta)d(z,z''))^2/2 - \theta(1-\theta)d(z,z'')^2/2 = 0.$$

Applying the principle of stationary phase (see [25]), it thus suffices to verify the identity

$$\det(\nabla^2_{z'}\Psi_\theta(z,z',z'')|_{z'=z'_0})^{-1/2} = \frac{a(z,z'')}{a(z,z'_0)a(z'_0,z'')\,dg(z'_0)}$$

or (by (1.12)) that
(11.10)
$$\det(\nabla^2_{z'}\Psi_\theta(z,z',z'')|_{z'=z'_0}) = \frac{\det(-\nabla_z\nabla_{z'}\Phi(z,z'))\det(-\nabla_{z'}\nabla_{z''}\Phi(z',z''))}{\det(-\nabla_z\nabla_{z''}\Phi(z,z''))}|_{z'=z'_0}.$$

To see this, we now let z and z'' vary (keeping θ fixed), so that z'_0 is now thought of as a function of z and z''. We rewrite (11.9) as

$$\theta(1-\theta)\Phi(z,z'') = [(1-\theta)\Phi(z,z') + \theta\Phi(z',z'')]|_{z'=z_0}$$

and observe from (2.8) that the right-hand side is stationary in z' at z_0:

(11.11) $$\nabla_{z'}\Psi(z,z',z'')|_{z'=z'_0} = \nabla_{z'}[(1-\theta)\Phi(z,z') + \theta\Phi(z',z'')]|_{z'=z'_0} = 0,$$

Differentiating both sides with respect to z'' (and using the previous observation to ignore the z' variation), we thus obtain

$$\theta(1-\theta)\nabla_{z''}\Phi(z,z'') = \theta\nabla_{z''}\Phi(z'_0,z'').$$

Differentiating again, this time in z, and using the chain rule, we obtain

$$\theta(1-\theta)\nabla_z\nabla_{z''}\Phi(z,z'') = \theta\nabla_{z'}\nabla_{z''}\Phi(z',z'')|_{z'=z'_0}\cdot\nabla_z z'_0(z,z'').$$

On the other hand, if we differentiate (11.11) in z using the chain rule, we obtain

$$((1-\theta)\nabla_z\nabla_{z'}\Phi(z,z') + \nabla^2_{z'}\Psi(z,z',z'')|_{z'=z'_0})\cdot\nabla_z z'_0(z,z'') = 0.$$

Combining this with the previous identity, we obtain after some algebra

$$-\nabla_z\nabla_{z''}\Phi(z,z'') = [-\nabla_{z'}\nabla_{z''}\Phi(z',z'')$$
$$\cdot\nabla^2_{z'}\Psi(z,z',z'')^{-1}$$
$$\cdot(-\nabla_z\nabla_{z'}\Phi(z,z'))]|_{z'=z'_0}.$$

Taking determinants of both sides we obtain (11.10) as desired[21]. This proves (11.7).

In light of (11.7), it will now suffice to obtain a two-term expansion of the form

$$f_{\hbar,\theta}(z,z'') = f_{0,\theta}(z,z'') + \frac{\hbar}{\theta(1-\theta)(\langle z\rangle + \langle z''\rangle)}e_{\hbar,\theta}(z,z'')$$

where $c_{0,\theta}(z,z'')$ is some function independent of $\hbar$, since by taking $\hbar \to 0$ we then see that $f_{0,\theta}(z,z'')$ equals the expression in (11.7). In fact by (2.17), (2.18) it will suffice to obtain a representation of the form

(11.12) $$f_{\hbar,\theta}(z,z'') = f_{0,\theta}(z,z'') + \frac{\hbar}{\langle z'_0\rangle^2}e_{\hbar,\theta}(z,z'')$$

for some product symbol $e_{\hbar,\theta}$ of order $(0,0)$, with bounds uniform in $\hbar$ and θ.

---

[21]Note that we did not really use any Riemannian geometry to prove this lemma, instead relying exclusively on the fact that the phase (11.9) vanished and was stationary at z'_0. Indeed this computation is really one in symplectic geometry rather than Riemannian geometry, and reflects the multiplicative law for the symbol of Fourier integral operators under composition, and the untruncated parametrix $S_1(t)$ has symbol 1 (see Remark 2.10).

Introduce a smooth cutoff $\varphi_{z,z''}(z')$ which equals 1 when $d(z', z_0')\rangle \leq C^{-1}\langle z_0'\rangle$ and is supported in the region $d(z', z_0') \leq 2C^{-1}\langle z_0'\rangle$ for some large $C \gg 1$. Consider first the non-local contribution to $f_{\hbar,\theta}(z, z'')$:

$$
\begin{aligned}
&f_{\hbar,\theta}(z, z'') - \tilde{f}_{\hbar,\theta}(z, z'') \\
&:= \frac{1}{(2\pi i \hbar)^{n/2}} \int_M e^{\frac{i}{\hbar}\Psi_\theta(z,z',z'')} ab(z, z', t) ab'(z', z'', t')(1 - \varphi_{z,z''}(z')) \, dg(z').
\end{aligned}
$$

From (11.8) and repeated integration by parts we obtain the bounds

$$
|(f_{\hbar,\theta} - \tilde{f}_{\hbar,\theta})(z, z'')| \leq C_N \hbar^N \langle z_0'\rangle^{-N}
$$

for any $N \geq 0$, and in fact we have the additional estimates

$$
|(\langle z\rangle \partial_z)^\alpha (\langle z''\rangle \partial_{z''})^\beta (f_{\hbar,\theta} - \tilde{f}_{\hbar,\theta})(z, z'')| \leq C_{N,\alpha,\beta} \hbar^N \langle z_0'\rangle^{-N}
$$

for any $\alpha, \beta \geq 0$ (mostly thanks to (2.18)). Thus this contribution to (11.12) is certainly acceptable.

It remains to consider the local contribution

$$
\tilde{f}_{\hbar,\theta}(z, z'') := \frac{1}{(2\pi i \hbar)^{n/2}} \int_M e^{\frac{i}{\hbar}\Psi_\theta(z,z',z'')} ab(z, z', t) ab'(z', z'', t') \varphi_{z,z''}(z') \, dg(z')
$$

But this contribution will be acceptable, thanks to the standard result in stationary phase that a oscillatory integral with a single non-degenerate critical point and with phase and amplitude smooth and smoothly depending on certain parameters will enjoy an asymptotic series, with all the main terms and error terms in the series depending smoothly on the parameters—see Theorem 7.7.6 of [14]. This concludes the proof of Theorem 3.9. $\qquad\square$

11.3. **The dispersive inequality.** We now prove Theorem 3.7.

When one of s or t is zero, the claim follows from (3.2) and (3.5), so assume s, t are non-zero. When t and s have opposite sign then the claim follows quickly from Lemma 3.5, Theorem 3.9, and (3.5). Thus we may assume that t and s have the same sign[22]. By duality and Lemma 3.5 we may in fact assume that $0 < s < t$. By expanding out the kernel of $S_b(t)^* S_{b'}(s)$ using (3.1), it thus suffices to show that

$$
(11.13) \quad |\int_M e^{i(\frac{\Phi(z_1,z_2)}{s} - \frac{\Phi(z_1,z_0)}{t})} \overline{ab(z_1, z_0, t)} ab'(z_1, z_2, s) \, dg(z_1)| \leq C \frac{t^{n/2} s^{n/2}}{|t - s|^{n/2}}
$$

for all $z_0, z_2 \in M$. Clearly we may restrict (via smooth truncation of b or b') to the region where

$$
(11.14) \qquad\qquad \langle z_1\rangle \geq C\left(\frac{ts}{|t - s|}\right)^{1/2}
$$

since the portion of the integral where (11.14) fails can be estimated by taking absolute values everywhere.

---

[22]These cases are not symmetric; the case of the equal signs is both more difficult and more important. The situation is analogous to that of the L^2 or L^p theory of the Hilbert transform kernel $\frac{1}{s-t}$; the portion of this kernel when s and t have opposing signs is easy to treat by Hardy's inequality (even when absolute values are placed on the kernel), but the portion when s and t have the same signs is more delicate.

The idea is to use the composition law (Theorem 3.9) to factorize $S_b(t)^* S_{b'}(s)$ into something resembling $S_{b''}(t-s)^* S_{b'''}(s)^* S_{b'}(s)$. Indeed, from Theorem 3.9 (with $b, b' := \chi$ for some suitable cutoff χ to a region Δ_ϵ) we have

$$\frac{(2\pi i t)^{\frac{n}{2}}}{(2\pi i s)^{\frac{n}{2}} (2\pi i (t-s))^{\frac{n}{2}}} \int_M e^{i(\frac{\Phi(z,z')}{s} + \frac{\Phi(z',z'')}{t-s})} a(z,z') a(z',z'') \chi(z,z') \chi(z',z'') \, dg(z')$$

$$= e^{i\Phi(z,z'')/t} (a(z,z'') \chi(z,z') \chi(z',z'') + \frac{t}{\langle z \rangle + \langle z'' \rangle} e(z,z''))$$

for some local product symbol e of order $(0,0)$. We may arrange matters so that $\chi(z,z') \chi(z',z'') = 1$ on the support of b. When t is small, or $\langle z \rangle$ or $\langle z'' \rangle$ is large, the error term $\frac{t}{\langle z \rangle + \langle z'' \rangle} e(z,z'')$ is dominated by the main term, and in particular the term in parentheses is bounded away from zero. When t is large and $\langle z \rangle$ or $\langle z'' \rangle$ is small, then this is not necessarily the case, but this is easily fixed in this case by modifying the amplitude $a(z,z') a(z',z'')$ on the left-hand side slightly in the region $\langle z \rangle, \langle z' \rangle, \langle z'' \rangle = O(1)$, replacing it with a local product symbol $\tilde{a}(z,z',z'')$ of order $(0,0,0)$ in the three variables. We can then multiply both sides by a suitable local product symbol of order $(0,0)$ in z and z'' to obtain a representation of the form

$$e^{i\frac{\Phi(z,z'')}{t}} ab(z,z'',t) = \frac{(2\pi i t)^{\frac{n}{2}}}{(2\pi i s)^{\frac{n}{2}} (2\pi i (t-s))^{\frac{n}{2}}} \int_M e^{i(\frac{\Phi(z,z')}{s} + \frac{\Phi(z',z'')}{t-s})} \tilde{b}(z,z',z'',s,t) \, dg(z')$$

for some local product symbol $\tilde{b}$ of order $(0,0,0)$, localized so that z, z', z'' are all close to each other in the compactified metric $d_{\overline{M}}$. Note that while $\tilde{b}$ will depend on both s and t, the local product symbol bounds are uniform in these parameters. Using this representation, we can rewrite (11.13) as[23]

$$s^{-n} | \int_M \int_M e^{i(\frac{\Phi(z_1,z_2)}{s} - \frac{\Phi(z_1,z')}{s} - \frac{\Phi(z_0,z')}{t-s})} \overline{\tilde{b}(z_1,z',z_0,s,t)} b'(z_1,z_2,s) \, dg(z_1) dg(z')| \leq C.$$

Note that all four points z_0, z', z_1, z_2 are constrained to lie close to each other in $d_{\overline{M}}$.

Let $m \geq 0$ be an integer, and denote by Q_m the quantity

$$s^{-n} \left| \int_{M^2} e^{i(\frac{\Phi(z_1,z_2)}{s} - \frac{\Phi(z_1,z')}{s} - \frac{\Phi(z_0,z')}{t-s})} \overline{\tilde{b}(z_1,z',z_0,s,t)} b'(z_1,z_2,s) \varphi_m(z_1) \, dg(z_1) dg(z') \right|$$

where φ_m is a suitable smooth cutoff to the region where $\langle z_1 \rangle$ is comparable to 2^m. From (11.14) we may assume that $2^{2m} \geq st/(t-s)$. It thus suffices to show that

$$(11.15) \qquad\qquad \sum_{m \geq 0 : 2^{2m} \geq st/(t-s)} Q_m \leq C.$$

Let us first observe that each Q_m is individually bounded. Indeed, from (2.15) we observe the phase oscillation estimate

$$\left| \nabla_{z_1} \left(\frac{\Phi(z_1,z_2)}{s} - \frac{\Phi(z_1,z')}{s} - \frac{\Phi(z_0,z')}{t-s} \right) \right| \geq C^{-1} d(z_2,z')/s,$$

---

[23]Semiclassically, what is going on is the following. The expression in (11.13) is sort of a quantum mechanical version of a trajectory consisting of the concatenation of the geodesics $\gamma_{z_2 \to z_1}$ and $\gamma_{z_1 \to z_0}$, where $\gamma'_{z_1 \to z_0}(0) = -\frac{t}{s} \gamma'_{z_2 \to z_1}(0)$ (so the path $\gamma_{z_1 \to z_0}$ overshoots past z_2). The point z' then corresponds to the point $\gamma_{z_1 \to z_0}(s/t) = z_2$, and so this factorization corresponds to a splitting of this path into a loop from z_2 to z_1 and back to $z' = z_2$, followed by a geodesic from $z' = z_2$ to z_0.

while for higher derivatives we see from (2.18) and Lemma 2.5 that

$$\left| \nabla_{z_1}^k \left(\frac{\Phi(z_1, z_2)}{s} - \frac{\Phi(z_1, z')}{s} - \frac{\Phi(z_0, z')}{t-s} \right) \right| \geq C_k \langle z_1 \rangle^{1-k} / s \text{ for } k \geq 2.$$

These estimates imply (after repeated integration by parts in z_1, and exploiting the local product symbol bounds for $\tilde{b}$, b' as well as the localization of φ_m) that for fixed $z' \in M$ we have

(11.16)
$$\left| \int_M e^{i(\frac{\Phi(z_1, z_2)}{s} - \frac{\Phi(z_1, z')}{s} - \frac{\Phi(z_0, z')}{t-s})} \overline{\tilde{b}} \, b' \, \varphi_m \, dg(z_1) \right| \leq C_N 2^{nm} (1 + 2^m d(z_2, z')/s)^{-N}$$

for any $N > 0$; taking $N = 100n$ (for instance) and integrating this estimate over z', we obtain the bound $Q_m = O(1)$ as claimed. A similar argument shows that the contribution of the integral to Q_m coming from the region where $\langle z' \rangle$ is not comparable to $\langle z_2 \rangle$ is extremely small (e.g. it is bounded by $O(2^{-100m})$ so we shall discard this portion and smoothly truncate to the region where $\langle z' \rangle$ is comparable to $\langle z_2 \rangle$.

To improve upon these bounds and obtain (11.15) we have to take advantage of oscillation in the z' variable as well as the z_1 variable. From (2.16) we have

$$(11.17) \ \left| \nabla_{z'} \left(\frac{\Phi(z_1, z_2)}{s} - \frac{\Phi(z_1, z')}{s} - \frac{\Phi(z_0, z')}{t-s} \right) \right| \geq C^{-1} d_M(z', \gamma_{z_1 \to z_0}(\frac{s}{t})) \frac{t}{s(t-s)}.$$

Since we have already localized z' to be close to z_2, it is now natural to introduce the quantity

$$\rho_m := \inf_{z_1 \in \text{supp } \varphi_m} d_M(z_2, \gamma_{z_1 \to z_0}(\frac{s}{t})).$$

This quantity is usually quite large:

Lemma 11.4. *We have $\rho_m \geq C^{-1} \frac{t-s}{t} 2^m$ for all but $O(1)$ values of m.*

Proof. Let $m \geq \tilde{m}$ be any two integers such that $\rho_m \leq C_0^{-1} \frac{t-s}{t} 2^m$ and $\rho_{\tilde{m}} \leq C_0^{-1} \frac{t-s}{t} 2^{\tilde{m}}$ for some large constant $C_0 > 1$ to be chosen later. By definition of ρ_m and the triangle inequality, we can thus find $z_1 \in \text{supp } \varphi_m$, $\tilde{z}_1 \in \text{supp } \varphi_{\tilde{m}}$ such that

$$d_M(\gamma_{z_1 \to z_0}(\frac{s}{t}), \gamma_{\tilde{z}_1 \to z_0}(\frac{s}{t})) \leq 2C_0^{-1} \frac{t-s}{t} 2^m.$$

But by (2.19), the left-hand side is comparable to

$$\frac{t-s}{t} |\gamma'_{z_0 \to z_1}(0) - \gamma'_{\tilde{z}_0 \to z_1}(0)|$$

which by (2.19) again is comparable to

$$\frac{t-s}{t} d_M(z_1, \tilde{z}_1).$$

Thus $d_M(z_1, \tilde{z}_1) \leq C C_0^{-1} 2^m$, which by the localization of $z_1, \tilde{z}_1$ forces $m = m' + O(1)$ if C_0 is chosen large enough. The claim follows. $\square$

We may assume that

$$(11.18) \qquad\qquad \rho_m \geq C^{-1} \frac{t-s}{t} 2^m,$$

since the contribution of the $O(1)$ exceptional values of m to (11.15) can be dealt with by the bound $Q_m = O(1)$ obtained previously. Since $2^{2m} \geq st/(t-s)$, this implies that

$$(11.19) \qquad \rho_m \geq C^{-1} s/2^m.$$

Now we smoothly truncate the z' integral into the region where $d_M(z', z_2) \leq \min(1, \rho_m)/2$ and where $d_M(z', z_2) \geq \min(1, \rho_m)/4$. The contribution of the latter case to Q_m is at most $C_N(2^m \min(1, \rho_m)/s)^{-N}$ for any $N \geq 0$ thanks to (11.16) and (11.19), which then gives an acceptable contribution to (11.15) thanks to (11.18). For the former case, we observe from (11.17) and the triangle inequality that

$$\left| \nabla_{z'} \left(\frac{\Phi(z_1, z_2)}{s} - \frac{\Phi(z_1, z')}{s} - \frac{\Phi(z_0, z')}{t-s} \right) \right| \geq C^{-1} \frac{\rho_m}{2} \frac{t}{s(t-s)},$$

while higher derivatives are controlled using Lemma 2.5 and (2.18) by the somewhat crude estmate

$$\left| \nabla_{z'}^k \left(\frac{\Phi(z_1, z_2)}{s} - \frac{\Phi(z_1, z')}{s} - \frac{\Phi(z_0, z')}{t-s} \right) \right| \leq C_k \frac{t}{s(t-s)} \text{ for all } k \geq 2$$

so by repeated integration by parts in z' (noting that every derivative in z' of a cutoff function or of phase derivatives costs us at most $O(1/\min(1, \rho_m))$) we see that the contribution of this case to Q_m is at most

$$C_N s^{-n} 2^{nm} \min(1, \rho_m)^n (C\rho_m \min(1, \rho_m) \frac{t}{s(t-s)})^{-N}$$

for any $N \geq 0$, which is in turn bounded by $C_N(2^m \min(1, \rho_m)/s)^{-N+O(1)}$ thanks to (11.19), and so as before this gives an acceptable contribution to (11.15). This completes the proof of Theorem 3.7. $\qquad \square$

References

[1] N. Burq, *Estimations de Strichartz pour des perturbations á longue portée de l'opérateur de Schrödinger*, preprint.

[2] N. Burq, *Lower bounds for shape resonances widths of long range Schrdinger operators*, Amer. J. Math. **124** (2002), 677–735.

[3] N. Burq, P. Gérard, N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*, Amer. J. Math. **126** (2004), 569–605.

[4] F. Cardoso and G. Vodev, *Uniform estimates of the resolvent of the Laplace-Beltrami operator on infinite volume Riemannian manifolds. II*, Ann. Henri Poincaré **3** (2002), 673–691.

[5] T. Cazenave, An introduction to nonlinear Schrödinger equations, Textos de Metodes Matematicos **22** (Rio de Janeiro), 1989.

[6] T. Cazenave, F.B. Weissler, *Critical nonlinear Schrödinger Equation*, Non. Anal. TMA, **14** (1990), 807–836.

[7] M. Christ, A. Kiselev, *Maximal functions associated to filtrations*, J. Funct. Anal. **179** (2001), 409–425.

[8] W. Craig, T. Kappeler, W. Strauss, *Microlocal dispersive smoothing for the Schrödinger equation*, Comm. Pure Appl. Math **48** (1995), 769–860.

[9] S. Doi, *Smoothing effects of Schrödinger evolution groups on Riemannian manifolds*, Duke Math J. **82** (1996), 679–706.

[10] G.I. Eskin, *Degenerate elliptic pseudodifferential operators of principal type*, Math. Sbornik **82** (1970), 585–628 (Russian); English transl. in Math. USSR Sbornik **11** (1970) 539–582.

[11] A. Hassell, J. Wunsch, *The Schrödinger propagator for scattering metrics*, to appear, Annals of Mathematics; math.AP/0301341.

[12] A. Hassell, T. Tao, J. Wunsch, *A Strichartz inequality for the Schrödinger equation on non-trapping asymptotically conic manifolds*, to appear, Comm. Part. Diff. Eq.; math.AP/0312225.

[13] L. Hörmander, *Fourier integral operators I*, Acta Math. **127** (1971), 79–183.
[14] L. Hörmander, *The analysis of linear partial differential operators I*, second edition, Springer-Verlag, Berlin 1990.
[15] M. S. Joshi and A. Sá Barreto, *Recovering asymptotics of metrics from fixed energy scattering data*, Invent. Math. **137** (1999), no. 1, 127–143.
[16] T. Kato, *On nonlinear Schrödinger equations*, Ann. Inst. H. Poincare Phys. Theor. **46** (1986), 113–129.
[17] M. Keel, T. Tao, *Endpoint Strichartz Estimates*, Amer. Math. J. 120 (1998), 955–980.
[18] R. B. Melrose, *Calculus of conormal distributions on manifolds with corners*, Internat. Math. Res. Notices **1992**, no. 3, 51–61.
[19] R. B. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces*, in Spectral and Scattering Theory (Sanda 1992), 85–130, Dekker, New York, 1994.
[20] S.J. Montgomery-Smith, *Time decay for the bounded mean oscillation of solutions of the Schrdinger and wave equations*, Duke Math. J. **91** (1998), no. 2, 393–408.
[21] L. Robbiano, C. Zuily, in preparation.
[22] P. Sjölin, *Regularity of solutions to the Schrödinger equation*, Duke Math. J. **55** (1987), no. 3, 699–715.
[23] H. Smith, C.D. Sogge, *Global Strichartz estimates for nontrapping perturbations of the Laplacian*, Comm. PDE **25** (2000), 2171–2183.
[24] G. Staffilani, D. Tataru, *Strichartz estimates for a Schrödinger operator with nonsmooth coefficients*, Comm. Part. Diff. Eq. **27** (2002), no. 7-8, 1337–1372.
[25] E. M. Stein, *Harmonic Analysis*, Princeton University Press, 1993.
[26] M. Sugimoto, *A smoothing property of Schrödinger equations along the sphere*, J. Anal. Math. **89** (2003), 15–30.
[27] T. Tao, *Spherically averaged endpoint Strichartz estimates for the two-dimensional Schrdinger equation*, Comm. Partial Differential Equations **25** (2000), no. 7-8, 1471–1485.
[28] D. Tataru, in preparation.
[29] A. Vasy, *Geometric scattering theory for long-range potentials and metrics*, Internat. Math. Res. Notices 1998, no. 6, 285–315.
[30] A. Vasy, M. Zworski, *Semiclassical estimates in asymptotically Euclidean scattering*, Comm. Math. Phys. **212** (2000), no. 1, 205–217.
[31] L. Vega, *Schrödinger equations: pointwise convergence to the initial data*, Proc. Amer. Math. Soc. **102** (1988), no. 4, 874–878.

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