THE MORSE INDEX THEOREM FOR MECHANICAL SYSTEMS WITH REFLECTIONS

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Abstract. We prove a Morse index theorem for action functionals on paths that are allowed to reflect at a hypersurface (either in the interior or at the boundary of a manifold). Both fixed and periodic boundary conditions are treated.

1. Introduction

The classical Morse index theorem [10] on a smooth Riemannian manifold (M, g) says that at a geodesic $\alpha(t)$, $t \in [0, T]$, the index of the second variation of the energy functional with fixed endpoints equals the total number (with multiplicity) of points on α conjugate to $\alpha(0)$. Indeed, Morse's celebrated book [10] treats a number of variations on this theme, allowing for different boundary conditions, such as periodicity, which entail interesting corrections to the count of conjugate points; in the periodic case, Morse refers to the additional term as the "order of concavity."

Here we are concerned with a generalization of these classical results, where we also allow *reflections*. We simultaneously treat two cases: either M is a manifold with boundary $Y = \partial M$, or Y is an embedded interior hypersurface of M. The paths under consideration are required to undergo reflection at Y in the case $Y = \partial M$, or permitted to undergo *either* reflection or transmission in the case of an interior hypersurface.

We are moreover concerned here not with the usual setting of Riemannian geometry most common in the literature, but rather with the more general case of a mechanical system: rather than just using an energy functional given by the Riemannian metric, we employ a Lagrangian $L \in \mathcal{C}^\infty(TM)$

$$L(x, v) = \frac{1}{2} g_{ij}(x) v^i v^j - V(x),$$

and associated action

$$J[\alpha] = \int_0^T L(\alpha(t), \dot{\alpha}(t))\, dt.$$

Here $V(x)$ is a real valued function that is not required to be globally smooth: our hypotheses are that V is smooth up to Y but in the interior hypersurface case is merely required to have matched values and first derivatives across the two sides. (The choice to work in this generality stems from our intended applications, described below.)

The main results of this paper are a Morse index theorem for the functional J, both for trajectories with fixed boundary points and for the problem of periodic trajectories; these are stated in Theorems 5.5 and 5.7 below.

Our interest in this problem was stimulated by the aim of proving a *Gutzwiller trace formula* that would relate the asymptotics of the trace of the Schrödinger propagator $e^{-it(\frac{1}{2}\Delta+V)/h}$ to the behavior of closed classical trajectories [5]. Typically in such trace formulae, one obtains a *Maslov factor*, a power of i that is the Morse index of the variational problem for closed trajectories. For instance, in the Riemannian geometric case related to the Duistermaat–Guillemin trace formula [3], this variational characterization of the Maslov factor was established in [2]. In mechanical systems with a non-smooth potential V, singular across Y as described above, it turns out that in addition to ordinary periodic physical trajectories, there are contributions to the trace asymptotics from periodic trajectories that are *reflected* at Y as well as those transmitted across Y [4]. We were dismayed that we could not find any existing account of the Morse index theorem for the periodic variational problem with reflected mechanical trajectories; since this problem seems a physically natural and important one, we have attempted to fill this gap in the literature here.

A simple invocation of the usual proof of the index theorem *mutatis mutandis* does not suffice to deal with the case of reflected paths. To begin with, the spaces of allowable paths and variations must be rather carefully set up: we must enforce compatibility conditions at the reflection times (as well as at times of transmission across Y), and this of course affects the space of allowable variation vector fields. The Jacobi fields, in turn, must satisfy interesting geometric compatibility conditions at the moment of reflection (involving the second fundamental form of the hypersurface), and much of our work here has been to tell the story of reflected Jacobi fields; the final proof of the index theorem is straightforward once the tools to deal with the Jacobi fields are in place.

Some results in this direction do appear in the existing literature, but not in the generality that we seek here. In particular, there are a number of treatments describing Jacobi fields for reflected geodesics by considering the first variation of a family of broken trajectories reflecting at the boundary according to Snell's law [7, 12, 13, 6, 8]. However, we have not been able to find analysis of the second variation, nor a proof of the index theorem (neither for periodic nor fixed boundary conditions). It has also proved impossible to find an account of the reflection conditions in the presence of a potential. Additionally, Morse's "order of concavity" arising in the periodic variational problem moreover makes a somewhat obscure appearance in [10], and is not easily suited to physical interpretation in the mechanical context; the version here is not one we have seen in the literature.

This paper is thus intended to provide a thorough and, we hope, readable account of the generalization of the classic theory of Jacobi fields and Morse indices to the general setting of mechanical systems with reflections.

Structure of the paper. We first work with paths with reflections off the hypersurface/boundary. In Section 2, we consider permissible path spaces and variation vector fields in our variations. In Section 3, we define the action of a path and consider the first variation of the action. In Section 4, we consider the second variation of the action and the corresponding Jacobi fields. In Section 5, we prove the Morse index theorem for the closed reflected paths as well as dealing with the more usual case of paths with fixed endpoints.

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2. Path space of reflected trajectories

Consider a smooth Riemannian manifold with boundary (M, g) and a compact embedded hypersurface Y, possibly disconnected. We assume that $\partial M \subset Y$. Near any point on Y, we may choose local Fermi coordinates $(x^1, \ldots, x^n)$ so that locally $Y = \{x^1 = 0\}$; thus x^1 is the signed distance to Y (or, near points where Y coincides with ∂M, simply the distance) and locally the metric is of the form

$$g = (dx^1)^2 + \sum_{i,j=2}^{n} h_{ij}(x) dx^i dx^j$$

For $W \in T_Y M$ we adopt the notation

$$W = W_\perp + W_\top$$

to denote the splitting of W into normal and tangent components, using the metric, i.e., if $W = \sum W^j \partial_{x^j}$ in Fermi coordinates, then

$$W_\perp = W^1 \partial_{x^1}, \quad W_\top = \sum_{j=2}^{n} W^j \partial_{x^j}$$

We will occasionally use the subscript 1 to denote the x^1 component of curves or vector fields; in particular, then, with a choice of an oriented unit normal to Y, e.g. $N = \partial_{x^1}$ in Fermi normal coordinates, we write

$$W_\perp = W_1 N.$$

Throughout this paper we assume that $V \in \mathcal{C}^\infty(M \backslash Y; \mathbb{R}) \cap \mathcal{C}^1(M)$ and that V is $\mathcal{C}^\infty$ smooth up to Y, separately from each side if Y is locally an interior hypersurface. As noted above, the allowed discontinuities of second or higher derivatives of V across Y are not especially interesting in the context of the geometric considerations here, and are included for the sake of future applications to Schrödinger operators (for which the derivative discontinuities of V reflect energy).

We will consider the variation problem for the action associated to the Lagrangian $L \in \mathcal{C}^\infty(TM)$ given by

$$L(x, v) = \frac{1}{2} g_{ij}(x) v^i v^j - V(x).$$

Implicitly, then, we are dealing with the Hamiltonian dynamics for the Hamiltonian function on T^*M given by Legendre transform:

$$\frac{1}{2} g^{ij}(x) \xi_i \xi_j + V(x).$$

2.1. **Path spaces and variations.** Fix a time T; this will be left implicit in our notation for path spaces. In what follows we will use the notation $\bullet(t\pm)$ for $\lim_{\epsilon\downarrow 0}\bullet(t\pm\epsilon)$, with $\bullet$ denoting a function, vector field, etc., depending on t.

Our path space is defined as follows:

Definition 2.1. *Let $0 = T_0 < T_1 < \cdots < T_m < T_{m+1} = T$, with $\{T_i : 1 \leq i \leq m\} = \mathcal{R}\cup\mathcal{K}$ a partition into a set of* reflection *times $\mathcal{R}$ and a set of* kink *times $\mathcal{K}$. A* reflected path *with reflection times $\mathcal{R}$ and kink times $\mathcal{K}$ is a continuous map $\alpha : [0, T] \to M$ such that*

1. *If $\alpha(t) \in \partial M$ then there exists i such that $t = T_i \in \mathcal{R}$.*
2. *For each $0 \leq i \leq m$, α restricted to $[T_i, T_{i+1}]$ is smooth (i.e., smooth in the interior with derivatives extending to the boundary of each subinterval).*
3. *If $T_i \in \mathcal{R}$, then $\alpha(T_i) \in Y$, and if x^1 is a defining function for Y and $I_i \ni T_i$ is a sufficiently small open interval then the sign of $x^1\circ\alpha$ is constant on $I_i\backslash\{T_i\}$.*
4. *If $T_i \in \mathcal{R}$, then $(x^1 \circ \alpha)'(T_i\pm) \neq 0$, i.e., α is not tangent to Y at T_i from either direction.*

More generally, a piecewise smooth path α equipped with a set $\mathcal{R}$ of reflection times is said to be a reflected path if there exist some choice of kink times $\mathcal{K}$ such that the above definition applies.

Remark 2.2.

- We allow $\alpha(t) \in Y$ even if $t \notin \mathcal{R}$; the importance of the reflection times arises in the requirement that the paths do intersect Y transversely at the reflection times and stay on the same side of Y before and after these times, and in the following definition of allowed variations, which will ensure that physical paths must be reflected rather than allowing transmission across Y at times in $\mathcal{R}$. The specification of the reflection times is part of the data of the path; the kink times, by contrast, are not.
- The kink times $\mathcal{K}$ should be thought of times, outside of $\mathcal{R}$, where α is allowed to fail to be smooth; such a set of times is in general not unique. Indeed, if α is a reflected path with reflection and kink times $\mathcal{R}$ and $\mathcal{K}$, and $\mathcal{K}'$ is any finite set with $\mathcal{K} \subset \mathcal{K}' \subset [0, T]\backslash\mathcal{R}$, then α is a reflected path with reflection and kink times $\mathcal{R}$ and $\mathcal{K}'$ as well. Note that α always admits a *minimal* kink time set, namely
$$\mathcal{K}_{min} = \{t \in (0, T)\backslash\mathcal{R} : \alpha \text{ is not } \mathcal{C}^\infty \text{ at } t\},$$
and any other set of kink times $\mathcal{K}$ satisfies $\mathcal{K} \supset \mathcal{K}_{min}$. As we see below, we will sometimes introduce additional kink times, where α is in fact smooth, in order to consider variations which develop kinks at those times.

Let α be a reflected path from $\alpha(0) = p$ to $\alpha(T) = p'$, with reflection times $\mathcal{R}$ and kink times $\mathcal{K}$ as in the definition above, so that $\mathcal{R}\cup\mathcal{K} = \{T_1, \ldots, T_m\}$.

Definition 2.3. *A variation of $\alpha(t)$ is a map $\alpha(t, \epsilon)$ from $[0, T]\times(-\epsilon_0, \epsilon_0)$ to M, together with a family of smooth functions $0 < T_1(\epsilon) < \cdots < T_m(\epsilon) < T$, divided into a family of reflected time functions $\tilde{\mathcal{R}}$ and kink time functions $\tilde{\mathcal{K}}$ with $T_i(\epsilon) \in \tilde{\mathcal{R}} \iff T_i \in \mathcal{R}$, such that*

(1) *$\alpha(t,0) = \alpha(t)$ for $t \in [0,T]$ and $T_i(0) = T_i$ for $i = 1,\dots,m$.*
(2) *For any fixed ϵ, $\alpha(\cdot,\epsilon)$ is a reflected path as defined in Definition 2.1, with reflection times $\{T_i(\epsilon) : T_i \in \mathcal{R}\}$ and kink times $\{T_i(\epsilon) : T_i \in \mathcal{K}\}$.*
(3) *$\alpha(t,\epsilon)$ is smooth, up to the boundary, on each set of the form*

$$\{(t,\epsilon) : T_i(\epsilon) \le t \le T_{i+1}(\epsilon), \epsilon \in (-\epsilon_0,\epsilon_0)\}, \quad 0 \le i \le m$$

(where we interpret $T_0(\epsilon) = 0$ and $T_{m+1}(\epsilon) = T$).

A two-parameter variation is defined analogously, with $\epsilon \in (-\epsilon_0,\epsilon_0)$ replaced by $(\delta,\epsilon) \in (-\delta_0,\delta_0) \times (-\epsilon_0,\epsilon_0)$.

Remark 2.4. Note that we allow the kink times to vary in ϵ, which is not the standard prescription e.g. in [9]. This is useful owing to the non-smoothness of physical paths at times of transmission across Y, which may vary in families.

Throughout this paper, we shall consider the following three families of paths (with possibly multiple reflections)

(1) the *space of reflected paths* $\Omega(M)$: see Definition 2.1.
(2) the *space of reflected paths with fixed endpoints* p, p':

$$\Omega_0(M;p,p') = \{\alpha \in \Omega(M) : \alpha(0) = p,\, \alpha(T) = p'\}$$

(3) the *space of periodic reflected paths* where the endpoints are not fixed but need to be equal, i.e.,

$$\Omega_{\mathrm{per}}(M) = \{\alpha \in \Omega(M) : \alpha(0) = \alpha(T)\}.$$

Note that we do not require the derivative to match for $t = 0, T$, but this is consistent with our convention that elements of $\Omega(M)$ are only piecewise smooth; the endpoint $\alpha(0)$ thus plays no distinguished role, as there may or may not be a derivative discontinuity there.

Then we have

$$\Omega_0(M;p,p) \subset \Omega_{\mathrm{per}}(M) \subset \Omega(M) \;\text{ and }\; \Omega_0(M;p,p') \subset \Omega(M).$$

For notational convenience, we introduce notation for jumps and averages of vector fields along α at reflections and kinks. Recall that for Z any vector field along α and $t \in [0,T]$, we let

$$Z(t\pm) := \lim_{\epsilon\to 0^+} Z(t \pm \epsilon);$$

sometimes we denote this $Z^{\pm}$ if the time of evaluation t is understood. We additionally set

$$\triangle Z(t) = Z(t+) - Z(t-) \text{ and } \overline{Z}(t) = \tfrac{1}{2}(Z(t+) + Z(t-)).$$

We now anticipate the outcome of our analysis of first variations by defining reflected physical paths; the relevance of this definition is demonstrated by Lemma 3.2. In the following definition (and henceforth) we denote the covariant derivative along a path by

$$D_t := \nabla_{\dot\alpha}.$$

Definition 2.5. *We say that a reflected path $\alpha(t)$, $t \in I = [0,T]$ is a* reflected physical path *if the following conditions hold:*

(1) $\alpha \in \Omega(M)$
(2) $D_t\dot{\alpha} + \nabla V(\alpha(t)) = 0$ *on* $I \backslash \{T_1, \dots, T_m\}$.
(3) *For all* $T_j \in \mathcal{R}$, $\dot{\alpha}_\perp(T_j) = 0$ *and* $\triangle \dot{\alpha}_\top(T_j) = 0$.
(4) *For all* $T_j \in \mathcal{K}$, $\triangle \dot{\alpha}(T_j) = 0$.

In addition, if $\alpha(t)$ *also satisfies endpoint conditions* $\alpha(0) = \alpha(T)$ *and* $\dot{\alpha}(0) = \dot{\alpha}(T)$*, it is called a* periodic reflected physical path.

Remark 2.6. Since α and $\dot{\alpha}$ are continuous at points in $\mathcal{K}$ and since α solves a second order ODE with smooth coefficients away from Y, a reflected physical path α is in fact smooth at *interior* kinks. Since $\nabla^2 V$ is allowed to be discontinuous at Y, however, third derivatives of α may be discontinuous at kinks in Y, i.e., at times $T_j \in \mathcal{K}$ with $\alpha(T_j) \in Y$. Nonetheless, α will be C^2 at such kinks, and hence for a reflected physical path α, the reflective times $\mathcal{R}$ can be characterized as the times where α is continuous, but not C^2.

We now show that, given a reflected physical path, we can perturb the initial position and velocity to uniquely produce another reflected physical path which reflects at similar times.

Lemma 2.7. *Let* $\alpha(t)$ *be a reflected physical path, with initial position and velocity* $(\alpha(0), \dot{\alpha}(0)) \in TM$*. For sufficiently small* $\epsilon > 0$*, there exists a neighborhood* V_ϵ *of* $(\alpha(0), \dot{\alpha}(0))$ *in* TM *with the property that for all* $(x, v) \in V_\epsilon$*, there exists a unique reflected physical path* $\alpha_{x,v}$ *such that* $(\alpha_{x,v}(0), \dot{\alpha}_{x,v}(0)) = (x, v)$*,* $\alpha_{x,v}$ *has reflection times* $\mathcal{R}_{x,v}$ *with* $|\mathcal{R}_{x,v}| = |\mathcal{R}|$*, and, if* $\mathcal{R} = \{T_1 < \cdots < T_r\}$ *and* $\mathcal{R}_{x,v} = \{\tilde{T}_1 < \cdots < \tilde{T}_r\}$*, we have* $|\tilde{T}_i - T_i| \leq \epsilon$ *for* $i = 1, \dots, r$*.*

Proof. The idea is that since a reflected physical path solves a second-order ODE classically up to reflection times, it is uniquely specified, up until the reflection time, by its initial position and velocity; at reflection times the path experiences a jump in its velocity uniquely specified by the reflection condition (3) in Definition 2.5, which uniquely specifies the path until the next reflection, and so on. We make this idea rigorous below.

Suppose for convenience that $\mathcal{R}$ consists of a single time T_1, i.e. α reflects just once; let x^1 be a boundary defining function of Y near $\alpha(T_1)$. Let $\epsilon > 0$ be sufficiently small, so that there exists a neighborhood W of $(\alpha(T_1), \dot{\alpha}(T_1-)) \in TM \backslash TY$ satisfying the following technical assumption: if β is a $C^2([0, 2\epsilon])$ solution to $D_t\dot{\beta}(t) + \nabla V(\beta(t)) = 0$ in $(0, 2\epsilon)$, with $\beta(0) \in Y$ and $(\beta(0), \dot{\beta}(0))$ or $(\beta(0), Q\dot{\beta}(0))$ is in W, where $Q : T_Y M \to T_Y M$ is the reflection across Y, then sgn $(x^1 \circ \beta)'(t)$ is constant on $(0, 2\epsilon)$. (That is, reflected physical paths starting on Y with initial velocity in W or $Q(W)$ will always move away from the boundary and will not return in time 2ϵ.) Such a neighborhood exists for ϵ sufficiently small by the non-tangency assumption of α at reflected times.

Given (x, v) near $(\alpha(0), \dot{\alpha}(0))$, we construct a nearby reflected physical path $\alpha_{x,v}$ as follows: we note that if (x, v) is sufficiently close to $(\alpha(0), \dot{\alpha}(0))$ (or any vector if M is complete), then there exists a unique C^2 solution on $[0, T_1 + \epsilon]$ to

$$D_t\dot{\tilde{\alpha}}(t) + \nabla V(\tilde{\alpha}(t)) = 0 \text{ in } (0, T_1 + \epsilon), \quad (\tilde{\alpha}(0), \dot{\tilde{\alpha}}(0)) = (x, v).$$

Note that such a solution, if it intersects Y, does not reflect off Y. Moreover, if (x, v) is sufficiently close to $(\alpha(0), \dot\alpha(0))$, then the corresponding path $\tilde\alpha$ intersects Y at some time in $[T_1 - \epsilon, T_1 + \epsilon]$; let $\tilde T_1$ denote the first such time. Finally, for (x, v) sufficiently close to $(\alpha(0), \dot\alpha(0))$, we can also arrange for $(\tilde\alpha(\tilde T_1), \dot{\tilde\alpha}(\tilde T_1-)) \in W$. We let V be a neighborhood of $(\alpha(0), \dot\alpha(0))$ such that its elements satisfy all of the conditions above.

Then, for $(x, v) \in V$, we construct $\alpha_{x,v}$ as follows. For $t \in [0, \tilde T_1]$, we set $\alpha_{x,v}(t) = \tilde\alpha(t)$ as above. For $t \in [\tilde T_1, T]$, we let $\alpha_{x,v}$ be the unique $C^2([\tilde T_1, T])$ solution to

$$D_t\dot\alpha_{x,v}(t) + \nabla V(\alpha_{x,v}(t)) = 0 \text{ in } (\tilde T_1, T), \quad (\alpha_{x,v}(\tilde T_1), \dot\alpha_{x,v}(\tilde T_1+)) = (\tilde\alpha(T_1), Q\dot{\tilde\alpha}(\tilde T_1-)).$$

Then, by construction, $\alpha_{x,v}$ is a reflected physical path, with one reflection at $\tilde T_1$ satisfying $|\tilde T_1 - T_1| \le \epsilon$ (note that $\dot\alpha_{x,v}(\tilde T_1-) = \dot{\tilde\alpha}(\tilde T_1-) \in W \subset TM\backslash TY$ guarantees the non-tangency condition). This shows the existence for $(x, v) \in V$.

For uniqueness, we note that if $\beta_{x,v}$ is another reflected physical path satisfying $(\beta_{x,v}(0), \dot\beta_{x,v}(0)) = (x, v)$ with exactly one reflection time τ_1 satisfying $|\tau_1 - T_1| \le \epsilon$, then $\beta_{x,v}$ is C^2 on $[0, \tau_1]$, and in particular it must agree with $\alpha_{x,v}$ up to time $T_1 - \epsilon$, after which it continues to agree with $\alpha_{x,v}$ until $\alpha_{x,v}$ hits Y, i.e. at time $\tilde T_1$. The only way $\beta_{x,v}$ does not agree with $\alpha_{x,v}$ after that is if $\beta_{x,v}$ transmits through Y instead of reflecting across Y, i.e. $\dot\beta_{x,v}(\tilde T_1+)$ equals $\dot\alpha_{x,v}(\tilde T_1-)$ instead of its reflection. However, by the technical assumption made above, this would force $\beta_{x,v}$ to not intersect Y again in $(\tilde T_1, \tilde T_1 + 2\epsilon]$, and in particular it will not reflect at a time within ϵ of T_1. This forces $\beta_{x,v}$ to reflect at $\tilde T_1$, and since there are no other reflections, this means $\beta_{x,v}$ is a C^2 solution on $[\tilde T_1, T]$ whose value and derivative agrees with those of $\alpha_{x,v}$ at $\tilde T_1$, forcing $\beta_{x,v} = \alpha_{x,v}$ on $[\tilde T_1, T]$ as well. This gives uniqueness as well.

The case for multiple reflections is similar, by performing the above technical constructions in a neighborhood of each reflection time.

□

Given a variation $\alpha(t, \epsilon)$ along a family of reflected paths, we can consider the tangent vector field

$$Z(t) = \frac{\partial\alpha}{\partial\epsilon}\Big|_{\epsilon=0}$$

along $\alpha(t)$. Note that the ϵ derivative is only well-defined on $[0, T]\backslash(\mathcal{R} \cup \mathcal{K})$; however the one-sided limits $Z(T_j\pm)$ exist for all $T_j \in \mathcal{R} \cup \mathcal{K}$. We use this notion to define corresponding tangent spaces $T_\alpha\Omega(M)$, $T_\alpha\Omega_0(M; p, p')$, and $T_\alpha\Omega_{\text{per}}(M)$. These spaces are characterized by our enforcement of the continuity conditions at the boundary, as follows.

Lemma 2.8. *Let $\alpha(t, \epsilon) \in \Omega(M)$ be a family of reflected paths with reflection time functions $\tilde{\mathcal{R}}$ and kink time functions $\tilde{\mathcal{K}}$, and let $\tilde{\mathcal{R}} \cup \tilde{\mathcal{K}} = \{T_i(\epsilon) : 1 \le i \le m\}$. Then, for each $1 \le i \le m$, the variation vector field $Z = \partial\alpha/\partial\epsilon$ satisfies the jump condition*

$$T_i'(0)\dot\alpha(T_i-) + Z(T_i-) = T_i'(0)\dot\alpha(T_i+) + Z(T_i+). \tag{1}$$

If in addition we have $T_i \in \mathcal{R}$, then we have the additional condition that

$$T_i'(0)\,\dot\alpha_\perp(T_i-) + Z_\perp(T_i-) = T_i'(0)\,\dot\alpha_\perp(T_i+) + Z_\perp(T_i+) = 0. \tag{2}$$

In particular, if α is $\mathcal{C}^1$ at T_i, then $Z(T_i-) = Z(T_i+)$, i.e. at such times we may view Z as being well-defined and continuous at T_i.

Proof. For each i we have the continuity equation

$$\alpha(T_i(\epsilon)-, \epsilon) = \alpha(T_i(\epsilon)+, \epsilon).$$

Differentiating in ϵ and evaluating at $\epsilon = 0$ yields (1). If, in addition, $T_i \in \mathcal{R}$, then as usual letting $\alpha_1(t)$ denote the signed distance from the boundary (first component in Fermi coordinates), for all ϵ,

$$\alpha_1(T_i(\epsilon)\pm, \epsilon) = 0$$

(since the path is in the boundary at time $T_i(\epsilon)$). Differentiating in ϵ and evaluating at $\epsilon = 0$ yields (2). □

Lemma 2.9. *Let $\alpha \in \Omega(M)$ be a reflected path with reflections and kinks at $\mathcal{R}$ and $\mathcal{K}$, respectively, with $\mathcal{R} \cup \mathcal{K} = \{T_i : 1 \leq i \leq m\}$. Let Z be a vector field along α defined on $[0,T]\backslash(\mathcal{R} \cup \mathcal{K})$ such that $Z|_{(T_i,T_{i+1})}$ extends smoothly to $[T_i, T_{i+1}]$ for all $0 \leq i \leq m$, and that for some $\mu_1, \dots, \mu_m \in \mathbb{R}$,*

$$\mu_i\dot{\alpha}(T_i-) + Z(T_i-) = \mu_i\dot{\alpha}(T_i+) + Z(T_i+) \tag{3}$$

and

$$\mu_i\dot{\alpha}_\perp(T_i-) + Z_\perp(T_i-) = \mu_i\dot{\alpha}_\perp(T_i+) + Z_\perp(T_i+) = 0 \quad \text{if } T_i \in \mathcal{R}. \tag{4}$$

Then there is a variation $\alpha(t, \epsilon) \in \Omega(M)$ of α with $Z = \partial\alpha/\partial\epsilon|_{\epsilon=0}$ and with $T_i'(0) = \mu_i$.

Remark 2.10. (3) can be rewritten as

$$\triangle Z(T_i) = -\mu_i \triangle \dot{\alpha}(T_i).$$

Moreover, at reflection times $T_i \in \mathcal{R}$, the two conditions (3) and (4) can be rephrased as coupled jump conditions on the tangential and normal components, via

$$Z_\perp(T_i\pm) = -\mu_i\dot{\alpha}_\perp(T_i\pm), \quad \triangle Z_\top(T_i) = -\mu_i\triangle\dot{\alpha}_\top(T_i).$$

Finally, given the corresponding endpoint conditions, we may obtain variations $\alpha(t, \epsilon) \in \Omega_0(M; p, p')$ or $\Omega_{\mathrm{per}}(M)$ from conditions (3) and (4) as in the lemma.

Proof. We construct such a variation explicitly. The idea is that we consider *some* variation whose derivative is Z, and then correct for reflection/kink conditions.

For each $T_i \in \mathcal{R} \cup \mathcal{K}$, choose a neighborhood I_i in t such that $I_i \cap (\mathcal{R} \cup \mathcal{K}) = \{T_i\}$, and $\alpha(I_i)$ is contained in an open set trivializable by local coordinates, where if $T_i \in \mathcal{R}$, then the local coordinates are Fermi coordinates oriented so that $\alpha_1 \geq 0$ for $t \in I_i$. Let $I = \cup I_i$. For $t \in [0,T]\backslash I$, we define

$$\alpha(t, \epsilon) := \exp_{\alpha(t)}(\epsilon Z(t)),$$

where exp is the exponential map with respect to some Riemannian metric (e.g. the metric g on M), smoothly extended across the boundary in the case where $Y = \partial M$ locally. Note that this is well-defined and smooth on $([0,T]\backslash I) \times (-\epsilon_0, \epsilon_0)$ for sufficiently small $\epsilon_0 > 0$. Moreover,

$$\frac{\partial\alpha}{\partial\epsilon}|_{\epsilon=0}(t) = Z(t) \text{ on } [0,T]\backslash I.$$

For $t \in I_i$, $1 \le i \le m$, we define α via local coordinates $(x^1, \dots, x^n)$ (which are local Fermi coordinates if $T_i \in \mathcal{R}$), i.e. we define the values of $\alpha_j(t,\epsilon) = x^j \circ \alpha(t,\epsilon)$ for $1 \le j \le n$. Write $\alpha^- = \alpha|_{(T_{i-1},T_i)}$ and $\alpha^+ = \alpha|_{(T_i,T_{i+1})}$, define $Z^\pm$ similarly, and extend $\alpha^\pm$, $Z^\pm$ smoothly in a neighborhood of T_i. Set $T_i(\epsilon) = T_i + \epsilon\mu_i$ so that $T_i'(0) = \mu_i$, and for $t \in I_i$, let

$$\alpha_j(t,\epsilon) = \begin{cases} (\exp_{\alpha^-(t)}(\epsilon Z^-(t)))_j + r_j^-(\epsilon)\varphi(t) & t \le T_i(\epsilon) \\ (\exp_{\alpha^+(t)}(\epsilon Z^+(t)))_j + r_j^+(\epsilon)\varphi(t) & t \ge T_i(\epsilon) \end{cases}$$

where $\varphi(t) \in C_c^\infty(I_i)$ is identically equal to 1 in a small neighborhood of T_i. The vector-valued functions $r^\pm(\epsilon)$ are defined depending on whether $T_i \in \mathcal{R}$ or $T_i \in \mathcal{K}$: if $T_i \in \mathcal{R}$, set

$$\begin{aligned} &r_1^-(\epsilon) = -(\exp_{\alpha^-(T_i(\epsilon))}(\epsilon Z^-(T_i(\epsilon))))_1, \quad r_j^-(\epsilon) = 0 \text{ for } j \ge 2, \\ &r_1^+(\epsilon) = -(\exp_{\alpha^+(T_i(\epsilon))}(\epsilon Z^+(T_i(\epsilon))))_1, \\ &r_j^+(\epsilon) = (\exp_{\alpha^-(T_i(\epsilon))}(\epsilon Z^-(T_i(\epsilon))))_j - (\exp_{\alpha^+(T_i(\epsilon))}(\epsilon Z^+(T_i(\epsilon))))_j \text{ for } j \ge 2. \end{aligned}$$

If $T_i \in \mathcal{K}$, set $r^-(\epsilon) = 0$ and

$$r_j^+(\epsilon) = (\exp_{\alpha^-(T_i(\epsilon))}(\epsilon Z^-(T_i(\epsilon))))_j - (\exp_{\alpha^+(T_i(\epsilon))}(\epsilon Z^+(T_i(\epsilon))))_j \text{ for } 1 \le j \le n.$$

We first verify that this construction produces paths $\alpha(\cdot,\epsilon)$ which are reflected paths for ϵ sufficiently small and that the construction makes sense in the case $Y = \partial M$, i.e., that the constructed family of paths stays in M rather than passing into an extension across the boundary. By construction, $\alpha(\cdot,\epsilon)$ is continuous on I_i and is smooth on $I_i \backslash \{T_i(\epsilon)\}$. Furthermore, if $T_i \in \mathcal{R}$, then taking $\operatorname{supp}\varphi$ sufficiently small, nonnegativity of α_1 and nontangency allow us to ensure

$$\operatorname{sgn} \dot\alpha_1^\pm = \pm 1 \quad \text{on } \operatorname{supp}\varphi.$$

Then taking $\epsilon > 0$ sufficiently small ensures that the same holds for the varied path, i.e.,

$$\operatorname{sgn} \dot\alpha_1^\pm(t,\epsilon) = \pm 1 \quad \text{on } \operatorname{supp}\varphi.$$

For small ϵ it is also the case that $\alpha_1 \neq 0$ for $t \in I_i \backslash \operatorname{supp}\varphi$. Since $\alpha_1^\pm(T_i(\epsilon),\epsilon) = 0$ by construction, this shows that $\alpha_1^\pm$ remains nonnegative for $t \in I_i$, and vanishes only at $t = T_i(\epsilon)$. In particular, then, the family of paths remains in M even when $Y = \partial M$ locally. Finally, we have

$$\dot\alpha_1^\pm(T_i(\epsilon),\epsilon) = \dot\alpha_1^\pm(T_i(\epsilon)) + \epsilon Z_1^\pm(T_i(\epsilon)) + r_1^\pm(\epsilon)\varphi'(T_i(\epsilon)) + O(\epsilon^2),$$

and this is nonzero for ϵ sufficiently small since $\dot\alpha_1^\pm(T_i) \neq 0$ and $r_1^\pm(\epsilon) \to 0$ as $\epsilon \to 0$. Hence, $\alpha(t,\epsilon)$ intersects Y transversely at $t = T_i(\epsilon)$. Thus, $\alpha(t,\epsilon)$ is a family of reflected paths.

Finally, need to check that the first derivative of α is in fact Z. We clearly have $\frac{\partial\alpha}{\partial\epsilon}|_{\epsilon=0} = Z$ on $[0,T]\backslash I$, while on I_i we have

$$\frac{\partial\alpha}{\partial\epsilon}|_{\epsilon=0} = \begin{cases} Z^-(t) + (r^-)'(0)\varphi(t) & t \in I_i, t < T_i \\ Z^+(t) + (r^+)'(0)\varphi(t) & t \in I_i, t > T_i \end{cases}.$$

Thus it suffices to show that $(r^{\pm})'(0) = 0$. If $T_i \in \mathcal{R}$, we have

$$(r_1^{\pm})'(0) = -(\mu_i \dot{\alpha}_1^{\pm}(T_0) + Z_1^{\pm}(T_0)) = 0,$$

with the last equality following from (4), while for $j \geq 2$, by (3) we have

$$(r_j^-)'(0) = 0, \quad (r_j^+)'(0) = (\mu_i(\dot{\alpha}_j^-(T_0) - \dot{\alpha}_j^+(T_0)) + (Z_j^-(T_0) - Z_j^+(T_0))) = 0.$$

It follows that $(r^{\pm})'(0) = 0$, as desired. Similar calculations show that $(r^{\pm})'(0) = 0$ in the case that $T_i \in \mathcal{K}$ as well. □

Remark 2.11. An analogous statement and proof holds for constructing two-parameter family of variations with admissible pairs of variation vector fields Z and W (say satisfying (3) and (4) with μ_i, ν_i), by defining $T_i(\epsilon, \delta) = T_i + \epsilon\mu_i + \delta\nu_i$, $\alpha(t, \epsilon, \delta) = \exp_{\alpha(t)}(\epsilon Z(t) + \delta W(t))$ away from reflections and kinks, and correcting analogously near the reflections/kinks. We omit the proof for brevity.

Corollary 2.12. *For any choice of $V_i \in T_{\alpha(T_i)}M$ such that $V_i \in T_{\alpha(T_i)}Y$ when $T_i \in \mathcal{R}$, there exists a variation $\alpha(t, \epsilon) \in \Omega(M)$ such that $\frac{\partial \alpha}{\partial \epsilon}|_{\epsilon=0}$ is continuous for all t, with*

$$\frac{\partial \alpha}{\partial \epsilon}|_{\epsilon=0}(T_i) = V_i.$$

Moreover, such a variation can be chosen so that the corresponding reflection times $T_i(\epsilon)$ satisfy $T_i'(0) = 0$.

Proof. Let Z be *any* smooth vector field with $Z(T_i) = V_i$. Then Z satisfies (3) and (4) with all μ_i equal to 0. □

Corollary 2.13. *Let $\mu_1, \ldots, \mu_m$ be any collection of numbers. Then there exists a variation $\alpha(t, \epsilon) \in \Omega(M)$ where the corresponding reflection times $T_i(\epsilon)$ satisfy $T_i'(0) = \mu_i$.*

Proof. Let Z be smooth on $[0, T] \backslash (\mathcal{R} \cup \mathcal{K})$ with limits at $T_i \in \mathcal{R}$ determined by

$$Z_\perp(T_i-) = -\mu_i \dot{\alpha}_\perp(T_i-), \quad Z_\top(T_i-) \text{ arbitrary},$$
$$Z(T_i+) = Z(T_i-) - \mu_i(\dot{\alpha}(T_i+) - \dot{\alpha}(T_i-)),$$

and with limits at $T_i \in \mathcal{K}$ determined by

$$Z(T_i-) \text{ arbitrary}, \quad Z(T_i+) = Z(T_i-) - \mu_i(\dot{\alpha}(T_i+) - \dot{\alpha}(T_i-)).$$

Then Z satisfies (3) and (4) with the prescribed values of μ_i. □

Corollary 2.14. *Let $\alpha(t)$ be a reflected physical path. Then a piecewise smooth vector field Z along α is a variation vector field if and only if:*

- *Z is smooth on $[0, T] \backslash (\mathcal{R} \cup \mathcal{K})$,*
- *At reflection times $T_i \in \mathcal{R}$, we have*

$$\triangle Z_\top(T_i) = 0, \quad \overline{Z}_\perp(T_i) = 0$$

(i.e. $\triangle Z$, resp. $\overline{Z}$, is normal, resp. tangent, to the hypersurface).

- *At kink times $T_i \in \mathcal{K}$, we have*

$$\triangle Z(T_i) = 0$$

(i.e. Z is continuous at kink times).

Proof. This follows by rewriting the conditions in Lemmas 2.8 and 2.9, using that in a reflected physical path we have the conditions $\overline{\dot\alpha_\perp(T_i)} = 0$ and $\triangle\dot\alpha_\top(T_i) = 0$ if $T_i \in \mathcal{R}$ and $\triangle\dot\alpha(T_i) = 0$ if $T_i \in \mathcal{K}$. □

Let α be a reflected path, with reflection times $\mathcal{R}$. By Remark 2.2, α admits a minimal kink time set $\mathcal{K}_{min}$. Let $\mathcal{V}(\alpha)$ denote the set of vector fields along α. By Lemmas 2.8 and 2.9, we may identify the tangent spaces to our various path spaces as follows:

$$\begin{aligned} T_\alpha\Omega(M) = \{&W \in \mathcal{V}(\alpha) : \exists\, \mathcal{K} \supset \mathcal{K}_{min} \text{ s.t. } W \text{ is smooth on } [0,T]\backslash(\mathcal{R}\cup\mathcal{K}) \\ &\text{satisfying jump conditions (3) and (4) at } \mathcal{R}\cup\mathcal{K}\}\,, \\ T_\alpha\Omega_0(M;p,p') = \{&W \in T_\alpha\Omega(M) : W(0) = W(T) = 0\} \\ T_\alpha\Omega_{\text{per}}(M) = \{&W \in T_\alpha\Omega(M) : W(0) = W(T)\}. \end{aligned}$$

If α is a periodic reflected path with $\alpha(0) = \alpha(T) = p$, then

$$T_\alpha\Omega_0(M;p,p) \subset T_\alpha\Omega_{\text{per}}(M) \subset T_\alpha\Omega(M),$$

while if α is a (reflected) path from p to p' (with p not necessarily equal to p') then

$$T_\alpha\Omega_0(M;p,p') \subset T_\alpha\Omega(M).$$

Remark 2.15. Note that the definition of $T_\alpha\Omega(M)$ allows for freedom in the choice of $\mathcal{K}$: in addition to points on α that fail to be $\mathcal{C}^\infty$, which by definition occur at times in $\mathcal{K}_{min}\cup\mathcal{R}$, we may always add "fictitious" extra kink points (i.e. $\mathcal{K}\backslash\mathcal{K}_{min}$) where we do not enforce continuity of derivatives of variation vector fields; note at those points that α is smooth, and hence the jump condition at those points reduces to the condition of continuity. Consequently an authentic kink is allowed to develop in the varied paths at such points.

Moreover, the space $T_\alpha\Omega(M)$ (and hence $T_\alpha\Omega_0(M;p,p')$ and $T_\alpha\Omega_{\text{per}}(M)$) is in fact a vector space. Indeed, given $Z_1, Z_2 \in T_\alpha\Omega(M)$ with kink times $\mathcal{K}_1$ and $\mathcal{K}_2$, we may view both of them as having kink times at $\mathcal{K}_1\cup\mathcal{K}_2$ by adding "fictitious" kink points, and hence it is clear that Z_1+Z_2 satisfies the jump conditions (3) and (4) at $\mathcal{R}\cup(\mathcal{K}_1\cup\mathcal{K}_2)$.

3. Actions along paths and first variations

In the following sections, we shall consider the action functional on a family of reflected paths $\{\alpha(\cdot,\epsilon) : \epsilon \in (-\epsilon_0,\epsilon_0)\} \subset \Omega_0(M;p,p')$. The action functional is given by

$$
\begin{aligned}
J[\alpha(\cdot,\epsilon)] &= \int_0^T \frac{1}{2}|\dot\alpha(t,\epsilon)|_g^2 - V(\alpha(t,\epsilon))\,dt \\
&:= \sum_{i=0}^{m} \int_{T_i(\epsilon)}^{T_{i+1}(\epsilon)} \frac{1}{2}|\dot\alpha(t,\epsilon)|_g^2 - V(\alpha(t,\epsilon))\,dt,
\end{aligned} \tag{5}
$$

where as usual we take $T_0(\epsilon) = 0$ and $T_{m+1}(\epsilon) = T$.

3.1. **First variations.** The following lemma gives the derivative of the actions (5).

Lemma 3.1. *The derivative* $\frac{d}{d\epsilon}J[\alpha(\cdot,\epsilon)]$ *is given by*

$$
-\int_0^T \left\langle D_t\dot\alpha + \nabla V(\alpha), \frac{\partial\alpha}{\partial\epsilon}(t,\epsilon)\right\rangle dt + \sum_{i=1}^m \left\langle -\triangle\dot\alpha(T_i(\epsilon),\epsilon), \overline{\frac{\partial\alpha}{\partial\epsilon}}(T_i(\epsilon),\epsilon)\right\rangle. \tag{6}
$$

Proof. For each term in the sum in the second line of equation (5), we differentiate in ϵ to obtain

$$
\begin{aligned}
&\frac{d}{d\epsilon}\left(\int_{T_i(\epsilon)}^{T_{i+1}(\epsilon)} \frac{1}{2}|\dot\alpha(t,\epsilon)|_g^2 - V(\alpha(t,\epsilon))\,dt\right) = \int_{T_i(\epsilon)}^{T_{i+1}(\epsilon)} \left\langle \dot\alpha, D_t\frac{\partial\alpha}{\partial\epsilon}\right\rangle - \left\langle \nabla V(\alpha), \frac{\partial\alpha}{\partial\epsilon}\right\rangle dt \\
&+ T_{i+1}'(\epsilon)\left(\frac{1}{2}|\dot\alpha(T_{i+1}(\epsilon)-,\epsilon)|_g^2 - V(\alpha(T_{i+1}(\epsilon),\epsilon))\right) - T_i'(\epsilon)\left(\frac{1}{2}|\dot\alpha(T_i(\epsilon)+,\epsilon)|_g^2 - V(\alpha(T_i(\epsilon),\epsilon))\right),
\end{aligned}
$$

where the integral term follows using the metric compatibility and the torsion-free property of the Levi-Civita connection (the latter yields $\frac{\partial}{\partial\epsilon}\dot\alpha = D_t\frac{\partial\alpha}{\partial\epsilon}$); note that we can evaluate α at $T_i(\epsilon)$ since α is continuous. We can then use the metric compatibility of ∇ to integrate by parts, yielding

$$
\begin{aligned}
&\int_{T_i(\epsilon)}^{T_{i+1}(\epsilon)} \left\langle \dot\alpha, D_t\frac{\partial\alpha}{\partial\epsilon}\right\rangle - \left\langle \nabla V(\alpha), \frac{\partial\alpha}{\partial\epsilon}\right\rangle dt \\
&= -\int_{T_i(\epsilon)}^{T_{i+1}(\epsilon)} \left\langle D_t\dot\alpha + \nabla V(\alpha), \frac{\partial\alpha}{\partial\epsilon}\right\rangle dt + \left[\left\langle \dot\alpha, \frac{\partial\alpha}{\partial\epsilon}\right\rangle\right]\Big|_{T_i(\epsilon)+}^{T_{i+1}(\epsilon)-}.
\end{aligned}
$$

Adding the terms together, noting that the boundary terms involving V cancel, $T_0'(\epsilon) = 0 = T_{m+1}'(0)$, and that $\frac{\partial\alpha}{\partial\epsilon}$ vanishes at $t = 0, T$ by assumption of fixed endpoints, we obtain

$$
\begin{aligned}
\frac{d}{d\epsilon}(J[\alpha(\cdot,\epsilon)]) &= -\int_0^T \left\langle D_t\dot\alpha + \nabla V(\alpha), \frac{\partial\alpha}{\partial\epsilon}\right\rangle dt \\
&\quad - \sum_{i=1}^m \left(\frac{1}{2}T_i'(\epsilon)\triangle\left(|\dot\alpha|_g^2\right) + \triangle\left(\left\langle \dot\alpha, \frac{\partial\alpha}{\partial\epsilon}\right\rangle\right)\right)\Big|_{t=T_i(\epsilon)}.
\end{aligned} \tag{7}
$$

We now rewrite each term in the second line of (7). Fix $1 \le i \le m$, and for convenience, let $\dot\alpha^\pm$ and $\frac{\partial\alpha}{\partial\epsilon}^\pm$ denote the values of each quantity at $T_i(\epsilon)\pm$. Using the algebraic identity $\langle a,b\rangle - \langle c,d\rangle = \frac{1}{2}\langle a+c, b-d\rangle + \frac{1}{2}\langle a-c, b+d\rangle$, we have

$$\begin{aligned}\triangle\left(\left\langle \dot\alpha, \frac{\partial\alpha}{\partial\epsilon}\right\rangle\right) &= \left\langle \dot\alpha^+, \frac{\partial\alpha}{\partial\epsilon}^+\right\rangle - \left\langle \dot\alpha^-, \frac{\partial\alpha}{\partial\epsilon}^-\right\rangle \\ &= \frac{1}{2}\left\langle (\dot\alpha^+ + \dot\alpha^-), \left(\frac{\partial\alpha^+}{\partial\epsilon} - \frac{\partial\alpha^-}{\partial\epsilon}\right)\right\rangle + \frac{1}{2}\left\langle (\dot\alpha^+ - \dot\alpha^-), \left(\frac{\partial\alpha^+}{\partial\epsilon} + \frac{\partial\alpha^-}{\partial\epsilon}\right)\right\rangle \\ &= \left\langle \overline{\dot\alpha}, \triangle\frac{\partial\alpha}{\partial\epsilon}\right\rangle + \left\langle \triangle\dot\alpha, \overline{\frac{\partial\alpha}{\partial\epsilon}}\right\rangle.\end{aligned}$$

Furthermore, from Lemma 2.8 we have

$$T_i'(\epsilon)\dot\alpha^- + \frac{\partial\alpha^-}{\partial\epsilon} = T_i'(\epsilon)\dot\alpha^+ + \frac{\partial\alpha^+}{\partial\epsilon} \implies \triangle\frac{\partial\alpha}{\partial\epsilon} = -T_i'(\epsilon)\triangle\dot\alpha. \tag{8}$$

Thus

$$\left\langle \overline{\dot\alpha}, \triangle\frac{\partial\alpha}{\partial\epsilon}\right\rangle = -T_i'(\epsilon)\langle \overline{\dot\alpha}, \triangle\dot\alpha\rangle = -\frac{1}{2}T_i'(\epsilon)\triangle(|\dot\alpha|_g^2),$$

and hence

$$\triangle\left(\left\langle \dot\alpha, \frac{\partial\alpha}{\partial\epsilon}\right\rangle\right) = \left\langle \overline{\dot\alpha}, \triangle\frac{\partial\alpha}{\partial\epsilon}\right\rangle + \left\langle \triangle\dot\alpha, \overline{\frac{\partial\alpha}{\partial\epsilon}}\right\rangle = -\frac{1}{2}T_i'(\epsilon)\triangle(|\dot\alpha|_g^2) + \left\langle \triangle\dot\alpha, \overline{\frac{\partial\alpha}{\partial\epsilon}}\right\rangle. \tag{9}$$

Substituting equation (9) into equation (7) yields (6), as desired. □

3.2. **Critical points of first variations.** We prove the following lemma in this subsection:

Lemma 3.2. *For any variation $\alpha(t,\epsilon) \in \Omega_0(M;p,p')$ with $\alpha(t,0) = \alpha(t)$, $\alpha(t)$ is a critical point of $J[\alpha(\cdot,\epsilon)]$, in the sense that $\frac{d}{d\epsilon}\Big|_{\epsilon=0}(J[\alpha(\cdot,\epsilon)]) = 0$ holds, if and only if $\alpha(t)$ is a reflected physical path.*

Proof. Suppose that $\frac{d}{d\epsilon}\Big|_{\epsilon=0}(J[\alpha(\cdot,\epsilon)]) = 0$ for all variations in $\Omega_0(M;p,p')$ with $\alpha(t,0) = \alpha(t)$. We first consider variations where $Z = \frac{\partial\alpha}{\partial\epsilon}|_{\epsilon=0}$ vanishes at all times of reflection and kinks $t = T_i(0) \in \mathcal{R}\cup\mathcal{K}$, with the corresponding time functions satisfying $T_i'(0) = 0$; such variations are possible by Corollary 2.12. Then all of the boundary terms in (6) vanish, and we obtain

$$0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0}(J[\alpha(\cdot,\epsilon)]) = -\int_0^T \langle D_t\dot\alpha(t) + \nabla V(\alpha(t)), Z(t)\rangle\, dt.$$

We note that we can arrange for $Z(t)$ to be any arbitrary smooth vector-valued function which vanishes at $t=0$, $t=T$, and all $t = T_i \in \mathcal{R}\cup\mathcal{K}$, and such vector fields are dense in L^2. It follows that $\alpha(t)$ must satisfy

$$D_t\dot\alpha(t) + \nabla V(\alpha(t)) = 0 \text{ on } [0,T]\backslash(\mathcal{R}\cup\mathcal{K}), \tag{10}$$

i.e. condition (2) in Definition 2.5. Hence, for a path $\alpha(t)$ where $J(\alpha(\cdot,\epsilon))$ is stationary at $\epsilon = 0$, (6) reduces to

$$0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0}(J[\alpha(\cdot,\epsilon)]) = \sum_{i=1}^m \langle -\triangle\dot\alpha(T_i), \overline{Z}(T_i)\rangle \tag{11}$$

for all variations $\alpha(t,\epsilon) \in \Omega_0(M;p,p')$ with $\alpha(t,0) = \alpha(t)$.

We then consider variation $\alpha(t,\epsilon)$ such that $Z(T_i) = 0$ and $T_i'(0) = 0$ for all reflection times $T_i \in \mathcal{R}$, but Z does not necessarily vanish at the kink times. For such variations, equation (11) becomes

$$0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0}(J[\alpha(\cdot,\epsilon)]) = \sum_{T_i \in \mathcal{K}} \big\langle -\triangle\dot{\alpha}(T_i), \overline{Z}(T_i)\big\rangle.$$

By Corollary 2.12, we may take Z to be continuous, with arbitrary values at $T_i \in \mathcal{K}$. Thus $\triangle\dot{\alpha}(T_i) = 0$ for all $T_i \in \mathcal{K}$, which is condition (4) in Definition 2.5. It follows that, for any variation, equation (11) further reduces to

$$0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0}(J[\alpha(\cdot,\epsilon)]) = \sum_{T_i \in \mathcal{R}} \big\langle -\triangle\dot{\alpha}(T_i), \overline{Z}(T_i)\big\rangle. \tag{12}$$

Each term in the above sum can be rewritten as

$$\begin{aligned}\big\langle -\triangle\dot{\alpha}(T_i), \overline{Z}(T_i)\big\rangle &= \big\langle -\triangle\dot{\alpha}_\top(T_i), \overline{Z_\top}(T_i)\big\rangle + \big\langle -\triangle\dot{\alpha}_\perp(T_i), \overline{Z_\perp}\big\rangle \\ &= \big\langle -\triangle\dot{\alpha}_\top(T_i), \overline{Z_\top}(T_i)\big\rangle - \frac{1}{2}T_i'(0)\triangle(|\dot{\alpha}|_g^2)|_{t=T_i},\end{aligned}$$

since $\overline{Z_\perp}(T_i) = -T_i'(0)\overline{\dot{\alpha}(T_i)}$ by equation (2). Hence equation (12) can be rewritten as

$$0 = -\sum_{T_i \in \mathcal{R}} \left(\big\langle \triangle\dot{\alpha}_\top(T_i), \overline{Z_\top}(T_i)\big\rangle + \frac{1}{2}T_i'(0)\triangle(|\dot{\alpha}|_g^2)|_{t=T_i}\right). \tag{13}$$

We now consider variations where $T_i'(0) = 0$ for $T_i \in \mathcal{R}$, in which case by Corollary 2.12 each $\overline{Z_\top}(T_i)$ can be chosen to be any arbitrary vector tangent to Y. Applying Equation (13) with $\overline{Z_\top}(T_i)$ attaining arbitrary tangent values, it follows that

$$\triangle\dot{\alpha}_\top(T_i) = 0$$

for all i, which is part of condition (3) in Definition 2.5. Finally, (12) now reduces to

$$0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0}(J[\alpha(\cdot,\epsilon)]) = -\sum_{T_i \in \mathcal{R}} \frac{1}{2}T_i'(0)\triangle(|\dot{\alpha}|_g^2)|_{t=T_i} = -\frac{1}{2}\sum_{T_i \in \mathcal{R}} T_i'(0)(|\dot{\alpha}_\perp(T_i+)|_g^2 - |\dot{\alpha}_\perp(T_i-)|_g^2) \tag{14}$$

(the last equality following since $\triangle(|\dot{\alpha}|_g^2) = \triangle(|\dot{\alpha}_\top|_g^2 + |\dot{\alpha}_\perp|_g^2) = \triangle(|\dot{\alpha}_\perp|_g^2)$ since we now know that $\triangle\dot{\alpha}_\top = 0$). By Corollary 2.13, we can find variations with arbitrary values of $T_i'(0)$, from which we conclude that

$$|\dot{\alpha}_\perp(T_i+)|_g^2 - |\dot{\alpha}_\perp(T_i-)|_g^2 = 0$$

for each i. Finally, since $\alpha_1^\pm \geq 0$ with $\alpha_1^\pm(T_i) = 0$, it follows that $\pm\dot{\alpha}_1^\pm(T_i) \geq 0$. For the above condition to hold, it must be the case that $\dot{\alpha}_\perp^-(T_i) + \dot{\alpha}_\perp^+(T_i) = 0$, which corresponds to the remaining part of condition (3) in Definition 2.5. Therefore, we conclude that if $J(\alpha(\cdot,\epsilon))$ is stationary for any variation $\alpha(t,\epsilon)$ with $\alpha(t,0) = \alpha(t)$, then $\alpha(t)$ must be a reflected physical path.

Conversely, suppose $\alpha(t) = \alpha(t,0)$ is a reflected physical path. Then the integral term in (6) vanishes by condition (2) in Definition 2.5, while the boundary terms over $T_i \in \mathcal{K}$ vanish by condition (4). By condition (3), at $T_i \in \mathcal{R}$, $\triangle\dot{\alpha}(T_i)$ is normal to the boundary, while by (2), $\overline{Z_\perp}(T_i) = -T_i'(0)\overline{\dot\alpha_\perp}(T_i) = 0$ by condition (3); hence the pairing vanishes for each T_i. This gives $\frac{d}{d\epsilon}\Big|_{\epsilon=0}(J[\alpha(\cdot,\epsilon)]) = 0$, as desired. $\square$

4. The Hessian of the action at a reflected physical path

4.1. **Second variations of reflected physical paths.** Now we take $\alpha(t,\epsilon,\delta) \in \Omega_0(M,p,p')$ to be a two-parameter variation with fixed endpoints of a reflected physical path $\alpha(t)$. Let

$$\begin{gathered}\partial_\epsilon\alpha(t,\epsilon,\delta) = Z(t,\epsilon,\delta), \quad \partial_\delta\alpha(t,\epsilon,\delta) = W(t,\epsilon,\delta)\\ \partial_\epsilon\alpha(t,0,0) = Z, \quad \partial_\delta\alpha(t,0,0) = W\end{gathered}$$

so that (using our jump and average notation from above), the first variation (6) now reads

$$\begin{aligned}\frac{d}{d\epsilon}(J[\alpha(\cdot,\epsilon,\delta)]) &= -\int_0^T \langle D_t\dot\alpha + \nabla V(\alpha), Z(t,\epsilon,\delta)\rangle\, dt\\ &\quad + \sum_{i=1}^n \big\langle -\triangle\dot\alpha(T_i(\epsilon,\delta),\epsilon,\delta), \overline{Z}(T_i(\epsilon,\delta),\epsilon,\delta)\big\rangle.\end{aligned} \tag{15}$$

Under the hypothesis that $\alpha(t) = \alpha(t,0,0)$ is a reflected physical path, we will now apply $\partial/\partial\delta$ to the various terms in (15) and evaluate at $\delta = 0$ to find the second variation.

We now examine the second variation $\frac{\partial^2}{\partial\epsilon\partial\delta}(J[\alpha(\cdot,\epsilon,\delta)])|_{\epsilon=0,\delta=0}$. We split

$$\frac{\partial^2}{\partial\epsilon\partial\delta}(J[\alpha(\cdot,\epsilon,\delta)])|_{\epsilon=0,\delta=0} := J''_\circ + J''_\partial$$

where $J''_\circ$, respectively J''_∂, denote the δ derivative (evaluated at 0) falling on the integral term in the first line of (15) (the "interior" term) and the derivative falling on the second line ("boundary" term).

Differentiating the integral term and evaluating at $\epsilon = 0, \delta = 0$ yields

$$J''_\circ = -\int_0^T \big\langle D_\delta D_t\dot\alpha + (\nabla^2 V)W, Z\big\rangle\, dt; \tag{16}$$

note that there are no boundary terms arising from differentiating $T_i(\epsilon,\delta)$ since the integrand $\frac{D}{dt}\dot\alpha + \nabla V(\alpha)$ equals zero by assumption of α being a reflected physical path. (We recall that by assumption $\nabla^2 V$ may have no worse than jump discontinuities across Y, so by our assumption that $\dot\alpha$ is transverse to Y, we may legitimately differentiate inside the integral by the Dominated Convergence Theorem.) We now note that

$$D_\delta D_t\dot\alpha = D_t D_\delta\dot\alpha + R\big(\dot\alpha, \frac{\partial\alpha}{\partial\delta}\big)\dot\alpha = D_t^2 W + R\big(\dot\alpha, W\big)\dot\alpha.$$

Hence this integral term becomes the standard interior Jacobi equation term

$$J''_\circ = -\int_0^T \big\langle D_t^2 W + R(\dot\alpha, W)\dot\alpha + (\nabla^2 V)W, Z\big\rangle\, dt. \tag{17}$$

Now we consider the boundary term J''_∂. Noting that $\partial_\delta\langle \bullet,\bullet\rangle = \nabla_W\langle\bullet,\bullet\rangle$, and that the covariant derivative may be brought inside the inner product by compatibility of the connection, we see that each term in the sum differentiates to

$$\begin{aligned}&\frac{\partial}{\partial\delta}\left(\left\langle -\triangle\dot\alpha(T_i(\epsilon,\delta),\epsilon,\delta),\overline{Z}(T_i(\epsilon,\delta),\epsilon,\delta)\right\rangle\right)\\ &=\underbrace{\left\langle -\partial_\delta T_i\triangle D_t\dot\alpha,\overline{Z}\right\rangle}_{\mathrm{I}}+\underbrace{\left\langle -\triangle\nabla_W\dot\alpha,\overline{Z}\right\rangle}_{\mathrm{II}}+\underbrace{\left\langle -\triangle\dot\alpha,\partial_\delta T_i\overline{D_tZ}\right\rangle}_{\mathrm{III}}+\underbrace{\left\langle -\triangle\dot\alpha,\overline{\nabla_W Z}\right\rangle}_{\mathrm{IV}};\end{aligned}\tag{18}$$

here, for brevity, we have omitted the evaluation of each term at $(T_i(\epsilon,\delta),\epsilon,\delta)$. Since α satisfies $D_t\dot\alpha = -\nabla V$ both before and after T_i, with ∇V continuous, the term I is zero, and we focus on II, III, IV. We split into the cases where $T_i\in\mathcal{K}$ and $T_i\in\mathcal{R}$.

If $T_i\in\mathcal{K}$, then $\triangle\dot\alpha=0$, i.e. the terms III, IV both vanish. Hence, for $T_i\in\mathcal{K}$, we get

$$\frac{\partial}{\partial\delta}\left(\left\langle -\triangle\dot\alpha(T_i(\epsilon,\delta),\epsilon,\delta),\overline{Z}(T_i(\epsilon,\delta),\epsilon,\delta)\right\rangle\right)\Big|_{\epsilon=0,\delta=0}=\left\langle -\triangle\nabla_W\dot\alpha,\overline{Z}\right\rangle|_{T_i}=\left\langle -\triangle D_tW(T_i),\overline{Z}(T_i)\right\rangle,$$

where we rewrite $\nabla_W\dot\alpha = D_tW$ using the vanishing of the torsion.

We now focus on $T_i\in\mathcal{R}$. Fixing this T_i for the moment, we will employ the more concise notation

$$\begin{aligned}\alpha^+ &= \alpha|_{t\in[T_i(\epsilon,\delta),T_{i+1}(\epsilon,\delta)]},\\ \alpha^- &= \alpha|_{t\in[T_{i-1}(\epsilon,\delta),T_i(\epsilon,\delta)]}.\end{aligned}\tag{19}$$

for the successive smooth segments of α. We will further abbreviate by writing simply $\dot\alpha^\pm$ for the evaluation of this time derivative at time $T_i\pm$. Recalling that by definition of reflected physical paths, we have $\triangle\dot\alpha = -2\dot\alpha^-_\perp$, we rewrite the remaining terms as

$$\underbrace{\left\langle -\triangle\nabla_W\dot\alpha,\overline{Z}\right\rangle}_{\mathrm{II}}+\underbrace{\left\langle 2\dot\alpha^-_\perp,\partial_\delta T_i\overline{D_tZ}\right\rangle}_{\mathrm{III}}+\underbrace{\left\langle 2\dot\alpha^-_\perp,\overline{\nabla_W Z}\right\rangle}_{\mathrm{IV}}.\tag{20}$$

Let

$$c(\delta,\epsilon) := \alpha^\pm(T_i(\epsilon,\delta),\epsilon,\delta)\in Y\tag{21}$$

(with choice of $\pm$ irrelevant) be the point at which α reflects. Note that as a consequence, $\partial_\delta c\in TY$. Recall that the second fundamental form is defined by

$$\mathrm{II}(X,Y) = -\langle Y,\nabla_X N\rangle N = \langle\nabla_X Y,N\rangle N$$

(with the second equality obtained using compatibility of the connection and vanishing of its torsion). For later use we also introduce the shape operator S, given by

$$\mathrm{II}(V,W) = \langle\mathsf{S}(V),W\rangle N.\tag{22}$$

Lemma 4.1. *The averaged normal components satisfy the following relation:*

$$\partial_\delta T_i\overline{D_tZ}_\perp+\overline{\nabla_Z W}_\perp = \partial_\delta T_i\partial_\epsilon T_i(\nabla V)_\perp - \partial_\epsilon T_i\overline{D_tW}_\perp + \mathrm{II}(\partial_\delta c,\partial_\epsilon T_i\dot\alpha_\top+\overline{Z}).\tag{23}$$

Proof. We recall that

$$\alpha^\pm(T_i(\epsilon,\delta),\epsilon,\delta)\in Y\quad\text{ for all }\epsilon,\delta.$$

Differentiating in δ yields

$$\langle\dot\alpha^\pm\partial_\delta T_i+W^\pm,N\rangle = 0.\tag{24}$$

Further differentiating in ϵ then gives (omitting a $\pm$ on all terms)

$$\begin{aligned}&\big\langle D_tZ\partial_\delta T_i + D_t\dot\alpha\partial_\epsilon T_i\partial_\delta T_i + \dot\alpha\partial^2_{\epsilon\delta}T_i + \nabla_Z W + D_tW\partial_\epsilon T_i, N\big\rangle\\ &+ \langle\partial_\delta c, \nabla_Z N + \partial_\epsilon T_i D_t N\rangle = 0.\end{aligned}$$

(Here we have used $\nabla_Z\dot\alpha = D_tZ$ by the vanishing of the torsion.) We now take the average of this equation over $\pm$ and recall that $D_t\dot\alpha = -\nabla V$, while $\langle\overline{\dot\alpha}, N\rangle = 0$ to obtain

$$\big\langle\overline{D_tZ}\partial_\delta T_i - \nabla V\partial_\epsilon T_i\partial_\delta T_i + \overline{\nabla_Z W} + \overline{D_tW}\partial_\epsilon T_i, N\big\rangle + \langle\partial_\delta c, \nabla_{\overline{Z}}N + \partial_\epsilon T_i D_t N\rangle = 0.$$

Now recalling the definition of the second fundamental form we rewrite our identity as

$$\big\langle\overline{D_tZ}\partial_\delta T_i - \nabla V\partial_\epsilon T_i\partial_\delta T_i + \overline{\nabla_Z W} + \overline{D_tW}\partial_\epsilon T_i, N\big\rangle - \langle \mathrm{II}(\partial_\delta c, \overline{Z} + \partial_\epsilon T_i\dot\alpha_\top), N\rangle = 0,$$

as desired. $\square$

Since $\nabla_Z W = \nabla_W Z$ we may now substitute (23) into (20), using it to replace the terms III and IV with terms involving the LHS of (23) and get

$$\big\langle -\triangle D_tW, \overline{Z}\big\rangle + \big\langle 2\dot\alpha^-_\perp, \partial_\delta T_i\partial_\epsilon T_i\nabla V - \partial_\epsilon T_i\overline{D_tW} + \mathrm{II}(\partial_\delta c, \overline{Z} + \partial_\epsilon T_i\dot\alpha_\top)\big\rangle.$$

Since $\alpha^\pm(T(\epsilon,\delta),\epsilon,\delta) = 0$,

$$\langle\dot\alpha^\pm\partial_\delta T_i + W^\pm, N\rangle = \langle\dot\alpha^\pm\partial_\epsilon T_i + Z^\pm, N\rangle = 0. \tag{25}$$

Substituting in the above yields for boundary terms in the second variation.

$$\big\langle -\triangle D_tW, \overline{Z}\big\rangle + \big\langle 2W^-_\perp, (Z^-_\perp/\dot\alpha^-_\perp)\nabla V\big\rangle + 2\langle Z^-_\perp, \overline{D_tW}\rangle + \langle 2\dot\alpha^-_\perp, \mathrm{II}(\partial_\delta c, \overline{Z} + \partial_\epsilon T_i\dot\alpha_\top)\rangle.$$

Now use (25) to eliminate $\partial_\epsilon T_i$ in favor of the variation vector field Z (and recall the definition (22) of the shape operator), to find that this sum equals

$$\big\langle -\triangle D_tW, \overline{Z}\big\rangle + 2\langle\overline{D_tW}, Z^-_\perp\rangle + \big\langle 2W^-_\perp, (Z^-_\perp/\dot\alpha^-_\perp)\nabla V\big\rangle + 2\dot\alpha_1\langle \mathsf{S}(\partial_\delta c), \overline{Z}\rangle - 2\,\mathrm{II}(\partial_\delta c, \dot\alpha_\top)Z^-_1.$$

Summarizing the above discussion, we obtain:

Theorem 4.2. *Let $\alpha(t,\epsilon,\delta) \in \Omega_0(M;p,p')$ be a two parameter variation of a reflected physical path $\alpha(t)$ with variational vector field $\partial_\epsilon\alpha(t,0,0) = Z, \partial_\delta\alpha(t,0,0) = W \in T_\alpha\Omega_0(M,p,p')$. Then the second variation $\frac{\partial^2}{\partial\epsilon\partial\delta}(J[\alpha(\cdot;\epsilon,\delta)])\big|_{\epsilon=0,\delta=0}$ is given by*

$$\begin{aligned}&-\int_0^T \big\langle D_t^2W + R(\dot\alpha, W)\dot\alpha + (\nabla^2 V)W, Z\big\rangle\,dt + \sum_{T_i\in\mathcal{R}}\Big(- \big\langle\triangle D_tW, \overline{Z}\big\rangle + 2\langle\overline{D_tW}, Z^-_\perp\rangle\\ &+ \big\langle 2W^-_\perp, (Z^-_\perp/\dot\alpha^-_\perp)\nabla V\big\rangle + 2\dot\alpha_1\langle\mathsf{S}(\partial_\delta c), \overline{Z}\rangle - 2\,\mathrm{II}(\partial_\delta c, \dot\alpha_\top)Z^-_1\Big)_{T_i} - \sum_{T_i\in\mathcal{K}}\big\langle\triangle D_tW(T_i), \overline{Z}(T_i)\big\rangle\end{aligned} \tag{26}$$

where

$$\partial_\delta c = -\frac{W_\perp}{\dot\alpha_\perp}\dot\alpha_\top + W_\top. \tag{27}$$

As written here, the second variation is manifestly linear in Z. It is also symmetric in Z and W owing to its definition as a second derivative. As every element in $T_\alpha\Omega_0(M,p,p')$ arises from differentiating a variation (cf. Lemma 2.9), we obtain a quadratic form on the tangent space:

Definition 4.3. *For $\alpha \in \Omega_0(M, p, p')$, we define a symmetric bilinear quadratic form*

$$J''(\cdot, \cdot) : T_\alpha\Omega_0(M, p, p') \times T_\alpha\Omega_0(M, p, p') \longrightarrow \mathbb{R} \tag{28}$$

using equation (26).

4.2. **Reflected Jacobi fields for reflected physical trajectories.** The result of Theorem 4.2 now motivates the following definition (the essential point being Proposition 4.7 below).

Definition 4.4. *A vector field $W \in T_\alpha\Omega(M)$ along a reflected physical path $\alpha(t)$ with reflection and kink times at $\mathcal{R}$ and $\mathcal{K}$, respectively, is called a* reflected Jacobi field *if it satisfies the Jacobi equation*

$$D_t^2 W + R(\dot\alpha, W)\dot\alpha + (\nabla^2 V)W = 0 \tag{29}$$

on $[0, T]\backslash(\mathcal{R} \cup \mathcal{K})$, as well as the reflection conditions

$$\triangle D_t W_\top = 2\dot\alpha_1^- \,\mathsf{S}(\partial_\delta c), \tag{30}$$

$$\overline{D_t W}_\perp = -(W_\perp^-/\dot\alpha_\perp^-)(\nabla V)_\perp + \mathrm{II}(\partial_\delta c, \dot\alpha_\top) \tag{31}$$

at $t = T_i \in \mathcal{R}$, where

$$\partial_\delta c = -\frac{W_\perp}{\dot\alpha_\perp}\dot\alpha_\top + W_\top, \tag{32}$$

and the kink conditions

$$\triangle D_t W = 0 \tag{33}$$

at $t = T_i \in \mathcal{K}$.

Note that a reflected Jacobi field is determined completely by its initial conditions $W(0)$, $D_tW(0)$ (and depends smoothly on them). Indeed, for $T_i \in \mathcal{R}\cup\mathcal{K}$, the values of $W(T_i-)$ and $D_tW(T_i-)$ uniquely determine the values of $W(T_i+)$ (via the requirement $W \in T_\alpha\Omega(M)$) and $D_tW(T_i+)$ (via the reflection/kink conditions above). As with reflected physical paths, passing over Y at points where $\dot\alpha$ is transverse to Y creates no difficulties with solvability or with smooth dependence on initial data. At internal kinks, note that Jacobi fields must be smooth.

Remark 4.5. Note that, in the definition, we do not require that a reflected Jacobi field vanish at endpoints. We allow the case in which no reflections occur, in which case the definition coincides with the usual definition of Jacobi fields.

Let $p = \alpha(a)$ and $q = \alpha(b)$ $(a \neq b)$ be two points on a reflected physical path $\alpha(t)$. In particular, we are allowing $p, q \in Y$ or $p = q$.

Definition 4.6. *The points p and q are* conjugate *along $\alpha(t)$ if there exists a non-vanishing reflected Jacobi field W along $\alpha(t)$ such that $W(a) = W(b) = 0$. The* multiplicity *of p and q as conjugate points is equal to the dimension of the vector space consisting of all such reflected Jacobi fields.*

Recall that the *null space* of the second variation $J'' : T_\alpha\Omega_0 \times T_\alpha\Omega_0 \to \mathbb{R}$ is the vector space consisting of those $W \in T_\alpha\Omega_0$ such that $J''(W,Z) = 0$ for all reflected variation vector field $Z \in T_\alpha\Omega_0$. The *nullity* ν of J'' is equal to the dimension of this null space. We say J'' is *degenerate* if $\nu > 0$. We have the following Proposition

Proposition 4.7. *A vector field $W \in T_\alpha\Omega_0$ belongs to the null space of J'' if and only if W is a reflected Jacobi field. Therefore J'' is degenerate if and only if the end points p and q are conjugate along $\alpha(t)$. The nullity of J'' is equal to the multiplicity of p and q as conjugate points.*

Proof. If W is a reflected Jacobi field and vanishes at p and q, comparing equation (26) with Definition 2.5, it is easy to see that $J''(W,Z)$ vanishes for all $Z \in T_\alpha\Omega_0$.

Assume now $W \in T_\alpha\Omega_0$ belongs to the null space of J''. Let Z_1 be any smooth vector field vanishing at all $T_i \in \mathcal{R}\cup\mathcal{K}$; note that $Z_1 \in T_\alpha\Omega_0$ by Corollary 2.14. Then in computing $J''(W,Z_1)$ we see that all boundary terms vanish, leaving

$$0 = J''(W,Z_1) = -\int_0^T \left\langle D_t^2 W + R(\dot\alpha, W)\dot\alpha + (\nabla^2 V)W, Z_1\right\rangle dt.$$

Applying this to arbitrary Z_1 vanishing on $\mathcal{R}\cup\mathcal{K}$, it follows that W satisfies $D_t^2 W + R(\dot\alpha, W)\dot\alpha + (\nabla^2 V)W = 0$ on $[0,T]\backslash(\mathcal{R}\cup\mathcal{K})$, i.e. W satisfies condition (29).

Next, we take a smooth Z_2 which vanishes on $\mathcal{R}$, but not necessarily on $\mathcal{K}$. For such Z_2, we have

$$0 = J''(W,Z_2) = -\sum_{T_i\in\mathcal{K}} \left\langle \triangle D_t W(T_i), \overline{Z_2}(T_i)\right\rangle$$

since all boundary terms at $T_i \in \mathcal{R}$ vanish, and the integral term vanishes since we already have (29). By Corollary 2.14, we can take Z_2 to take on arbitrary values at $T_i \in \mathcal{K}$. Thus we obtain $\Delta D_t W(T_i) = 0$ for all $T_i \in \mathcal{K}$. Finally, we consider Z_3 which do not vanish at $T_i \in \mathcal{R}$. Specifically we take Z_3 to vanish at $t = 0$ and $t = T$ and satisfy

$$Z_{3,\top}(T_i\pm) = \triangle D_t W_\top(T_i) - 2\dot\alpha_1^-(T_i)\,\mathsf{S}(\partial_\delta c)(T_i)$$

and

$$Z_{3,\perp}(T_i\pm) = \pm\left(\overline{D_t W}_\perp(T_i) + (W_\perp^-(T_i)/\dot\alpha_\perp^-(T_i))(\nabla V)_\perp(\alpha(T_i)) + \mathrm{II}(\partial_\delta c, \dot\alpha_\top)(T_i)\right)$$

when $T_i \in \mathcal{R}$. Note that $Z_3 \in T_\alpha\Omega_0(M)$ by Corollary 2.14, since $\triangle Z_3(T_i)$, resp. $\overline{Z_3}(T_i)$, is normal, resp. tangent, to Y at $\alpha(T_i)$. Then

$$\begin{aligned}0 = J''(W,Z_3) = -\sum_{i=1}^m \Big(&\|\triangle D_t W_\top - 2\dot\alpha_1^-\,\mathsf{S}(\partial_\delta c)|_{T_i}\|_g^2\\ &+\|\overline{D_t W}_\perp + (W_\perp^-/\dot\alpha_\perp^-)(\nabla V)_\perp + \mathrm{II}(\partial_\delta c, \dot\alpha_\top)|_{T_i}\|_g^2\Big),\end{aligned}$$

which yields condition (30) and (31). Therefore, W must be reflected Jacobi field by Definition 4.4. $\square$

4.3. **Reflected Jacobi fields as variation of physical paths.** Let $\alpha(t,\epsilon) \in \Omega(M)$ be a 1-parameter variation of $\alpha(t)$ in the sense that $\alpha(t,0) = \alpha(t)$, not necessarily keeping the endpoints fixed, and such that each $\alpha(\cdot,\epsilon)$ is a reflected physical path for any fixed $\epsilon \in (-\epsilon_0, \epsilon_0)$. In fact, such variation is given by a family of reflected physical path.

Proposition 4.8. *If $\alpha(t,\epsilon) \in \Omega(M)$ is variation of a reflected physical path as above, then the corresponding variation vector field $W(t) = \frac{\partial \alpha}{\partial \epsilon}(t,0) \in T_\alpha \Omega(M)$ is a reflected Jacobi field along $\alpha(t)$.*

Remark 4.9. Note that this result (and hence the reflection conditions for Jacobi fields) was previously known in the Riemannian geometry case $V = 0$—see [7, Theorem 3.13], [12, Equation (2)], [13, Lemma 16], [6, Lemma 12], etc.

Proof. The idea to derive the conditions for reflected Jacobi fields is to take the conditions for a reflected physical path (cf. Definition 2.5) and differentiate.

For the condition (29) on the interior points, note that as reflected physical paths in the interior point satisfying $D_t\dot\alpha + \nabla V(\alpha(t,\delta)) = 0$, we have

$$0 = D_\delta(D_t\dot\alpha + \nabla V(\alpha(t,\delta))) = D_t D_\delta \dot\alpha + R\big(\dot\alpha, \frac{\partial \alpha}{\partial \delta}\big)\dot\alpha + \nabla^2 V(\alpha(t,\delta))\frac{\partial \alpha}{\partial \delta}, \tag{34}$$

where the RHS of the above equation is indeed (29) if we evaluate at $\delta = 0$.

For the conditions (30)(31) at the reflection times $T_i \in \mathcal{R}$, write $\alpha^\pm(t,\delta)$ as the restriction of $\alpha(t,\delta)$ to $t > T_i(\delta)$ or $t < T_i(\delta)$, extended smoothly to a neighborhood of $T_i(\delta)$. Note that we have the reflection conditions

$$\dot\alpha^-_\perp(T_i(\delta),\delta) + \dot\alpha^+_\perp(T_i(\delta),\delta) = 0, \quad \dot\alpha^-_\top(T_i(\delta),\delta) - \dot\alpha^+_\top(T_i(\delta),\delta) = 0,$$

since $\alpha(\cdot,\delta)$ is a reflected physical path for each δ. Differentiating in δ gives

$$\begin{aligned}
\left(\partial_\delta T_i D_t^- + D_\delta^-\right)\dot\alpha^-_\perp + \left(\partial_\delta T_i D_t^+ + D_\delta^+\right)\dot\alpha^+_\perp = 0,\\
\left(\partial_\delta T_i D_t^- + D_\delta^-\right)\dot\alpha^-_\top - \left(\partial_\delta T_i D_t^+ + D_\delta^+\right)\dot\alpha^+_\top = 0,
\end{aligned}$$

where $D^\pm$ denotes taking the covariant derivative along $\alpha^\pm$, and all terms are evaluated at $t = T_i(0)$, $\delta = 0$. Recalling the notation $c(\delta) = \alpha^\pm(T_i(\delta),\delta)$ (in which case $c(\delta) \in Y$ and $\partial_\delta c \in TY$, with $\partial_\delta c = \partial_\delta T_i \dot\alpha^\pm + \frac{\partial \alpha^\pm}{\partial \delta}$), we have

$$\begin{aligned}
\left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right)\dot\alpha^\pm_\perp &= \left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right)\left(\langle \dot\alpha^\pm, N\rangle N\right)\\
&= \left(\left\langle \left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right)\dot\alpha^\pm, N\right\rangle + \left\langle \dot\alpha^\pm, \left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right) N\right\rangle\right) N\\
&\quad + \langle \dot\alpha^\pm, N\rangle \left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right) N\\
&= \left(\left\langle \left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right)\dot\alpha^\pm, N\right\rangle + \left\langle \dot\alpha^\pm, \nabla_{\partial_\delta c} N\right\rangle\right) N + \dot\alpha^\pm_1 \nabla_{\partial_\delta c} N\\
&= \left\langle \left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right)\dot\alpha^\pm, N\right\rangle N - \mathrm{II}(\dot\alpha^\pm_\top, \partial_\delta c) - \dot\alpha^\pm_1 \mathsf{S}(\partial_\delta c).
\end{aligned} \tag{35}$$

Since

$$D_t^\pm \dot\alpha^\pm + \nabla V(\alpha^\pm) = 0,$$

it follows that

$$\left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right)\dot\alpha^\pm = D_\delta^\pm \dot\alpha^\pm - \partial_\delta T_i \nabla V = \dot W^\pm - \partial_\delta T_i \nabla V. \tag{36}$$

Thus, combining (35) and (36) yields

$$\begin{aligned}\left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right)\dot\alpha_\perp^\pm &= \left\langle \dot W^\pm - \partial_\delta T_i \nabla V, N\right\rangle N - \mathrm{II}(\dot\alpha_\top^\pm, \partial_\delta c) - \dot\alpha_1^\pm\,\mathsf{S}(\partial_\delta c)\\ &= \dot W_\perp^\pm - \partial_\delta T_i (\nabla V)_\perp - \mathrm{II}(\dot\alpha_\top^\pm, \partial_\delta c) - \dot\alpha_1^\pm\,\mathsf{S}(\partial_\delta c).\end{aligned}$$

Averaging the above equation over $\pm$, and using that $\sum_\pm \dot\alpha_\perp^\pm = 0$, yields

$$0 = \sum_\pm \left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right)\dot\alpha_\perp^\pm \implies \overline{\dot W_\perp} = \partial_\delta T_i (\nabla V)_\perp + \mathrm{II}(\dot\alpha_\top, \partial_\delta c) = -\frac{W_\perp^-}{\dot\alpha_\perp^-}(\nabla V)_\perp + \mathrm{II}(\dot\alpha_\top, \partial_\delta c),$$

thus giving (30). Moreover,

$$\begin{aligned}\left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right)\dot\alpha_\top^\pm &= \left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right)\dot\alpha^\pm - \left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right)\dot\alpha_\perp^\pm\\ &= \dot W^\pm - \partial_\delta T_i \nabla V - \left(\dot W_\perp^\pm - \partial_\delta T_i (\nabla V)_\perp - \mathrm{II}(\dot\alpha_\top^\pm, \partial_\delta c) - \dot\alpha_1^\pm\,\mathsf{S}(\partial_\delta c)\right)\end{aligned}$$

so we have

$$0 = \Delta\left(\left(\partial_\delta T_i D_t^\pm + D_\delta^\pm\right)\dot\alpha_\top^\pm\right) = \Delta\dot W - \Delta\dot W_\perp + \Delta\dot\alpha_1^\pm\,\mathsf{S}(\partial_\delta c)$$

since $\partial_\delta T_i \nabla V$ and $\dot\alpha_\top$ are the same across the jump. Note that $\Delta\dot W - \Delta\dot W_\perp = \Delta\dot W_\top$. Hence we have

$$\Delta(\dot W_\top) = -\Delta\dot\alpha_1^\pm\,\mathsf{S}(\partial_\delta c) = 2\dot\alpha_1^-\,\mathsf{S}(\partial_\delta c),$$

thus giving (31).

Finally, (33) follows similarly by differentiating the condition $\triangle\dot\alpha(T_i(\delta), \delta) = 0$ in δ whenever $T_i \in \mathcal{K}$. ☐

Proposition 4.10. *Every reflected Jacobi field along a reflected physical path $\alpha : [0,T] \to M$ may be obtained by a variation through reflected physical paths.*

Proof. Let W be a reflected Jacobi field along α. Let $\alpha(t,\epsilon)$ be a family of reflected physical paths with $\alpha(t,0) = \alpha(t)$, $\partial_\epsilon\alpha(0,0) = W(0)$, $D_t\partial_\epsilon\alpha(0,0) = \dot W(0)$. Then $Z := \partial\alpha/\partial\epsilon|_{\epsilon=0}$ is a reflected Jacobi field satisfying the same initial conditions as W, hence $W = Z$. ☐

We record the following proposition which will be useful later.

Proposition 4.11. *If $p, p' \in M$ are non-conjugate along a reflected physical path $\alpha(t)$ with $\alpha(0) = p, \alpha(T) = p'$, then for any pair of vectors $(V_0, V_T) \in T_pM \times T_{p'}M$, there exists a unique reflected Jacobi field $W(t)$ along $\alpha(t)$ such that $W(0) = V_0$ and $W(T) = V_T$.*

Proof. Recall that a reflected Jacobi field along α exists and is unique given its initial data. Now given $V, Z \in T_{\alpha(0)}M$, let $W_{V,Z}(t)$ denote the Jacobi field with

$$W_{V,Z}(0) = V, \quad D_t W_{V,Z}(0) = Z;$$

note that this depends bilinearly on (V, Z). Consider the map $\Phi : T_{\alpha(0)}M \to T_{\alpha(T)}M$ defined by

$$\Phi(Z) := W_{0,Z}(T).$$

This linear map is injective since $\alpha(0)$ and $\alpha(T)$ are non-conjugate, hence is an isomorphism. Now given V_0 and V_T, let

$$Z_0 = \Phi^{-1}(V_T - W_{V_0,0}(T)).$$

Then $W_{V_0,Z_0}(T) = V_T$ as desired. $\square$

By Lemma 2.7, given a reflected physical path $\alpha(t)$, there is a neighborhood of $(\alpha(0), \dot\alpha(0)) \in TM$ such that for (x, v) in this neighborhood, there exists a unique reflected physical path, with initial location and velocity (x, v), and with reflection times close to that of α. Let $\alpha_{x,v}(t)$ denote this path.

Proposition 4.12. *If $p, p' \in M$ are non-conjugate along a reflected physical path $\alpha(t)$ with $\alpha(0) = p,\ \alpha(T) = p'$, then the map*

$$\Psi = (x, v) \mapsto (x, \alpha_{x,v}(T))$$

is a local diffeomorphism from a neighborhood of $(p, \dot\alpha(0))$ in TM to a neighborhood of (p, p') in $M \times M$.

Proof. The derivative of Ψ is invertible by Proposition 4.11, hence the result follows from the Inverse Function Theorem. $\square$

Given x and y in the range of the local diffeomorphism defined by Proposition 4.12, we let $\alpha_{x,y}(t)$ denote the resulting reflected physical path from x to y.

5. The index theorem

We are ready to prove the Morse index theorem in the case of fixed boundary conditions. Some further setup is needed for the case of periodic trajectories, however, so we begin with some further discussion of variations and Jacobi fields in the periodic case.

5.1. **Periodic paths.**

As we can freely choose a starting point for a periodic reflected physical path, we can without loss of generality let $\alpha(t)$ denote a periodic reflected physical path with $\alpha(0) = \alpha(T) \notin Y$, and let $\alpha(t, \epsilon, \delta)$ be a two-parameter family of periodic reflected paths with $\alpha(t, 0, 0) = \alpha(t)$. Denoting $\mathcal{R}$ and $\mathcal{K}$ the reflection and kink times of α, it will be convenient to consider $\mathcal{K}_0 := \mathcal{K} \cup \{0\}$, i.e. to consider $t = 0$ as an additional kink time, owing to the possibility of α or its variations being $\mathcal{C}^0$ but not $\mathcal{C}^\infty$ at $t = 0$ (equivalently $t = T$) when viewed as a periodic path. Correspondingly, for a vector field Z along α, we write

$$\triangle Z(0) := Z(0) - Z(T), \quad \overline{Z}(0) := \frac{1}{2}(Z(0) + Z(T)).$$

As we do not have vanishing of the endpoints at $t = 0, T$ for variational vector field $Z(t, \epsilon, \delta) \in T_\alpha \Omega_{\mathrm{per}}(M)$, in contrast to equation (15), we obtain the first variation formula

$$
\begin{aligned}
\frac{d}{d\epsilon}(J[\alpha(\cdot,\epsilon,\delta)]) &= -\int_0^T \langle D_t\dot\alpha + \nabla V(\alpha), Z(t,\epsilon,\delta)\rangle\, dt \\
&\quad + \sum_{i=0}^m \big\langle -\triangle\dot\alpha(T_i(\epsilon,\delta),\epsilon,\delta), \overline{Z}(T_i(\epsilon,\delta),\epsilon,\delta)\big\rangle,
\end{aligned} \tag{37}
$$

where we use the convention that $T_0 = 0$ is independent of ϵ, δ. The proof is similar to the proof for fixed endpoints, noting in this case that integration by parts *does* produce boundary terms at $t = 0$ and $t = T$; those boundary terms can be manipulated into the desired form in the same way as the terms at the other T_i.

The second variation $J'' := \frac{\partial^2 J}{\partial\epsilon\partial\delta}\big|_{\substack{\epsilon=0\\ \delta=0}}$, in contrast to equation (26), is given by

(38)

$$
\begin{aligned}
J''(W,Z) = &-\int_0^T \big\langle D_t^2 W + R(\dot\alpha, W)\dot\alpha + (\nabla^2 V)W, Z\big\rangle\, dt + \sum_{T_i\in\mathcal{R}} \bigg(-\big\langle \triangle D_t W, \overline{Z}\big\rangle + 2\langle \overline{D_t W}, Z_\perp^-\rangle \\
&+ \big\langle 2W_\perp^-, (Z_\perp^-/\dot\alpha_\perp^-)\nabla V\big\rangle + 2\dot\alpha_1\langle \mathsf{S}(\partial_\delta c), \overline{Z}\rangle - 2\,\mathrm{II}(\partial_\delta c, \dot\alpha_\top) Z_1^-\bigg)_{T_i} - \sum_{T_i\in\mathcal{K}_0} \big\langle \triangle D_t W(T_i), \overline{Z}(T_i)\big\rangle.
\end{aligned}
$$

Remark 5.1. $J''(W,Z)$ is a quadratic form on the space $T_\alpha\Omega_{\mathrm{per}}(M)$.

Note that $\alpha(t)$ is a periodic reflected physical path (cf. Definition 2.5). Let

$$
\mathcal{J}(\alpha) = \{W \in T_\alpha\Omega(M) \,:\, W \text{ is a reflected Jacobi field along } \alpha|_{(0,T)}\},
$$

i.e. the space of reflected Jacobi fields with no boundary conditions at 0, T. Let

$$
\begin{aligned}
\mathcal{J}_{C^0}(\alpha) &= \{W \in T_\alpha\Omega_{\mathrm{per}}(M) \,:\, W \text{ is a reflected Jacobi field along } \alpha|_{(0,T)}\} \\
&= \{W \in \mathcal{J}(\alpha) \,:\, W(0) = W(T)\}.
\end{aligned}
$$

This is a finite-dimensional vector space. (In general the vector fields in $\mathcal{J}_{C^0}$ are continuous but not C^1 at the endpoint $\alpha(0) = \alpha(T)$, hence the C^0 notation.) Note that if p is not conjugate to itself along α, i.e. there does not exist a nonzero $W \in \mathcal{J}(\alpha)$ with $W(0) = 0 = W(T)$, then

$$
\mathcal{J}(\alpha) \cong T_p(M) \times T_p(M) : \; W \mapsto (W(0), W(T)) \tag{39}
$$

is an isomorphism (since both spaces have dimension $2n$), and hence for any $w \in T_p(M)$, there exists a unique $W \in \mathcal{J}_{C^0}(\alpha)$ with $W(0) = w = W(T)$. This then gives the following:

Lemma 5.2. *Suppose p is not conjugate to itself along α. Then*

$$
T_\alpha\Omega_{per}(M) = \mathcal{J}_{C^0}(\alpha) \oplus T_\alpha\Omega_0(M; p, p),
$$

where the direct sum is orthogonal with respect to J''.

Proof. Given any $V \in T_\alpha\Omega_{\mathrm{per}}(M)$, by the above isomorphism, its value V_p uniquely determines a closed reflected Jacobi field $W \in \mathcal{J}_{C^0}(\alpha)$. There exists a unique variational vector field $Z \in T_\alpha\Omega_0(M; p, p)$ such that $V = Z + W$. The orthogonality condition $J''(W,Z) = 0$ is true as $W \in \mathcal{J}_{C^0}(\alpha)$ kills all terms in equation (38) except the term $\big\langle \triangle D_t W(0), \overline{Z}(0)\big\rangle$, which is killed by $Z \in T_\alpha\Omega_0(M; p, p)$. □

As the *index* of J'', denoted by $\mathrm{ind}(J'')$, is defined as the maximum dimension of a subspace of $T_\alpha\Omega_{\mathrm{per}}(M)$ on which J'' is negative definite, we obtain the following corollary:[1]

Corollary 5.3. *The index of J'' on periodic paths is given by*

$$\mathrm{ind}(J''|_{T_\alpha\Omega_{per}(M)}) = \mathrm{ind}(J''|_{\mathcal{J}_{C^0}(\alpha)}) + \mathrm{ind}(J''|_{T_\alpha\Omega_0(M;p,p)}).$$

We thus set out to compute the two indices in the sum above. The second arises simply as a special case (for a closed trajectory) of the Morse index theorem with fixed boundary conditions. Hence we now turn to this more standard problem of the index problem with fixed endpoints.

5.2. **The Morse index theorem for fixed endpoints.** We follow Milnor's treatment [9] with minor modifications.

We begin with a lemma insuring that sufficiently short (reflected) paths are locally action-minimizing.

Lemma 5.4. *Fix a reflected physical path $\alpha(t)$, $t \in [0,T]$. For $\epsilon > 0$, let*

$$\begin{aligned} V_\epsilon = \{ & Z \in T_\alpha\Omega_0(M;p,p') : \textit{for all } t \in [0,T], \textit{ there exists } t' \in [0,T] \\ & \textit{with } |t'-t| < \epsilon \textit{ such that } Z(t') = 0\}. \end{aligned}$$

For $\epsilon > 0$ sufficiently small, J'' is positive definite on V_ϵ.

Proof. By equation (26), we have $J''(Z,Z) = I(Z,Z) + B(Z,Z)$ with

$$I(Z,Z) = -\int_0^T \left\langle D_t^2 Z + R(\dot\alpha, Z)\dot\alpha + (\nabla^2 V)Z, Z\right\rangle dt,$$

[1]This follows from the following general fact: if Q is a symmetric bilinear form on a (possibly infinite-dimensional) vector space V, and $V = V_1 \oplus V_2$, where the direct sum is orthogonal with respect to Q, then

$$\mathrm{ind}(Q|_V) = \mathrm{ind}(Q|_{V_1}) + \mathrm{ind}(Q|_{V_2}),$$

assuming $\mathrm{ind}(Q|_{V_1}), \mathrm{ind}(Q|_{V_2}) < \infty$. If V is finite dimensional, this follows by diagonalizing Q on V_1 and V_2 with appropriate choices of inner product on V and bases on V_1, V_2; this is is a special case of the *Haynsworth inertia additivity formula*; see [11] for a reference. We were unable to find a proof in the literature in the case that V is not finite-dimensional; however, the result follows from the finite-dimensional case, as follows: if W is a finite-dimensional subspace of V on which Q is negative definite, and W_1, W_2 are subspaces of V_1, V_2 on which Q is negative definite with maximal dimension, consider

$$\tilde V = W + W_1 + W_2, \quad \tilde V_1 = \pi_1(W) + W_1, \quad \tilde V_2 = \pi_2(W) + W_2$$

where π_1, π_2 are the projections from V onto V_1, V_2. Then $\mathrm{ind}\, Q|_{\tilde V} = \mathrm{ind}\, Q|_{\tilde V_1} + \mathrm{ind}\, Q|_{\tilde V_2}$ since $\tilde V$ is finite dimensional. Moreover

$$\dim(W) \le \mathrm{ind}\, Q|_{\tilde V} = \mathrm{ind}(Q|_{\tilde V_1}) + \mathrm{ind}(Q|_{\tilde V_2}) = \mathrm{ind}(Q|_{V_1}) + \mathrm{ind}(Q|_{V_2}),$$

where the last equality follows since $\tilde V_1$, $\tilde V_2$ already contain a maximal-dimensional subspace of V_1, V_2 where Q is negative definite. This shows $\mathrm{ind}(Q|_V)$ is finite and is at most $\mathrm{ind}(Q|_{V_1}) + \mathrm{ind}(Q|_{V_2})$; the other inequality $\mathrm{ind}(Q|_V) \ge \mathrm{ind}(Q|_{V_1}) + \mathrm{ind}(Q|_{V_2})$ is obvious.

and $B(Z,Z)$ the "boundary terms" aside from the integral. Integrating by parts on each interval of the form $[T_i, T_{i+1}]$ yields

$$I(Z,Z) = \int_0^T \langle D_t Z, D_t Z\rangle - \left\langle R(\dot\alpha, Z)\dot\alpha + (\nabla^2 V)Z, Z\right\rangle dt + \sum_{i=1}^n \triangle\left(\langle D_t Z, Z\rangle\right)(T_i);$$

note that we can also write

$$\triangle\left(\langle D_t Z, Z\rangle\right)(T_i) = \left\langle \triangle D_t Z(T_i), \overline{Z}(T_i)\right\rangle + \left\langle \overline{D_t Z}(T_i), \triangle Z(T_i)\right\rangle.$$

For $T_i \in \mathcal{K}$, we have $\triangle Z(T_i) = 0$ since α is a reflected physical path. Hence,

$$\sum_{i=1}^n \triangle\left(\langle D_t Z, Z\rangle\right)(T_i) = \sum_{T_i\in\mathcal{R}} \left(\left\langle \triangle D_t Z, \overline{Z}\right\rangle + \left\langle \overline{D_t Z}, \triangle Z\right\rangle\right)\Big|_{T_i} + \sum_{T_i\in\mathcal{K}} \left\langle \triangle D_t Z, \overline{Z}\right\rangle\Big|_{T_i}.$$

It follows that

$$J''(Z,Z) = \int_0^T \langle D_t Z, D_t Z\rangle - \left\langle R(\dot\alpha, Z)\dot\alpha + (\nabla^2 V)Z, Z\right\rangle dt + \tilde{B}(Z,Z)$$

where

$$\begin{aligned}
\tilde{B}(Z,Z) &= B(Z,Z) + \sum_{T_i\in\mathcal{R}} \left(\left\langle \triangle D_t Z, \overline{Z}\right\rangle + \left\langle \overline{D_t Z}, \triangle Z\right\rangle\right)\Big|_{T_i} + \sum_{T_i\in\mathcal{K}} \left\langle \triangle D_t Z, \overline{Z}\right\rangle\Big|_{T_i} \\
&= \sum_{T_i\in\mathcal{R}} \left(2\langle Z_\perp^-, \overline{D_t Z}\rangle + \left\langle \overline{D_t Z}, \triangle Z\right\rangle + \left\langle 2Z_\perp^-, (Z_\perp^-/\dot\alpha_\perp^-)\nabla V\right\rangle + \langle 2\dot\alpha_\perp^-, \mathrm{II}(\partial_\epsilon c, \overline{Z} + T'_\epsilon \dot\alpha_\top)\rangle\right)\Big|_{T_i} \\
&= \sum_{T_i\in\mathcal{R}} \left(\left\langle 2Z_\perp^-, (Z_\perp^-/\dot\alpha_\perp^-)\nabla V\right\rangle + \langle 2\dot\alpha_\perp^-, \mathrm{II}(\partial_\epsilon c, \overline{Z} + T'_\epsilon \dot\alpha_\top)\rangle\right)\Big|_{T_i}
\end{aligned}$$

since $\triangle Z = -2Z_\perp^-$. Recalling that $\partial_\epsilon T_i$ and $\partial_\epsilon c$ are determined by the values of Z at the reflection points via

$$\triangle Z = -\partial_\epsilon T_i \triangle\alpha, \quad \partial_\epsilon c = Z_\top + \partial_\epsilon T_i \alpha_\top,$$

it follow there exists a constant C independent of Z such that

$$|\tilde{B}(Z,Z)| \le C \max_{[0,T]} \langle Z, Z\rangle.$$

Since there also exists C' such that

$$|\left\langle R(\dot\alpha, Z)\dot\alpha + (\nabla^2 V)Z, Z\right\rangle| \le C'\langle Z, Z\rangle,$$

it follows that

$$J''(Z) \ge \|D_t Z\|^2_{L^2([0,T])} - (C + C'T)\|Z\|^2_{L^\infty([0,T])}$$

where $\|W\|_{L^p([0,T])} := \||W(t)|_g\|_{L^p([0,T])}$. The claim is that this quantity is non-negative (and strictly positive if $Z \not\equiv 0$) if ϵ is sufficiently small. To verify this, we first take $\epsilon < \min_{i=0,\dots,m}(T_{i+1} - T_i)$, i.e. ϵ to be smaller than the width of any subinterval on which Z is smooth, in which case every $t \in [0,T]$ satisfies the property that there exists $t' \le t$, with $t - t' < \epsilon$, such that $Z(t') = 0$ and $Z|_{(t',t)}$ is smooth. Then the Fundamental Theorem of Calculus holds on $[t', t]$, and we can write

$$\langle Z(t), Z(t)\rangle = \langle Z(t'), Z(t')\rangle + \int_{t'}^{t} 2\langle D_t Z(s), Z(s)\rangle\, ds = \int_{t'}^{t} 2\langle D_t Z(s), Z(s)\rangle\, ds.$$

It then follows that

$$\begin{aligned} |\langle Z(t), Z(t)\rangle| = \left| \int_{t'}^{t} 2\langle D_t Z(s), Z(s)\rangle\, ds \right| &\leq 2\|D_t Z\|_{L^2([0,T])}\|Z\|_{L^2([t',t])} \\ &\leq 2\epsilon^{1/2}\|D_t Z\|_{L^2([0,T])}\|Z\|_{L^\infty([0,T])} \\ &\leq 2\epsilon\|D_t Z\|^2_{L^2([0,T])} + \frac{1}{2}\|Z\|^2_{L^\infty([0,T])}, \end{aligned}$$

from which taking supremums and absorbing $\frac{1}{2}\|Z\|^2_{L^\infty([0,T])}$ into the LHS yields

$$\|Z\|^2_{L^\infty([0,T])} \leq 4\epsilon\|D_t Z\|^2_{L^2([0,T])}.$$

Thus,

$$J''(Z) \geq (1 - 4(C + C'T)\epsilon)\|D_t Z\|^2_{L^2([0,T])},$$

with $1 - 4(C + C'T)\epsilon > 0$ for ϵ sufficiently small. Moreover, for such small ϵ, we have $J''(Z) = 0$ only if $D_t Z \equiv 0$, which when combined with Z equaling 0 at some times would happen only when $Z \equiv 0$. □

Theorem 5.5 (Morse Index Theorem with fixed endpoints)**.**

$$\operatorname{ind}(J''|_{T_\alpha\Omega_0(M;p,p')}) = \textit{number of conjugate points along } \alpha \textit{ with respect to } p$$

Proof. We follow the classic treatment of [9, §15] in the standard setting, mainly noting where our setting of reflected trajectories (and mechanical rather than geometric Lagrangian function) requires changes.

Fix times $0 = t_0 < t_1 < \cdots < t_k < T$ such that (for simplicity) $\alpha(t_j) \in M\backslash Y$ for all j and sufficiently closely spaced that each pair t_j, t_{j+1} can play the role of t, t' in Lemma 5.4 above.

Now let $T_\alpha\Omega(t_0, \ldots, t_k) \subset T_\alpha\Omega_0(M; p, p')$ denote the subspace of "piecewise reflected Jacobi fields," i.e. vector fields W along α such that on each interval $[t_{j-1}, t_j]$, W is a reflected Jacobi field and such that $W(0) = W(T) = 0$. Let T' denote the space of $W \in T_\alpha\Omega_0(M; p, p')$ vanishing at t_j for all $j = 0, \ldots, k$.

By the same reasoning as in [9], we now find that

$$T_\alpha\Omega_0(M; p, p') = T_\alpha\Omega(t_0, \ldots, t_k) \oplus T',$$

that the sum is orthogonal with respect to the quadratic form J'', and that on T' the form J'' is positive definite. (The first assertion follows from Theorem 4.2; the latter assertion is where Lemma 5.4 is essential.)

Thus, the index of J'' is the same as the index of its restriction to $T_\alpha\Omega(t_0, \ldots, t_k)$. As in [9], we now set $\lambda(\tau)$ to be the value of the index J'' at α restricted to $t \in [0, \tau]$; this is nondecreasing and zero for sufficiently small τ (using Lemma 5.4) by the same reasoning on employed in [9].

Note that we may identify $T_\alpha\Omega(t_0,\dots,t_k)$ with $T_{\alpha(t_1)}\oplus\dots T_{\alpha(t_k)}$. The quadratic form J'' on $T_\alpha\Omega(t_0,\dots,t_k)$ restricted to the time interval $[0,\tau]$ varies continuously in τ even when τ equals T_i for some i, since the boundary terms in the second variation $J''(W,W)$ given by (26) vanish when W is a Jacobi field. On any subspace of $T_\alpha\Omega(t_0,\dots,t_k)$ on which the index form is negative definite, it remains negative definite on that space under small variations of τ; since $\lambda(\tau)$ is nondecreasing, then, we have $\lambda(\tau-\epsilon)=\lambda(\tau)$ whenever $\epsilon>0$ is sufficiently small.

We claim further that if $\alpha(\tau)$ is conjugate to $\alpha(0)$ with multiplicity ν then for $\epsilon>0$ sufficiently small,

$$\lambda(\tau+\epsilon)=\lambda(\tau)+\nu;$$

this will suffice to establish the theorem. The fact that $\lambda(\tau+\epsilon)\leq\lambda(\tau)+\nu$ proceeds just as in [9], as it depends just on the continuity of the index form. It thus suffices to establish the reverse inequality.

If $\alpha(\tau)\notin Y$, then we also obtain $\lambda(\tau+\epsilon)\geq\lambda(\tau)+\nu$ as in Milnor; we thus make a few remarks on the case $\alpha(\tau)\in Y$. In this case, let $W_1,\dots,W_{\lambda(\tau)}$ denote the reflected piecewise Jacobi fields vanishing at the endpoints $t=0$ and $t=\tau$ (i.e., elements of $T_\alpha\Omega(t_0,\dots,t_k)$) on which J'' is negative definite. Let $Q_1,\dots,Q_\nu$ be independent reflected Jacobi fields vanishing at the endpoints $t=0$ and $t=\tau$. Pick X_k to be variation vector fields along α between times 0 and $\tau+\epsilon$ vanishing at the endpoints $t=0$ and $t=\tau+\epsilon$ so that

$$\langle D_tQ_i(\tau-),X_j(\tau-)\rangle_g=\delta_{ij}.$$

Extend the Q_i and W_j by zero on the interval $[\tau,\tau+\epsilon]$ (i.e. subsequent to reflection). We note that just as in the interior case, considering the second variation on $[0,\tau+\epsilon]$ we obtain

$$J''(Q_i,W_j)=0$$

and

$$J''(Q_i,X_j)=2\delta_{ij}.$$

Here we crucially use the fact that the second variation form (26) in the case where $\alpha(\tau)\in Y$ with $Q_i(\alpha(\tau))=0$, Q_i a reflected Jacobi field, and $Q_i=0$ for $t>\tau$ yields

$$J''(Q_i,X_j)=\langle D_tQ_i(\tau-),X_j(\tau-)\rangle_g.$$

The rest of the proof proceeds as usual: for c sufficiently small, the quadratic form J'' is negative definite on the span of

$$W_1,\dots,W_{\lambda(\tau)},c^{-1}Q_1-cX_1,\dots c^{-1}Q_\nu-cX_\nu.$$

□

5.3. **The Morse index for periodic paths.** Fix a periodic reflected physical path α with period T, and fix $p=\alpha(0)$. Assume that p is not conjugate to itself along α. Then given any x,y in a small neighborhood of p, recall from Proposition 4.12 that there exists a unique reflected physical path $\alpha_{x,y}(t)$ close to α with

$$\alpha_{x,y}(0)=x,\ \alpha_{x,y}(T)=y.$$

We then as usual define the *action*

$$S(x,y) := J[\alpha_{x,y}].$$

Lemma 5.6.

$$\operatorname{ind}(J''|_{\mathcal{J}_{C^0(\alpha)}}) = \operatorname{ind}(\mathsf{Hess}\,|_{x=p}[S(x,x)])$$

Proof. Let $\alpha_{x,y}(t)$ be defined as above; employing Riemann normal coordinates near p to make sense of the following expressions, consider

$$\beta(t,\epsilon) := \alpha_{p+\epsilon w, p+\epsilon w}(t),$$

a physical reflected path from $p+\epsilon w$ to itself. Then

$$W := \partial_\epsilon \beta(t,\epsilon)|_{\epsilon=0}$$

is a reflected Jacobi field in $\mathcal{J}_{C^0(\alpha)}$ by Proposition 4.8, with $W(0) = W(T) = w$. Hence

$$\mathsf{Hess}\,|_{x=p}[S(x,x)](w,w) = \partial^2_{\epsilon\epsilon} J[\beta(t,\epsilon)]|_{\epsilon=0} = J''(W,W),$$

and the indices of these forms thus coincide (since the map $w \mapsto W$ is an isomorphism of T_pM and $\mathcal{J}_{C^0}(\alpha)$). $\square$

Combining Corollary 5.3, Theorem 5.5 and Lemma 5.6, we obtain

Theorem 5.7 (Morse Index Theorem for periodic paths)**.**

$$\begin{aligned}\operatorname{ind}(J''|_{T_\alpha \Omega_{per}(M)}) &= \operatorname{ind}(\mathsf{Hess}\,|_{x=p}[S(x,x)]) \\ &\quad + \textit{ number of conjugate points along } \alpha \textit{ with respect to } p.\end{aligned} \tag{40}$$

Note that while the sum is an invariant of α, each of the two terms on the right-hand-side of (40) may individually depend on the choice of p along α [1, Section IV.B].

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