CONVERGENCE OF CONZE-LESIGNE AVERAGES

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ABSTRACT. We study the convergence of $\frac{1}{N}\sum f_1(T^{a_1 n}x)f_2(T^{a_2 n}x)f_3(T^{a_3 n}x)$, for a measure preserving system $(X,\mathcal{B},\mu,T)$ and $f_1,f_2,f_3 \in L^\infty(\mu)$. This generalizes the theorem of Conze and Lesigne on such expressions and simplifies the proof. As well, we obtain a description of the limit.

1. Introduction

1.1. **Background.** An open problem is the existence of limits of expressions of the form

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{a_1 n}x)f_2(T^{a_2 n}x)\dots f_\ell(T^{a_\ell n}x), \tag{1}$$

where T is a measure preserving automorphism of a probability space $(X,\mathcal{B},\mu)$, $f_1,f_2,\dots,f_\ell \in L^\infty(X)$ and $a_1,a_2,\dots,a_\ell$ are distinct integers. Limits of such expressions arise in Furstenberg's proof of Szeméredi's Theorem and have been studied in various forms by Bourgain[2], Bergelson[1], Furstenberg and Weiss[7] and Conze and Lesigne[5].

If one assumes that T is weak mixing, Bergelson[1] proved a convergence theorem for more general expressions. However, without the assumption of weak mixing, one can easily show that the limit need not be constant and proving convergence becomes much more difficult. The existence of limits for the case with $\ell = 3$ and with the added hypothesis that the system is totally ergodic was proved by Conze and Lesigne in a series of papers (see [3], [8], [9], [5], [4] and [10]). Similar expressions were considered by Furstenberg and Weiss[7] in order to study the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^n x)f_2(T^{n^2}x)\ .$$

1.2. **Statement of results.** We reprove the convergence obtained by Conze and Lesigne directly, without needing the elaborate machinery they used. Furthermore, we eliminate the reliance on the hypothesis of total ergodicity. As for the other known methods for analyzing expressions of the form of Equation (1), we are unable to extend our proof to more than three terms. In offering a new and simpler proof of the convergence for three terms, we hope to gain insights into limits of more general expressions.

In Section 4.3, we show:

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Theorem 1.1. *Let $(X,\mathcal{B},\mu,T)$ be a measure preserving system, a_1,a_2,a_3 three distinct integers and $f_1,f_2,f_3\in L^\infty(\mu)$. Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{3}f_i(T^{a_in}x)$$

exists in $L^2(\mu)$.

In fact we prove more than just existence, giving an description of the limit in Section 4.4. A more precise value for the limit will be given in a forthcoming paper.

1.3. **Organization of the paper.** Our proof, like those of Conze and Lesigne[5] and Furstenberg and Weiss[7], is roughly split into two parts. First we reduce the problem to studying convergence on a simpler system. We follow classical methods, using ideas introduced by Furstenberg[6] in the proof of the Szeméredi theorem, such as the Van der Corput lemma[1] and isometric extensions. However, we have no need for the detailed structure of the modified system, as used by Conze and Lesigne, nor of the normal systems introduced by Furstenberg and Weiss. We include few details of this portion of the proof, referring the reader to the literature. This is carried out in Section 2.

The second part of the proof is the demonstration of the convergence in the modified system. Our method is more elementary than that previously known, and we obtain a description of the limit. One of our main tools is a lemma in harmonic analysis, proved in Section 3. In Section 4, we prove the convergence and then in Section 4.4, we give the actual formula.

2. Reduction to a simpler system

We can always assume ergodicity of the system, without loss of generality, by using ergodic decomposition.

We plan to modify the original measure preserving system three times, showing each time that proving the theorem for the new system implies the result for the old system. First, we clarify the ideas needed for such reductions.

2.1. **Characteristic factors.**

Definition. *Given distinct integers $a_1,a_2,\dots,a_\ell$ and a factor $(Y,\mathcal{Y},\nu,S)$ of a system $(X,\mathcal{B},\mu,T)$, we say that Y is a* **characteristic factor** *of X for the scheme $a_1,a_2,\dots,a_\ell$ if for any $f_1,f_2,\dots f_\ell\in L^\infty(\mu)$, whenever there is some $i\in\{1,2,\dots,\ell\}$ with $\mathbb{E}(f_i\mid\mathcal{Y})=0$, the limit*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{\ell}f_i\circ T^{a_in}$$

exists in $L^2(\mu)$ and is equal to 0.

This property implies that for $f_1,f_2,\dots,f_\ell\in L^\infty(\mu)$

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{\ell}f_i\circ T^{a_in}-\frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{\ell}\mathbb{E}(f_i\circ T^{a_in}\mid\mathcal{Y})\right)=0$$

in $L^2(\mu)$.

Therefore, finding a characteristic factor Y for a system X allows us to restrict to functions defined only on Y, and this restriction simplifies computations when Y has a simple form.

2.2. **The Kronecker is a characteristic factor for two terms.** Throughout the sequel, $(X,\mathcal{B},\mu,T)$ is an ergodic measure preserving system and $(Z,\mathcal{Z},m,S)$ denotes its Kronecker factor. More specifically, $S:Z\to Z$ is the rotation defined by $Sz=z+\alpha$, and we use $\pi:X\to Z$ for the natural projection. For $f\in L^2(\mu)$, we write $\tilde f$ the function on Z defined by $\tilde f\circ\pi=\mathbb{E}(f\mid\mathcal{Z})$.

Using the Van der Corput lemma (see Bergelson[1]), Furstenberg and Weiss showed that the Kronecker factor is characteristic for two arbitrary terms b_1,b_2, and deduced:

Theorem 2.1 (Furstenberg and Weiss[7])**.** *Let $(X,\mathcal{B},\mu,T)$ be a measure preserving system, with notations as above. Let b_1, b_2 be integers. Then for any $f_1,f_2\in L^\infty(\mu)$*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}f_1(T^{b_1n}x)f_2(T^{b_2n}x)$$

exists in $L^2(\mu)$ and equals

$$\int_Z\tilde f_1(z+b_1\theta)\tilde f_2(z+b_2\theta)\,dm(\theta),$$

where $z=\pi(x)$.

2.3. **Two joinings.** We assume here that a_1,a_2,a_3 are given integers. Let $\tilde Z$ be the closed subgroup

$$\tilde Z=\big\{(z+a_1t,z+a_2t,z+a_3t):z,t\in Z\big\}$$

of Z^3 and let $\tilde m$ be its Haar measure. We write $\tilde z=(z_1,z_2,z_3)$ for an element $\tilde z\in\tilde Z$. These notations will be used throughout the sequel.

By Theorem 2.1, for $f_1,f_2,f_3\in L^\infty(\mu)$,

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\int_X\prod_{i=1}^{3}f_i(T^{a_in}x)\,dm(x)&=\int\prod_{i=1}^{3}\tilde f_i(z+a_i\theta)\,dm(z)\,dm(\theta)\\ &=\int_{\tilde Z}\prod_{i=1}^{3}\tilde f_i(z_i)d\tilde m\,(\tilde z)\ .\end{aligned}\tag{2}$$

The subgroup $\tilde Z$ is invariant under the transformation $\tilde S=S^{a_1}\times S^{a_2}\times S^{a_3}$.

Writing

$$\tilde\alpha=(a_1\alpha,a_2\alpha,a_3\alpha)\ ,$$

the transformation $\tilde S$ is given by

$$\tilde S\tilde z=\tilde z+\tilde\alpha\ .$$

Thus, $(\tilde Z,\tilde m,\tilde S)$ is a joining of the systems (Z,m,S^{a_i}), for $i=1,2,3$, and each of these is a factor of the corresponding (X,μ,T^{a_i}). Therefore we can form the "conditionally independent product" $(\tilde X,\tilde\mu,\tilde T)$ of these systems over this joining. It is a joining of (X,μ,T^{a_i}) for $i=1,2,3$. (See Furstenberg[6] and Furstenberg and Weiss[7].)

Using this, we rewrite Equation (2) and have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_X \prod_{i=1}^{3} f_i(T^{a_i n} x)\, d\mu(x) = \int_{\tilde{X}} \prod_{i=1}^{3} f_i(x_i)\, d\tilde{\mu}(x_1, x_2, x_3)\ .$$

2.4. **Group extensions.** We recall some basic facts about group extensions.

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving system and let $(Y, \mathcal{D}, \nu, T)$ be a factor. By definition, X is an **isometric extension** of Y if there exists a homogeneous space $H = L/K$ of a metrizable compact group L and a measurable map $\sigma : Y \to L$ so that $(X, \mathcal{B}, \mu, T)$ is isomorphic to the skew product $(Y \times H, \mathcal{B} \otimes \mathcal{B}_H, \nu \times m_H, T_\sigma)$, where $\mathcal{B}_H$ is the Borel σ-algebra of H, m_H is the L-invariant measure on H and $T_\sigma(y, u) = (Ty, \sigma(y)u)$. σ is called the **cocycle** of the extension. If $H = L$, we say that X is a **group extension** of Y.

We note for later use that by Lemma 7.2 in Furstenberg and Weiss, given an ergodic isometric extension $X = Y \times H$, one can express $H = L/K$ and $X = Y \times L/K$ so that the group extension $X_1 = Y \times L$ defined by the same cocycle is also ergodic.

2.5. **Reduction to an isometric extension of the Kronecker.** We now use these structures to make our first change in the measure preserving system, allowing us to assume that X is an isometric extension of the Kronecker factor Z.

Lemma 2.2 (Conze and Lesigne[3])**.** *Let a_1, a_2, a_3 be distinct integers. Assume that $f_1, f_2, f_3 \in L^\infty(\mu)$ and let $F(z_1, z_2, z_3) = f_1(z_1) f_2(z_2) f_3(z_3)$. If F is orthogonal to the space of $T^{a_1} \times T^{a_2} \times T^{a_3}$-invariant functions in $L^2(\tilde{\mu})$, then the averages*

$$\frac{1}{N} \sum_{n=0}^{N-1} \prod_{i=1}^{3} f_i(T^{a_i} x)$$

converge to 0 *in $L^2(\mu)$.*

Proof. The lemma is proved via an application of the Van der Corput Lemma, with $u_n(x) = f_1(T^{a_1 n} x) f_2(T^{a_2 n} x) f_3(T^{a_3 n} x)$. For details, see Furstenberg and Weiss[7]. □

For an ergodic system $(X, \mathcal{B}, \mu, T)$ with Kronecker factor (Z, α), let $(\widehat{Z}, \widehat{\mathcal{D}}, \widehat{\nu}, T)$ denote the maximal isometric extension of (Z, α) in (X, T).

Theorem 2.3 (Furstenberg and Weiss[7])**.** *$\widehat{Z}$ is a characteristic factor of X for all schemes $\{a_1, a_2, a_3\}$.*

Proof. The proof uses Lemma 2.2, the fact that $\tilde{\mu}$ defines a joining of the systems (X, μ, T^{a_i}) and a theorem on joinings of Furstenberg[6]: invariant functions on a conditionally independent product factorize through the conditionally independent product of the maximal isometric extension. Again, we refer to Furstenberg and Weiss[7] for the details. □

Thus in order to prove the existence of

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{i=1}^{3} f_i(T^{a_i n} x)$$

in $L^2(m)$, it suffices to show the convergence for functions defined on the isometric extension $\widehat{Z}$ of Z. We express the extension $\widehat{Z} = Z \times L/K$ in such a way that the corresponding group extension $X_1 = Z \times L$ is ergodic. Clearly it suffices to prove the convergence for this system X_1. Let Z_1 be the Kronecker factor of X_1. By Lemma 7.3 of Furstenberg and Weiss, X_1 is a group extension $Z_1 \times L_1$ of Z_1. The following diagram explains these relations:

$$\begin{array}{ccc} X & & \\ \downarrow & & \\ \widehat{Z} = Z \times L/K & \longleftarrow & Z \times L = X_1 = Z_1 \times L_1 \\ \downarrow & & \downarrow \\ Z & \longleftarrow & Z_1 \end{array}$$

For simplification of notation, from now on we can forget the initial system and assume that X is itself a group extension $Z \times L$ of its Kronecker factor Z.

2.6. **The Mackey groups.** The notion of the Mackey group (also referred to as the group of essential values) is not completely classical and so we recall the basic facts.

We consider an ergodic system (Y, ν, S) with an extension by a compact group H, defined by a cocycle σ. Thus σ is a measurable map from Y to H. We say that σ is a **coboundary** if there exists a cocycle ϕ with $\sigma(y) = \phi(Sy)\phi(y)^{-1}$. Two cocycles σ and σ' are said to be **cohomologous** if there exists a cocycle ϕ so that $\sigma'(y) = \phi(Sy)\sigma(y)\phi(y)^{-1}$.

Proposition 2.4. *For each cocycle σ, there is associated a closed subgroup M of H (uniquely determined up to conjugacy) satisfying:*

ı) σ is cohomologous to some cocycle σ' with values in M and M is a minimal closed subgroup of H with this property.

ıı) For all $m \in M$, each $S_{\sigma'}$-invariant function f on $Y \times H$ satisfies $f(y, mh) = f(y, h)$ for almost every $(y, h) \in Y \times H$.

Given a cocycle σ, the associated closed subgroup M is called its **Mackey group**.

Proof. The proof, again, is outlined in Furstenberg and Weiss[7]. □

Property (ıı) combined with Lemma 2.2 explains the important role of the Mackey group in our setup.

2.7. **The subgroup M of L^3.** Recall that we have defined $\tilde{X}$ and $\tilde{Z}$ in Section 2.3. We have that $\tilde{X} = \tilde{Z} \times L^3$ is a group extension of $\tilde{Z}$ with cocycle

$$\tilde{\sigma}(\tilde{z}) = \left(\sigma^{(a_1)}(z_1), \sigma^{(a_2)}(z_2), \sigma^{(a_3)}(z_3)\right) .$$

Thus the transformation $\tilde{T}$ on $\tilde{X}$ is given by

$$\begin{aligned} &\tilde{T}(z_1, z_2, z_3, \ell_1, \ell_2, \ell_3) = \\ &\qquad \left(z_1 + a_1\alpha, z_2 + a_2\alpha, z_3 + a_3\alpha, \sigma^{(a_1)}(z_1)\ell_1, \sigma^{(a_2)}(z_2)\ell_2, \sigma^{(a_3)}(z_3)\ell_3\right) . \end{aligned} \tag{3}$$

We can not immediately apply the theory of Mackey groups to the group extension $(\tilde{X}, \tilde{\mu}, \tilde{T})$ of $(\tilde{Z}, \tilde{m}, \tilde{S})$, as the second system is not ergodic, and so we need some preliminaries.

For every $z \in Z$ the subset

$$\tilde{Z}_z = \{(z + a_1 t, z + a_2 t, z + a_3 t) \;:\; t \in Z\}$$

of $\tilde{Z}$ is invariant under $\tilde{S}$. The uniform measure $\tilde{m}_z$ on this set is invariant by $\tilde{S}$ and is ergodic. It gives an ergodic decomposition

$$\tilde{m} = \int_Z \tilde{m}_z \, dm(z)$$

of $\tilde{m}$. (This is not the standard ergodic disintegration, as the measures m_z are not necessarily distinct.)

Consider the corresponding decomposition

$$\tilde{\mu} = \int_Z \tilde{\mu}_z \, dm(z)$$

of $\tilde{\mu}$. For each z the system $(\tilde{X}, \tilde{\mu}_z, \tilde{T})$ is an extension of $(\tilde{Z}_z, \tilde{m}_z, \tilde{S})$ by the group L^3, with the restriction of the cocycle $\tilde{\sigma}$, and so has a Mackey group M_z defined up to conjugacy. We write $[M_z]$ for the conjugacy class of M_z. The family of conjugacy classes of closed subgroups of the compact metrizable group L^3 can be endowed with a structure of Polish space so that the map $z \mapsto [M_z]$ is Borel. Moreover, the measure $\tilde{\mu}$ is invariant under the transformation $T \times T \times T$, which commutes with $\tilde{T}$. It follows that $[M_{z+\alpha}] = [M_z]$ for all $z \in Z$. By ergodicity of the rotation $S : z \mapsto z + \alpha$, the class $[M_z]$ is constant almost everywhere. Thus we can take M_z equal for almost all z to a fixed subgroup M of L^3. We call M the **Mackey group of the cocycle $\tilde{\sigma}$ on $\tilde{Z}$**. As for the true Mackey groups, it satisfies the properties of Propositions 2.4.

2.8. **A final reduction.** The final step is to reduce to an abelian group extension.

Theorem 2.5 (Furstenberg and Weiss[7]). *X has a characteristic factor for all schemes a_1, a_2, a_3 that is an abelian group extension of its Kronecker factor Z.*

Proof. We give a sketch of the proof given by Furstenberg and Weiss[7]. Recall that the transformation $\tilde{T}$ on $\tilde{X} = \tilde{Z} \times L^3$ is given by Formula (3). Using that Z is the Kronecker factor of X, they deduce that for all $i \neq j \in \{1, 2, 3\}$ and almost all z, the (i, j)-two dimensional marginal of $\tilde{\mu_z}$ is $T^{a_i} \times T^{a_j}$-ergodic. Therefore, each two dimensional projection $M \to L \times L$ is surjective. By an algebraic lemma, they prove that M contains $L' \times L' \times L'$, where L' is the commutator subgroup of L. Applying Lemma 2.2 and property ($\imath\imath$) of Proposition 2.4, the factor $Z \times L/L'$ of $X = Z \times L$ is characteristic for all schemes a_1, a_2, a_3. □

From now on, we write $G = L/L'$ and write this group additively.

At this point, we simplify without loss of generality and assume that our system X itself is $Z \times G$, a compact abelian group extension of its Kronecker.

Our new X is a factor of an extension of a factor of the original X. Many properties of the original system may be lost in this construction, for example total ergodicity. This has no implication for our present work.

2.9. **The Mackey group in an abelian extension.** We have reduced our original system to an abelian group extension of its Kronecker, and in this set-up we can say more about the Mackey group. We make frequent use of some elementary results about the duality in compact abelian groups and we review the necessary facts here.

Let H be a compact abelian group. The characters of H are the continuous group homomorphisms from H to the circle group $\mathcal{S}^1$. They form a multiplicative group $\widehat{H}$ called the dual group of H. For a closed subgroup M of H, its annihilator $M^\perp$ is the subgroup of $\widehat{H}$ given by

$$M^\perp = \{\chi \in \widehat{H} \ : \ \chi(m) = 1 \text{ for all } m \in M\} \ .$$

Also,

$$M = M^{\perp\perp} = \{h \in H \ : \ \chi(h) = 1 \text{ for all } \chi \in M^\perp\} \ .$$

Let (Y, μ, S) be an ergodic system and σ a cocycle with values in a compact, abelian group H. The Mackey group M associated to σ is unique. This means that in Proposition 2.4, the minimal, closed subgroup of H satisfying property ($\imath$) is actually the smallest.

Proposition 2.6. *Using notations as above:*

$\imath$) A character $\chi \in \widehat{H}$ belongs to the annihilator $M^\perp$ of M if and only if $\chi \circ \sigma$ is a multiplicative coboundary (as a cocycle with values in the circle group $\mathcal{S}^1$).

$\imath\imath$) For all $m \in M$, each S_σ-invariant function f on $Y \times H$ satisfies $f(y, h+m) = f(y, h)$ for almost all $(y, h) \in Y \times H$.

Proof. Part $\imath$): By Proposition 2.4, property ($\imath$), σ is cohomologous to a cocycle σ' with values in M and there is a function $b : Z \to H$ with

$$\sigma(z) = \sigma'(z) + b(Sz) - b(z) \ .$$

For $\chi \in M^\perp$,

$$\chi \circ \sigma(z) = \chi(b(Sz))\, \overline{\chi(b(z))}$$

and $\chi \circ \sigma$ is a coboundary.

Conversely, let $\chi \in \widehat{H}$ and assume that $\chi \circ \sigma$ is the coboundary of a function b . That is, $\chi \circ \sigma(z) = b(Sz)\overline{b(z)}$. The function $B(z, h) = \overline{b(z)}\chi(h)$ defined on $X = Z \times H$ is invariant by T_σ. By property ($\imath\imath$) of Proposition 2.4, for every $m \in M$, $B(z, h + m) = B(z, h)$ for almost all z and h. Thus $\chi(m) = 1$ and $\chi \in M^\perp$.

Part $\imath\imath$): The proof follows immediately from Part $\imath\imath$) of Proposition 2.4. □

2.10. **Conclusion of the reduction.** We summarize the results of our modifications. We have $X = Z \times G$ for some compact abelian metrizable group G (written additively) and the natural projection $X \to Z$ is given by $\pi(z, g) = z$. The transformation T on X is given by the cocycle $\sigma : Z \to G$ and can be written

$$T(z, g) = (Sz, g + \sigma(z)) \ .$$

The measure $\mu = m \times m_G$, where m_G is the Haar measure of G.

As usual, we write $\sigma^{(0)}(z) = 0$ and for $n > 0$,

$$\sigma^{(n)}(z) = \sigma(z) + \sigma(Sz) + \cdots + \sigma(S^{n-1}z),$$

with a similar formula for $n < 0$. For every integer n we have

$$T^n(z, g) = (S^n z, g + \sigma^{(n)}(z))$$

and for integers m, n we have the "cocycle equation"

$$\sigma^{(n+m)}(z) = \sigma^{(n)}(S^m z) + \sigma^{(m)}(z) \ .$$

As before, $\tilde{X} = \tilde{Z} \times G^3$. For $i = 1, 2, 3$ the i-th projection of $\tilde{X}$ on X is given by $(z_1, z_2, z_3, g_1, g_2, g_3) \mapsto (z_i, g_i)$. The measure $\tilde{\mu}$ is the product $\tilde{m} \times m_G \times m_G \times m_G$ of the Haar measures. The transformation $\tilde{T} = T^{a_1} \times T^{a_2} \times T^{a_3}$ on $\tilde{X}$ is given by

$$\tilde{T}(\tilde{z}, g_1, g_2, g_3) = \big(\tilde{z} + \tilde{\alpha}, g_1 + \sigma^{(a_1)}(z_1), g_2 + \sigma^{(a_2)}(z_2), g_3 + \sigma^{(a_3)}(z_3)\big) \ .$$

This means that the system $(\tilde{X}, \tilde{\mu}, \tilde{T})$ is a compact abelian group extension of $(\tilde{Z}, \tilde{m}, \tilde{S})$ by the group G^3, given by the cocycle $\tilde{\sigma} : \tilde{Z} \to G^3$, where $\tilde{\sigma} = \sigma^{(a_1)} \times \sigma^{(a_2)} \times \sigma^{(a_3)}$.

The results of Proposition 2.6 remain valid for the cocycle $\tilde{\sigma}$ and the subgroup M of G^3 defined as in Section 2.7.

3. A lemma in Harmonic Analysis

Here we stop following the paper of Furstenberg and Weiss[7]. Our main technical tool in proving Theorem 1.1 is a result in harmonic analysis and its corollaries.

Unless otherwise noted, all L^2-norms $\| \cdot \|_2$ are relative to the the measure m, Haar measure on Z, and are assumed to be taken with respect to the variable z.

We recall that Z is a compact monothetic group and so $\mathbb{Z}\alpha$ is dense in Z. We say that a function ω is affine if $\omega = c\gamma$, the product of a constant c and a character γ on Z.

Lemma 3.1. *Let f be a function of modulus 1 on Z so that the following two conditions are satisfied:*

1. $\int_Z |f(z+\alpha) - f(z)|^2 \, dm(z) < 2/9$
2. *For all $s \in Z$ there exists an affine function ω_s on Z such that*

$$\int_Z |f(z+s) - \omega_s(z) f(z)|^2 \, dm(z) < \delta^2 \tag{4}$$

for some $\delta > 0$.

Then there exists an affine function ω so that $\|f - \omega\| < 3\sqrt{2}\delta$.

Proof. If $\delta > \sqrt{2}/3$, there is nothing to prove. We assume that $\delta \leq \sqrt{2}/3$.

Write $\omega_s(z) = c_s \gamma_s(z)$ for a constant c_s and a character γ_s on Z. Since $\delta \leq \sqrt{2}/3$, the character γ_s is uniquely defined by the bound (4). Moreover, by the first hypothesis, $\gamma_\alpha = 1$.

The constant c_s is not defined by the bound (4) and we can choose it so that the integral in (4) is minimal. By the continuity of translations on $L^2(Z)$, the map $s \mapsto c_s$ is continuous on Z.

For $s, t \in Z$, applying bound (4) with s, t and $s + t$, we have

$$\|c_{s+t}\gamma_{s+t} - c_s c_t \gamma_s(t)\gamma_s\gamma_t\|_2 < 3\delta \leq \sqrt{2} \ .$$

Thus $\gamma_{s+t} = \gamma_s\gamma_t$. Furthermore, if s is sufficiently close to 0 in Z, by again using the same continuity argument we have that $\|f(z+s) - f(z)\|_2 < \sqrt{2}/3$ and so $\gamma_s = 1$. Thus the map $s \mapsto \gamma_s$ is a continuous group homomorphism from Z to $\widehat{Z}$. Since $\gamma_\alpha = 1$, $\gamma_s = 1$ for all $s \in Z$ by continuity and density. The bound (4) becomes

$$\|f(z+s) - c_s f(z)\|_2 < \delta \ .$$

Taking the Fourier transform with respect to z and integrating with respect to s in this bound, we get

$$\sum_{\theta \in \widehat{Z}} |\widehat{f}(\theta)|^2 \big(1 - \mathrm{Re}(\widehat{c}(\theta))\big) < \delta^2/2 \ ,$$

where

$$\widehat{c}(\theta) = \int c_s \bar{\theta}(s)\, dm(s) \ .$$

Since $\sum_{\theta\in\widehat{Z}} |\widehat{f}(\theta)|^2 = 1$, there exists $\gamma \in \widehat{Z}$ such that $1 - \mathrm{Re}(\widehat{c}(\gamma)) < \delta^2/2$. Thus $|\widehat{c}(\gamma)|^2 > 1 - \delta^2$. Since $\sum_{\theta\in\widehat{Z}} |\widehat{c}(\theta)|^2 = 1$, for all $\theta \neq \gamma$ we have $|\widehat{c}(\theta)| < \delta$ and thus $1 - \mathrm{Re}(\widehat{c}(\theta)) > 1 - \delta$. We get

$$\sum_{\theta\neq\gamma} |\widehat{f}(\theta)|^2 < \frac{\delta^2}{2(1-\delta)} < \delta^2 \ .$$

That is, $\|f - \widehat{f}(\gamma)\gamma\|_2 < \delta$. Taking $c = \widehat{f}(\gamma)/|\widehat{f}(\gamma)|$ and we get the statement of the lemma. □

Proposition 3.2. *Let a be a non-zero integer. There exists a constant $C > 0$ (depending only on a) such that whenever f is a function of modulus 1 on Z so that for some $\delta > 0$ the following two properties are satisfied:*

1. $\int_Z |f(z+\alpha) - f(z)|^2\, dm(z) < \delta^2$
2. *For all $s \in Z$ there exists an affine function ω_s on Z such that*

$$\int_Z f(z+as) - \omega_s(z) f(z)|^2\, dm(z) < \delta^2, \tag{5}$$

then there exists an affine function ω on Z so that $\|f - \omega\|_2 < C\delta$.

Proof. We use C to denote any positive constant depending only on a. Without loss of generality we can assume that δ is as small as needed.

Let K be the open subgroup aZ of Z and let k be its index in Z. As $\mathbb{Z}\alpha$ is dense in Z, $(k\alpha)\mathbb{Z}$ is dense in K, $K = kZ$, and the K-cosets are $K, \alpha+K, \dots, (k-1)\alpha+K$. As K is monothetic with generator $k\alpha$, we can apply Lemma 3.1 with K substituted for Z and $k\alpha$ substituted for α.

Fix $j \in \{0, \dots, k-1\}$, define a function f_j on K by $f_j(z) = f(j\alpha + z)$ and so

$$\int_K |f_j(z+k\alpha) - f_j(z)|^2\, dm_K(z) < k^3\delta^2 \ .$$

Furthermore, for all $s \in K$,

$$\int_K |f_j(z+s) - f_j(z)\omega_s(z+j\alpha)|^2\, dm_K(z) < k\delta^2 \ .$$

The restriction of the function $\omega_s(z+j\alpha)$ to K is affine and so we can apply Lemma 3.1 with the function f_j. As δ is small, there exists a constant c_j and a character $\gamma_j \in \widehat{Z}$ with

$$\int_K |f_j(z) - c_j\gamma_j(z)|^2\, dm_K(z) < C^2\delta^2$$

for some constant C. The character γ_j is defined only modulo $K^\perp$.

By the first hypothesis, we have

$$\sum_{j=0}^{k-2} \int_K |c_{j+1}\gamma_{j+1}(z) - c_j\gamma_j(z)|^2\, dm_K(z) + \int_K |c_0\gamma_0(z+k\alpha) - c_{k-1}\gamma_{k-1}(z)|^2\, dm_K(z) < C\delta^2$$

for some constant C. Thus $\gamma_{j+1} = \gamma_j \mod K^\perp$ for $0 \le j < k-1$, and we can choose $\gamma \in \widehat{Z}$ with $\gamma_j = \gamma \mod K^\perp$ for all j. Moreover we have $|c_{j+1} - c_j| < C\delta$ for $0 \le j < k-1$ and $|c_0\gamma(k\alpha) - c_{k-1}| < C\delta$. Thus there exists c with $|c| = 1$ and $|c - c_j| < C\delta$ for $0 \le j \le k-1$, and we have also $|1 - \gamma(\alpha)^k| < C\delta$.

There exists a complex number ξ with $\xi^k = 1$ and $|\gamma(\alpha) - \xi| < C\delta$. But, since the index of K in Z is k, there exists a character $\theta \in K^\perp$ with $\theta(\alpha) = \xi$.

It is now immediate that the affine function $\omega = c\bar{\theta}\gamma$ satisfies $\|f - \omega\|_2 < C\delta$ for some constant C. □

Although we only use the following results for three terms, we state them more generally for ℓ terms.

Lemma 3.3. *Let $\ell \ge 1$ be an integer and let $a_1, a_2, \dots, a_\ell$ be distinct integers. There exists a constant $C > 0$ such that whenever $f_1, f_2, \dots, f_\ell$ are functions of modulus 1 on Z so that*

$$\int_Z |f_i(z+\alpha) - f_i(z)|^2 \, dm(z) < \delta^2 \tag{6}$$

for $i = 1, \dots, \ell$ and

$$\int_{Z\times Z} \Big|1 - \prod_{i=1}^{\ell} f_i(z + a_i t)\Big|^2 \, dm(z)\, dm(t) < \delta^2 \tag{7}$$

for some $\delta > 0$, then there exist affine functions $\omega_1, \omega_2, \dots, \omega_\ell$ with

$$\|1 - \omega_i f_i\|_2 < C\delta \tag{8}$$

for $i = 1, 2, \dots, \ell$ and

$$\prod_{i=1}^{\ell} \omega_i(z + a_i t) = 1 \tag{9}$$

for all $z, t \in Z$.

Proof. The result is obvious for $\ell = 1$ and we proceed by induction. Let $\ell > 1$ and assume that the statement holds for any choice of $a_1, a_2, \dots, a_{\ell-1}$.

Fix $a_1, \dots, a_\ell$. As before, we write C for any constant depending only on the data $a_1, \dots, a_\ell$ and all L_2 norms are assumed to be taken with respect to z.

Let $f_1, \dots, f_\ell$ be functions of modulus 1 such that conditions (6) and (7) hold for some $\delta > 0$. Clearly we can assume that δ is as small as needed.

For $i = 1, \dots, \ell$, set $b_i = a_i - a_\ell$. Fix $s \in Z$. Substituting $z - a_\ell s$ for z and $t + s$ for t in (7) we get

$$\int_{Z\times Z} \Big|1 - \prod_{i=1}^{\ell} f_i(z + b_i s + a_i t)\Big|^2 \, dm(z)\, dm(t) < \delta^2 \ .$$

Setting $g_i(z) = f_i(z + b_i s)\overline{f_i(z)}$, using the bound (7) again we get

$$\int_{Z\times Z} \Big|1 - \prod_{i=1}^{\ell-1} g_i(z + a_i t)\Big|^2 \, dm(z)\, dm(t) < 4\delta^2$$

because $b_\ell = 0$. Moreover, for each i,

$$\|g_i(z+\alpha) - g_i(z)\|_2 < 2\delta \ .$$

Using the induction hypothesis, for $i = 1, \dots, \ell - 1$ and for all $s \in Z$ there exists an affine function $\omega_{s,i}$ such that

$$\|f_i(z + b_i s) - \omega_{s,i}(z) f_i(z)\|_2 < C\delta \tag{10}$$

for some constant C. By Proposition 3.2, there exists an affine function $c_i\gamma_i$ such that

$$\|f_i - c_i\gamma_i\|_2 < C\delta \tag{11}$$

for some constant C. Exchanging the role played by the indices ℓ and $\ell - 1$ we find an affine function $c_\ell\gamma_\ell$ satisfying the relation (11) for $i = \ell$.

Using (7) again we have

$$\int_{Z\times Z} \Big|1 - \prod_{i=1}^{\ell} c_i\gamma_i(z + a_i t)\Big|^2 \, dm(z)\, dm(t) < C\delta^2 \ . \tag{12}$$

Since δ is small,

$$\prod_{i=1}^{\ell} \gamma_i = \prod_{i=1}^{\ell} \gamma_i^{a_i} = 1$$

and $|1 - \prod_{i=1}^{\ell} c_i| < C\delta$. Modifying c_ℓ by an amount less than $C\delta$ so that $\prod_{i=1}^{\ell} c_i = 1$ and setting $\omega_i = c_i\gamma_i$, we have the affine functions satisfy the announced properties. □

Lemma 3.4. *Let $a_1, a_2, \dots, a_\ell$ be distinct integers. For $1 \le i \le \ell$ and $k \in \mathbb{N}$, let $f_{k,i}$ be a function of modulus 1 on Z such that the following two properties are satisfied as $k \to \infty$:*

1. *For $i = 1, \dots, \ell$, $f_{k,i}(z + \alpha)\overline{f_{k,i}(z)}$ converges in $L^2(Z)$*
2. *$\displaystyle\prod_{I=1}^{\ell} f_{k,i}(z + a_i t)$ converges in $L^2(Z \times Z)$.*

Then, for $i = 1, \dots, \ell$ and $k \in \mathbb{N}$, there exists an affine function $\omega_{k,i}$ such that the following two statements hold:

1. *$\omega_{k,i} f_{k,i}$ converges in $L^2(Z)$ as $k \to \infty$*
2. *For all $k \in \mathbb{N}$ and for all $z, t \in Z$,*

$$\prod_{i=1}^{\ell} \omega_{k,i}(z + a_i t) = 1 \ . \tag{13}$$

In particular, $\displaystyle\prod_{i=1}^{\ell} f_{k,i}(z)$ converges in $L^2(Z)$ as $k \to \infty$.

Proof. Let $\{k_j\}$ be an increasing sequence of integers such that for all j and all $k > k_j$,

$$\int_{Z\times Z} \Big|1 - \prod_{i=1}^{\ell} f_{k_j,i}(z + a_i t)\overline{f_{k,i}(z + a_i t)}\Big|^2 \, dm(z)\, dm(t) < 4^{-k}$$

and for all $i \in \{1, \dots, \ell\}$,

$$\int_Z \Big|f_{k,i}(z + \alpha)\overline{f_{k_j,i}(z + \alpha)} - f_{k,i}(z)\overline{f_{k_j,i}(z)}\Big|^2 \, dm(z) < 4^{-k} \ .$$

Proceeding by induction and using Lemma 3.3 at each step, for each $i \in \{1, \dots, \ell\}$ and each $j \in \mathbb{N}$ there exists an affine function $\omega_{k_j,i}$ such that relation (13) is valid for $k = k_j$ and

$$\left\|\omega_{k_{j+1},i} f_{k_{j+1},i} - \omega_{k_j,i} f_{k_j,i}\right\|_2 < C2^{-j} \ .$$

For $k_j < k < k_{j+1}$ we use Lemma 3.3 applied to the functions $f_{k,i}\overline{f_{k_j,i}}$ and obtain affine functions $\omega_{k,i}$ such that the relation (13) is valid and

$$\|\omega_{k,i} f_{k,i} - \omega_{k_j,i} f_{k_j,i}\|_2 < C2^{-j} \ .$$

The affine functions $\omega_{k,i}$ now defined for all values of k satisfy the required properties. □

We note that it follows immediately from Lemma 3.3 that, if all the limits arising in the hypothesis of Lemma 3.4 are equal to the constant 1, then the limits arising in the conclusion can all be taken equal to the constant 1 too.

4. Proof of Theorem 1.1

We now return to our original problem. We assume that a_1, a_2, a_3 are fixed, distinct and non-zero integers and that we are given $f_1, f_2, f_3 \in L^\infty(\mu)$. We prove the existence in $L^2(\mu)$ of

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{i=1}^{3} f_i(T^{a_i n}x) \ . \tag{14}$$

However, it now sufficies to prove the existence in $L^2(\mu)$ of the limit (14) for the modified system described in Section 2.10.

By density, it suffices to consider the case when the functions f_i are of the form

$$f_i(z, g) = w_i(z)\chi_i(g) \tag{15}$$

for $i = 1, 2, 3$, where $w_i \in L^\infty(m)$, $\chi_i \in \widehat{G}$ and $x = (z, g)$. We consider two cases, depending on whether or not the character $\tilde{\chi} = (\chi_1, \chi_2, \chi_3)$ belongs to $M^\perp$.

4.1. The easy case.

For $\tilde{\chi} \notin M^\perp$, the proof is straightforward.

Lemma 4.1. *Let the functions f_i be given by Formula* (15) *and assume that $\tilde{\chi} \notin M^\perp$. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{i=1}^{3} f_i(T^{a_i n}x)$$

exists in $L^2(\mu)$ and equals 0.

Proof. Let m_M denote the Haar measure of M. For $z \in Z$ and $g \in G$ we have

$$\int_M \prod_{i=1}^{3} f_i(z_i, g_i + u_i)\, dm_M(u_1, u_2, u_3) = \prod_{i=1}^{3} w_i(z_i)\chi_i(g_i) \int_M \prod_{i=1}^{3} \chi_i(u_i)\, dm_M(u_1, u_2, u_3) = 0$$

since $\tilde{\chi} \notin M^\perp$. On the other hand, each $\tilde{T}$-invariant function F in $L^2(\tilde{\mu})$ satisfies for all $(u_1, u_2, u_3) \in M$,

$$F(\tilde{z}, g_1 + u_1, g_2 + u_2, g_3 + u_3) = F(\tilde{z}, g_1, g_2, g_3) \quad \tilde{\mu}\text{-a.e.}$$

by property ($\imath\imath$) of Proposition 2.6. Consequently, the function $\prod_{i=1}^{3} f_i(z_i, g_i)$ is orthogonal in $L^2(\tilde{\mu})$ to the space of invariant functions. Thus by Lemma 2.2 (see the remark at the end of Section 2.10), the averages converge to 0 in $L^2(\mu)$. □

4.2. **The function ϕ_t.** For $\tilde{\chi} \in M^{\perp}$, the proof is a bit more involved, and we use the following lemma to express the limit as a continuous map.

Lemma 4.2. *Let $\tilde{\chi} \in M^{\perp}$. There exists a continuous map $s \mapsto \phi_s(.)$ from Z to $L^2(m)$ such that*

$$\phi_{n\alpha}(z) = \prod_{i=1}^{3} \chi_i\big(\sigma^{(na_i)}(z)\big)$$

for all $n \in \mathbb{Z}$.

We note that ϕ_s is of modulus one for all t.

Proof. Let $\{n_k\}$ be a sequence of integers such that $\{n_k\alpha\}$ converges to some $s \in Z$. For $1 \leq i \leq 3$ and $k \in \mathbb{N}$ we write

$$f_{k,i}(z) = \chi_i\big(\sigma^{(n_k a_i)}(z)\big) \ .$$

We have to prove that $\prod_{i=1}^{3} f_{k,i}(z)$ converges in $L^2(Z)$ as $k \to \infty$.

By property ($\imath$) of Proposition 2.6, the multiplicative $\mathcal{S}^1$-cocycle

$$\tilde{\chi} \circ \tilde{\sigma}(\tilde{z}) = \prod_{i=1}^{3} \chi_i\big(\sigma^{(a_i)}(z_i)\big)$$

is a multiplicative coboundary of $(\tilde{Z}, \tilde{S})$. Therefore, there exists a function $b(z,t)$ of modulus 1 on $Z \times Z$ such that for all $k \in \mathbb{N}$

$$\prod_{i=1}^{3} f_{k,i}(z + a_i t) = \prod_{i=1}^{3} \chi_i\big(\sigma^{(n_k a_i)}(z + a_i t)\big) = b(z, t + n_k\alpha)\overline{b(z,t)}$$

for $m \times m$-almost all $(z,t) \in Z \times Z$. As the translations act on $L^2(Z \times Z)$ in a continuous way,

$$\prod_{i=1}^{3} f_{k,i}(z + a_i t) \to b(z, t+s)\overline{b(z,t)}$$

in $L^2(Z \times Z)$ as $k \to \infty$. Moreover, for $i = 1, 2, 3$ and $k \in \mathbb{N}$,

$$f_{k,i}(z+\alpha)\overline{f_{k,i}(z)} = \chi_i\big(\sigma^{(n_k a_i)}(z+\alpha) - \sigma^{(n_k a_i)}(z)\big) = \chi_i\big(\sigma(z + n_k a_i \alpha) - \sigma(z)\big)$$

by the cocycle relation, and it follows as above that

$$f_{k,i}(z+\alpha)\overline{f_{k,i}(z)} \to \chi_i\big(\sigma(z + a_i s) - \sigma(z)\big)$$

in $L^2(Z)$ as $k \to \infty$.

The result follows now immediately from Lemma 3.4 □

4.3. **Proof of convergence and a first expression of the limit.** In order to show that the limit in Equation (14) exists, we are left with considering the case $\tilde{\chi} \in M^\perp$. For $x = (z, g)$ and $f_i(z, g) = w_i(z)\chi_i(g)$, we have

$$\frac{1}{N}\sum_{i=1}^{N-1}\prod_{i=1}^{3} f_i(T^{a_i n}x) = \prod_{i=1}^{3}\chi_i(g)\,\frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{3} w_i(z + na_i\alpha)\chi_i\big(\sigma^{(na_i)}(z)\big)\ .$$

The function $\prod_{i=1}^{3}\omega_i(z+na_i\alpha)\chi_i(\sigma^{(na_i)}(z))$ is exactly the value at $t = n\alpha$ of the mapping

$$t \mapsto \phi_t(z)\prod_{i=1}^{3} w_i(z + a_i t)\ ,$$

a continuous map from Z to $L^2(m)$ by Lemma 4.2.

Lemma 4.3. *Let Z be a compact metric space, $S : Z \to Z$ a homeomorphism so that Z is uniquely ergodic with invariant measure m. Let $f : Z \to \mathcal{H}$ be a continuous map into a Hilbert space $\mathcal{H}$. Then for all $z \in Z$,*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(T^n z) = \int_Z f(u)\,dm(u)$$

in $\mathcal{H}$.

Proof. Without loss, we can assume that $\int_Z f(u)\,dm(u) = 0$. For an integer k, we consider the continuous, complex valued function $z \mapsto \langle f(S^k z), f(z)\rangle$ on Z, where $\langle\ ,\ \rangle$ denotes the inner product on $\mathcal{H}$. By unique ergodicity,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\langle f(S^{n+k}z), f(S^n z)\rangle = \int_Z \langle f(S^k u), f(u)\rangle\,dm(u)\ .$$

Using γ_k to denote this limit, by the Hilbert space Van der Corput lemma (see Bergelson[1]) it suffices to show that $\frac{1}{K}\sum_{k=0}^{K-1}\gamma_k \to 0$ as $K \to \infty$, and this follows from the ergodic theorem. □

By the lemma and the fact that (Z, S) is uniquely ergodic,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{3}\omega_i(z+na_i\alpha)\chi_i(\sigma^{(na_i)}(z)) = \int_Z \phi_t(z)\prod_{i=1}^{3} w_i(z + a_i t)\,dm(t)$$

in $L^2(m)$.

Thus, the limit in Equation (14) exists and equals

$$\int_Z \phi_t(z)\prod_{i=1}^{3}\chi_i(g)w_i(z + a_i t)\,dm(t)\ .$$

The proof of Theorem 1.1 is complete.

4.4. **A global expression of the limit.** The function $\phi_t(z)$ constructed in Section 4.2 depends on the character $\tilde{\chi} \in M^\perp$ used to decompose the f_i. To take this into account, we write $\phi_t^{\tilde{\chi}}(z)$. By construction, the dependence on $\tilde{\chi}$ is multiplicative. More specifically, for $\tilde{\chi}, \tilde{\theta} \in M^\perp$

$$\phi_t^{\tilde{\chi}\tilde{\theta}}(z) = \phi_t^{\tilde{\chi}}(z)\phi_t^{\tilde{\theta}}(z)$$

for all $z \in Z$. As $M^\perp$ is the dual group of the compact group G^3/M, it follows that there exists a measurable map $F_t(z)$ with values in G^3/M such that $\phi_t^{\tilde\chi}(z) = \tilde\chi(F_t(z))$ for all $\tilde\chi \in M^\perp$. We can lift this map to a measurable map

$$\tilde\Phi_t(z) = \big(\Phi_{t,1}(z), \Phi_{t,2}(z), \Phi_{t,3}(z)\big)$$

taking values in G^3. Then for $\tilde\chi = (\chi_1, \chi_2, \chi_3) \in M^\perp$,

$$\phi_t^{\tilde\chi}(z) = \prod_{i=1}^{3} \chi_i\big(\Phi_{t,i}(z)\big)\ .$$

For $f_1, f_2, f_3 \in L^\infty(\mu)$, consider the function on Ψ on $X = Z \times G$ given by

$$\Psi(z,g) = \int \prod_{i=1}^{3} f_i\big(z + a_i t, g + g_i + \Phi_{t,i}(z)\big)\, dm(t)\, dm_M(g_1, g_2, g_3)$$

where m_M is the Haar measure of M.

We first consider the case that for each i, the function $f_i(z,g)$ is of the form $w_i(z)\chi_i(g)$, with $w_i \in L^\infty(m)$ and $\chi_i \in \widehat{G}$. Then $\Psi(z,g)$ equals

$$\int \prod_{i=1}^{3} \chi_i(g) w_i(z + a_i t)\phi_t^{\tilde\chi}(z)\, dm(t)\ \int \prod_{i=1}^{3} \chi_i(g_i)\, dm_M(g_1, g_2, g_3)\ .$$

If $\tilde\chi \notin M^\perp$ the last integral equals 0. Therefore, $\Psi(z,g) = 0$ for all z and g and so $\Psi = 0$. Thus, Ψ is the limit in Theorem 1.1 by the discussion of Section 4.1. If $\tilde\chi \in M^\perp$, we also have that $\Psi(z,g)$ is equal to the limit already obtained.

By density, Ψ equals the limit for all choices of the functions f_i. We have proven the following theorem, with Theorem 1.1 as a corollary.

Theorem 4.4. *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system, a_1, a_2, a_3 distinct integers and $f_1, f_2, f_3 \in L^\infty(\mu)$. Assume that $X = Z \times G$, a compact abelian group extension of the Kronecker Z, and let $M \subset G^3$ be the Mackey group constructed in Section 2.7. There exists a measurable map $\tilde\Phi_t(z) = \big(\Phi_{t,1}(z), \Phi_{t,2}(z), \Phi_{t,3}(z)\big) : Z \to G^3$ so that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{i=1}^{3} f_i(T^{a_i n}(x))$$

exists in $L^2(m)$ and equals

$$\int \prod_{i=1}^{3} f_i\big(z + a_i t, g + g_i + \Phi_{t,i}(z)\big)\, dm(t)\, dm_M(g_1, g_2, g_3)$$

at the point $x = (z,g)$.

References

[1] V. Bergelson. Weakly mixing PET. *Erg. Th. & Dyn. Sys.*, 7:337–349, 1987.

[2] J. Bourgain. Pointwise ergodic theorems for arithmetic sets. *Inst. Hautes. Études Sci. Publ. Math.*, 69:5–45, 1989.

[3] J.-P. Conze and E. Lesigne. Théorèmes ergodiques pour des mesures diagonales. *Bull. Soc. Math. France*, 112:143–175, 1984.

[4] J.-P. Conze and E. Lesigne. Sur un théorème ergodique pour des mesures diagonales. *Publications de l'Institut de Recherche de Mathématiques de Rennes, Probabilités*, 1987.

[5] J.-P. Conze and E. Lesigne. Sur un théorème ergodique pour des mesures diagonales. *C. R. Acad. Sci. Paris, Série I*, 306:491–493, 1988.

[6] H. Furstenberg. Ergodic behavior of diagonal measures and a theorem of Szeméredi on arithmetic progressions. *J. d'Analyse Math.*, 31:204–256, 1977.

[7] H. Furstenberg and B. Weiss. A mean ergodic theorem for $\frac{1}{N}\sum_{n=1}^{n} f(T^n x)g(T^{n^2} x)$. *Convergence in Ergodic Theory and Probability*, Eds.:Bergelson/March/Rosenblatt, Walter de Gruyter & Co, Berlin, New York:193–227, 1996.

[8] E. Lesigne. Résolution d'une équation fonctionelle. *Bull. Soc. Math. France*, 112:177–196, 1984.

[9] E. Lesigne. Théorèmes ergodiques pour une translation sur une nilvariété. *Erg. Th. & Dyn. Sys.*, 9(1):115–126, 1989.

[10] E. Lesigne. Équations fonctionelles, couplages de produits gauches et théorèmes ergodiques pour mesures diagonales. *Bull. Soc. Math. France*, 121:315–351, 1993.

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