A PRIME SYSTEM WITH MANY SELF-JOININGS

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ABSTRACT. We construct a rigid, rank 1, prime transformation that is not quasi-simple and whose self-joinings form a Paulsen simplex. This seems to be the first example of a prime system whose self-joinings form a Paulsen simplex and the first example of a prime system that is not quasi-simple.

1. INTRODUCTION

A natural question is to find indecomposable structures, and we study this question in the setting of measurable dynamics. More precisely, we consider a measure preserving dynamical system \((Z, M, \mu, T)\), where \(Z\) is a set endowed with a \(\sigma\)-algebra \(M\), \(\mu\) is a measure on the measure space \((Z, M)\), and \(T : Z \to Z\) is a measurable transformation that preserves the measure \(\mu\). Throughout this article, we assume that \((Z, M, \mu)\) is a (non-atomic) Lebesgue space. A factor of a measure preserving system \((Z, M, \mu, T)\) is a measure preserving system \((Z', M', \mu', T')\) and a measurable map \(\pi : Z \to Z'\) such that \(\mu \circ \pi^{-1} = \mu'\) and \(T' \circ \pi(x) = \pi \circ T(x)\) for \(\mu\)-almost all \(x \in Z\). In this setting, the indecomposable structures are the prime transformations, which are transformations with no nontrivial (measurable) factors. Historically, showing systems are prime has largely been accomplished by understanding the self-joinings of the system, that is, the \(T \times T\) invariant measures on \(Z \times Z\) with marginals \(\mu\) on each of the coordinates. Our main result is that there exists a prime transformation with many self-joinings (the self-joinings form a Paulsen simplex) and the self-joinings can be large (there is a self-joining that does not arise as a distal extension of the system):

Theorem 1.1. There exists a prime system \((Y, B, \nu, T)\) that is rank 1, rigid, and has a self-joining \(\eta\) such that \((Y \times Y, B \times B, \eta, T \times T)\) is not a distal extension of \((Y, B, \nu, T)\). Moreover, the set of self-joinings of \(Y\) is a Paulsen simplex.

To highlight the novelty of our construction, this is the first example of a rank 1 system that is not quasi-simple (recall that a system is quasi-simple if all of its ergodic self-joinings are either the product measure or isometric extensions of the base system). Being a distal extension is a milder condition than being an isometric extension, meaning that the self-joining not being a distal extension is a stronger result. Additionally, our methods show that being not quasi-simple is a residual property in the space of measure preserving transformations (endowed with the weak topology), and this is a strengthening a result of Ageev [2]. Furthermore, our result answers a question posed by Danilenko [5, Section 7, Question (iii)] if the

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The set of quasi-simple transformations and the set of distal-simple transformations are both meager.

1.1. Context of the results. The first systematic family of prime systems was introduced by Rudolph [26], based on Ornstein’s counterexample machinery, and have been studied extensively since; for example, see [7, 9, 21, 15, 18, 27]. A system $(Y, \nu, T)$ has 2-fold minimal self-joinings if all of the ergodic self-joinings are either $\nu \times \nu$ or are concentrated on the graph $\{(x, T^j x)\}$ for some integer $j \geq 0$. Defining the natural generalization for $k$-fold minimal self-joinings for all $k \geq 2$, Rudolph showed that any system having minimal self-joinings is prime.

However, having minimal self-joinings is quite special, and so there was interest in more general criteria for obtaining prime systems. In this direction, Veech showed that a 2-simple system is prime if it has no compact subgroups in its centralizer. Recalling that a system is 2-simple (which Veech called property S) if the only ergodic self-joinings arise from the product measure and measures carried on graphs of transformations in the centralizer of the system. Simple systems have since been studied in a variety of contexts (see for example [24, 5, 14, 2, 11, 13, 28]). Veech’s criterion gave rise to the first example of a rigid prime system, with the construction by del Junco and Rudolph [8] of a specific rigid, simple system that had no nontrivial compact subgroups in its centralizer. Glasner and Weiss [14] constructed an example of a prime system that is not simple, by taking a simple system and considering the factor corresponding to a non-normal maximal compact subgroup, again using Veech’s criteria to show that the factor is prime since it arises from a maximal compact subgroup. In this example, as the subgroup is not normal, the factor itself is not simple, but the self-joinings of the factor of a simple system are always isometric extensions of the factor.

There are a few other known examples of prime systems. For example, King [20, Section 2] showed that the (proper) factors of rank 1 systems are rigid and so mildly mixing rank 1 systems are prime. Continuing in this vein, Thouvenot asked if mildly mixing rank 1 transformations have minimal self-joinings, and this difficult question remains open. Parreau and Roy [25] gave a construction of prime systems for some Poisson suspensions of (infinite measure preserving) prime systems. In the same article, Parreau and Roy write “it is yet unknown whether prime rank one maps are always factors of simple systems.” Our construction resolves this by producing a rank 1 prime system that is not the factor of a simple system (it is not quasi-simple).

This short list of examples basically includes all known prime systems, and one motivation for this work is to give a new construction of prime systems not relying on a paucity of joinings (as in the minimal self-joinings, simple or factor of simple systems) or soft restrictions on the prime factors (as in the mildly mixing rank 1 or Poisson suspension of prime infinite measure preserving systems with additional properties).

Turning to the second conclusion of Theorem 1.1, we note that it is well known that a residual set of measure preserving systems is rank 1 and rigid. King [22] showed that for a typical measure preserving transformation, its self-joinings form a Poulsen simplex (recall that a Poulsen simplex is a simplex such that the extreme points are dense). Putting this in context, Lindenstrauss, Olsen, and Sternfeld [23] proved that a Poulsen simplex is unique up to affine homeomorphism. Aseev showed that the typical transformation is not prime [1] and is not simple [2].
1.2. A brief outline of the proof. The idea of the explicit construction of our system carried out in Section 3 is to start with a particular odometer system on which we carefully control the dynamics, in a way such that the alphabet size is a fixed size most of the time, but is perturbed on a sequence of times tending to infinity quickly. We then choose a particular set and study the first return map to this set. The system constructed in this way has many self-joinings, and this would, a priori, seem like an obstruction to constructing prime transformations. However, by the introduction of times in the underlying odometer that disrupt the regularity of the returns, we build asymmetry into the system, guaranteeing that none of these self-joinings arise from factors (Sections 5 and 6).

Our methods for building self-joinings and building self-joinings that cannot be distal extensions of the base system are fairly soft and general (if a little involved). To do this we build on earlier results of the first named author and Eskin [4], where they show that for almost every 3-IET, its self-joinings form a Paulsen simplex. Our proof that the transformation is prime is more combinatorial, making heavy use of the specific construction. This should not be surprising, because being prime is a meager property [1] in the space of measure preserving transformations (with the weak topology). Nevertheless, an ideology of this work is that it may still be a fairly common property. In particular, we conjecture that in some families of measure preserving transformations almost every system is prime. To be specific:

**Conjecture 1.** Almost every 3-IET is prime.

Although this may hold more generally for a \( k \)-IET, such a conjecture is out of reach at this point, both in terms of the methods of [4] and other ingredients that would be needed to prove this.

A second conjecture, closer to the work of this paper, is stated in Section 3, after we have developed some further background.

2. Definitions and notation

2.1. Systems and joinings. By a measure preserving system \((X, \mathcal{B}, \mu, T)\), we mean that \(\mathcal{B}\) is the Borel \(\sigma\)-algebra for some compact metric topology on \(X\), \((X, \mathcal{B}, \mu)\) is a probability space and \(T : X \to X\) is a measurable, measure preserving map. Throughout the paper, we generally omit the associated \(\sigma\)-algebra from the notation, assuming that any measure preserving system is endowed with the Borel \(\sigma\)-algebra. We say that the measure preserving system \((Y, \nu, S)\) is a factor of \((X, \mu, T)\) if there exists a measurable map \(\pi : X \to Y\) such that \(\pi \circ T = S \circ \pi\) and \(\mu \circ \pi^{-1} = \nu\).

A joining of the measure preserving systems \((X_i, \mu_i, T_i)\) for \(i = 1, 2\) is a \((T_1 \times T_2)\)-invariant measure \(\alpha\) on \(X_1 \times X_2\) such that \(\alpha\) projects to \(\mu_1\) on the first coordinate and to \(\mu_2\) on the second coordinate. A self-joining of a system is a joining of two copies of the same system. If \((X, \mu, T)\) is a measure preserving system, \(J(n)\) denotes the off diagonal joining on \(\{(x, T^n x)\}\), meaning that \(J(n)\) is the measure on \(X \times X\) so that for all \(f \in C(X \times X)\)

\[
\int f(x, y) \, dJ(n) = \int f(x, T^n x) \, d\mu.
\]

If \((X, \mu, T)\) is an ergodic measure preserving system, we say that the bounded linear operator \(P : L^2(\mu) \to L^2(\mu)\) is a Markov operator if it satisfies:

(i) For all \(f \in L^2(\mu)\) with \(f \geq 0\), we have \(Pf \geq 0\) and \(P^*f \geq 0\).
with Conditions (i) and (iii), we have a rigidity sequence.

(ii) $P^1_X = 1_X$ and $P^* 1_X = 1_X$, where $1_A$ denotes the indicator function of the set $A$.

(iii) $PU_T = U_T P$, where $U_T : L^2(\mu) \to L^2(\mu)$ by $U_T f = f \circ T$.

Markov operators can be defined more generally for an operator mapping one measure preserving system to another, but our interest is when it arises as an integral of fibers of a factor and so we can take the map from a system to itself; see, for example, Glasner [12] for more on such operators. More precisely, if $(X, \mu, T)$ has a factor $(Y, \nu, S)$ with factor map $\pi$, then by integrating out the fibers of the factor map, we obtain a bounded linear operator $P : L^2(\mu) \to L^2(\mu)$, satisfying Properties (i)–(iii) and we call this the Markov operator defined by $\pi$.

2.2. Rigid rank one by cylinders. As above, we assume that each system is endowed with its Borel $\sigma$-algebra, but we omit it from the notation.

**Definition.** An invertible ergodic system $(Z, \lambda, R)$, where $Z \subset [0, 1]$ and $\lambda$ denotes normalized (probability) Lebesgue measure restricted to $Z$, is rigid rank one by cylinders if there exist a sequence of intervals $(I_i)_{i \in \mathbb{N}}$, which we call cylinders, and a sequence of positive integers $(n_i)_{i \in \mathbb{N}}$ such that:

(i) For $0 \leq i < n_k$, the iterate $R^i I_k$ is a cylinder having the same measure as $I_k$.

(ii) The cylinders $R^i I_k$ and $R^j I_k$ are pairwise disjoint for all $k$ and for $0 \leq i < j < n_k$.

(iii) The measure $\lambda(\bigcup_{i=0}^{n_k-1} R^i I_k)$ tends to 1 as $k \to \infty$.

(iv) The ratio $\frac{\lambda(\bigcup_{i=0}^{n_k-1} R^i I_k)}{\lambda(I_k)}$ tends to 0 as $k \to \infty$.

Note that cylinders in this setting are intervals in $[0, 1]$, but we refer to them as cylinders in analogy with the symbolic setting. By a symbolic system $(X, T)$, we mean an infinite sequence space $X \subset \prod_{i=1}^{\infty} A_i$, where each $A_i$ is a finite alphabet, and $T : X \to X$ is a measurable map. We denote elements of the space as $x = (x_i)_{i \in \mathbb{N}} \in X$, with the convention that a bold face letter $x$ has its entries denoted as $x_i$. In a symbolic system $X$, a cylinder set $[w]$ determined by a word $w = w_1 \ldots w_n$ is defined to be

$[w] = \{x \in X : x_i = w_i \text{ for all } 1 \leq i \leq n\}$

We also consider cylinders defined only by some entries $a_{i_1} \in A_{i_1}, \ldots, a_{i_k} \in A_{i_k}$ defining the cylinder

$\{x \in X : x_{i_j} = a_{i_j} \text{ for all } 1 \leq j \leq k\}$

and we refer to the $i_j$ as defining indices of the cylinder. The collection of cylinder sets forms a basis for the topology of $X$. When working with a symbolic system $(X, T)$, fixing initial entries corresponds to an interval in $[0, 1]$.

The first three conditions in the definition of rigid rank one by cylinders imply that $G$ is rank one, but in the general setting of a rank one transformation there is no requirement that the subsets $I_i$ are intervals. The fourth condition gives a sequence of times under which the transformation $R$ is rigid, meaning that along these times the iterates of $R$ approach the identity. Indeed, Condition (iv) implies that $\frac{\lambda(\bigcup_{i=0}^{n_k-1} R^i I_k)}{\lambda(I_k)}$ is close to 1 for all large $k$ and $0 \leq i < n_k$, and so using this with Conditions (i) and (iii), we have a rigidity sequence.
2.3. Distal extensions. We review the definitions of (measurable) isometric and distal extensions, as introduced by Parry [24]. These extensions were key in Furstenberg’s proof [10] of Szemerédi’s Theorem (see [11] for further background), and the definition we use comes from Zimmer [31, 32], who showed that a measurably distal system is equivalent to a (possibly transfinite) inverse limit of a tower of isometric extensions.

If \( G \) is a compact group, \( H \subset G \) is a closed subgroup, and \((X, \mu, T)\) is a Borel probability system, then a measurable map \( \phi: X \to G \) is called a cocycle and the extension of \( G \) by \( G/H \) given by the cocycle \( \phi \) is defined to be the system \((X \times G/H, \mu \times m_{G/H}, T_\phi)\), where \( T_\phi(x, \tilde{g}) = (Tx, \phi(x) \cdot \tilde{g}) \) for \( x \in X \) and \( \tilde{g} \in G/H \) and \( m_{G/H} \) is the Haar measure on \( G/H \) (we use the convention that cosets in \( G/H \) are denoted by \( \gamma \)). Defining the topology of the group \( G \) by a distance \( d_G \) that is invariant under right translation, and of course continuous with respect to translation on either side, we have an induced distance \( d_{G/H} \) on \( G/H \) and we have that the restriction of \( T_\phi \) to each fiber of the natural projection map \( X \times G/H \to X \) is continuous. The system \((X \times G/H, \mu \times m_{G/H}, T_\phi)\) is an isometric extension of the system \((X, \mu, T)\).

If \((X, \mu, T)\) and \((Y, \nu, S)\) are ergodic systems, then \((X, \mu, T)\) is a distal extension of \((Y, \nu, S)\) if it has a sequence of factors \( X_\eta \) indexed by ordinals \( \eta \leq \eta_0 \) for some countable ordinal \( \eta_0 \) such that \( X_0 = Y \), \( X_{\eta_0} = X \), \( X_{\eta+1} \) is an isometric extension of \( X_\eta \) for each \( \eta \), and for each limit ordinal \( \zeta \leq \eta \) the system \( X_\zeta \) is an inverse limit of the systems \( X_\eta \) with \( \eta \leq \zeta \). The system \((X, \mu, T)\) is (measurably) distal if it is a distal extension of the trivial system.

**Notation.** We use \( d \) to denote the metric in various settings, with a subscript indicating the space as needed. Thus \( d_G \) denotes the right invariant metric on the group \( G \), \( d_{G/H} \) denotes the induced distance on \( G/H \), and \( d \) without any subscripts denotes the distance \( d \) on the product on \( X \times Y \).

3. Construction of the system

3.1. Definition of the transformation \( T \). We construct a symbolic system. Set

\[
X = \prod_i \{0, \ldots, a_i \} - 1,
\]

where

\[
a_i = \begin{cases} 
8 & \text{if } i \notin \{10^k: k \geq 2\} \\
8 & \text{if } i = 10^k \text{ for some } k \geq 2.
\end{cases}
\]

We write elements \( x \in X \) as \( x = (x_i)_{i \in \mathbb{N}} \). Let \( S \) denote the odometer on \( X \), meaning that \( S \) is addition by \((1, 0, 0, \ldots)\) with carrying to the right. Thus

\[
S(x) = S(x_1, x_2, \ldots, x_k, x_{k+1}, \ldots) = (0, 0, \ldots, 0, x_k + 1, x_{k+1}, \ldots),
\]

where \( k \) is the least entry such that \( x_k < a_k - 1 \) and if there is no such \( k \), then the odometer turns over and outputs the point \( 0 = (0, 0, \ldots) \).

Set

\[
Z_k = \{ x \in X : x_k = 7 \text{ and } x_i = a_i - 2 \text{ for all } i < k \}
\]

and

\[
W_k = \{ x \in X : x_i = a_i - 2 \text{ for all } i < 10^k \text{ and } x_{10^k} < a_{10^k}/2 \}.
\]
Define

\[ Y = X \setminus \left( \bigcup_{t \in \{10^k : k \geq 2\}} Z_t \cup \bigcup_{k=1}^{\infty} W_k \right) \]

and define \( T : Y \to Y \) to be the first return map of \( S \) to \( Y \). Throughout this paper, \( T \) refers to this map. As usual, we denote elements \( y \in Y \) as \( y = (y_i)_{i \in \mathbb{N}} \).

Define \( D_k \) to be the cylinder sets with largest defining index \( k \) in \( X \setminus Y \). More explicitly, this means that:

\[ D_k = \begin{cases} Z_k & \text{if } k \notin \{10^k : k \geq 2\} \\ W_t & \text{if } k = 10^{2\ell} \text{ for some } \ell \geq 1 \\ \emptyset & \text{if } k = 10^{2\ell+1} \text{ for some } \ell \geq 1. \end{cases} \]

The following result is standard:

**Lemma 3.1.** The odometer \( S \) is uniquely ergodic with respect to a probability measure \( \mu \), and thus the first return map \( T \) is uniquely ergodic with respect to the measure \( \nu = \mu(Y)^{-1} : \mu|_Y \).

**Notation** (for the systems we study throughout this article). Throughout this article, \( X \) is the space defined by \( [1] \), \( S \) is the odometer defined on \( X \) as in \([2]\), \( \mu \) is the unique ergodic measure on this system, and \( (X, \mu, S) \) is the odometer system thus defined. The space \( Y \) is defined by \([3]\) and \((Y, \nu, T)\) is the associated uniquely ergodic system defined by the first return map.

Both \( (X, S) \) and \( (Y, T) \) are measurable maps of compact metric spaces. The remainder of this paper is devoted to studying the properties of the system \((Y, \nu, T)\).

### 3.2. A further conjecture.

Maintaining the notation of this section, we state a conjecture closely related to this subject:

**Conjecture 2.** Let \( a_1, a_2, \ldots \in \mathbb{N} \) with \( a_i \geq 2 \) for all \( i \in \mathbb{N} \). Let \( T \) be the corresponding odometer viewed as a measure preserving map of \([0,1]\), meaning that if \( x = \sum_{j=1}^{\infty} b_j \frac{1}{a_1 \cdots a_j} \) with \( b_i \in \{0, \ldots, a_i-1\} \), then \( Tx = \sum_{j=1}^{\infty} c_j \frac{1}{a_1 \cdots a_j} \) where \( c_k = b_k + 1 \) if \( k = \min \{ j : b_j < a_j - 1 \} \), \( c_i = 0 \) for all \( i < k \) and \( c_i = b_i \) for all \( i > k \). For almost every \( x \in [0,1] \), the first return map of \( T \) to \([0,x]\) is prime.

### 3.3. Weak mixing of the transformation \( T \).

Our first goal is to show that the transformation \( T \) is weakly mixing, and we start with a sufficient (but not necessary) condition for a transformation to be weakly mixing.

**Lemma 3.2.** Assume that \((Z_1, \lambda, T_1)\) is an ergodic measure preserving system with respect to the Lebesgue measure \( \lambda \). If there exist a constant \( c > 0 \), a sequence of integers \((n_i)_{i \in \mathbb{N}}\), and sequences of measurable sets \((A_i)_{i \in \mathbb{N}}\) and \((B_i)_{i \in \mathbb{N}}\) such that

(i) the measures \( \lambda(A_i) > c \) for all \( i \in \mathbb{N} \),
(ii) the limit \( \lim_{i \to \infty} \int_{A_i} |T_1^n x - x| \, d\lambda(x) = 0 \), and
(iii) the limit \( \lim_{i \to \infty} \int_{B_i} |T_1^n x - T_1 x| \, d\lambda(x) = 0 \),

then \( T_1 \) is weakly mixing.

**Proof.** Assume that \( f \) is an eigenfunction of \( T_1 \) with eigenvalue \( \gamma \neq 1 \). By Lusin’s Theorem, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) and a measurable set \( U \) with \( \lambda(U) > 1 - \varepsilon \) such that if \( |x - y| < \delta \) and \( x, y \in U \), then \( |f(x) - f(y)| < \varepsilon \). Choose
\[ \varepsilon < \min\left\{ \frac{1-\gamma}{2} \right\}, \]
where \( c \) is the constant given in the statement. For all sufficiently large \( i \in \mathbb{N} \), by hypothesis there exists a measurable set \( A_i' \) with measure at least \( \frac{\varepsilon}{5} \)
and integer \( n_i \) such that if \( x \in A_i' \), then \( |T_1^n x - x| < \varepsilon \). It follows that there exists \( x \in A_i' \cap U \) and \( T_1^n x \in U \) and so
\[ |f(x) - f(T_1^n x)| = |(1 - \gamma^n)| \cdot |f(x)| = |1 - \gamma^n| < \varepsilon. \]
Similarly there exists \( y \in B_i \cap U \) such that \( |f(y) - T_1^n y| = |1 - \gamma^n+1| < \varepsilon. \) If these two inequalities hold simultaneously, this contradicts the choice of \( \varepsilon \), and so \( \gamma = 1 \). Since \( T_1 \) is ergodic, it follows that \( f \) is constant almost everywhere and so \( T_1 \) is weakly mixing.

Set
\[ q_i = \prod_{j=1}^{i-1} a_j. \]
Then \( S^n(x) \) fixes the first \( i-1 \) positions of \( x \) and increments the entry in \( x_i \) position by 1. All other entries remain the same unless the \( i^{th} \) position was exactly \( a_i - 1 \), in which case the carrying continues until this process terminates.

Given \( n \in \mathbb{N} \), we choose \( c_i(n) \) such that
\[ n = \sum c_i(n)q_i \quad \text{with} \quad c_i(n) \in \left\{ \frac{-a_i+1}{2}, \ldots, \frac{a_i+1}{2} \right\}. \]
Note that there is no unique choice of these coefficients, but we can make a canonical choice by using the greedy algorithm to define the coefficients \( c_i \). That is, we choose \( i \) and \( c_i \) such that \( |n - c_i q_i| \) is minimal out of all possible \( i \in \mathbb{N} \) and \( c_i \in \left\{ \frac{-a_i+1}{2}, \ldots, \frac{a_i+1}{2} \right\} \), and then iteratively choose the next coefficient. If there is a tie, that is if \( |n - c_i q_i| = |n - c_{i'} q_{i'}| \) is minimal, we choose \( i = \min\{j, j'\} \). Once such a representation is fixed, our constructions depend on this choice.

We define two functions from \( \mathbb{N} \) to itself that allow us to move between studying properties of the odometer \( S \) and those of the first return map \( T \):

**Notation.** For \( y \in Y \), define \( \zeta_y : \mathbb{N} \to \mathbb{N} \) to be the map taking the integer \( n \) to the integer \( m \) such that \( S^m y = T^n y \).

For \( y \in Y \), define \( \zeta_y : \mathbb{N} \to \mathbb{N} \) to be the map taking the integer \( n \) to the least integer \( m \) such that there exists \( \ell \geq n \) satisfying \( T^m y = S^\ell y \).

Let \( 0 \in Y \) denote the point consisting of all 0's. To keep track of the iterates of \( S \) that fix the first \( i \) positions, as determined by the \( q_i \) defined in (7) and the expansion of any integer in the base determined by the sequence \( q_i \), as defined in (8), we define
\[ r_i = \zeta_0(q_i) \]
and define
\[ d_i(n) = c_i(\zeta_0(n)). \]
Thus the map \( \zeta_y \) maps an iterate of \( T \) to an iterate of \( S \) and the coefficients \( c_i \) are changed into \( d_i \), while the map \( \zeta_y \) reverses this, taking an iterate of \( S \) to an iterate of \( T \). However they are not precisely inverses, as one can not regain all of the odometer \( S \) from the first return \( T \) if \( S^n(x) \notin Y \), then there is no corresponding \( T \) time.

An easy analysis of the return times for the odometer \( S \) leads to (we omit the proof):
Lemma 3.3. The sum $\sum_{j=0}^{q_i-1} 1_{Z_j}(S^jx)$ does not depend on $x$ for any $\ell$ such that $\ell < i$. Similarly, $\sum_{j=0}^{q_i-1} 1_{W_i}(S^jx)$ does not depend on $x$ for any $\ell$ such that $10^{2\ell} < i$.

We use this to show:

Proposition 3.4. The system $(Y, \nu, T)$ is weakly mixing.

Proof. Assume $i = 10^k - k$ and set

$$U_i = \left( \bigcup_{j=0}^{q_i} S^{-j} \left( \bigcup_{k=i+1}^{\infty} Z_k \cup \bigcup_{j=10^i+1}^{\infty} W_j \right) \right)^c.$$

We claim that if $i \geq 10^k$, then $\mu(U_i) \geq 1 - \frac{1}{8} - \sum_{k=i+1}^{\infty} \frac{1}{8^k}$. Indeed, under this assumption $a_k \geq 8$ and so $\mu(W_i) = \frac{\ell}{q_{10^k+i}}$ for $\ell \geq i$ and $\mu(Z_k) = \frac{1}{q_{k+1}}$ for all $k \geq i$. Thus, $q_i \mu(Z_k) \leq 8^{k-i-1}$ for all $k \geq i$. By the assumption on $i$, it follows that $\sum_{j: 10^j > k} q_j \frac{1}{8} < \frac{1}{8}$.

Set

$$A_i = \{ x \in U_i : x_i = 4 \}$$

and so $\nu(A_i) > \frac{1}{2} - \frac{1}{8}$. For $x \in A_i$, we have $S^j x \notin D_\ell$ for any $0 \leq j \leq q_i$ and $\ell \geq i$ (recall that the sets $D_k$ are defined in \textit{[6]} and $(S^jx)_i \neq a_i - 2$). Thus by Lemma 3.3 and the definition of $T$, there is some $r_i \in \mathbb{N}$ such that $T^{r_i} x = S^{r_i} x$, which by choice of $q_i$ is close to $x$ (note that $r_i$ is defined in \textit{[9]}). Set

$$B_i = \{ x \in U_i : x_i = 7, x_{i-1} \neq j, x_i = 5, x_{i-1} < 6, x_1 < 5 \}.$$

Then $\mu(B_i) > \frac{1}{64}$ and $\nu(B_i \cap Y) > \frac{1}{128}$. For $x \in B_i$, we have $S^j x \in D_i = Z_{10^k} - k$ for some $0 < j < q_i$ and by definition $S^j x \notin D_i$ for all $\ell > i$ (because $(S^j x)_i \neq a_i - 2$). Thus Lemma 3.3 implies that $T^{r_i} x = S^{r_i} x$ (by our assumption that $x_1 < 5$, and we have $S^{r_i+1} x \in Y$). Thus the assumptions of Lemma 3.2 are verified for the measurable sets $A_i, B_i$, and sequence of integers $n_i = r_i$ with $i \in \{10^k - k : k \geq 8\}$.

3.4. $T$ is rigid rank one by cylinders. We now show that the constructed system is rigid rank one by cylinders, using information on the odometer system $(X, S)$ to study the system $(Y, T)$. Recall that since the system $(X, S)$ is an odometer, fixing initial entries corresponds to an interval in $[0, 1)$.

Lemma 3.5. The system $(Y, \nu, T)$ is rigid rank one by cylinders.

Proof. Let $I_k$ be the cylinder set determined by the word of all 0’s up to $10^{2k+1}$ and with any value between 0 and $2k + 1 - 5 = 2k - 4$ in the entry at $10^{2k+1}$. Let $n_k = r_{10^{2k+1}}$, as defined in \textit{[7]}. If $y \in \bigcup_{i=0}^{2n_k-1} T^i(I_k)$, then $y_{2k+1} < 2k + 1 - 3$ and so $S^i y)_{10^{2k+1}} < 2k+1-2$ for $0 \leq i \leq n_k$. Thus, $S^i y \notin \bigcap_{I_{10^{2k+1}}} D_{10^i+1}$ for $0 \leq i \leq n_k$ and so

$$\bigcup_{i=0}^{2n_k-1} T^i(I_k) \cap \left( \bigcup_{\ell=10^{2k+1}+1}^{\infty} Z_{\ell} \cup \bigcup_{j=k+1}^{\infty} W_j \right) = \emptyset.$$

Additionally, by Lemma 3.3 we have that $\sum_{j=0}^{q_{10^{2k+1}+1} - 1} 1_{\cup_{i \leq 10^{2k+1}}} D_{10^i} S^j x$ is constant on $X$. Therefore $\xi_y(1_{10^{2k+1}})$ is constant (and equal to $n_k = r_{10^{2k+1}}$) on this set.
For any \( x \in I_k \), we have that \((T^{n_k}(x))_i = x_i\) for all \( i \neq 10^{2k+1} \) and \((T^{n_k}(x))_{10^{2k+1}} = x_{10^{2k+1}} + 1\). Thus
\[
\mu(T^{n_k}I_k \cap I_k) = (1 - \frac{1}{2k-4})\mu(I_k),
\]
establishing condition (iv) (after passing from \( \mu \) to \( \nu \)) of the definition of rigid rank one by cylinders. For any \( x \in I_k \) and \( 0 < i < n_k \), we have \( T^i(x)_j \neq 0 \) for some \( j < 10^{2k+1} \), and so condition (v) follows. Since each \( T^iI_k \) is either contained in \( Z_\ell \) or is disjoint from \( Z_\ell \) and \( W_s \) for \( \ell < 10^{2k+1} \) and \( s < 2k+1 \), and furthermore is disjoint from all other \( Z_\ell \) and \( W_s \), we have that \( T^iI_k \) is a cylinder set for all \( 0 \leq i < q_k \), establishing condition (i). Finally condition (iii) follows since \( \bigcup_{k=0}^{q_k-1} T^iI_k \) contains all of \( Y \) other than the cylinder sets defined by having entries at least \( 2k-4 \) in the position \( 10^{2k+1} \).

\[
\square
\]

4. Joinings

4.1. Isometric and distal extensions. Given systems \((Z_1, \zeta_1, T_1)\) and \((Z_2, \zeta_2, T_2)\), if \( \eta \) is a measure on \( Z_1 \times Z_2 \), we make a mild abuse of notation and let \( \eta_x \) denote the measure on \( Z_2 \) that is defined for almost all \( x \) by disintegrating the measure \( \eta \) on the fiber \( \{ x \} \times Z_2 \).

**Theorem 4.1.** Assume \((Z_1, \zeta_1, T_1)\) and \((Z_2, \zeta_2, T_2)\) are ergodic, Borel probability systems such that \( Z_1 \) and \( Z_2 \) are compact metric spaces. Let \( \eta \) be an ergodic joining of the systems \((Z_1, \zeta_1, T_1)\) and \((Z_2, \zeta_2, T_2)\), and let \( c > 0 \). Assume that there exists \( \delta > 0 \), a sequence of integers \((n_i)_{i \in \mathbb{N}} \) tending to infinity, a sequence of integers \((L_i)_{i \in \mathbb{N}} \) such that \( L_i > \delta n_i \), and measurable sets \( A_i \subset Z_1 \) satisfying
\[
(i) \quad \zeta_1(A_i) = c \text{ for all } i \in \mathbb{N}.
\]

Further assume that for each \( x \in A_i \), there exist sets \( C_i(x), E_i(x) \subset Z_2 \), and \( j_x \in [-n_i, n_i] \) (all depending on \( x \)) satisfying the following conditions:
\[
(ii) \sup_{x \in A_i} \left| \sup_{y \in C_i(x)} \frac{1}{L_i} \sum_{t=0}^{L_i-1} d_{Z_2}(T_{2}^t y, T_{2}^t y) \right| \to 0.
\]

(iii) For all \( y \in E_i(x) \), we have
\[
\frac{1}{L_i} \left| \{ 0 \leq \ell \leq L_i - 1 : d_{Z_2}(T_{2}^\ell y, T_{2}^\ell y) > c \} \right| > c.
\]

(iv) For all \( x \in A_i \), \( \eta_x(C_i(x)) \), \( \eta_x(E_i(x)) > c \).

(v) For any \( c' > 0 \), there exists \( i_0 \) so that \( i \geq i_0 \) and \( x \in A_i \) if we have balls \( B(p_x, c') \subset Z_2 \) so that \( \eta_x(E_i(x) \cap \bigcup B(p_x, c) > c - c' \) then \( \eta_x(E_i(x) \cap \bigcup B(p_x, 2c') > c - 2c' \).

Then \( \eta \) is not a distal extension of \((Z_1, \zeta_1, T_1)\).

Note that this is a general result, holding for arbitrary measure preserving systems whose underlying spaces are compact metric spaces, and this result does not depend on the particular constructions we have for the systems \((Z_1, \zeta_1, T_1)\) and \((Z_2, \zeta_2, T_2)\).

The proof of Theorem 4.1 proceeds by contradiction. We assume \((Z_1 \times Z_2, \eta, T_1 \times T_2)\) is an isometric extension of \((Z_1, \zeta_1, T_1)\), meaning that there exists a (measurable) isomorphism \( \Psi : (Z_1 \times Z_2, \eta, T_1 \times T_2) \rightarrow (Z_1 \times G/H, \zeta_1 \times m_{G/H}, T_\phi) \) that is the identity on the first coordinate, and use this to derive a contradiction. Since a distal system can be decomposed as a tower of isometric extensions, we conclude that it can not be a distal extension.
Before turning to this proof, we start with some preliminaries and a lemma used to derive the contradiction.

Let \( \mathcal{K} \) be a compact continuity set for \( \Psi \) with \( \eta(\mathcal{K}) > 1 - \frac{4}{10}c^4 \). Thus \( \mathcal{K} \) is also a continuity set for \( \pi_2 \circ \Psi \), where \( \pi_2 : Z_1 \times G/H \to G/H \) is the projection on the second coordinate. Choose \( \delta > 0 \) such that \( d_{G/H}(\bar{h}, \bar{h}') < \frac{\varepsilon}{8} \) whenever \( d_{G/H}(\bar{h}, \bar{h}') < \delta \) and \( g \in G \). Choose \( \frac{\varepsilon}{5} > \delta' > 0 \) such that \( d_{G/H}(\pi_2 \circ \Psi(x, y), \pi_2 \circ \Psi(x', y')) < \delta' \) whenever \( (x, y), (x', y') \in \mathcal{K} \) and \( d_{Z_1 \times Z_2}((x, y), (x', y')) < \delta' \).

**Lemma 4.2.** Under the assumptions of the theorem, there exist a pair of points \( (x, y), (x', y') \in (Z_1 \times Z_2)^2 \) and \( b \in \mathbb{Z} \) such that

- (i) \( (x, y), (x', y'), (T_1^b x, T_2^b y), (T_1^b x, T_2^b y') \in \mathcal{K} \);
- (ii) \( d_{Z_2}(y, y') < \delta' \);
- (iii) \( d_{Z_2}(T_2^b y, T_2^b y') > \frac{\varepsilon}{5} - \frac{\varepsilon}{5} > \frac{\varepsilon}{5} \).

**Proof.** For any \( L \geq 1 \), we have that \( \eta((x, y) : \sum_{i=0}^{L-1} 1_{K}((T_1^i x, T_2^i y)) < L - \frac{L}{10}c) < \frac{1}{10}c^3 \). Choosing \( c' = \frac{\varepsilon}{8} \), for all large \( i \), we can pick \( x \in A_i, (x, y) \in C_i(x), (x', y') \in E_i(x) \) satisfying \( d_{Z_2}(y, y') < \delta' \) and the conditions

\[
\sum_{i=0}^{L_i-1} 1_{K}((T_1^i x, T_2^i y')) > (L_i + n_i) - \frac{L_i}{10}c, \\
\sum_{i=0}^{L_i+n_i-1} 1_{K}((T_1^i x, T_2^i y)) > L_i + n_i - \frac{L_i}{10}c.
\]

By Conditions (ii) and (iii), there exists \( \ell \) such that the points \( (T_1^{\ell+n_i} x, T_2^{\ell+n_i} y), (T_1^{\ell+n_i+j_x} x, T_2^{\ell+n_i+j_x} y), (T_1^{\ell+n_i} x, T_2^{\ell+n_i+j_x} y), (T_1^{\ell+n_i+j_x} x, T_2^{\ell+n_i+j_x} y) \) all lie in the set \( \mathcal{K} \), while at the same time \( d_{Z_2}(T_2^b y, T_2^b y') < \delta' \) and \( d_{Z_2}(T_2^b y, T_2^b y') > c \). Thus we can take \( b \) to be one of \( \ell + j_x \) or \( \ell + n_i \).

**Proof of Theorem 4.1.** We first show that \( \eta \) is not an isometric extension. Observe that if \( g = \phi(T_1^{b-1} x) \cdot \ldots \cdot \phi(x) \), then \( \pi_2 \circ \Psi(T_1^b x, T_2^b y) = g \pi_2 \circ \Psi(x, y) \) and \( \pi_2 \circ \Psi(T_1^b x, T_2^b y) = g \pi_2 \circ \Psi(x, y) \). Because all four of the points \( (T_1^{\ell+n_i} x, T_2^{\ell+n_i} y), (T_1^{\ell+n_i} x, T_2^{\ell+n_i+j_x} y), (T_1^{\ell+n_i+j_x} x, T_2^{\ell+n_i+j_x} y), (T_1^{\ell+n_i+j_x} x, T_2^{\ell+n_i+j_x} y) \) lie in the set \( \mathcal{K} \), Conclusion (ii) of Lemma 4.2 implies that \( d_{Z_2}(T_2^b y, T_2^b y') < \frac{\varepsilon}{7} \), a contradiction of Conclusion (iii) of Lemma 4.2.

Now assume that \( \eta \) is a distal extension of \( (Z_1, \zeta_1, T_1) \). By the structure theorem for distal flows of Furstenberg [10] and Zimmer [32], the system \( (Z_1 \times Z_2, \eta, T_1 \times T_2) \) is an inverse limit of systems, each of which is an isometric extension of the preceding one. Thus there is a factor of our distal extension, which is an isometric extension of \( (Z_1, \zeta_1, T_1) \), and which satisfies the assumptions of Theorem 4.1 (with different \( c \)). Indeed, by the definition of inverse limits, we can embed our distal extension into the product defining the inverse limit. This contradicts the previous paragraph. \( \square \)

### 4.2. A self-joining that is not quasi-simple

We apply Theorem 4.1 to establish part of Theorem 4.1.

**Theorem 4.3.** There exists a self-joining of \( (Y, \nu, T) \) that is not a distal extension of \( (Y, \nu, T) \).
Before turning to the proof, we start with some preliminaries. If \((Y, T)\) is a compact metric space, let \(d_{M(Y \times Y)}(\cdot, \cdot)\) denote the Kantorovich-Rubenstein metric, defined for Borel probability measures \(\mu, \nu \in \mathcal{M}(Y \times Y)\) as

\[
d_{M(Y \times Y)}(\mu, \nu) := \sup \{ \int f \, d\mu - \int f \, d\nu : f \text{ is 1-Lipschitz function on } Y \times Y \}.
\]

This metric endows the set of Borel probability measures \(\mathcal{M}(Y \times Y)\) on \(Y \times Y\) with the weak*-topology. Similarly, define \(d_{\mathcal{M}(Y)}\) to be the Kantorovich-Rubenstein metric on the \(\mathcal{M}(Y)\).

Recall that \(J(n)\) denotes the off diagonal joining on \(\{(x, T^n x)\}\), meaning that \(J(n)\) is the measure on \(X \times X\) defined by

\[
\int f(x, y) \, dJ(n) = \int f(x, T^n x) \, d\mu.
\]

Recall that if \(\sigma\) is a self-joining of \((Y, \nu, T)\), we let \(\sigma_x\) denote the disintegration of \(\sigma\) given by projection to the first coordinate, thought of as a measure on \(Y\). Note that this is only defined \(\nu\)-almost everywhere and is slightly different than the usual disintegration of measures: it defines a measure on \(Y\), rather than a measure on \(Y \times Y\) that this is only defined -almost everywhere and is slightly different than the usual disintegration of measures: it defines a measure on \(Y\), rather than a measure on \(Y \times Y\) that gives full measure to \(\{x\} \times Y\).

The main tool in establishing Theorem 4.3 is the following proposition:

**Proposition 4.4.** For any \(\varepsilon > 0\) and \(k_1, \ldots, k_r \in \mathbb{Z}\), there exist \(\ell_1, \ldots, \ell_{2r}\), \(N, M, L \in \mathbb{Z}\), a set \(A\) with \(\nu(A) > \frac{1}{99}\), and for each \(x \in A\) there exists \(j_x \in [-M, M]\) such that

(a) \(\nu(\{x : d_{M(Y)}((\frac{1}{r} \sum_{n=1}^{2r} J(\ell_n))x, (\frac{1}{r} \sum_{n=1}^{r} J(k_n))x) > \varepsilon\}) < \varepsilon\).

(b) \(\nu(\{x \in A : \text{there exist reorderings } i_1, \ldots, i_r \text{ of } 1, \ldots, r \text{ and } i_{1+r}, \ldots, i_{2r} \text{ of } r+1, \ldots, 2r \text{ such that for all } 1 \leq s \leq r, \)

\[
d_Y(T^{k_s}x, T^{i_s}x) < \varepsilon \text{ and } d_Y(T^{k_s}x, T^{i_s}x) < \varepsilon\} > \nu(A) - \varepsilon.
\]

(c) \(d_{M(Y \times Y)}(\sum_{i=1}^{N} d(T^{k_i}x, T^{i}x), \frac{1}{r} \sum_{i=1}^{r} J(k_i)) < \varepsilon \text{ for all } n \leq 2r \text{ and } x \in A.
\]

(d) \(\frac{1}{r} \sum_{i=0}^{L-1} d(T^{M+i}T^{\ell}x, T^{i}x) < \varepsilon \text{ for all } x \in A \text{ and } n \leq r.
\]

Moreover, if we assume that there exist \(a, b \in \mathbb{N}\) and \(c > 0\) such that \(d(T^a x, T^b x) > 4c + \varepsilon\) for a set \(W\) of \(x\) with \(\nu(W) = \frac{1}{2}\) and

\[
(11) \quad |\{1 \leq n \leq r : d(T^{k_n}x, T^a x) < c\}| = |\{1 \leq n \leq r : d(T^{k_n}x, T^b x) < c\}| = \frac{r}{2}
\]

for all \(x \in W\), then

(e) \(\frac{1}{r} |\{0 \leq i \leq L-1 : d(T^{M+i}T^{\ell}x, T^{i}T^{a}x) > c \text{ for all } x \in A \text{ and } n < d \leq r + 2|\frac{1}{16} r|\} | > \frac{r}{9}.
\]

The proof of this proposition occupies the rest of this section, starting with finding the first half of the \(\ell_i\) and then the second half. Before we turn to this, we comment on the role that this proposition plays. We use this to show that before passing to a limit, we have various properties in the system. Conditions \(\text{(a)}\) and \(\text{(b)}\) are used to prove ergodicity, and Conditions \(\text{(d)}\) and \(\text{(e)}\) lead to a version of Theorem 4.1 all before passing to the limit: more precisely, the sets \(C_i(x)\) of the limiting object are approximated by \(\{T^{k_n} x\}_{n=1}^{r}\) in the sense that \(\eta_x\) restricted to
Corollary 4.7. For all \(k\), we have that 
\[
0 < r_j(x) \leq \mathcal{E}_j(x),
\]
where \(\mathcal{E}_j(x)\) is close to \(r_j(x)\). Similarly, if \(r_j(x) < r_{j+1}\), then \(d(T^{\ell r_{j+1}} x, x) < \varepsilon\).

4.2. Finding \(\ell_1, \ldots, \ell_r\). We now construct \(\ell_1, \ldots, \ell_r\) satisfying the conclusions of Proposition 4.3.

Lemma 4.5. For all \(\varepsilon > 0\), there exists \(k_0 \in \mathbb{N}\) such that for all \(k > k_0\) and \(1 \leq \ell \leq k\):

(i) If \((T^\ell x)_{10^{2k}} < k\) for all \(0 \leq i \leq \ell r_{10^{2k}}\), then \(d(T^{\ell r_{10^{2k}}} x, x) < \varepsilon\).

Similarly, if \((T^\ell x)_{10^{2k}} < k\) for all \(0 \geq i \geq -\ell r_{10^{2k}}\), then \(d(T^{\ell r_{10^{2k}}} x, x) < \varepsilon\).

(ii) If \(k \leq (T^\ell x)_{10^{2k}} \leq 2k - 2\) for all \(0 \leq i \leq \ell(r_{10^{2k}} + 1)\), then \(d(T^{\ell r_{10^{2k}} + 1} x, x) < \varepsilon\).

Similarly, if \(k \leq (T^\ell x)_{10^{2k}} \leq 2k - 2\) for all \(0 \geq i \geq -\ell(r_{10^{2k}} + 1)\), then \(d(T^{\ell r_{10^{2k}} + 1} x, x) < \varepsilon\).

Proof. We only include the proof of the first part of (i), as the proofs of all four statements are similar. Thus we need to show that under the assumptions, 
\[
(T^{\ell r_{10^{2k}} + 1} x) = x_i \quad \text{for all} \quad i < 10^{2k}.
\]

This statement immediately follows once we show that 
\[
T^{\ell r_{10^{2k}} + 1} x = S^\ell q_{10^{2k}} x.
\]

To prove (12), note that by assumption, 
\[
(T^\ell x)_{10^{2k}} < 2k - 2 \quad \text{for all} \quad 0 \leq i \leq \ell(r_{10^{2k}} + 1),
\]
and so \(S^\ell x \notin \bigcup_{j > 10^{2k}} D_j\) for all \(0 \leq i \leq \ell q_{10^{2k}}\). (Recall that \(D_j\) are defined in (b).) Similarly, by the assumption that \(k \leq (T^\ell x)_{10^{2k}} \leq 2k - 2\) for all \(0 \leq i \leq \ell(r_{10^{2k}} + 1)\), we have that \(S^\ell x \notin D_{10^{2k}}\) for all \(0 \leq i \leq \ell q_{10^{2k}}\). Thus for any such \(x\), \(\zeta_x(q_{10^{2k}}) = \zeta_0(q_{10^{2k}}) + 1 = r_{10^{2k}} + 1\). Iterating this process for \(x\), we obtain that \(S^\ell q_{10^{2k}} x = T^{\ell r_{10^{2k}} + 1} x, \ldots, S^{\ell q_{10^{2k}}} x = T^{\ell r_{10^{2k}} + 1} x\), thus proving the claim.

Lemma 4.6. Let \(n = u + v r_{10^{2k}}\). If \(x_{10^{2k}} \in [|u| + |v| + 1, k - |u| - |v| - 1]\), then 
\[
(T^n x)_j = (T^u x)_j \quad \text{for all} \quad j \neq 10^{2k}.
\]

Similarly, if \(x_{10^{2k}} \in [|u| + |v| + k - |u| - |v| - 3]\), then 
\[
(T^n x)_j = (T^{u - |v|} x)_j \quad \text{for all} \quad j \neq 10^{2k}.
\]

Proof. These results follow from Lemma 4.5 and again we only prove the first part as the others are analogous. If \(x_{10^{2k}} \in [|u| + |v| + 1, k - |u| - |v| - 1]\), then we apply the first part of Lemma 4.5 with \(\nu = v\) to \(T^u x\).

Corollary 4.7. For all \(\varepsilon > 0\) and \(b, b' \in \mathbb{Z}\), there exists \(k_0\) such that for all \(\ell > k_0\) there exists \(p_\ell \in \mathbb{Z}\), disjoint sets \(A_\ell, B_\ell\), and an interval \(J_\ell\) satisfying

(i) \(T^{\nu(A_\ell)}(x)_j = (T^{\nu(b, b')})_j\) for all \(j \neq 10^{2\ell}\) and \(x \in A_\ell\).

(ii) \(T^{\nu(b, b')}_j(x)_j\) for all \(j \neq 10^{2\ell}\) and \(x \in B_\ell\).

(iii) \(\nu(A_\ell), \nu(B_\ell) > \frac{1}{2} - \varepsilon\).

(iv) \(A_\ell = \bigcup_{i=0}^{r_{10^{2k}}} T^{i r_{10^{2k}}} J_\ell\)

Proof. We apply Lemma 4.6 to \(n = b + (b - b')\). Since \(b - (b - b') = b'\), by choosing
\[
J_\ell = \{x: x_j = 0\} \quad \text{for all} \quad j < 10^{2\ell} \quad \text{and} \quad x_j = |u| + |v| + 1,
\]
the corollary follows with \(p_\ell = b + (b - b')\).
Lemma 4.8. For any \( \varepsilon > 0 \) and \( b \in \mathbb{Z} \), there exists \( N \in \mathbb{N} \) such that for all \( x \in Y \) and \( n \geq N \),

\[
d_m(Y \times Y) \left( \sum_{i=1}^{n} \delta(T^i x, T^i T^s x), J(b) \right) < \varepsilon.
\]

Proof. Since \( T \times T \) is continuous and uniquely ergodic on the compact metric space \( \{(x, T^b x)\} \), we have uniform convergence of the Birkhoff averages (see, for example, Walters [30]).

Applying this result in our system, we obtain:

Corollary 4.9. For all \( \varepsilon > 0 \), there exists \( \ell \geq 1 \) such that

\[
d_m(Y \times Y) \left( \frac{1}{\ell} \sum_{i=1}^{L} \delta(T^i x, T^i T^s x), J(b) \right) < \varepsilon
\]

for all \( x \in Y \) and all \( L \geq (\ell - 2(|v| + |v| + 2))r_{1025} \). The analogous result holds for \( b' \).

We now combine these results with the strategy developed in [4] to build off diagonal joinings close to the barycenter of other off diagonal joinings. We begin by summarizing the results of [4], where the input is a sequence of numbers and sets with certain properties.

We assume that \( c > 0 \), \( J_j \) is a sequence of intervals, \( m_j \) is a sequence of natural numbers, \( b_j^{(1)}, \ldots, b_j^{(d)} \) are sequences of integers, and \( \tilde{A}_j, \tilde{B}_j \) and \( U_j \) are sequences of sets, and \( \varepsilon_j > 0 \) satisfy the following properties:

(i) For all \( j \), \( \tilde{A}_j = \bigcup_{i=1}^{m_j} T^i J_j \setminus U_j \).

(ii) For all \( j \), \( \tilde{B}_j = Y \setminus (A_j \cup U_j) \).

(iii) For all \( j \), \( \nu(\tilde{A}_j), \nu(\tilde{B}_j) > c \).

(iv) The minimal return time to \( J_j \) is at least \( \frac{1}{2} m_j \).

(v) For all \( j \), \( \nu(U_j) \leq \varepsilon_j \).

(vi) For all \( j \), \( m_j \sum_{\ell=j+1}^{\infty} \nu(J_\ell) < \varepsilon \).

(vii) For all \( j \), \( \varepsilon_{j+1} \leq \varepsilon_j \) and \( \sum_{j=1}^{\infty} \varepsilon_j < \infty \).

(viii) For any \( x \in \tilde{A}_j \), we have \( d(T^{b_j^{(p)}} x, T^{b_j^{(p+1)}} x) < \varepsilon_j \) and for any \( x \in \tilde{B}_j \), we have \( d(T^{b_j^{(p)}} x, T^{b_j^{(p-1)}} x) < \varepsilon_j \). (Note that \( b_j^{(p+1)} \) is interpreted to be \( b_j^{(d)} \) if \( p = 1 \).)

(ix) \( d_m(Y \times Y) \left( \frac{1}{\ell} \sum_{i=1}^{L} (T \times T)^i (J(b_j^{(p)} x), J(b_j^{(p)})), J(b_j^{(p)}) \right) < \varepsilon_i \) for all \( x \in Y \), all \( L \geq m_i^{m_{i+1}} \), and any \( p \in \{1, \ldots, d \} \).

Theorem 4.10. ([Chaika-Eskin [4] Proposition 3.1 and (the proof of) Corollary 3.3]) Assuming sequences of numbers and sets satisfying (i)-(ix), there exist \( p < 1 \), \( C' > 0 \) (depending only on \( c \) and \( d \)) such that

\[
d_m(Y \times Y) \left( J(b_k^{(p)}), \frac{1}{r} \sum_{p=1}^{r} J(b_j^{(p)}), J(b_j^{(p)}) \right) \leq C' \sum_{q=i}^{k} \varepsilon_q + C' \rho^{k-i},
\]

whenever \( k \geq i \) and \( p \in \{1, \ldots, r \} \). Moreover, if \( x \notin \bigcup_{q=i}^{k} U_q \), there is a reordering (which is allowed to depend on \( x \)) \( p_1, \ldots, p_d \) with \( d(T^{b_j^{(p)}} x, T^{b_j^{(p)}} x) < \sum_{q=1}^{k} \varepsilon_q \) for all \( 1 \leq j \leq d \).
Remark 1. The last statement of this theorem is not in the statement of Corollary 3.3, but follows by iterating (viii). The condition in (viii) is a slightly simpler condition than that in [H], where the conditional measure of an off diagonal joining on a fiber is used instead of the distance between points, but the condition in [H] follows immediately by using the definition of the Kantorovich-Rubinstein metric.

Remark 2. We iteratively apply the result of Theorem 4.10 for different (decreasing) choices of $\varepsilon_i$ and (increasing) $d$, with each choice satisfying all of the properties [1]–[x].

Corollary 4.11. For any $\varepsilon > 0$ and integers $b_1, \ldots, b_d$, there exist integers $\hat{b}_1, \ldots, \hat{b}_d$ such that

\begin{equation}
\label{eq:corollary11}
    d_{\mathcal{M}(Y \times Y)}(J(\hat{b}_j), \frac{1}{d} \sum_{j=1}^{d} J(b_j)) < \varepsilon
\end{equation}

for all $\ell \in \{1, \ldots, d\}$. Moreover, we may assume that there is a set $\hat{W}$ of measure $1 - \varepsilon$ such that for every $x \in \hat{W}$, there is a reordering $p_1, \ldots, p_d$ with $d(T^b_j x, T^{\hat{b}_j} x) < \varepsilon$ for all $1 \leq j \leq d$.

We note that the reordering in the second part of this statement depends on the particular $x$.

Proof. By Corollaries [1.7] and [4.9] the corollary holds when $d = 2$. Moreover, we claim that we can simultaneously apply these results to $d$ different pairs $b_1, b_1', \ldots, b_d, b_d'$ (the resulting common sets become $\hat{A}$ and $\hat{B}$). To see this, choose

$\hat{A} = \{x: x_{10\alpha \varepsilon} \in [\max\{|b_i|\} + \max\{|b_i - b_i'\}| + 1, \ell - (\max\{|b_i|\} + \max\{|b_i - b_i'\}| - 1)]\}$

and

$\hat{B} = \{x: x_{10\alpha \varepsilon} \in [\ell + \max\{|b_i|\} + \max\{|b_i - b_i'\}| + 1, 2\ell - (\max\{|b_i|\} + \max\{|b_i - b_i'\}| - 3)]\}$.

We apply this argument for the $d$ pairs $b_1^{(1)}, b_1^{(2)}; \ldots; b_1^{(d)}, b_1^{(1)}$ to produce measures $b_2^{(1)}, \ldots, b_2^{(d)}$. Note that on $\hat{A}$, there is a reordering of $1, \ldots, d$, call it $p_1, \ldots, p_d$, such that $d(T^b_j x, T^{\hat{b}_j} x) < \varepsilon$ for all $1 \leq j \leq d$ and $x \in \hat{A}$. (In fact, this is the reordering $p_j = j$.) There is a similar reordering on $\hat{B}$ (this is the reordering $p_j = j - 1$ for $j \neq 1$ and $p_1 = d$). Inductively, given $b_j^{(1)}, \ldots, b_j^{(d)}$ we apply this to the corresponding pairs $b_j^{(1)}, b_j^{(2)}; \ldots; b_j^{(d)}, b_j^{(1)}$. Let $A_1, \ldots, A_j$ and $\hat{B}_1, \ldots, \hat{B}_j$ denote the corresponding sets, as above. By Theorem 4.10 there exists $j \in \mathbb{N}$ such that $b_j^{(1)}$ satisfy (13) for all $i = 1, \ldots, d$.

Define

\begin{equation}
\label{eq:14}
    \hat{W} = \bigcap_{i=1}^{j} (A_i \cup \hat{B}_i)
\end{equation}

to be the intersection of the sets obtained at each step, and this satisfies the desired conclusion.

We now combine these results to find $\ell_1, \ldots, \ell_r$. Given $\varepsilon > 0$, applying Corollary 4.11 to $k_1, \ldots, k_r$, we obtain $\ell_1, \ldots, \ell_r$ such that

$\nu(\{x: d_{\mathcal{M}(Y)}(\frac{1}{r} \sum_{i=1}^{r} J(\ell_i x), \frac{1}{r} \sum_{i=1}^{r} J(k_i x)) > \varepsilon/2\}) < \varepsilon/2$.  

Lemma 4.13. Assume there exist $i$ inductively pick any $\nu$ in the proof of Corollary 4.11 and note that\( \nu(15) \) (Condition (a) for the first $r$) and also satisfy the reordering condition on $\ell_1, \ldots, \ell_r$ in $W$ where for each $x \in \hat{W}$, the reordering is given by $p_k' = k - |\{1 \leq i \leq j : x \in \hat{B}_i\}|$ and this difference is taken modulo $d$. (Note that as we have not yet introduced $\ell_{r+1}, \ldots, \ell_{2r}$, we have not yet fully established (a) or (b)) Towards obtaining Conclusion (c) for each $\ell_i$, choose $N_i$ such that for all $L \geq N_i$ we have\[
 d_{M(Y \times Y)} \left( \frac{1}{L} \sum_{j=0}^{L-1} \delta_{(T^j \times T^j)(x, T^j x), J(\ell_i)} \right) < \varepsilon.\]

4.2.2. Finding $\ell_{r+1}, \ldots, \ell_{2r}$. We start first by finding $\ell_{r+1}, \ldots, \ell_{2[3r]}$.

Lemma 4.12. Under the assumptions of Proposition 4.4 including the additional assumption, there exists $J \subset \{1, \ldots, r\}$ with $|J| = 2\lfloor \frac{r}{100} \rfloor$ and an order $2$ bijection $\phi : J \to J$ such that\[
 \nu((x : d(T^{\ell_i} x, T^{\ell_{\phi(i)}} x) > c)) > \frac{1}{8} \quad \text{for all} \quad i \in J.
\]

Proof. First we claim that for each $i \leq r$, we have that\[
 |\{j \leq r : \nu((x : d(T^{\ell_i} x, T^{\ell_j} x) > c)) > \frac{1}{8} \}| \geq \frac{r}{8}.
\]

To justify (16), we limit our consideration to $W \cap \hat{W}$, where $\hat{W}$ is defined as in (14) as given in the proof of Corollary 4.11 and note that $\nu(W \cap \hat{W}) \geq \frac{1}{2} - \varepsilon > \frac{49}{100}$.

If (16) does not hold,\[
 \int_{W \cap \hat{W}} \frac{1}{r}|\{j : d(T^{\ell_i} x, T^{\ell_j} x) \leq c\}| d\nu \geq \left( \frac{49}{100} - \frac{1}{8} \right) > \frac{r}{8} \nu(W \cap \hat{W}).
\]

It follows that there exists $x \in (W \cap \hat{W})$ such that\[
 |\{j : d(T^{\ell_i} x, T^{\ell_j} x) < c\}| > \frac{r}{2}.
\]

Since $x \in \hat{W}$, it follows that\[
 |\{j : d(T^{\ell_j} x, T^{\ell_i} x) < c + \varepsilon\}] > \frac{r}{2}.
\]

But since $x \in W \cap \hat{W}$, we have that $d(T^{\ell_i} x, T^c x)$ or $d(T^{\ell_j} x, T^b x)$ is less than $c + \varepsilon$, all of these $T^b x$ are at least $2c$ away from whichever of $T^a x$ or $T^b x$ that $T^{\ell_i} x$ is not close to. This contradicts the fact that $x \in W$.

Given (16), we can obtain our set of $J$, because until $|J| = \lfloor \frac{1}{8} r \rfloor$, we can always inductively pick any $i \notin J$ and find $j \notin J$ satisfying (15) and add them both into $J$, letting $\phi(i) = j$ and $\phi(j) = i$. Thus we can obtain a set $J$ whose cardinality is the smallest even number that is at least $\frac{1}{8} r$. □

Lemma 4.13. Assume there exist $a, b \in \mathbb{Z}$ and $c > 0$ such that\[
 \nu((x : d(T^a x, T^b x) > c)) > \frac{1}{8}.
\]

Let\[
 G_k = \{x \in Y : x_{102k} \in \left[ \frac{1}{3} k + |a - b|, \frac{1}{2} k - 2 - |a - b| \right] \text{ and } x_{102k-1} = 3\}
\]

and set $d_k = a + (a - b)r_{102k}$. Then for every $\varepsilon > 0$, there exists $k_0$ such for all $k \geq k_0$ and $x \in G_k$ there exists $j_k$ satisfying
for all $\ell \in [-r_{10^{2k}-1}, r_{10^{2k}-1}]$. Moreover, for all but a set of such $x$ with measure at most $\varepsilon$, we have

\begin{equation}
\left| \{ \ell \in [-r_{10^{2k}-1}, r_{10^{2k}-1}]: d(T^{a+\ell}x, T^{b+\ell}x) > c \} \right| > \frac{1}{9} 2r_{10^{2k}-1}.
\end{equation}

Proof. We apply the proof of Corollary 4.7 with $n = d_k$ to obtain the first 2 conditions. More precisely, by construction $G_k$ is a subset of $A_k \cup B_k$ where $A_k$ and $B_k$ given by the proof of Corollary 4.7. Choose $j_k = \sum_{i=0}^{\zeta_k\ell} 1_{D_{10^{2k}}}(S^i x) - \frac{b}{2}$. (Recall that $D_j$ is defined in 3.) Then since $x_{10^{2k}-1} = 3$, it follows that $j_k = j_{T^k x}$ for all such $x$ and $\ell' \in [-\frac{3}{2} r_{10^{2k}-1}, \frac{3}{2} r_{10^{2k}-1}]$. Indeed, because $y \in D_{10^{2k}}$ implies $y_{10^{2k}-1} = 6$, for all $x$ with $x_{10^{2k}-1} = 3$ and $\ell' \in [-\frac{3}{2} r_{10^{2k}-1}, \frac{3}{2} r_{10^{2k}-1}]$ we have

\begin{equation}
\sum_{i=0}^{\zeta_k\ell} 1_{D_{10^{2k}}}(S^i x) - \sum_{i=0}^{\zeta_k\ell'} 1_{D_{10^{2k}}}(S^i x) = 0.
\end{equation}

Choosing $k_0$ such that $|a|, |b| < r_{10^{2k_0}-1}$, the first 2 conditions hold. For the final condition, let $V = \{x: d(T^a x, T^b x) > c\}$.

By the (mean) ergodic theorem, there exists $N \in \mathbb{N}$ such that

\begin{equation}
\nu(\{x: \frac{1}{M} \sum_{i=0}^{M-1} 1_{V}(T^i x) > (1 - \varepsilon)\nu(V)) > 1 - \varepsilon
\end{equation}

for all $M \geq N$. Choosing $r_{10^{2k}-1} > N$ we have \ref{eq:17}.

We now use this to define $\ell_j$ for $j \in \{r + 1, \ldots, r + 2\frac{r}{10}\}$. Choose $J$ as in Lemma 4.12 and enumerate the elements of the set $J$ as $a_1, \ldots, a_{2\frac{r}{10}}$. Let $\ell_{r+j}$ be given by Lemma 4.13 applied with $a = a_j$ and $b = \ell_{\phi(a_j)}$ and where $k$ is chosen larger than the $k_0$ needed for the $2\frac{r}{10}$ applications of Lemma 4.13 as well as sufficiently large such that $\ell_i < \frac{1}{2} r_{10^{2k}-1}$ for all $1 \leq i \leq r$. For each such $j$, we define $\ell_{r+j}$ to be the corresponding $d_k$.

We now define $\ell_{2\frac{r}{10}+1}, \ell_{2r'-1}, \ldots, \ell_{2r'}$. For each $j \in J^c$, define $\ell_j = \ell_j + r_{10^{2k}+1}$ for sufficiently large $a$ such that each $j \in J^c$ corresponds to a unique $i$ with $r + 2\frac{r}{10} < i \leq 2r$. Let $\hat{V}$ be the set of measure at least $1 - \varepsilon$ such that $d(T^{\ell_j} x, T^{\ell_j + r_{10^{2k}+1}} x) < \varepsilon$ for all $j \in J^c$ and $x \in \hat{V}$.

We use this to prove the proposition:

Proof of Proposition 4.4. Choose $M = \frac{k}{2} r_{10^{2k}}$ and set

\begin{equation}
B = \left\{ \frac{1}{3} k + 2 \max_{j=1,\ldots,\frac{r}{10}} \{ |a_j| \}, \frac{1}{2} k - 2 - 2 \max_{j=1,\ldots,\frac{r}{10}} \{ |a_j| \} \right\}
\end{equation}

and

\begin{equation}
A = \{ x: x_{10^{2k}-1} = 3 \} \cap \hat{W} \cap \hat{V}.
\end{equation}

By Conclusion \ref{a} of Lemma 4.13 and our choice that $|\ell_j| < r_{10^{2k}-1}$, we have Conclusion \ref{a} with $\lambda_j$ as in Conclusion \ref{a}. By Corollary 4.11 (see also the last paragraph of Section 4.2.1) we have Conclusion \ref{a} and Conclusion \ref{b} for $\{\ell_j\}_{j=1}^{r_{10^{2k}}-1}$. Since each $\ell_{a_1}, \ldots, \ell_{a_{\frac{r}{10}}}$ appears as both $j$ and $\phi(j)$ in our construction of $\{\ell_j\}_{j=r+1}^{\frac{r}{10}+1}$, it follows outside a set of $\mathbf{x}$ with small measure, for each such $\mathbf{x}$ there exist, $p_1, \ldots, p_{\frac{r}{10}}$.\]
a reordering of $\ell_i$ for $r < i \leq r + 2\left\lceil \frac{r}{10} \right\rceil$ such that $d_Y(T^{\ell_i}x, T^{\ell_{i+1}}x) < \varepsilon$. For the remaining $r < j \leq 2r$, the off diagonal joining $\ell_j$ is built to be $\varepsilon$ close to the corresponding $\ell_i$. Thus Conclusion (b) follows for $\{\ell_i\}_{i=r+1}^{2r}$. We have (c) for $N = \max \{N_i\}_{i=1}^r$ (see the end of Section 4.2.1). Finally, by (17) and (6) of Lemma 4.13 we have (e) Indeed, by our choice of $A$ we can apply (ii) and by (17) this gives the desired distance bound of $T^{M+\ell_k+i}x$ from $T^{\ell_k+i}x$.

\begin{proof}

4.3. **Proof of Theorem 4.3** Let $J_\nu$ denote the self-joinings of $(Y,T,\nu)$. Recall that $\sigma_x$ denotes a measure on $Y$, and not on $\{x\} \times Y$.

**Lemma 4.14.** Let $\varepsilon > 0$, $k_1, \ldots, k_r \in \mathbb{Z}$, $\ell_1, \ldots, \ell_{2r}, L, M \in \mathbb{Z}$, the set $A$, and $j_x \in [-M,M]$ be as in Proposition 4.4.

There exists $\frac{1}{20} > \delta > 0$ such that if for some $\sigma \in J_\nu$ we have

\begin{equation}
\nu\left( \{x : d_M(Y)\sigma_x, (\sum_{n=1}^{2r} J(\ell_n))x) > \delta \} \right) < \delta, \tag{18}
\end{equation}

then there exists $\tilde{A} \subset A \subset Y$ and $\nu(\tilde{A}) > \frac{1}{999}$ such that

(i) $\sigma(\{(x,y) \in \tilde{A} \times Y : d_M(Y \times Y)(\frac{1}{N} \sum_{i=1}^{N} \delta(T^x_i, T^y_j), \frac{1}{r} \sum_{i=1}^{r} J(k_i)) < 2\varepsilon \}) > \frac{9}{10}\nu(\tilde{A})$,

(ii) for all $x \in \tilde{A}$, there exists $C_x$ with $\sigma_x(C_x) > \frac{1}{9999}$ and $\frac{1}{L} \sum_{i=0}^{L-1} d(T^{M+i}y, T^{j_x+i}y) < 2\varepsilon$ for all $y \in C_x$.

Moreover, under the additional assumption that there exist $a,b \in \mathbb{N}$ and $c > 0$ such that $d(T^ax, T^b x) > 3c$ for a set of $x$ with measure $\frac{1}{2}$, then there exists $E_x$ with $\sigma_x(E_x) > \frac{1}{9999}$ satisfying

(iii) $\frac{1}{L} \{ 0 \leq i \leq L : d(T^{M+i}y, T^{j_x+i}y) > \frac{\varepsilon}{2} \} > \frac{\varepsilon}{2}$ for all $x \in \tilde{A}$ and $y \in E_x$.

**Proof.** Choose a compact set $G$ with $\nu(G) > 1 - \frac{\varepsilon}{10000}$ such that $T^i|_G$ is (uniformly) continuous for all $|i| \leq \max\{N,L,M\}$. Let $\tilde{G} = G \cap \bigcap_{n=1}^{2r} T^{-\ell_n}G$. There exists $\delta > 0$ such that if $x \in \tilde{G}$ and $d(x,y) < \delta$, then

\begin{equation}
\min\{\varepsilon, 10^{-7}\} \quad \text{for all } |i| \leq \max\{N,L,M\}. \tag{19}
\end{equation}

Thus we can choose $A_1 = A \cap \tilde{G}$. If $x \in A_1$, $y,y' \in \tilde{G}$, $d(y,y') < \delta$, and

\begin{equation}
\frac{1}{N} \sum_{i=1}^{N} \delta(T^x_i, T^y_j), \frac{1}{r} \sum_{i=1}^{r} J(k_i)) < \varepsilon,
\end{equation}

then by (19) the definition of $d_M(Y \times Y)$ we have

\begin{equation}
\frac{1}{N} \sum_{i=1}^{N} \delta(T^x_i, T^{y'}_j), \frac{1}{r} \sum_{i=1}^{r} J(k_i)) < 2\varepsilon.
\end{equation}

Thus Condition (c) follows from Condition (e) of Proposition 4.4 (as well as (18) and the measure bound on $G$). Setting

$$C_x = \tilde{G} \cap \bigcup_{n=1}^{r} B(T^{\ell_n}x, \delta),$$
Thus \( \lim \) for all \( \delta \) \( N \) \( \mu \) for \( t \)

\[
E_x = \bar{G} \cap \bigcup_{n=r+1}^{r+2[\frac{r+1}{\ell}]_1} B(T^n x, \delta),
\]

then \( \text{iii} \) (without the measure bound) follows from \( \text{iv} \) Setting

\[
\nu\left(\{x: d_M(Y) \left(\sum_{i=1}^{2^r} J(k_i^{(r)})_x, \sum_{i=1}^{2^n} J(k_i^{(n)})_x \right) > \delta_\ell(1 - \frac{1}{2^{r-\ell}})\}\right) < \delta_\ell(1 - \frac{1}{2^{r-\ell}})
\]

for all \( n < r \). Applying Lemma 4.14 to obtain \( \delta_r \), which again we take to be bounded by \( \frac{k_i^{(r)}}{2r} \) for all \( n < r \), we can repeat the application of Proposition 4.4 but with \( \varepsilon_{r+1} = \min\{\frac{\varepsilon_r}{2_r}, \frac{1}{2}\delta_r\} \).

We pass to the weak*-limit of \( \frac{1}{2^r} \sum_{i=1}^{2^r} J(k_i^{(r)}) \), which we denote \( \sigma \). By construction, we have that

\[
\nu\left(\{x: d_M(Y) (\sigma_x, \sum_{i=1}^{2^n} J(k_i^{(n)})_x) > \delta_\ell\}\right) \leq \delta_n
\]

for all \( \ell \). From this, it follows that

\[
\sigma\left\{(x, y) \in Y \times Y: d_M(Y \times Y) \frac{1}{N_r} \sum_{i=1}^{N_r} \delta(s^{(r)}_i, s^{(r')}_y), \frac{1}{2^{r-1}} \sum_{i=1}^{2^{r-1}} J(k_i^{(r-1)}) > \varepsilon_r\right\}
\]

\[
< 2\varepsilon_r \leq \frac{1}{2^{r-1}}.
\]

Thus \( \lim \frac{1}{N_r} \sum_{i=1}^{N_r} \delta(s^{(r)}_i, s^{(r')}_y) \) is the weak*-limit of \( \frac{1}{2^r} \sum_{i=1}^{2^r} J(k_i^{(r)}) \), which is \( \sigma \), and this holds for \( \sigma \)-almost every \( (x, y) \in Y \times Y \). By the criterion given in \( \text{iv} \), it follows that \( \sigma \) is ergodic.

Thus to complete the proof of the theorem, it suffices to show that the assumptions for Theorem 4.1 are satisfied. To see this, by \( \text{iv} \), we have a set \( \tilde{A}_m \) such that
for every $x \in \tilde{A}_m$ we have sets $C_x^{(m)}$, $E_x^{(m)}$ such that $\sigma_x(C_x^{(m)})$, $\sigma_x(E_x^{(m)}) > \frac{1}{2}$ and

(i) $\frac{1}{L} \sum_{l=0}^{L-1} d(T^{M+l}y, T^{j+l}y) < 2\varepsilon$ for all $x \in \tilde{A}_m$ and $y \in C_x$ giving Theorem 4.1 (ii) as $m \to \infty$ and we can choose $\varepsilon \to 0$. (This uses Proposition 4.4 part (d) and Lemma 4.14 part (ii).)

(ii) $\frac{1}{L} \sum_{i=0}^{L-1} |\{0 \leq i < L : d(T^{M+i}y, T^{j+i}y) > \frac{1}{L}\}| > \frac{1}{2}$ for all $x \in \tilde{A}_m$ and $y \in E_x$ giving Theorem 4.1 (iii). (This uses Proposition 4.4 part (d) and Lemma 4.14 part (iii).)

(iii) The assumption that $\sigma_x(C_x), \sigma_x(E_x) > \frac{1}{2}$ giving Theorem 4.1 (iv).

(iv) Proposition 4.4 part (b) applied to $r < d = r + 2\lceil \frac{L}{M} \rceil$ combined with (18) imply Theorem 4.1 (iv).

(Not that strictly speaking $M, L, C_x, E_x, j_x$ depend on $m$, but we omit the dependency for the sake of readability.) Thus we have proven the assumptions needed to apply Theorem 4.1.

4.4. These properties are residual.

**Theorem 4.15.** A residual set of measure preserving transformations are not quasi-simple.

If $(h_j)_{j \in \mathbb{N}}$ is a sequence of positive integers, we say a system $(X, T, \mu)$ admits special linked approximation of type $(h_j, h_j + 1)$ if for each $j \in \mathbb{N}$, there exist sets $A_j, C_j \subset X$ satisfying the following five conditions:

(i) $\lim_{j \to \infty} \mu(\bigcup_{i=0}^{h_j-1} T^i A_j) = \frac{1}{2} = \lim_{j \to \infty} \mu(\bigcup_{i=0}^{h_j} T^i C_j)$;

(ii) The sets $A_j, \ldots, T^{h_j-1}A_j, C_j, \ldots, T^{h_j}C_j$ are pairwise disjoint;

(iii) $\lim_{j \to \infty} \frac{\mu(T^{h_j}A_j \cap A_j)}{\mu(A_j)} = 1 = \lim_{j \to \infty} \frac{\mu(T^{h_j}C_j \cap C_j)}{\mu(C_j)}$;

(iv) Defining $R_A^{(j)} = \bigcup_{i=0}^{h_j-1} T^i A_j$ and $R_C^{(j)} = \bigcup_{i=0}^{h_j} T^i C_j$, there exist measurable sets $J_i \subset A_j$ and $a, b \in \mathbb{N}$ such that $J_j, \ldots, T^{a+b-1}J_j$ are all pairwise disjoint, $T^i J_j \subset R_A^{(j)}$ for all $0 \leq i \leq a-1$, and $T^i J_j \subset R_C^{(j)}$ for all $a \leq i \leq a + b - 1$ and $\lim_{j \to \infty} \mu(\bigcup_{i=0}^{a+b-1} T^i J_j) = 1$;

(v) For all $\varepsilon > 0$, there exist measurable sets $B_0^{(j)}, \ldots, B_{h_j-1}$ and $\hat{B}_0^{(j)}, \ldots, \hat{B}_{h_j}^{(j)} \in X$ of diameter at most $\varepsilon$ such that

$\lim_{j \to \infty} \sum_{i=0}^{h_j-1} \mu(T^i A_j \setminus B_i^{(j)}) = 0 = \lim_{j \to \infty} \sum_{i=0}^{h_j} \mu(T^i C_j \setminus \hat{B}_i^{(j)})$.

Condition (iv) distinguishes this from usual linked approximation, and is needed to carry out the arguments of Section 4.2. This property is a residual property in the space of measure preserving transformations. Indeed, it is conjugacy invariant, and nonempty. Halmos [16] Theorem 1] showed that the conjugacy class of any aperiodic measure preserving transformation is dense. Our conditions (i)-(v) are the intersection of a countable number of open conditions and so the property holds on a $G_\delta$ set. Thus it is a dense $G_\delta$, that is residual, property.

We say a system $(X, T, \mu)$ is rigid rank 1 if there exist numbers $n_j$ and sets $I_j$ such that
Lemma 4.13 is less explicit) and the construction of positive integers. By the rigidity of all of the rationals have a common denominator, writing

\[ \lim_{j \to \infty} \mu(\bigcup_{i=0}^{n_j-1} T^i I_j) = 1; \]

(ii) The sets \( I_j, \ldots, T^{n_j-1} I_j \) are pairwise disjoint;

(iii) \( \lim_{j \to \infty} \frac{\mu(T^i I_j \setminus B^{(j)}_i)}{\mu(T^i I_j)} = 1; \)

(iv) For all \( \varepsilon > 0 \), there exist measurable sets \( B^{(j)}_0, \ldots, B^{(j)}_{n_j-1} \in X \) of diameter at most \( \varepsilon \) such that

\[ \lim_{j \to \infty} \sum_{i=0}^{n_j-1} \mu(T^i I_j \setminus B^{(j)}_i) = 0. \]

Note that this property is stronger than being both rigid and rank 1. Similarly to the property of admitting a special linked approximation, rigid rank 1 is also a residual property in the space of measure preserving transformations.

Any transformation that both admits a special linked approximation of type \((h_j, h_{j+1})\) and is rigid rank 1 has a self-joining that is not a distal extension of \((X, T, \mu)\). Indeed, these transformations have the following property: for any pair of integers \( a, b \in \mathbb{N} \) and \( \varepsilon > 0 \), there exists \( m \in \mathbb{N} \) and a pair of sets \( C, D \) with measure at least \( \frac{1}{2} - \varepsilon \) so that

\[ \mu(\{x \in C : d(T^a x, T^m x) > \varepsilon\}) < \varepsilon \quad \text{and} \quad \mu(\{x \in D : d(T^b x, T^m x) > \varepsilon\}) < \varepsilon. \]

Using this property, rank 1 rigidity, and the ergodicity of \( \mu \), our construction of the joining that is not a distal extension of \((X, T, \mu)\) proceeds similarly to Sections 4.2 and 4.3. More precisely, for sufficiently large \( j \), we can choose \( C = \bigcup_{i=0}^{h_j-1} T^i A_j, D = \bigcup_{i=0}^{h_j} T^i B_j \), and \( m = a + (a-b)h_j \). The inductive construction of \( \ell_1, \ldots, \ell_r \) proceeds verbatim. Similarly for \( \ell_{r+1}, \ldots, \ell_{r+2[\frac{m}{a}]} \) is almost verbatim (the described set in Lemma 4.13 is less explicit) and the construction of \( \ell_{r+2[\frac{m}{a}]} \) is almost verbatim (making use of the property that our transformation is rigid).

Because rigid rank 1 transformations have “flat stacks,” King proved the following:

**Theorem 4.16. (King [20] EJCL Theorem)** If \( \eta \) is a self-joining of \((Y, \nu, T)\) and \( \eta \) is rigid rank 1, then there exist real numbers \( \alpha_i^{(k)} > 0 \) such that \( \sum_i \alpha_i^{(k)} J(i) \) converges in the weak*-topology to \( \eta \).

In fact, he establishes that ergodic self-joinings of transformations with flat stack lie in the weak closures of off diagonal joinings. A different proof of this result is given in [4] Corollary 2.3 (see also [3] Corollary 0.3).

**Proposition 4.17.** The self-joinings of \((Y, \nu, T)\) form a Paulsen simplex.

**Proof.** By King’s Theorem, it suffices to show that for any integers \( n_1, \ldots, n_k \) and positive rationals \( \beta_1, \ldots, \beta_k \) such that \( \sum \beta_i = 1 \), there exists \( m \) such that \( d_{M(Y \times Y)}(J(m), \sum \beta_i J(n_i)) < \varepsilon \). Without loss of generality, we can assume that all of the rationals have a common denominator, writing \( \beta_i = \frac{m_i}{\ell_i} \) where all \( m_i \) are positive integers. By the rigidity of \( T \), for each \( n_i \) there exist \( a_{i,1}, \ldots, a_{i,\ell_i} \) such that \( d_{M(Y)}(J(a_{i,\ell_i}^i), J(n_i)) < \frac{\varepsilon}{\ell_i} \) for \( \ell_i = 1, \ldots, m_i \). By Corollary 4.11, there exists \( m \) such that \( d_{M(Y \times Y)}(J(m), \frac{1}{\tau} \sum_{i \in I} J(a_{i,\ell_i}^i)) < \frac{\varepsilon}{\tau} \). Thus \( d_{M(T \times T)}(J(m), \sum \beta_i J(n_i)) < \varepsilon. \)
Remark 3. Analogously, Proposition 4.17 can be generalized for any rigid rank 1 transformation that admits special linked approximation of type \((h_j, h_j + 1)\). Using this, it follows that there is a residual set of measure preserving transformations such that their self-joinings form a Paulsen simplex.

5. Coding and results

5.1. The mechanism for showing \((Y, T, \nu)\) is prime. Throughout this section, we continue to assume that \((X, \mu, S)\) and \((Y, \nu, T)\) are the systems defined in Section 3, maintaining all of the notation introduced in that section.

The proof that \((Y, T, \nu)\) is prime is based on showing that a factor map is either an isomorphism or a map to the one point system. The first step is relating factor maps to linear combinations of powers of \(T\):

Theorem 5.1. (\cite{H} Theorem 2.2) If \(\tau\) is a self-joining of \(T\) and \(P\) is the corresponding Markov operator then \(P\) is the strong operator topology limit of linear combinations of powers of \(U_T\) with non-negative coefficients.

Note that by considering \(P1_Y\), we may assume the (non-negative) coefficients add up to 1. That is, \(P\) is a strong operator topology limit of a convex combination of \(\{U_T^i\}_{i \in \mathbb{Z}}\). We use this trivial strengthening throughout.

With this theorem in hand, we seek to understand when \(T^n\) is either close to the identity or causes nearby points to diverge in a controlled fashion. The former corresponds to a factor that is an isomorphism and the latter to a factor to the one point system. We start by understanding the second mechanism.

5.2. A condition for a factor to be the one point system. Recall that \(Z_k\) and \(W_k\) are defined in (3) and (4). Given \(n \in \mathbb{N}\), we say that \(x, y\) are \(n\)-friends if

\[
\sum_{j=0}^{n} 1_{Z_k}(S^j x) = \sum_{j=0}^{n} 1_{Z_k}(S^j y)
\]

for all but one \(k \in \mathbb{N}\),

\[
\left| \sum_{j=0}^{n} 1_{Z_k}(S^j x) - \sum_{j=0}^{n} 1_{Z_k}(S^j y) \right| = 1
\]

for exactly one \(k \in \mathbb{N}\), and

\[
\sum_{j=0}^{n} 1_{W_\ell}(T^j x) = \sum_{j=0}^{n} 1_{W_\ell}(T^j y)
\]

for all \(\ell \in \mathbb{N}\).

Lemma 5.2. If \(x\) and \(y\) are \(\zeta(x)(n)\)-friends, then \(0 < |\zeta_y(n) - \zeta_x(n)| \leq 3\).

Proof. Since \(Y = X \setminus \left( \bigcup_{\ell \in \{10^k : k \geq 2\}} Z_\ell \cup \bigcup_{k=1}^{\infty} W_k \right)\) and since \(x\) and \(y\) are \(n\)-friends, it follows that

\[
\left| \sum_{j=0}^{n} 1_Y(S^j x) - \sum_{j=0}^{n} 1_Y(S^j y) \right| = 1.
\]
Assume \( \sum_{j=0}^{\zeta_x(n)} 1_Y(S^j y) = \sum_{j=0}^{\zeta_y(n)} 1_Y(S^j x) - 1 \), and so \( \zeta_y(n) = \zeta_x(n) + m \) where \( m \) is the least integer such that
\[
\sum_{j=1}^{m} 1_Y(S^j S^{\zeta_x(n)} y) = 1.
\]

To prove the statement, we are left with showing that \( m \leq 3 \). If \( z \in X, \ell \in \mathbb{Z} \), and \( S^\ell z, S^{\ell+1} z \notin Y \), then one of the two iterates lies in \( Z_1 \) (the only \( D_\ell \) with 1st index not 6) and the other lies in \( \bigcup_{k=2}^{\infty} D_k \), and so \( (S^{\ell+2} z)_1 \notin \{6,7\} \) which means it lies in \( Y \). It follows that \( m \leq 3 \), completing the proof.

We record part of the proof for future reference:

**Corollary 5.3.** For every \( x \in Y \), \( n \in \mathbb{Z} \) we have \(|\zeta_x(n)| \leq 3|n|\).

**Notation 5.4.** Given \( j \in \mathbb{N} \), let \( G_j : A_j \to B_j \) be a measure preserving bijection of disjoint measurable sets \( A_j \) and \( B_j \) such that \( x \) and \( G_j(x) \) are \( \zeta_x(j) \)-friends. Given \( n \in \mathbb{N} \) and \( \varepsilon > 0 \), set

\[
(22) \quad \mathcal{H}_{n,\varepsilon} = \{ j : \text{there exists } A_j, G_j \text{ as above such that } \mu(A_j) > \varepsilon \text{ and } x_k = G(x)_k \text{ for all } k \leq n \}.
\]

In the next lemma we approximate a non-explicit measurable set by cylinders. As such, we have a non-explicit smallness condition on \( \varepsilon \), which we denote as \( \hat{\varepsilon} \), and we reserve this symbol to denote this constant for the remainder of the paper.

**Lemma 5.5.** Assume that \((Y, \nu, T)\) has a nontrivial factor \((Z, \rho, R)\) with associated factor map \( P : Y \to Z \). Let \( F : L^2(\nu) \to L^2(\nu) \) be a Markov operator defined by integrating the fibers of the factor map \( P \) and further assume that \( F \) is the limit (as \( k \to \infty \)) of \( \sum \alpha_i^{(k)} U_{T^k} \), in the strong operator topology where \( \alpha_i^{(k)} \geq 0 \) for all \( i, k \) and \( \sum_i \alpha_i^{(k)} = 1 \) for all \( k \). Then there exists \( \hat{\varepsilon} > 0 \) and \( N_0 = N_0(F) \) such that for all \( N \geq N_0 \) and sufficiently large \( m \),
\[
\sum_{j \in \mathcal{H}_{N,\varepsilon}} \alpha_j^{(m)} < \hat{\varepsilon}.
\]

**Proof.** Since \( T \) is weakly mixing, \( R \) is aperiodic and by Rokhlin’s Lemma, for any \( \delta > 0 \), there exists \( V \subset Z \) such that \( \rho(V) > \frac{1}{2} - \delta \) and such that \( V, RV, R^2 V, R^3 V \) are pairwise disjoint. Set
\[
g = 1_V + \sqrt{-1} 1_{RV} - 1_{R^2 V} - \sqrt{-1} 1_{R^3 V}
\]
and let \( f = g \circ P \) be the pullback of \( g \) to \( Y \). Choose \( \tilde{f} \), taking values in \( \{ \sqrt{-1} \}_{j=0}^{3} \), that is a finite linear combination of characteristic functions of cylinder sets such that \( \nu(\{ x : \tilde{f}(x) \neq f(x) \}) < \delta \), and let \( k \) be the largest defining index out of all of these cylinder sets.

We claim that if \( N > r_{k+1} \) and \( n \in \mathcal{H}_{N,\varepsilon} \), then
\[
\nu(\{ x : |U_{T^n} \tilde{f}(x) - \hat{f}(x)| > \frac{1}{\sqrt{2}} \}) > \hat{\varepsilon} - 11\delta.
\]
To prove the claim, assume that \( G_n : A_n \to B_n \) is the measure preserving bijection given in the definition of \( H_{n, \varepsilon} \) in (22) and define

\[
G(x) = \begin{cases} 
G_n(x) & \text{if } x \in A_n \\
G_n^{-1}(x) & \text{if } x \in B_n \\
x & \text{otherwise.}
\end{cases}
\]

We restrict our attention to the set of points \( y \) of measure at least \( \varepsilon - 10\delta \) that satisfy the following properties:

(i) the points lie in \( A_n \)
(ii) the points satisfy \( \bar{f}(y) = f(y) \) and \( \bar{f}(G(y)) = f(G(y)) \).
(iii) the points satisfy \( P(S^n y) \) and \( P(G(S^n y)) \) lie in \( V \cup R_v \cup R^2 \cup R^3 \).

Then for any such point \( y \), we have that \( \bar{f}(y) = f(G(y)) \) and furthermore for some \( 1 \leq m \leq 3 \) (which may depend on \( y \)) we have

\[
\bar{f}(T^n y) = f(T^n y) = \sqrt{-1}^m f(T^n G(y)) = \sqrt{-1}^m \bar{f}(T^n G(y))
\]

(see the second equality follows from Lemma 5.2). Thus either \( \bar{f}(y) \neq \bar{f}(T^n y) \) or \( \bar{f}(G(y)) \neq \bar{f}(T^n G(y)) \). Since \( \bar{f} \) takes values in \( \{\sqrt{-1}\}^3 \), if \( \bar{f}(x) \neq \bar{f}(z) \) then \( |\bar{f}(x) - \bar{f}(z)| \geq \sqrt{2} \) and the claim follows.

By construction \( \|F(\bar{f}) - \bar{f}\|_2 < 4\delta \). However, if

\[
\|U_{T^n}, \bar{f} - \bar{f}\|_2 > c, \quad \gamma_j \geq 0, \quad \text{and} \quad \sum \gamma_j \leq 1,
\]

then by taking a convex combination it follows that

\[
\|\sum \gamma_j U_{T^n}, \bar{f} - \bar{f}\|_2 > Cc^2.
\]

Similarly if

\[
\sum \{j : \|U_{T^n}, \bar{f} - \bar{f}\|_2 > c\} \gamma_j > \hat{\varepsilon},
\]

then \( \|\sum \gamma_j U_{T^n}, \bar{f} - \bar{f}\|_2 > C\hat{\varepsilon}^2 c^2 \). Since \( \delta \) is arbitrary, the lemma follows. \( \square \)

5.3. Recoding of time scales. We introduce some new notation.

**Notation.** Let

\[
\sigma_n = \max\{i : d_i(n) = 0\},
\]

where \( d_i \) is defined as in (10).

Set

\[
E = \{10^k : k \geq 2\}.
\]

Now if \( \sigma_m \notin E \) it is relatively easy to see that \( m \in H_{\sigma_m-1, \varepsilon} \) (Lemma 5.10). If \( \sigma_m \in E \) we seek to relate \( m \) to \( m' \) where \( \sigma_m' < \sigma_m \). We now consider two motivating examples: \( T^{10^{2k+1} + 2} \approx T^2 \), because off of a small measure set \( (T^{10^{2k+1} + 2} x)_j = (T^2 x)_j \) for all \( j < 10^{2k+1} \). There is a more complicated situation, \( T^{10^{2k+2}} \) is roughly \( T^2 \) on \( \{y \in Y : y_{10^{2k}} < k\} \) (off of a set of small measure) and \( T^{10^{2k+2}} \) is roughly \( T \) on \( \{y \in Y : y_{10^{2k}} \geq k\} \) (off of a set of small measure). We make this recoding precise below by triples which keep track of the new powers in the first coordinate, the set where this approximation is relevant in the second coordinate and the measure of the set where this approximation fails in the third coordinate. Note, the third coordinate can also be related to friends (Lemma 5.11).
Definition 5.6. Fix $N \in \mathbb{N}$ and $\varepsilon > 0$. For $r \geq 1$ and a set of triples $\mathcal{F}_r(N, \varepsilon) \subset (\mathbb{Z}, B(Y), [0,1])$, we inductively define the set of triples $\mathcal{F}_{r+1}(N, \varepsilon)$ as follows: if $(j, A, \rho) \in \mathcal{F}_r(N, \varepsilon)$ and at least one of the following conditions holds

(i) $\sigma_j \notin E$
(ii) $j = 0$
(iii) $\sigma_j \leq N$
(iv) $\rho > \varepsilon$,

then $(j, A, \rho) \in \mathcal{F}_{r+1}(N, \varepsilon)$. Otherwise we modify the triple, depending on the value of $\sigma_j$:

(i) If $\sigma_j \in \{10^{2^l+1} : l \geq 1\}$, then
\[(j - d_{\sigma_j}(j)r_j, A, \rho + \frac{|d_{\sigma_j}(j)|}{a_{\sigma_j}}) \in \mathcal{F}_{r+1}(N, \varepsilon).\]

(ii) If $\sigma_j \in \{10^{2^l} : l \geq 1\}$, then both
\[(j - d_{\sigma_j}(j)r_j, A \cap \bigcup_{\ell < \frac{\sigma_j}{10}} C_{\sigma_j}(\ell), \rho + \frac{|d_{\sigma_j}(j)|}{a_{\sigma_j}}) \in \mathcal{F}_{r+1}(N, \varepsilon)\]
and
\[(j - d_{\sigma_j}(j)r_j + d_{\sigma_j}(j), A \cap \bigcup_{\ell \geq \frac{\sigma_j}{10}} C_{\sigma_j}(\ell), \rho + \frac{|d_{\sigma_j}(j)|}{a_{\sigma_j}}) \in \mathcal{F}_{r+1}(N, \varepsilon).\]

Definition 5.7. Fix $N \in \mathbb{N}$ and $\varepsilon > 0$.
Define $\mathfrak{F}(N, \varepsilon)$ to be the collection of triples $\mathcal{F}_r(N, \varepsilon)$ that stabilize with respect to $r$, meaning that
\[\mathfrak{F}(N, \varepsilon) = \mathcal{F}_r(N, \varepsilon)\text{ when } \mathcal{F}_r(N, \varepsilon) = \mathcal{F}_{r+1}(N, \varepsilon).\]
Define $\mathcal{F}(N, \varepsilon)$ to keep track of the measure of the sets in $\mathfrak{F}(N, \varepsilon)$, meaning that
\[\mathcal{F}(N, \varepsilon) = \{(n, \nu(A), \rho) : (n, A, \rho) \in \mathfrak{F}(N, \varepsilon)\}.\]
If $\mathcal{F}_0(N, \varepsilon) = (i, [0,1], 0)$ for some $i \in \mathbb{N}$, we define $\mathcal{F}_i(N, \varepsilon)$ to be the set $\mathcal{F}(N, \varepsilon)$.
Similarly, define $\mathcal{F}_r,i(N, \varepsilon)$ to be $\mathcal{F}_r(N, \varepsilon)$ if $\mathcal{F}_0(N, \varepsilon) = \{(i, [0,1], 0)\}$. We similarly define $\mathfrak{F}_i(N, \varepsilon)$ to be $\mathfrak{F}(N, \varepsilon)$ when $\mathcal{F}_0(N, \varepsilon) = \{(i, [0,1], 0)\}$.

Note that $\sigma_n$ is defined in (23), $a_i$ are defined in (1), and $r_i$ are defined in (0). We state a lemma that motivates the sets given in Definition (5.7):

Lemma 5.8. Given $n \in \mathbb{N}$, let $C$ be a cylinder defined by positions in $E$ that are greater than $\sigma_n$.

(i) Assume $\sigma_n$ is an odd power of 10. Setting $\tilde{n} = n - d_{\sigma_n}(n)r_{\sigma_n}$, we have
\[\nu(\{x \in C : (T^{\tilde{n}}x)_i \neq T^n(x)_i \text{ for some } i < \sigma_j\}) < 4\nu(C)\frac{|d_{\sigma_n}(n)|}{a_{\sigma_n}}.\]

(ii) Assume $\sigma_n$ is an even power of 10. Setting $n' = n - d_{\sigma_n}(n)(r_{\sigma_n} - 1)$ and defining $A_1 = \{x : x_{\sigma_n} \geq \frac{\varepsilon_n}{2}\}$, we have
\[\nu(\{x \in A_1 : (T^{n'}x)_i \neq T^n(x)_i \text{ for some } i < \sigma_{n'}\}) < 4\nu(C \cap A_1)\frac{|d_{\sigma_n}(n)|}{a_{\sigma_n}}.\]
Furthermore, setting \( n'' = n - d_{\sigma_n}(n)r_{\sigma_n} \), and defining \( A_2 = \{ x : x_{\sigma_n} < \frac{\sigma_n}{2} \} \), we have

\[ \nu(\{ x \in A_2 : (T^{n''}x)_i \neq T^n(x)_i \text{ for some } i < \sigma_{n''} \}) < 4\nu(C \cap A_2)\frac{|d_{\sigma_n}(n)|}{a_{\sigma_n}}. \]

Proof. For convenience, in this proof we assume \( d_{\sigma_n} > 0 \) (the case \( d_{\sigma_n} < 0 \) is similar). Recall that \( D \) is defined in (26). Observe that if \( T^n(x) = S^{d_{\sigma_n}(n)q_{\sigma_n}}(T^n(x)) \), then \( T^n(x)_j = T^n(x)_j \) for all \( j < \sigma_n \), and (by Lemma 3.3) this holds if

\[ \sum_{i=0}^{d_{\sigma_n}(n)q_{\sigma_n} - 1} 1_{\bigcup_{i=\sigma_n}^{n} D_j}(S^i x) = 0. \]  

(25)

First we consider the case of \( \sigma_n = 10^j \) for \( j \) odd.

Since \( \sigma_n \in \{10^{2k+1} : k \geq 1\} \), we have that \( D_{\sigma_n} = \emptyset \) and so (25) fails for a set of \( x \) of \( \mu \) measure at most

\[ \frac{d_{\sigma_n}(n)}{a_{\sigma_n}} q_{\sigma_n} \mu(\bigcup_{j=\sigma_n+1}^{\infty} D_j) \leq \frac{d_{\sigma_n}(n)}{\log_1(\sigma_n)}. \]

Furthermore,

\[ \nu(\{ x \in Y : \sum_{i=0}^{d_{\sigma_n}(n)q_{\sigma_n} - 1} 1_{\bigcup_{i=\sigma_n}^{n} D_j}(S^i x) \neq 0 \}) \leq 3d_{\sigma_n}(n)q_n \mu(Z_{\sigma_n+1}) = 3 \frac{1}{8} \frac{d_{\sigma_n}(n)}{a_{\sigma_n}}. \]

( Restricting to \( x \in Y \) and converting from \( \mu \) to \( \nu \) changes this by a factor of less than 3. )

The next two cases are similar, but a bit more complicated as \( D_{\sigma_n} \) is not empty, but is equal to \( W_{1} \) for some \( \ell \). If \( x \in A_1 \), then the conclusion holds if

\[ \sum_{i=0}^{d_{\sigma_n}(n)q_{\sigma_n} - 1} 1_{\bigcup_{i=\sigma_n}^{n} D_j}(S^i T^{n''} x) = 0. \]

Indeed, if \( \sum_{i=0}^{d_{\sigma_n}(n)q_{\sigma_n} - 1} 1_{\bigcup_{i=\sigma_n}^{n} D_j}(S^i y) = 0 \), then this follows from Lemma 3.3 and the fact that \( \sum_{i=0}^{d_{\sigma_n}(n)q_{\sigma_n} - 1} 1_{\bigcup_{i=\sigma_n}^{n} D_j}(S^i 0) = 1. \) So,

\[ T^{n''}(y)_j = \begin{cases} (T^{-1}y)_j & \text{for } j \neq \sigma_n \\ (T^{-1}y)_j + 1 & \text{for } j < \sigma_n. \end{cases} \]

Thus this case follows analogously to (26) above after estimating

\[ \mu(\{ y \in A_1 : S^i y \in A_1 \setminus (\bigcup_{j=\sigma_n+1}^{\infty} D_j) \text{ for all } i \leq d_{\sigma_n}(n)q_{\sigma_n} \}). \]

This is at most \( \frac{d_{\sigma_n}(n)}{a_{\sigma_n}} \).

The third case is similar: if \( x \in A_2 \), then the conclusion holds if

\[ \sum_{i=0}^{\sigma_n-1} 1_{\bigcup_{i=\sigma_n+1}^{n} D_j}(S^i x) = 0 \]

and

\[ \sum_{j=0}^{d_{\sigma_n}(n)q_{\sigma_n} - 1} 1_{D_{\sigma_n}}(S^j x) = d_{\sigma_n}(n), \]

where \( 10^{2\ell} = \sigma_n \). The remainder of the proof is analogous to the first case. \( \square \)
Motivated by the sets in Lemma 5.8, we make a few more definitions. If \((n, A, \rho) \in \mathcal{S}_{r,i}(N, \varepsilon)\), let
\[
P_r(n, A, \rho) = \{x \in A: (T^n x)_j \neq (T^i x)_j \text{ for some } j \leq \sigma_n\}
\]
and
\[
Q_r(n, A, \rho) = \{x \in A: (T^n x)_j = (T^i x)_j \text{ for all } j \leq \sigma_n\}.
\]
Define
\[
\mathcal{P}_r = \bigcup_{(n, A, \rho) \in \mathcal{D}_r(N, \varepsilon)} P_r(n, A, \rho)
\]
and
\[
\mathcal{Q}_r = \bigcup_{(n, A, \rho) \in \mathcal{D}_r(N, \varepsilon)} Q_r(n, A, \rho).
\]

**Lemma 5.9.** Assume \(\sigma_1 \in E\) and let \(A\) be a cylinder with all defining indices at least \(\sigma_1\). Let \(\mathcal{S}_0(N, \varepsilon) = \{(i, [0, 1], 0)\}\). There exist cylinders \(C_1, \ldots, C_\ell\) defined in positions greater than or equal to \(\sigma_1\) such that the following hold:

1. \(A \cap \mathcal{P}_1 \subset \bigcup_{j=1}^{\ell} C_j\).
2. \(99\nu(A \cap \mathcal{P}_1) > \nu(\bigcup_{j=1}^{\ell} C_j)\).

**Proof.** We treat \(i\) with \(\sigma_i \in \{10^{2k+1}\}\). Consider the set of \(y \in Y\) such that (25) fails. We cover this set by cylinders and show that \(\nu(\mathcal{P})\) is proportional to the union of these cylinders. The set \(D_r\) requires that \(x_j = a_{j-2}\) for all \(j < \ell\), and so \(S^{-d_{\sigma_i}(i)\sigma_i}(\bigcup_{\alpha=\sigma_i+1}^{\infty} D_\alpha)\) is contained in at most \(d_{\sigma_i}(\sigma_i) + 1\) cylinders defined by the position \(\sigma_i\). Furthermore,
\[
\mathcal{P} \supset \{y \in Y: \sum_{j=0}^{d_{\sigma_i}(i)\sigma_i-1} 1_{D_{\sigma_i+1}}(S^j y) = 1 \text{ and } \sum_{j=0}^{n_y-1} 1_{D_{\sigma_i+1}}(S^j y) = 0\}
\]
where \(n_y\) is the first coordinate of \((n_y, B, \rho) \in \mathcal{S}_1(N, \varepsilon)\) and \(y \in B\). This set has measure at least \((1 - \frac{1}{10})^{d_{\sigma_i}(i)\sigma_i}\).

The argument for \(i\) with \(\sigma_i \in \{10^{2k}\}\) is similar, but slightly complicated analogously to the proof of Lemma 5.8 because \(D_{\sigma_i} = W_{\frac{1}{2}\log_{10}(\sigma_i)}\). \(\square\)

### 5.4. Obtaining friends.

**Lemma 5.10.** If \(\sigma_m \notin E\), then \(m \in \mathcal{H}_{\sigma_m-1, \frac{1}{10}}\). Furthermore, if \(G: \mathcal{A}_m \to \mathcal{B}_m\) is the measure preserving bijection associated to \(\mathcal{H}_{\sigma_m-1, \frac{1}{10}}\) as defined in (22), then \(\mathcal{A}_m\) and \(G(\mathcal{A}_m)\) can be chosen to be a union of cylinders whose defining indices are a subset of \(\sigma_m - 1, \sigma_m, \text{ and } \sigma_m + 1\).

**Proof.** Assume \(\sigma_m \notin E\) and set \(k = \sigma_m\). Recall that \(d_k = d_k(m)\) is defined in (10). Assume that \(d_k \in \{1, 2, 3, 4\}\) (the case that \(d_k \in \{-1, -2, -3, -4\}\) is analogous). Set \(x_k = 0, x_{k-1} = 5\), and \(x_{k+1} = 4\) for whichever of \(k-1\) and \(k+1\) do not lie in \(E\). Whenever \(k-1\) or \(k+1\) lies in \(E\), we stipulate that \(x_{k-1}\) or \(x_{k+1}\) is in \((\frac{2k-1}{2}, a_{k-1} - 3)\). Set \(y_k = 7\) and \(x_j = y_j\) for all other \(j\).

We claim that \(x\) and \(y\) are \(\zeta_x = \zeta_y(m)\)-friends. We first check that
\[
\sum_{j=0}^{\zeta_x} 1_{Z_t}(S^j y) = \sum_{j=0}^{\zeta_x} 1_{Z_t}(S^j x)
\]
for all $\ell < k$. To see this, note that the inclusion $S^j z \in Z_k$ depends only on $z_1, \ldots, z_j$ and we have that $x_j = y_j$ for all $j < k$. Likewise if $10^{2\ell} < k$, then $
olimits \sum_{j=0}^{\zeta_k} 1_{W_j}(S^jy) = \sum_{j=0}^{\zeta_k} 1_{W_j}(S^jx)$. Also note that because $S^j(x)_{k+1}, S^j(y)_{k+1} \neq a_{k+1} - 2$ for all $j \leq \zeta_k$, we have

$$\sum_{j=0}^{\zeta_k} 1_{Z_j}(S^jy) = \sum_{j=0}^{\zeta_k} 1_{Z_j}(S^jx) = 0$$

for all $\ell > k + 1$ and

$$\sum_{j=0}^{\zeta_k} 1_{W_j}(S^jy) = \sum_{j=0}^{\zeta_k} 1_{W_j}(S^jx) = 0$$

for all $10^{2\ell} > k + 1$. Now, if $k + 1 \notin E$, then since $(S^jx)_{k+1}, (S^jy)_{k+1} \neq a_{k+1} - 1$ we have

$$\sum_{j=0}^{\zeta_k} 1_{Z_{k+1}}(S^jy) = \sum_{j=0}^{\zeta_k} 1_{Z_{k+1}}(S^jx) = 0.$$  

If $k + 1 \in E$, then since $y_{k+1} = x_{k+1} > \frac{a_{k+1} - 1}{2}$, we have

$$\sum_{j=0}^{\zeta_k} 1_{W_j}(S^jy) = \sum_{j=0}^{\zeta_k} 1_{W_j}(S^jx) = 0,$$

where $10^\ell = k + 1$.

Lastly, since $\zeta_k > \frac{5}{2} a_k q_{k-1}$, we have that by the condition on the digits $k$ and $k - 1$ of $y$,

$$\sum_{j=0}^{\zeta_k} 1_{Z_j}(S^jy) = 1.$$

But since $\zeta_k < 5q_k$, using that $x_k = 1$ we have that $(S^jx)_k < 7$ for all $0 \leq j \leq \zeta_k$ and so $\sum_{j=1}^{\zeta_k} 1_{Z_j}(S^jx) = 0$. This proves the claim that $x$ and $y$ are $\zeta_k = \zeta_k(m)$-friends and $G$ is the bijection taking $x$ to $y$. □

**Lemma 5.11.** If $\sigma_m \in E$, then there exist cylinder sets $K_1, \ldots, K_r$ defined on the entries $\sigma_{m+1}, \sigma_m$, and $\sigma_{m-1}$ such that

$$\nu(\bigcup_{j=1}^r K_j) > \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{d_{\sigma_m}(m)}{a_{\sigma_m}} \cdot \frac{1}{64}$$

and there exists a measure preserving map $G$: $\bigcup_{j=1}^r K_j \to Y \setminus \bigcup_{j=1}^r K_j$ defined by changing the $\sigma_{m+1}$ entry such that if $x \in \bigcup_{j=1}^r K_j$, then $x$ and $G(x)$ are $m$-friends.

**Proof.** Let $10^k = \sigma_m$ and assume that $d_{10^k}(m) > 0$ (the case that $d_{10^k}(m) < 0$ is similar). Let $x_{10^k+1} = 0$ and $y_{10^k+1} = 7$. Let

$$y_{10^k} \in \{k-2, k-3, \ldots, k-1 - \min\{d_{10^k}(m), \frac{1}{3} a_k\}\}$$

and set $y_{10^k+1} = 5$. Furthermore, set $x_\ell = y_\ell$ for all $\ell \neq 10^k+1$. It is straightforward that $\sum_{j=0}^{\zeta_k} 1_{Z_{10^k+1}}(S^jx) = 0$ and $\sum_{j=0}^{\zeta_k} 1_{Z_{10^k+2}}(S^jx) = 1$.

We claim that $1_V(S^jx) = 1_V(S^jy)$ for all $|j| \leq \zeta_k(m)$, where $V$ is either $Z_\ell$ for $\ell \neq 10^k + 1$ or $V$ is any $W_\ell$. To see this, for $Z_\ell$ with $\ell < 10^k$ and $W_\ell$ with $\ell \leq k$, this holds since $y$ and $x$ agree in the relevant entries. Furthermore, $(S^jx)_{10^k+1}$ and

...
(S/\mathbf{y})_{10^\ell+1}$ are not 6 in this range and so we never land in $Z_\ell$ for $\ell > 10^k + 1$ or in $W_\ell$ for $\ell > k$, proving the claim. Assuming $K_j$ is a cylinder set as in the statement, define $G(x)$ to change the $10^k + 1$ entry from 0 to 7 (leaving all the other entries unchanged). Thus $G$ satisfies all of the announced properties.

Now

$$
\mu(\{x: x_{10^k} \in \{k - 2, k - 3, \ldots, k - 1 - \min\{d_{10^k}(m), \frac{1}{3}\sigma_k\}\} \),
\begin{align*}
x_{10^k+1} = 0, \quad \text{and} \quad x_{10^k-1} = 5^n
\end{align*}
$$

Considering the set of such $x \in Y$ so that $\mathbf{y} \in Y$ as well with $x_i = y_i$ for all $i \neq \sigma_m + 1 = 10^k + 1$ and $y_{10^k+1} = 7$ and (trivially) converting to $\nu$ establishes \cite{29}. \hfill \square

**Lemma 5.11.** (This argument only depends on the cylinders with defining entries in $\tilde{B}$ and these entries are the same for $\hat{B}$, so we will not prove the claim for $\hat{B}$.)

**Proof.** Let $\hat{B}_1, \ldots, \hat{B}_\ell$ be the cylinders and $\hat{G}$ be the function given by Lemma \ref{5.11} applied with $m = n$. Set $B_j = \hat{B}_j \cap A$ and let $\hat{B}$ be the set of points in $\bigcup_{j=1}^\ell B_j \cap Q_r \cap G^{-1}(Q_r)$. Let $(n', A, \rho') \in \tilde{S}_{r-1}(N, \varepsilon)$ be the predecessor of $(n, A, \rho)$. Since $\sigma_{n'} > \sigma_n + 1$, we have that $x \in \tilde{B}$ and $x, y \in Q_r$ then for all $k$, we claim that

$$
\sum_{j=0}^{\zeta_x(n)} 1_{Z_k(S^jx)} - \sum_{j=0}^{\zeta_x(n)} 1_{Z_k(S^jy)} = \sum_{j=0}^{\zeta_y(n)} 1_{Z_k(S^jx)} - \sum_{j=0}^{\zeta_y(n)} 1_{Z_k(S^jy)}
$$

We first consider the case of $k < n'$. The sums on the left hand side of \ref{31} are either $\lfloor \frac{n}{q_k} \rfloor$ or $\lceil \frac{n}{q_k} \rceil$, while on the right hand side they are either $\lfloor \frac{n'}{q_k} \rfloor$ or $\lceil \frac{n'}{q_k} \rceil$. The choice of $\lfloor \cdot \rfloor$ or $\lceil \cdot \rceil + 1 = \lceil \cdot \rceil$ depends on comparing $x_j$ and $(S^jx)_j$ for the left sums on each side, and similarly $y_j$ and $(S^jy)_j$ for the right sums on each side, for $j \leq k$. By our assumption that $x, y \in Q_r$, we have that $(S^jx)_j$ is the same as $(S^jy)_j$ for all $j \leq k < n'$ and so whether the first sum on the left hand side is the floor function or one more is the same for the first sum on the right hand side. The case of $y$ is identical.

Next consider the case of $k > n + 1$ (since $n < n'$ this covers $k \geq n'$). We have that $\sum_{j=0}^{\zeta_x(n)} 1_{Z_k(S^jx)} = \lceil \frac{\zeta_x(n)}{q_k} \rceil$ and $\sum_{j=0}^{\zeta_y(n)} 1_{Z_k(S^jy)} = \lceil \frac{\zeta_y(n)}{q_k} \rceil$ by the argument in Lemma \ref{5.11} (This argument only depends on the cylinders with defining entries in positions $\sigma_n - 1, \sigma_n$, and $\sigma_n + 1$ that define the cylinders in the proof of Lemma \ref{5.11} and these entries are the same for $\tilde{B}$.) For the right hand side, as above $(S^jx)_j = (S^jy)_j$ for $j \in \{n, n + 1\}$, so whether we take the floor or ceiling in the summands on the right hand side depends on $j > n + 1$. These are the same for $x$ and $y$ by construction, giving \ref{31}. So in the left hand side both summands take the floor
and for the right hand side they either both take the floor or both take the ceiling, establishing (31).

A similar computation yields

\[
(32) \quad \sum_{j=0}^{\zeta_{n}(x)} 1_{W_{i}(S^{j}x)} - \sum_{j=0}^{\zeta_{n}(y)} 1_{W_{i}(S^{j}y)} = \sum_{j=0}^{\zeta_{n}(x)} 1_{W_{i}(S^{j}x)} - \sum_{j=0}^{\zeta_{n}(y)} 1_{W_{i}(S^{j}y)}.
\]

To complete the lemma we are left with establishing (30). To check this, we claim that it suffices to show that \( P_{r} \) can be chosen to be unions of cylinders defined by entries with positions at least \( 10^{k_{0}(\sigma_{n})+1} \). This follows from the following:

**Claim.** For all \( \delta > 0 \), there exists \( k \in \mathbb{N} \) such that if \( C_{1}, C_{2} \) are cylinders with the smallest entry defining \( C_{2} \) at least \( k \) larger than the largest entry defining \( C_{1} \), then

\[
(33) \quad \frac{\nu(C_{1} \cap C_{2})}{\nu(C_{1}) \nu(C_{2})} \in [1 - \delta, 1 + \delta].
\]

**Proof of Claim:** To check that the claim holds, let \( L \) be the smallest entry defining \( C_{2} \). Let \( U_{1}, \ldots, U_{m} \) be the cylinders given by proscribing the first \( L - 1 \) terms that intersect \( Y \). All but one of these cylinders are also cylinders in \( X \), and so they all have the same \( \nu \) measure. If \( U_{i} \) is the one cylinder set in \( Y \) that is not also a cylinder set in \( X \), then \( U_{i} \) has smaller \( \nu \) measure than the other \( m - 1 \) cylinders. Assume \( U_{1}, \ldots, U_{m'} \) are those cylinders that are contained in \( C_{1} \). If \( i \in \{1, \ldots, m'\} \), then \( \nu(C_{1} \cap C_{2}) < \nu(C_{1}) \nu(C_{2}) \), but it is at least \( \frac{m'-1}{m} \nu(C_{1}) \nu(C_{2}) \). Similarly, if \( i \not\in \{1, \ldots, m'\} \), then \( \nu(C_{1} \cap C_{2}) < \nu(C_{1}) \nu(C_{2}) \), but it is at most \( \frac{m}{m'-1} \nu(C_{1}) \nu(C_{2}) \). Since \( \hat{B} \) is a union of the sets \( B_{i} \) that pairwise satisfy this condition, the claim follows.

The sufficient condition, that \( P_{r} \) can be chosen to be unions of cylinders defined by entries with positions at least \( 10^{k_{0}(\sigma_{n})+1} \), then follows by iterating Lemma 5.9 and our assumption that \( \varepsilon < \frac{1}{8^{99}} \) and so \( \nu(P_{r}) > \frac{1}{2} \); indeed, by Lemmas 5.8 and 5.9 at least \( \nu(A \cap P_{r}) > \nu(A)(1 - 99 \cdot 4 \cdot \frac{1}{8^{99}}) \).

**Lemma 5.13.** Assume \( r \geq 1, \varepsilon < \frac{1}{8^{99}}, \mathcal{B}_{0}(N, \varepsilon) = \{(i, [0,1], 0)\}, (n, A, \rho) \in \mathcal{B}_{r}(N, \varepsilon), and \sigma_{n} \notin E. \) Let \( (n', A', \rho') \in \mathcal{B}_{r-1}(N, \varepsilon) \) so that \( A \subset A' \). Then there exist cylinders \( B_{1}, \ldots, B_{t} \subset A' \) defined by positions whose entries are at least \( \sigma_{n} - 2 \), such that there exists \( \hat{B} = \bigcup_{j=1}^{t} B_{j} \) with \( \nu(\hat{B}) > \frac{1}{8^{99}} \nu(A) \) and a map \( \hat{G}: \hat{B} \to Y \) such that \( x \) and \( G(x) \) are \( \zeta_{n}(i) \)-friends for all \( x \in \bigcup_{j=1}^{t} B_{j} \). Moreover, \( x_{j} = G(x)_{j} \) for all \( j < \sigma_{n} \).

**Proof.** We first prove the statement under the assumption that \( \sigma_{n} \neq \sigma_{n}' - 1 \). In this case, let \( B_{1}, \ldots, B_{t} \) be the cylinders and let \( \hat{G}: A \to \hat{B} \) be the map given by Lemma 5.10 for \( m = n \). Let \( B_{j} = B_{j} \cap A \) and \( \hat{B} = \bigcup_{j=1}^{t} B_{j} \cap Q_{r} \cap \hat{G}^{-1}Q_{r} \).

Repeating the proof used to derive (33), we obtain cylinders defined in entries at most \( \sigma_{n-2} \), with the entry before the last place defining the cylinders in \( B_{j} \) (and also smaller than the cylinders defining \( Q_{r} \) and \( A \)). One of these cylinders differs from the cylinder with the same defining entries in \( X \). On all the other cylinders, \( B_{j} \) intersects \( A \cap Q_{r} \) as expected and the lemma follows using an argument analogous to the proof of (33).

Now we treat the remaining case, \( \sigma_{n} = \sigma_{n}' - 1 \) and the largest entry of the cylinders defining \( B_{j} \) overlaps with the smallest entry of the cylinders defining \( A \). In this case, we consider \( A' \) whose defining entries are all larger than \( \sigma_{n} \) (they are at least \( 10^{k+1} \) where the smallest entry defining \( A \) is \( 10^{k} \)). Then \( (n', A', \rho') \) has two
descendants in $H_r(N, \varepsilon)$, $(n, A, \rho)$ and $(\hat{n}, \hat{A}, \hat{\rho})$. One of these is $A \cap \{x: x_{10^k} \geq \frac{k}{2}\}$ and so by the definition of $B_j$ in Lemma 5.10 has nonempty intersection with the cylinders $\hat{B}_1, \ldots, \hat{B}_\ell$. The proof then follows as above, via the same arguments used to conclude the proof of Lemma 5.12. □

For approximating, we make use of a metric giving rise to the strong operator topology on $L^2(\mu)$. While any such metric suffices for our purposes, it is convenient to choose one that simplifies the computations:

**Notation.** Let $B = B(X)$ denote the Borel $\sigma$-algebra of $L^2(\nu)$ and let $\{f_i\}_{i=1}^\infty$ be an orthonormal basis for $L^2(\nu)$ such that $\|f_i\|_\infty = 1$ for all $i$. Set $D: B \times B \to [0, \infty)$ to be the metric defined by

$$D(U, V) = \sum_{i=1}^{\infty} 2^{-i} \|Uf_i - Vf_i\|_2.$$  

(34)

Note that restricting $D$ to the set (the choice of 10 is any arbitrary positive real)

$$\{(U, V) \in B \times B: \|U\|_{op} + \|V\|_{op} \leq 10\}$$

endows this set with the strong operator topology.

**Proposition 5.14.** Assume that $(Y, T, \nu)$ has a nontrivial factor with associated factor map $P: Y \to Z$ and let $F: L^2(\nu) \to L^2(\nu)$ be a Markov operator defined by integrating the fibers of the factor map $P$ and

$$\lim_{k \to \infty} D(\sum_i \alpha_i^{(k)}F, F) = 0$$

where $\alpha_i^{(k)} \geq 0$ for all $i, k$ and $\sum_i \alpha_i^{(k)} = 1$ for all $k$. Then for all $\varepsilon > 0$, there exists $N_0$ such that for all $N > N_0$ and all large enough $k$,

$$\sum_i \alpha_i^{(k)} \sum_{\{n, \beta, \rho \in F_i(N, \varepsilon): \sigma_n > N\}} \beta < \varepsilon.$$  

(35)

We record an immediate corollary for later use:

**Corollary 5.15.** Assume that $(Y, T, \nu)$ has a nontrivial factor with associated factor map $P: Y \to Z$ and let $F: L^2(\nu) \to L^2(\nu)$ be a Markov operator defined by integrating the fibers of the factor map $P$, then for all $\varepsilon > 0$ there exists $N_0$ and $\delta > 0$ so that if $a_i \geq 0$, $\sum a_i = 1$ and

$$D(\sum_i a_i U_{T^i}, F) < \delta$$

then for all $N \geq N_0$

$$\sum_i a_i \sum_{\{n, \beta, \rho \in F_i(N, \varepsilon): \sigma_n > N\}} \beta < \varepsilon.$$  

We proceed by contradiction, and show that if

$$\sum_i \alpha_i^{(k)} \sum_{\{n, \beta, \rho \in F_i(N, \delta): \sigma_n > N\}} \beta > 10c,$$

then

(36)

$$\sum_{i \in \mathcal{H}_r(N, \varepsilon)} \alpha_i^{(k)} > c.$$  

Then by taking $N$ sufficiently large, we obtain a contradiction via Lemma 5.5.
Let $J \subset \mathbb{N}$ be the set of indices $i$ such that
\[
\sum_{\{(n,\beta,\rho) \in F_i(N,\delta) : \sigma_n > N\}} \beta > c.
\]
Since $\sum_{i \in J} \alpha_i^{(k)} \sum_{\{(n,\beta,\rho) \in F_i(N,\delta) : \sigma_n > N\}} \beta \leq c$, it follows that $\sum_{i \in J} \alpha_i^{(k)} > c$.

**Claim 5.16.** For any $i \in J$, we have $i \in \mathcal{H}_{N,\{\frac{c}{\mu}\}}$.

To check this, the triples defined in Definition [5.6] give the two possible reasons for $(n, A, \rho) \in \mathcal{F}(N,\delta)$ with $\sigma_n > N$. The first is that $\sigma_n \notin E$, in which case Lemma [5.13] gives a set of points, contained in $A$, which have $i$-friends of measure at least $\frac{c}{\mu} \nu(A)$ and a map $G$ defined on these symbols, identifying the point with its friend so that $G(x)_j = x_j$ for all $j < \sigma_n$. The second is that $\rho > \delta$, in which case Lemma [5.12] similarly gives cylinders with measure at least $\frac{1}{64} \delta \cdot (1 - 0.999) \nu(A)$.

So if $i$ is such that $\sum_{\{(n,\beta,\rho) \in F_i(N,\delta) : \sigma_n > N\}} \beta > c$, then since the $\beta$ are the measure of the sets $A$ mentioned in the previous 2 sentences, we have $i \in \mathcal{H}_{N,\{\frac{c}{\mu}\}}$. □

### 5.5. Restricting factors. We use the next two lemmas to prove the main result of this section characterizing conditions that rule out nontrivial factors.

**Lemma 5.17.** Assume that $(Y, T, \nu)$ has a nontrivial factor with associated factor map $P : Y \to Z$ and assume that $F : L^2(\nu) \to L^2(\nu)$ is the Markov operator defined by $P$. Then for all $c, \varepsilon > 0$, there exists $N_0$ such that for all $N > N_0$ we have that
\[
D(U_T^N, \sum_{(n,A,\beta) \in F_1(N,\varepsilon)} 1_A \cdot U_T^N) < 10\sqrt{\varepsilon} + c
\]
for all $i$.

**Proof.** We claim that for all $c > 0$, there exists $N$ such that if $S_1, S_2$ are measure preserving transformations such that $\nu(\{(x : (S_1x)_j = (S_2x)_j) \text{ for all } j < N\}) > 1 - \varepsilon$ then $D(U_{S_1}, U_{S_2}) < c + 2\sqrt{\varepsilon}$.

To prove the claim, given $N \in \mathbb{N}$, set
\[
A(N) = \{x \in X : (S_1x)_j = (S_2x)_j \text{ for } j < N\}.
\]
Then by Lusin’s Theorem and uniform integrability, for any $f \in L^2(\mu)$, there exists $N \in \mathbb{N}$ such that if $A = A(N)$, then $\|f \circ S_1 - f \circ S_2\|_1 < \varepsilon$. Let $\{f_i\}_{i=1}^\infty$ be an orthonormal basis of $L^2(\mu)$ with $\|f_i\|_\infty = 1$ for all $i \in \mathbb{N}$. Given $\varepsilon > 0$, choose $k$ such that $2^{-k} < \varepsilon$ and pick $N$ sufficiently large such that the associated set $A = A(N)$ ensures that $\|f_i \circ S_1 - f_i \circ S_2\|_1 < c$ for $i = 1, \ldots, k$. Then $\nu(A) > (1 - \varepsilon)$ and so (the definition of the metric $D$ is given in (54))
\[
D(U_{S_1}, U_{S_2}) \leq c + \sum_{i=1}^\infty 2^{-i} \left( \int_{A_i} \|f_i \circ S_1 - f_i \circ S_2\|_1^2 \, d\nu \right)^{1/2} \leq c + 2\sqrt{\varepsilon},
\]
proving the claim.

Take $N_0 = N$ where $N$ is sufficiently large such that this claim holds. Then for any $N > N_0$, if $D(U_T^N, \sum_{(n,A,\beta) \in \mathcal{F}(N,\varepsilon)} 1_A \cdot U_T^N) > c + 10\sqrt{\varepsilon}$, the claim implies that
\[
\nu(\{(x : (T^ix)_j \neq (T^n x)_j) \text{ for } j > N \text{ and } x \in A \text{ where } (n, A, \rho) \in \mathcal{F}(N,\varepsilon)\}) > 5\varepsilon.
\]
So by Lemma [5.8]
\[
\sum_{\{(n,\beta,\rho) \in \mathcal{F}(N,\varepsilon)\}} \rho \beta > \varepsilon.
\]
Thus if $N$ is large enough (depending on $\varepsilon$), then by Proposition 5.14 we have that $F$ is given by integrating out the fiber to the one point system. □

**Lemma 5.18.** Let $F$ be the Markov operator on $L^2(\nu)$ defined by a factor map that is not an isomorphism, then

$$\limsup_{\delta \to 0^+} \sum_{i \in R_\delta} \alpha_i^{(k)} \leq \frac{1}{2},$$

where $R_\delta = \{i : D(S^i, \text{Id}) < \delta\}$.

**Proof.** If $F \neq \text{Id}$, then almost every $x \in X$ lies on a fiber with at least one other point. The lemma follows. □

Combining Lemmas 5.17 and 5.18, we have shown:

**Proposition 5.19.** If $F$ is the Markov operator defined a factor map that is neither an isomorphism nor a map to the 1 point system, then for all $\delta, \varepsilon > 0$ there exists $N_0$ such that for any $N > N_0$ and all sufficiently enough $k$, we have that

$$\sum_{i \in \mathcal{F}(N/2, \varepsilon^2)} \beta > \frac{1}{2} - 10\sqrt{\varepsilon} - \delta.$$

6. The behavior of a projection

6.1. Overview of the proof that $(Y, \nu, T)$ is prime. In this section, we show that our constructed system is prime:

**Theorem 6.1.** The system $(Y, \nu, T)$ is prime.

We start with an overview of the proof and then proceed to study different cases. We assume that $(Y, \nu, S)$ has a nontrivial factor $Z$ with factor map $P : Y \to Z$ and assume that $F : L^2(\nu) \to L^2(\nu)$ is a Markov operator defined by integrating the fibers of the factor map $P$. We further assume that $F$ is the limit, as $k \to \infty$, of $\sum \alpha_i^{(k)} U_{T^i}$ in the strong operator topology. Given $\varepsilon > 0$, by Proposition 5.14 we can assume that there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$ and sufficiently large $k$, we have that reducing to $F_{i}^{(N/2, \varepsilon^2)}$ gives a good approximation to $\sum \alpha_i^{(k)} U_{T^i}$ (which in turn leads to a good approximation for $F$). The general idea in the proof of Theorem 6.1 is that we rule out the possibility that $\sum \alpha_i^{(k)} U_{T^i}$ is close to a nontrivial projection. The key facts used are that the composition of projections is still a projection and by properties of the strong operator topology, we may assume that for any fixed $M$, for all large enough $k$

$$\left(\sum \alpha_i^{(k)} U_{T^i}\right) \circ \left(\sum \alpha_i^{(k)} U_{T^i}\right) \cdots \circ \left(\sum \alpha_i^{(k)} U_{T^i}\right)$$

is close to $F^M = F$, and this is also close to $\sum \alpha_i U_{T^i}$. We then use the fact that (38) is

$$\sum_{(i_1, \ldots, i_M)} \left(\prod_{m=1}^M \alpha_i^{(k)} U_{T^i_{\Sigma m}}\right),$$

and apply Definition 5.6 to (39). Treating 3 different cases, this allows us to produce friends and obtain a contradiction via Lemma 5.5. We now make this precise.
6.2. Set up for the proof of Theorem 6.1

We begin a proof by contradiction, assuming that there is a Markov operator $F$ coming from a non-trivial factor map. By Theorem 5.1, there exists $\alpha^{(k)}_i \geq 0$ with $\sum_i \alpha^{(k)}_i = 1$ for all $i$ such that $\sum_i \alpha^{(k)}_i U_T$, converges in the strong operator topology to $F$.

We assume that $\varepsilon > 0$ is sufficiently small such that all of the Lemmas and Propositions in Section 5 hold. That is, $\varepsilon < \min\{\hat{\varepsilon}, \frac{1}{8}, \frac{1}{9999}\}$. We also assume that

\begin{equation}
(40) \quad \varepsilon < \frac{1}{10^6 \cdot 9999},
\end{equation}

which is to be used in Lemma 6.6. Furthermore, we choose $N_1 > 6$ (this choice is made to simplify the analysis in the third case we consider) to be sufficiently large such that Lemma 5.5 holds for $\frac{N_1}{2}$ and $\varepsilon^8$ and such that

\begin{equation}
(41) \quad 2^{-\frac{N_1}{2}} < \varepsilon^4.
\end{equation}

Setting

$$G_N = \{n: \sum_{(i,c,\gamma) \in F_n(\frac{1}{2} \cdot r^2): \sigma_i > \frac{1}{2})} c < \varepsilon^4\},$$

Claim 5.16, Lemma 5.5, and our choices imply that for all sufficiently large $k$,

\begin{equation}
(42) \quad \sum_{n \in G_N} \alpha_n^{(k)} > \frac{3}{4}.
\end{equation}

Set $s = \min\{10^j : 10^{j-1} \geq N\}$, set $s' = \min\{10^j : 10^j \geq N\}$, and recall that $r_i$ is defined in (9). Define

\begin{equation}
(43) \quad M = M_N = \frac{r_{s-2}}{r_N}.
\end{equation}

Although $M = M_N$ depends on $N$, as $N$ is fixed at this point, we omit it from the notation. The motivation behind this definition of $M$ is given by the following lemma (this plays a role in the proof of Proposition 6.3):

**Lemma 6.2.** For all sufficiently large $N$, we have that $M_N^{\frac{1}{2}} > r_{s'+1}$, where $M$ is defined as in (43). Moreover, for any $\varepsilon > 0$, for all sufficiently large $N$ we have $8^{(1-\varepsilon)10^k} < r_{10^k-1} < 8^{(1+\varepsilon)10^k}$.

**Proof.** We first claim that for all $\varepsilon > 0$, there exists $k$ such that

\begin{equation}
(44) \quad 8^{(1-\varepsilon)10^k} < r_{10^k-1} < 8^{(1+\varepsilon)10^k}.
\end{equation}

For all $\ell \notin E$, we have that $r_{\ell+1} = a_{\ell+1} r_{\ell} - 1$. Thus there exists $\ell_0$ with $r_{\ell+1} \geq 8^{1-\frac{\varepsilon}{2}} r_{\ell}$ for all $\ell \geq \ell_0$ and the lower bound follows. For the upper bound, $r_{\ell+1} < 8 \ell_\varepsilon$ for all $\ell \notin E$ and so there exists $\ell_0$ such that $\ell < (8\varepsilon^{9}10^{\ell-1})$ for all $\ell \geq \ell_0$. For any such $\ell$, we have $r_{10^{\ell-1}+j} < (8^{1+\varepsilon})^{j} r_{10^{\ell-1}}$ and the upper bound follows, completing the proof of (44).

For all large enough $N$, by (44) we have

$$M = M_N = \frac{r_{s-1}}{r_N} > \frac{1}{5} \frac{r_s}{r_{s'}} > \frac{8^{(1-\varepsilon)10^k}}{8^{(1+\varepsilon)10^k-1}} > 8 \cdot 8^{5(1+\varepsilon)10^k-1} > r_{s'+1}^8,$$

where the second to last inequality holds for all sufficiently large $N$. \[\square\]
6.3. The three cases: This leads us to consider three possibilities for the behavior of the projection $F$ on $L^2(\mu)$ (recall that $D$ is the metric defined in (34)):

Case 1:

(45) \[ D\left( \sum_{(i_1, \ldots, i_M)} \prod_{m=1}^{M} \alpha_{i_m}^{(k)} U_{T^{i_m}}, \sum_{\{i: \sigma_i \not\in \{10^j\}\}} \sum_{\{i: \sigma_i \in \{10^j\}\}} \sum_{\{j, A, \rho\} \in \mathcal{A}(N, \varepsilon)} \alpha_{i}^{(k)} U_{T^{j}} 1_{A} \right)^M < \frac{\varepsilon}{M}. \]

Case 2: Case 1 does not hold and

(46) \[ \sum_{n} \alpha_{n}^{(k)} \sum_{\{j, \beta, \rho\} \in \mathcal{F}_n(N, \varepsilon^2): \sigma_j > N} \beta < \frac{\varepsilon^2}{M}. \]

Case 3: Case 1 does not hold and

(47) \[ \sum_{n} \alpha_{n}^{(k)} \sum_{\{j, \beta, \rho\} \in \mathcal{F}_n(N, \varepsilon^2): \sigma_j > N} \beta \geq \frac{\varepsilon^2}{M}. \]

We analyze each of these cases separately.

6.4. Case 1: Fix $\varepsilon > 0$ and assume that (45) holds. Roughly speaking, we assume that when we iterate the approximation of our transformation given by Definition 5.6 up to $M$ times, we remain close to the original map.

Proposition 6.3. There exists $N_0$ such that if $N > N_0$ and $\gamma$ is the probability measure supported on $\{-2r_N, -2r_N + 1, \ldots, -1, 0, 1, \ldots, 2r_N\}$ with $\gamma(\{0\}) < \frac{5}{9}$, and $M$ corresponds to $N$ as in (43), then

\[ \gamma^M \left( \{(i_1, \ldots, i_M): r_{s-2} > \sum_{j=1}^{M} i_j > r_{s+1}\} \right) > \frac{1}{9}. \]

Proof. Let $F_1, \ldots, F_M: (\Omega, \nu) \to \mathbb{Z}$ be a sequence of independent $\gamma$ distributed random variables, let $Z = \sum_{i=1}^{M} F_i - M\mathbb{E}_\nu(F_1)$, and let $\sigma$ be the variance of $F_1$. Then the variance of $Z$ is $M\sigma$.

By choice of $M = M_N$, we have that $|\sum_{j=1}^{M} i_j| < r_{s-2}$ for all $i_j$ in the support of $\gamma$ and the remainder of the proof is devoted to showing the lower bound.

By Lemma 6.2 for sufficiently large $N$, we have that $M^\frac{1}{2}$ is bounded from below. We start by computing $\mathbb{E}(F_1)$, and we consider two cases.

First assume that $\frac{1}{99} < \mathbb{E}(F_1)$. We compute $\mathbb{E}(Z^2)$ and apply Chebyshev’s inequality. By independence of the $F_i$, we have that $\mathbb{E}(Z^2)$ is $M\mathbb{E}((F_1^2 - 2F_1\mathbb{E}(F_1) + \mathbb{E}(F_1)^2))$. Applying Hölder’s Inequality, we bound $\|F_1^2\|$ by $\|F_1\|_\infty \cdot \|F_1\|$. By Lemma 6.2 we have that $\|F_1\| \leq 2r_N < M^{\frac{1}{2}}$ and similarly, $\|F_1\|_\infty < M^{\frac{1}{2}}$. So $\mathbb{E}(Z^2) \leq MM^{\frac{1}{2}}$ and by Chebyshev’s Inequality, we have

\[ \nu(\{\omega: \|Z(\omega)\| < 4M^{\frac{1}{2}}\}) > \frac{1}{2}. \]

Since $\sum F_i = M\mathbb{E}(F_1) + Z$ and $|M\mathbb{E}(F_1)| > \frac{M}{99}$, we have that if $|Z(\omega)| < 4M^{\frac{1}{2}}$, then

\[ \left| \sum_{i=1}^{M} F_i(\omega) \right| > \frac{M}{99} - 4M^{\frac{1}{2}} > M^{\frac{1}{2}}. \]
Now we consider the case that \(|E(F_1)| \leq \frac{1}{99}\). Under this assumption, then since \(\gamma(\{0\}) < \frac{5}{9}\) and is supported on integers, we have that

\[
|E(F_1 - E(F_1)^2)| \leq \frac{4}{9}(1 - \frac{1}{99})^2,
\]

which implies that the variance \(\sigma\) is at least \(\frac{1}{9}\).

Let \(\varepsilon > 0\). There exists \(N_0\) such that for all \(N > N_0\), there exists \(M\) with \(\sqrt{M} > r_{s'+1}\), \(2r_N < r_{s-6}\) and

\[
\frac{1}{M^{\frac{1}{2}}/t} \int |t|^k d\nu(t) < \frac{\varepsilon}{2^k}
\]

for all \(k \geq 3\). It follows that \(\phi_t = \mathbb{E}e^{\frac{M^t}{\sqrt{M}\sigma}}\) is bounded by \(-\frac{t^2}{2} + c(t)\), where \(|c(t)| < \varepsilon\).

We use Lévy’s Continuity Theorem to complete the proof. Namely, we repeat this process for a sequence of \(N_j\) tending to infinity and obtain \(Z_j, M_j, \text{ and } \sigma_j\) such that \(\phi_t^{(j)} = \mathbb{E}e^{\frac{M_j^t}{\sqrt{M_j}\sigma_j}} \to e^{-\frac{t^2}{2}}\) pointwise. Since \(e^{-t^2/2}\) is continuous at 0, it follows that \(\frac{Z_j}{\sqrt{M_j}\sigma_j}\) converges in distribution. Thus for sufficiently large \(j\), it follows that the probability that \(|Z_j| > \frac{\sqrt{M_j}}{16}\) is at least 1/9. If the expectation of \(F_1\) is nonnegative, this implies that

\[
\gamma^M \left( \{(i_1, \ldots, i_M) : \sum_{j=1}^{M} i_j > r_{s'+1}\} \right) > \frac{1}{9},
\]

Otherwise,

\[
\gamma^M \left( \{(i_1, \ldots, i_M) : \sum_{j=1}^{M} i_j < -r_{s'+1}\} \right) > \frac{1}{9},
\]

and the result follows.

\[\square\]

**Notation:** Let \(\tilde{\varepsilon} < \min\{\varepsilon, \frac{1}{10}\}\) as in Section 6.2. Let \(N_2\) be chosen according to Corollary 5.15 for \(\varepsilon\). Let \(\tilde{N} \geq \max\{N_0, N_1, N_2\}\) and \(M = M_{\tilde{N}}\). Let \(k\) be chosen large enough such that

- \(D((\sum \alpha^{(k)}_j U_j)^M, F) < \delta\) where \(\delta\) is as in Corollary 5.15 for \(\varepsilon = \tilde{\varepsilon}\) and \(N_0 = \tilde{N}\).
- \((41)\) with \(G_{\tilde{N}}\) and \((42)\) hold.

**Concluding the proof of Theorem 6.1 in case 1:** Let

\[A = \{(i_1, \ldots, i_M) : \sigma \sum i_j > \tilde{N} \text{ and } \sigma_i \notin E\}.
\]

By Proposition 6.3, \(\sum A \prod_{j=1}^{M} \alpha^{(k)}_{i_j} > \frac{1}{2}\). By \((89)\) and our choice of \(k\), we obtain a contradiction of Corollary 5.15. Thus this case can not occur.

**6.5. Case 2:** In the absence of the first case, we are left with showing that for at least \(\tilde{\varepsilon}^2\) of the sums \(\sum_{j=1}^{M} i_j\), for at least \(\tilde{\varepsilon}^2\) points we have \(\sum_{j=1}^{M} i_j\)-friends.

Roughly speaking, the idea is that \(\prod_{j=1}^{M} U_j^k\) under iteration does not stay close to \(\prod_{j=1}^{M} U_j^k_{\tilde{N}, \tilde{\varepsilon}}\sum_{(\ell, A, \rho) \in \tilde{A}_{i_j}} 1_A U_j^k\), and so the sum

\[
\sum_{(i_1, \ldots, i_M) \in \mathbb{Z}^M} \left( \prod_{j=1}^{M} \alpha^{(k)}_{i_j} \right) \sum_{(j, \beta, \rho) \in F_{\tilde{N}, \tilde{\varepsilon}}}(\tilde{N}, \tilde{\varepsilon); \sigma_j > \tilde{N})
\]
becomes significant. To make this precise, we deal with two cases separately, depending on the sizes of the sums in (46) and (47).

Thus as we are in Case 2, we assume that

$$\sum_n \alpha_n(k) \sum_{\{(j, \beta, \rho) \in F_n(N, \varepsilon^2) : \sigma_j > \tilde{N}\}} \beta < \frac{\varepsilon^2}{M}.$$  

Given $n \in \mathbb{N}$, set $\mathcal{S}_0 = \{(n, [0, 1], \rho)\}$ and define the reduction $\text{red}_k(n)[x] = (m, A, \rho)$, where

- $(m, A, \rho) \in \mathcal{S}_r(\tilde{N}, \tilde{\varepsilon})$ for some (smallest) $r$,
- $x \in A$, and
- $\sigma_m \leq 10^k$ or $(m, A, \rho) \in \mathcal{S}(N, \varepsilon)$ (that is, $m \notin E$ or $\rho > \tilde{\varepsilon}$).

We say that the sum $\sum_j i_j$ is treatable if

$$\sum_{\{(k, \beta, \rho) \in F_j(N, \varepsilon^2) : \sigma_k > N\}} \beta < \tilde{\varepsilon}$$

for all choices of $i_j$ and the sum $\sum_j i_j$ is $x$-treatable if for all $i_j$, the elements $(n, A, \rho) \in \mathcal{S}_j(N, \varepsilon)$ satisfy $x \in A$ has $\sigma_n \leq N$.

**Lemma 6.4.** Any $x$-treatable sum such that $\text{red}_k(\sum_{j=1}^M i_j)[x] \neq \sum_{j=1}^M \text{red}_k(i_j)[x]$ has friends, and the measure of the set of friends is at least proportional to $\nu(A)$, where $(n, A, \rho) = \text{red}_k(\sum_{j=1}^M i_j)[x]$.

**Proof.** Choose $k$ maximal such that $\text{red}_k(\sum_{j=1}^M i_j)[x] \neq \sum_{j=1}^M \text{red}_k(i_j)[x]$. Set

$$m = \sum_{j=1}^M \text{red}_{k+1}(i_j)[x] = \text{red}_{k+1}(\sum_{j=1}^M i_j)[x].$$

We show that $\text{red}_{k+1}(m)[x]$ or $\text{red}_k(m)[x]$ is a stopping point for the algorithm. Once this is shown, by Lemmas 5.12 and 5.13 we obtain a set of friends proportional to $\nu(A)$.

First, because $i_j$ is $x$-treatable, we have

$$\text{red}_{k+1}(i_j)[x] - \text{red}_k(i_j)[x] \in \{d_{10^k+1}(\text{red}_{k+1}(i_j)[x])r_{10^k+1},$$

$$d_{10^k+1}(\text{red}_{k+1}(i_j)[x])r_{10^k+1} + d_{10^k+1}(\text{red}_{k+1}(i_j)[x])\}$$

for all $j$. Let $n = \sum_j (\text{red}_{k+1}(i_j)[x] - \text{red}_k(i_j)[x])$ and by our assumption on $k$ and $k+1$, $n \neq \text{red}_{k+1}(m)[x] - \text{red}_k(m)[x]$. Now if $\sigma_{\text{red}_{k+1}(m)[x]} > 10^{k+1}$, then because $|\text{red}_{k+1}(m)[x]| \leq Mr_{10^k+1} < r_{10^{k+2}+1}$ the reduction algorithm halts and the lemma follows. If $\sigma_{\text{red}_{k+1}(m)[x]} = 10^{k+1}$, then by choice of $M$ and $N$ we have

$$|\sum_j \text{red}_k(i_j)[x]| < r_{10^{k+1}+2}.$$  

Also, because $\text{red}_{k+1}(m)[x] - \text{red}_k(m)[x], \text{red}_{k+1}(i_j)[x] - \text{red}_k(i_j)[x]$ are all multiples of either $r_{10^k+1}$ or $r_{10^k+1} + 1$ (depending on $x_{10^k+1}$), we have that $\text{red}_{k+1}(m)[x] - \text{red}_k(m)[x] = n + p$ where $|p| \geq r_{10^k+1}$. By the algorithm for representing numbers in terms of $d_i$, we have that if $|p| \geq 7\sigma_{10^k+1}$ then $|p| > 5|\ell|$ and so

$$\sigma_{\ell+p} \geq \sigma_{p} - 1.$$  

Therefore, the reduction is complete.
Thus by Equation (49),
\[ \sigma_{red_k(m)|x} = \sum_{j} \text{red}_k(i_j)|x| \geq 10^{k+1} - 1 \]
and the algorithm halts in this case as well. In the final case, \( \sigma_{red_k+1(m)|x} \leq 10^k \) in which case \( \text{red}_k(m)|x| = \text{red}_k+1(m)|x| \) and this is \( \sum \text{red}_k(i_j)|x| \) a contradiction, or \( 10^k < \sigma_{red_k+1(m)|x} \) and so the algorithm stops. □

Concluding the proof of Theorem 6.1 in case 2. Let
\[ I(x, M) = \{ (i_1, ..., i_M) : \sum_i i_j \text{ is } x\text{-treatable and } \sum_{j=1}^M \text{red}(i_j)|x| \neq \text{red}(\sum_{j=1}^M i_j)|x| \}. \]
We assume that we are not in the first case and moreover (46) holds. Thus for a set of \( x \) of measure at least \( \tilde{\varepsilon} \), we have
\[ \sum \{ (i_1, ..., i_M) : (i_1, ..., i_M) \in I(x, M) \} \prod_{j=1}^M \alpha_{i_j}^{(\tilde{k})} \geq \tilde{\varepsilon}. \]
For each such sum, we apply Lemma 6.4 and produce a set of friends at least proportional to \( \tilde{\varepsilon}^2 \). By Lemma 5.5, Corollary 5.15, and our choice of \( \tilde{k} \), this establishes case 2. □

6.6. Case 3 (we assume neither of the conditions in Case 1 or in Case 2 holds). We say that \( n \) is **good for reduction** if
\[ \sum_{(i, c, \gamma) \in F_n(\tilde{N}_2, \tilde{\varepsilon}^4)} c < \frac{\tilde{\varepsilon}}{2} \]
and we say \( n \) is **bad for reduction** if
\[ \sum_{(i, c, \gamma) \in F_n(\tilde{N}, \tilde{\varepsilon}^2)} c > \tilde{\varepsilon}. \]

By our assumption on \( \tilde{k} \) and the estimate for sufficiently large \( \tilde{k} \) given in (42), we have:

**Lemma 6.5.** Let \( G = \{ \vec{i} \in \mathbb{Z}^M : i_j \text{ is good for reduction for at least } \frac{M}{2} \text{ choices of } j \} \), then
\[ \sum_{\vec{i} \in G} \prod_{j=1}^M \alpha_{i_j}^{(\tilde{k})} \geq \frac{1}{2}. \]

**Lemma 6.6.** If \( j \) is bad for reduction and \( k \) and \( m \) are good for reduction, then
\[ j - k + m \in \mathcal{H}_{\tilde{N}, \tilde{\varepsilon}^4}. \]
Similarly, if \( j \) and \( m \) are good for reduction and \( k \) is bad for reduction, then
\[ j - k + m \in \mathcal{H}_{\tilde{N}, \tilde{\varepsilon}^4}. \]

Note that we separate the roles of the terms \( k \) and \( m \) to make it easier to apply the lemma (see Corollary 6.7).

**Proof.** We establish the first claim, as the second is similar. First, if \( j \) is bad for reduction, then by Claim 5.16 we have that \( j \in \mathcal{H}_{\tilde{N}, \tilde{\varepsilon}^4} \). Taking \( A_j \) and \( G_j \) as in Notation 5.4 we have that when \( x \in A_j \) there exists \( 0 < |\ell| \leq 3 \) such that \((T^\ell T^j x)_i = (T^j (G_j x))_i \) for all \( i \leq \tilde{N} \). Recall that the sets \( \mathcal{D}_a \) are defined in (3).
Now if $(T^{n+j}x)_i \neq (T^{n+j}(G_jx))_i$ for some $i \leq \tilde{N}$, then there exists $a > \tilde{N}$ such that either $S^bT^{n+j}x \in D_a$ for $|b| \leq 3|n| \leq |\zeta_{T^{n+j}x}|(b)$ or $S^bT^jG_jx \in D_a$ for $|b| \leq 3|n| \leq |\zeta_{T^jG_jx}|(b)$. This uses Corollary 5.3. If $\sigma_n \leq \frac{\tilde{N}}{2}$, then the measure of such points is at most

$$4 \cdot 3n \sum_{\sigma_n > \tilde{N}} \mu(D_n) < 2^{-\frac{n}{2}}.$$

Let $d \in \mathbb{Z}$, $(n, A, \rho) \in \mathfrak{F}_d\left(\frac{\tilde{N}}{2}, \tilde{\varepsilon}^4\right)$, and $x \in A_j \cap A$ (which implies that $G_j(x) \in A$). If $(T^{n+j}x)_i = (T^j(G_jx))_i$, then there exists $\tilde{N}$ for some $r$. Since $k$ and $m$ are good for reduction, by iterating Lemma 5.9 when $d = k$ or $m$, we have that the measure of the set of such points is at most $40\varepsilon^4$. Combining these two estimates and considering $(n, A, \rho) \in \mathfrak{F}_d\left(\frac{\tilde{N}}{2}, \tilde{\varepsilon}^4\right)$ with $\sigma_n > \frac{\tilde{N}}{2}$, we obtain that

$$j-k+m \in \mathcal{H}_{\tilde{N}}, \frac{\tilde{\varepsilon}^4}{\rho_n} - 2(\tilde{\varepsilon}^4 + 40\varepsilon^4 + 2^{-\frac{n}{2}}).$$

By the assumptions \[40] and \[41] on $\tilde{N}$ and $\tilde{\varepsilon}$, the lemma follows. \[\square\]

**Corollary 6.7.** Assume $\sum_{\ell=1}^M i_\ell$ is good for reduction and $j_\ell$ is such that $j_\ell = i_\ell$ except at one place where $i_\ell$ is good for reduction and $j_\ell$ is bad for reduction. Then $\sum_{\ell=1}^M j_\ell \in \mathcal{H}_{\tilde{N}, \tilde{\varepsilon}^4}$. Similarly if $j_\ell = i_\ell$ except at one place where $i_\ell$ is bad for reduction and $j_\ell$ is good for reduction, then $\sum_{\ell=1}^M j_\ell \in \mathcal{H}_{\tilde{N}, \tilde{\varepsilon}^4}$.

**Proof.** We prove the first case and the second is similar. For concreteness we assume that $j_\ell \neq i_\ell$. So $\sum_{\ell=1}^M j_\ell = j_1 - i_1 + \sum_{\ell=1}^M i_\ell$ satisfies the assumptions of Lemma 6.6 completing the proof. \[\square\]

Set $A = \{i: i \text{ is good for reduction}\}$ and set $B = \{i: i \text{ is bad for reduction}\}$. Set $u_1 = \sum_{i \in A} a_i^{(k)}$ and let $u_2 = \sum_{i \in B} a_i^{(k)}$. Let $C = \mathbb{Z} \setminus (A \cup B)$ and $u_3 = 1 - (a+b) = \sum_{i \in C} a_i^{(k)}$. Consider the set of triples of non-negative integers $(a, b, c)$ such that $a + b + c = M$. For each such triple $(a, b, c)$, let $\mathcal{N}_{a,b,c}$ denote the set of all numbers that can be formed by summing $a$ terms for $A$, $b$ terms from $B$ and $c$ terms from $C$. To each $j \in \mathcal{N}_{a,b,c}$, we assign the probability of obtaining it as a term in $(\sum_{i} a_i^{(k)} U_{T^j})^M$.

**Proposition 6.8.** If the conditional probability that $j \in \mathcal{N}_{a,b,c}$ is good for reduction is greater than $\frac{1}{2}$, then the conditional probability that $j \in \mathcal{N}_{a+1,b-1,c}$ is not good for reduction is at least $\varepsilon^8$.

The proof of this proposition follows from some lemmas that we develop. We start with an explanation of the setting. Observe that $\mathcal{N}_{a,b,c}$ can be partitioned into elements of $A^\alpha \times B^\beta \times C^\gamma$ for $\alpha \leq a, \beta \leq b, \gamma \leq c$ by grouping the elements that have the same initial (ordered) $\alpha$-tuple of elements of $A$, and similarly for $B$ and $C$. (That is we keep track of the order of the initial $\alpha$ elements within $A, B, C$ but not how these $A$ elements fall relative to the $B$ and $C$ elements and so forth.)

One can calculate the conditional probability of being in such a partition element given that one lies in $\mathcal{N}_{a,b,c}$ in the following way: given a partition element $\tilde{v} \times \tilde{w} \times \tilde{x}$,
it has conditional probability

\[
\prod_{i=1}^{a-1} \frac{v_i}{u_i} \prod_{i=1}^{b} \frac{w_i}{u_i} \prod_{i=1}^{c} \frac{x_i}{u_3},
\]

We claim that we can consider elements of \(A^{a-1} \times B^b \times C^c\) as partitions of both \(N_{a,b+1,c}\) and \(N_{a-1,b,c}\). Indeed, elements of \(N_{a,b,c}\) are determined by the first \(a-1\) elements of \(A\), all the elements of \(B\), and all the elements of \(C\) (taken in the appropriate order within \(A,B,C\)). Similarly, elements of \(N_{a-1,b+1,c}\) are determined by the elements of \(A\), the first \(b\) elements of \(B\), and all the elements of \(C\).

\[\textbf{Lemma 6.9.}\] For \((\vec{v}, \vec{w}, \vec{z}) \in A^{a-1} \times B^b \times C^c\), let \(D\) denote the corresponding partition element of \(N_{a,b,c}\) and \(D'\) be the corresponding element of \(N_{a-1,b+1,c}\). The conditional probability of an element in \(N_{a,b,c}\) being in \(D\) is equal to the conditional probability of an element of \(N_{a-1,b+1,c}\) being in \(D'\).

\[\textbf{Proof.}\] Both of these conditional probabilities are

\[
\prod_{i=1}^{a} \frac{v_i}{u_i} \prod_{i=1}^{b} \frac{w_i}{u_i} \prod_{i=1}^{c} \frac{x_i}{u_3}.
\]

\[\textbf{Lemma 6.10.}\] For \((\vec{v}, \vec{w}, \vec{z}) \in A^{a-1} \times B^b \times C^c\), let \(D\) be the corresponding partition element of \(N_{a,b,c}\) and \(D'\) be the corresponding partition element of \(N_{a-1,b+1,c}\). If any element of \(D\) is good for reduction, then no element of \(D'\) is good for reduction, and the analogous statement holds when the roles of \(D\) and \(D'\) are exchanged.

\[\textbf{Proof.}\] Let \(x, y \in D \cup D'\). They differ by at most one entry. Moreover, if \(x \in D\) and \(y \in D'\), then this is a change from a good for reduction element to a bad for reduction element. By Corollary 6.7 if one is good for reduction, then the other is not good for reduction. If there exists one element in \(D\) that was good for reduction, then this argument shows every \(y \in D'\) is not good for reduction and similarly vice versa.

Combining Lemmas 6.9 and 6.10 Proposition 6.8 follows.

If \((\Omega, \mu)\) be a probability space and \(H: \Omega \rightarrow \{0, 1, \ldots\}\) is \(\mu\) measurable, we say \(i\) is \((H, \delta)-\text{spread}\) if

\[
\max\{\mu(H^{-1}(i + 1)), \mu(H^{-1}(i - 1))\} > \delta \mu(H^{-1}(i)).
\]

We say \(H\) is \(\delta\)-spread if \(\mu(\bigcup_{i,(H,\delta)} H^{-1}(i)) > \delta\).

\[\textbf{Lemma 6.11.}\] There exists \(C\) such that if \(F_i: (\Omega, \mu) \rightarrow \{0, 1\}\) are independent, identically \(\mu\) distributed random variables satisfying \(\frac{\delta}{K} \leq \mu(F_i^{-1}(0)) \leq 1 - \frac{\delta}{K}\), then \(H(\omega) = \sum_{i=1}^{K} F_i(\omega)\) is \(\min\{\frac{\delta^2}{K^2}, \frac{1}{3}\}\)-spread.

\[\textbf{Proof.}\] If \(\delta < \frac{\varphi}{K}\), this is straightforward and so we assume \(\delta > \frac{\varphi}{K}\). Let \(p = \mu(F_i^{-1}(0))\). Due to the symmetry, we can assume that \(p \leq \frac{1}{2}\), and by the assumption on \(\delta\), we can assume that \(p > \frac{\varphi}{K}\). Thus

\[
\frac{\mu(\{\omega: \sum_{i=1}^{K} F_i(\omega) = n + 1\})}{\mu(\{\omega: \sum_{i=1}^{K} F_i(\omega) = n\})} = \frac{\binom{K}{n+1}(p)^{n+1}(1-p)^{K-n-1}}{\binom{K}{n}(p)^n(1-p)^{K-n}} \left(\frac{K}{n}\right)^{p/(K)^n(1-(p/K))^{K-n}}
\]

\[= \frac{K-n-1}{n-1}p(1-p)^{-1}.
\]
If \( n \in \left[ \frac{1}{3} Kp, \frac{2}{3} Kp \right] \), then this is greater than \( \min \left\{ \frac{1}{99}, \frac{1}{99} Kp \right\} \). Since \( \frac{1}{99} Kp \geq \frac{1}{99} \delta \) the result follows if at least half of the \( \omega \) lie in this range. To check this, note that we have
\[
\int_{\Omega} (\sum_{i=1}^{K} F_i(\omega) - p)^2 \, d\omega = K \left( 1 - \frac{p}{K} \right)^2 \frac{Kp}{p} + \left( \frac{p}{K} \right)^2 \left( 1 - \frac{p}{K} \right) < \frac{3}{2} nKp.
\]
Thus by Chebyshev’s inequality, \( \mu \left( \{ \omega : | \sum_{i=1}^{K} F_i(\omega) - Kp | > 2\sqrt{pK} \} \right) < \frac{1}{2} \). Since \( pK \geq 9 \) we have that \( 2\sqrt{pK} \leq \frac{2}{3} Kp \) establishing the necessary condition.

**Concluding the proof of Theorem 6.7 in Case 3.** In this proof only, we introduce some terminology for clarity: we say \( i \) is decisive if it is either good or bad for reduction. Let \( \gamma^{(k)} \) be the probability measure on \( \{0, 1\} \) defined by
\[
\gamma^{(k)}(\{0\}) = \frac{\sum \text{is bad for reduction} \alpha_i^{(k)}}{\sum \text{is decisive} \alpha_i^{(k)}},
\]
and
\[
\gamma^{(k)}(\{1\}) = \frac{\sum \text{is good for reduction} \alpha_i^{(k)}}{\sum \text{is decisive} \alpha_i^{(k)}}.
\]
Note that \( \gamma^{(k)}(\{0\}) \) is the conditional probability that \( i \) is bad for reduction given that it is decisive and \( \gamma^{(k)}(\{1\}) \) is the conditional probability that \( i \) is good for reduction given that it is decisive. As we are not in Case 2, it follows that \( \gamma^{(k)}(\{0\}) > \frac{2^2}{M} \).

Thus by Lemma 6.11 \( \sum_{j=0}^{M} \gamma^{(k)}(j) \) is at least \( \frac{2^2}{M} \) spread. We partition \( \mathbb{Z}^M \) into sets \( \mathcal{N}(a,b,c) \cup \mathcal{N}(a+1,b-1,c) \) where \( a \) is even. We further partition these elements \( \mathcal{D} \cup \mathcal{D}' \) as in Lemma 6.9. By Lemma 6.10 for each such element \( \mathcal{D} \cup \mathcal{D}' \), one of these two pieces contains no good for reduction elements. By Lemma 6.9 and the fact that \( \sum_{i=0}^{M-1} \gamma^{(k)}(i) \) is \( \frac{2^4}{M} \) spread (so long as \( c > \frac{M}{2} \)), it follows from Lemma 6.9 that at least \( \frac{1}{2} \frac{2^6}{M} \) of our points are not good for reduction. Once again this contradicts Corollary 5.15 and our choices. \( \square \)

**References**


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