UNIFORMITY SEMINORMS ON $\ell^\infty$ AND AN INVERSE THEOREM

SUMMARY OF RESULTS

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Abstract. For each integer $k \geq 1$, we define seminorms on $\ell^\infty(\mathbb{Z})$, analogous to the seminorms defined by the authors on bounded functions in a measure preserving system (associated to the averages in Furstenberg’s proof of Szemerédi’s Theorem) and to the norms on $\mathbb{Z}/N\mathbb{Z}$ defined by Gowers (in his proof of Szemerédi’s Theorem). For the seminorms on $\ell^\infty(\mathbb{Z})$, we prove an inverse theorem. For the Gowers norms on $\mathbb{Z}/N\mathbb{Z}$, such an inverse theorem was proven for $k = 2$ and $k = 3$ by Green and Tao.

This is a short summary of some of the results that will eventually be contained in a paper with the above title. Some of these results are related to the analysis in [4].

1. Seminorms

Definition 1. Let $\mathbf{a} = (a_n : n \in \mathbb{Z})$ be a bounded sequence. We say that the averages of the sequence $\mathbf{a}$ converge if for all sequences of intervals $\mathbf{I} = (I_j : j \geq 1)$ whose lengths $|I_j|$ tend to infinity, the limit

$$\lim_{j \to +\infty} \frac{1}{|I_j|} \sum_{n \in I_j} a_n$$

exists; by definition, this limit does not depend on the sequence of intervals and we denote it by $\lim \text{averages}(a_n)$.

The upper limit of the averages of the sequence $\mathbf{a}$ is defined to be

$$\limsup |\text{averages}(a_n)| = \lim_{N \to +\infty} \sup_{I \text{ is an interval of length } N} \left| \frac{1}{N} \sum_{n \in I} a_n \right|.$$ We use the same definition and notation for sequences $\mathbf{a} = (a_{n_1}, \ldots, a_{n_k} : n_1, \ldots, n_k \in \mathbb{Z})$ indexed by $\mathbb{Z}^k$; sequences of intervals whose lengths tend to infinity are replaced by sequences of subsets of $\mathbb{Z}^k$ of the form $\mathbf{R} = (R_j : j \geq 1)$, where $R_j = I_{j,1} \times \cdots \times I_{j,k}$ and $I_{j,1}, \ldots, I_{j,k}$ are intervals with $\min_i |I_{j,i}| \to +\infty$ as $j \to +\infty$. In fact, we could replace this sequence of sets by any Følner sequence in $\mathbb{Z}^k$.

Notation. We write $z \mapsto Cz$ for the complex conjugation in $\mathbb{C}$. Thus $C^k z = z$ if $k$ is an even integer and $C^k z = \bar{z}$ if $k$ is an odd integer.

For $\mathbf{e} = (\epsilon_1, \ldots, \epsilon_k)$ and $\mathbf{h} = (h_1, \ldots, h_k)$, we define $|\mathbf{e}| = \epsilon_1 + \cdots + \epsilon_k$ and $\mathbf{e} \cdot \mathbf{h} = \epsilon_1 h_1 + \cdots + \epsilon_k h_k$. 

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**Theorem 1** (and definition). Let \( k \geq 1 \) be an integer and \( a \) be a bounded sequence. We say that a sequence of intervals \( I = (I_i : i \geq 1) \) whose lengths tend to infinity satisfies property \((P_k)\) for the sequence \( a \) if

for all \( h = (h_1, \ldots, h_k) \in \mathbb{Z}^k \), the limit

\[
    c_h(I, a) := \lim_{i \to +\infty} \frac{1}{|I_i|} \sum_{n \in I_i} \prod_{\epsilon \in \{0,1\}^{k-1}} C^{(\epsilon)}\sigma_{n+h_{\epsilon}}
\]

exists.

Let \( k \geq 2 \) be an integer and let \( a \) be a bounded sequence.

If \( I = (I_i : i \geq 1) \) is a sequence of intervals satisfying property \((P_{k-1})\) for \( a \), then the averages of the sequence \( (|c_h(J, a)|^2 : h \in \mathbb{Z}^{k-1}) \) converge to a limit \( \text{cor}_k(J, a) \).

If \( J = (J_j : j \geq 1) \) is a sequence of intervals satisfying property \((P_k)\) for \( a \), then the averages of the sequence \( (c_h(J, a) : h \in \mathbb{Z}^k) \) converge to a limit \( \text{cor}_k^*(J, a) \geq 0 \).

We have

\[
    \sup_I \text{cor}_k(I, a) = \sup_J \text{cor}_k^*(I, a)
\]

where the upper bounds are taken for all sequences \( I \) of intervals satisfying \((P_{k-1})\) and all sequences \( J \) of intervals satisfying \((P_k)\) for the sequence \( a \).

Writing

\[
    \|a\|_{U(k)} = \sup_I \text{cor}_k(I, a)^{1/2^k}
\]

we have that the map \( a \mapsto \|a\|_{U(k)} \) is a seminorm on \( \ell^\infty(\mathbb{Z}) \).

Furthermore, for every integer \( k \geq 3 \), \( \|a\|_{U(k)} \geq \|a\|_{U(k-1)} \).

This definition and result can be extended to the case \( k = 1 \). Property \((P_0)\) means that the averages of the sequence \( a \) over the intervals \( I_j \) converge to some limit \( c(I, a) \). We have \( \|a\|_{U(1)} = \limsup |\text{averages}(a_n)| \).

For clarity, we explain what the definitions mean when \( k = 2 \). Property \((P_1)\) says that for every \( h \in \mathbb{Z} \), the averages (over \( n \)) on the intervals \( I_j \) of \( a_{n+h\bar{a}_n} \) converge to a limit \( c_h(I, a) \). Property \((P_2)\) means that for all \( h, \ell \in \mathbb{Z} \), the averages (over \( n \)) on the intervals \( J_j \) of \( a_{n+h\bar{a}_{n+h} \bar{a}_{n+\ell\bar{a}_n}} \) converge to a limit \( c(h, \ell)(J, a) \). We have that

\[
    \text{cor}_2(I, a) = \lim_{H \to +\infty} \frac{1}{H} \sum_{h=0}^{H-1} |c_h(I, a)|^2
\]

and

\[
    \text{cor}_2^*(J, a) = \lim_{H \to +\infty} \frac{1}{H^2} \sum_{h, \ell=0}^{H-1} c(h, \ell)(J, a)
\]

**Remark 1.** The seminorms \( \|\cdot\|_{U(k)} \) are linked to the seminorms \( \|\cdot\|_k \) of \([1]\).

However there are important differences between the two different kind of seminorms. For example we have

\[
    \|f\|_{k+1}^{2^{k+1}} = \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \|\bar{f}(T^n f)|^2_k
\]

while

\[
    \|a\|_{U(k+1)}^{2^{k+1}} \leq \liminf_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \|\bar{a}\sigma^n a\|_{U(k)}^{2^k}
\]

where \( \sigma \) denotes the shift; in general the \( \liminf \) is not a limit and equality does not hold.
Proposition 1 (Cauchy-Schwartz-Gowers Inequality). Let $k \geq 2$ be an integer and for every $\epsilon \in \{0,1\}^k$, let $a(\epsilon) = (a_n(\epsilon) : n \in \mathbb{Z})$ be a bounded sequence. Let $I = (I_j : j \geq 1)$ be a sequence of intervals with $|I_j| \to +\infty$ and assume that for every $h = (h_1, \ldots, h_k) \in \mathbb{Z}^k$, the limit

$$\rho_h := \lim_{j \to +\infty} \frac{1}{|I_j|} \sum_{n \in I_j, \epsilon \in \{0,1\}^k} C^{[\epsilon]} a_{n+\epsilon} h(\epsilon)$$

exists.

Then the averages of $\rho_h$ (for $h \in \mathbb{Z}^k$) converge and

$$|\lim \text{averages } (\rho_h)| \leq \prod_{\epsilon \in \{0,1\}^k} \|a(\epsilon)\|_{U(k)}.$$

2. Inverse Theorem

We recall the standard definition of a nilsystem and that of a nilsequence (see [2]):

Definition 2. If $G$ is a $k$-step nilpotent Lie group and $\Gamma \subset G$ is a discrete and cocompact subgroup, then the compact manifold $X = G/\Gamma$ is called a $k$-step nilmanifold. The action of $G$ on $X$ by left translation is written $(g, x) \mapsto g \cdot x$ for $g \in G$ and $x \in X$. A $k$-step nilmanifold $X = G/\Gamma$, endowed with the translation $T: x \mapsto \tau \cdot x$ by some fixed element $\tau$ of $G$ is called a $k$-step nilsystem. If $f: X \to \mathbb{C}$ is a continuous function, $\tau \in G$ and $x_0 \in X$, then $(f(\tau^n \cdot x_0) : n \in \mathbb{Z})$ is called a basic $k$-step nilsequence. Without loss we can restrict to the case that the nilsystem $(X, T)$ is minimal. A $k$-step nilsequence is defined to be a uniform limit of basic $k$-step nilsequences.

We are now ready to state the main theorem:

Theorem 2 (Inverse Theorem). Let $k \geq 2$ be an integer. For every bounded sequence $a = (a_n : n \in \mathbb{Z})$, the following properties are equivalent:

(i) $\|a\|_{U(k)} = 0$;
(ii) For every $(k-1)$-step nilsequence $b = (b_n : n \in \mathbb{Z})$, the averages of the sequence $(a_n b_n)$ converge to 0.

Remark 2. Assume that a bounded sequence $a$ satisfies $\|a\|_k > 0$. This means that there exists a sequence of intervals $I = (I_j : j \geq 1)$ such that the averages $c_h(I, a)$ are “large” for sufficiently many values of $h$. Theorem 2 says that there exist a $(k-1)$-step nilsequence $b$ and a sequence of intervals $J = (J_\ell : \ell \in \mathbb{Z})$ such that the averages of the sequence $(a_n b_n : n \in \mathbb{Z})$ on these intervals are “large,” but it says nothing about the relation between the intervals $I_j$ and $J_\ell$ and in particular between their lengths. It is easy to build examples showing that Theorem 2 cannot be improved in this direction and that this limitation is intrinsic in this question.

The implication (i) $\implies$ (ii) of Theorem 2 follows from:

Proposition 2. Let $k \geq 2$ be an integer and let $b = (b_n : n \in \mathbb{Z})$ be a $(k-1)$-step nilsequence. For every $\delta > 0$, there exists a constant $C = C(b, \delta)$ such that for every bounded sequence $a = (a_n : n \in \mathbb{Z})$ with $\|a\|_\infty \leq 1$,

$$\limsup |\text{averages}(a_n b_n)| \leq \delta + C\|a\|_{U(k)}.$$
3. Quantitative results: The dual norm

Proposition 3 (and definition). Let\(^1\) \((X, T, \mu)\) be a minimal \((k - 1)\)-step nilsystem and let \(f\) be a smooth function on \(X\). Then there exists a constant \(c\) such that
\[
\left| \int f \cdot h \, d\mu \right| \leq c \|h\|_k \text{ for all } h \in \mathcal{C}(X),
\]
where \(\mathcal{C}(X)\) denotes continuous functions on \(X\).

We set \(\|f\|_k^*\) to be the smallest constant \(c\) satisfying this condition.

The hypothesis of smoothness of \(f\) is too strong: the result still holds if \(f\) is Lipschitz (for some smooth metric on \(X\)). A similar result for the finite case appears in [3].

Proposition 4. Let \((X, T, \mu)\) be a minimal \((k - 1)\)-step nilsystem, \(x_0 \in X\) and \(f\) be a smooth function on \(X\). Then for all bounded sequences \(a = (a_n : n \in \mathbb{Z})\), we have
\[
\limsup \left| \text{averages}(a_n f(T^n x_0)) \right| \leq \|f\|_k^* \|a\|_{U(k)}.
\]

Theorem 3 (Quantitative Inverse Theorem). Let \(a = (a_n : n \in \mathbb{Z})\) be a bounded sequence with \(\|a\|_{U(k)} > 0\). Then for all \(\delta > 0\), there exists a \(k - 1\) step nilsystem, \(x_0 \in X\) and a smooth function \(f\) on \(X\) such that
\[
\|f\|_k = 1 \text{ and } \limsup \left| \text{averages}(a_n f(T^n x_0)) \right| \geq \|a\|_{U(k)} - \delta.
\]

4. The case \(k = 2\)

For \(k = 2\), we have more explicit (and easier) results. We write \(T = \mathbb{R}/\mathbb{Z}\) and for \(t \in \mathbb{T}\), we write \(e(t) = \exp(2\pi it)\). Recall that 1-step nilsequences are almost periodic sequences.

Let \(f\) be a smooth function on a 1-step nilmanifold, that is, a compact abelian Lie group \(\mathbb{Z}\) endowed with a minimal translation \(T\). Then \(f\) can be written
\[
f(x) = \sum_{\chi \in \hat{\mathbb{Z}}} \hat{f}(\chi) \chi(x) \text{ with } \sum_{j=1}^{\infty} |\hat{f}(\chi)| < +\infty
\]
and we have
\[
\|f\|_{U(2)} = \left( \sum_{j=1}^{\infty} |\hat{f}(\chi)|^4 \right)^{1/4} \text{ and } \|f\|_{U(2)}^* = \left( \sum_{j=1}^{\infty} |\hat{f}(\chi)|^{4/3} \right)^{3/4}.
\]

Proposition 2 for \(k = 2\) can be deduced from:

Lemma 1. For every bounded sequence \(a = (a_n : n \in \mathbb{Z})\) and every sequence of intervals \(I = (I_j : j \in \mathbb{Z})\) with \(|I_j| \to +\infty\), we have
\[
\limsup_{j \to +\infty} \sup_{t \in I_j} \left| \frac{1}{|I_j|} \sum_{n \in I_j} a_n e(nt) \right| \leq \|a\|_{U(2)}.
\]

Corollary 1. For \(k = 2\), the equivalent properties (i) and (ii) of Theorem 2 are also equivalent to

(iii) For every \(t \in \mathbb{T}\), \(\limsup \left| \text{averages}(a_n e(nt)) \right| = 0\).

\(^1\)We adopt the convention of writing a measure preserving system as \((X, \mu, T)\), omitting mention of the \(\sigma\)-algebra.
(iv) For every sequence $I = (I_j : j \in \mathbb{Z})$ of intervals with $|I_j| \to +\infty$,

$$\lim_{j \to +\infty} \sup_{t \in \mathbb{T}} \left| \frac{1}{|I_j|} \sum_{n \in I_j} a_n e(nt) \right| = 0.$$ 

For $k = 3$ a similar result holds, where the family of exponential sequences is replaced by a family of special nilsequences (this is the class $M$ of [4]).

5. An application to ergodic theory

**Proposition 5.** Let $k \geq 2$ be an integer and let $a$ be a bounded sequence. If $\|a\|_{U(k)} = 0$, then for all measure preserving systems $(X, \mu, T)$ and all functions $f_1, \ldots, f_{k-1} \in L^\infty(\mu)$, the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} a_n T^n f_1 T^{2n} f_2 \cdots T^{(k-1)n} f_{k-1}$$

converge to zero in $L^2(\mu)$ as $N \to +\infty$.

The converse implication holds for $k = 2$. We believe that the converse implication holds also for $k = 3$, but the proof is not yet written (perhaps will be in the next few days); we conjecture that it holds for all $k$.

**References**


