A short proof of Erdős's $B + C$ conjecture

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Dedicated to Vitaly Bergelson on his 75th birthday.

Abstract We give a short proof of the fact that every set of natural numbers with positive upper Banach density contains the sum of two infinite sets. The approach simplifies earlier existing proofs.

1 Introduction

In the late 1970s and early 1980s, Erdős sought to characterize the types of subsets of integers that contain infinite sumsets, a line of inquiry that culminated in two noteworthy conjectures. Both have since been resolved and we state them as theorems below.

A *Følner sequence* is a sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ of finite subsets of $\mathbb{N}$ satisfying

$$\lim_{N\to\infty} \frac{|\Phi_N \cap (\Phi_N + 1)|}{|\Phi_N|} = 1.$$

A set $A \subset \mathbb{N}$ has *positive upper Banach density* if

$$\lim_{N\to\infty} \frac{|A \cap \Phi_N|}{|\Phi_N|} > 0$$

for some Følner sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$.

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We begin with the weaker conjecture, which was the second conjecture in chronological order.

Theorem 1 (Erdős's 2nd sumset conjecture)**.** *For any $A \subset \mathbb{N}$ with positive upper Banach density, there exist infinite sets $B, C \subset \mathbb{N}$ such that $B+C = \{b+c : b \in B,\ c \in C\} \subset A$.*

Theorem 1 was first proved in [MRR19]. The argument was later simplified significantly by Host in [Hos19]. Another proof, following a different and partly easier argument, was given in Section 3 of [KMRR24a].

Theorem 2 (Erdős's 1st sumset conjecture)**.** *For any $A \subset \mathbb{N}$ with positive upper Banach density, there exist $b_1 < b_2 < b_3 < \ldots \in \mathbb{N}$ and $t \in \mathbb{N}$ such that $\{b_i + b_j : i, j \in \mathbb{N},\ i \neq j\} \subset A - t$.*

Note that Theorem 2 contains Theorem 1 as a special case. Theorem 2 was proved in [KMRR24b].

The purpose of this note is to provide a new and relatively short proof of Theorem 1. The ideas behind this proof are based on recent advances made on Theorem 2 and its generalizations in [KMRR25, KMRR26].

2 Preliminaries from ergodic theory

Throughout, by a *measure-preserving system* we mean a tuple $(X, \mathscr{B}_X, \mu, T)$ where X is a compact metric space, $\mathscr{B}_X$ is the Borel σ-algebra on X, $T\colon X \to X$ is a homeomorphism, and $\mu\colon \mathscr{B}_X \to [0,1]$ is a Borel probability measure on X invariant under the transformation T.

Given a Følner sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$, a point $x \in X$ is *generic for μ along Φ* if

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f(T^n x) = \int_X f \,\mathsf{d}\mu \qquad \text{for all } f \in C(X), \tag{1}$$

where $C(X)$ denotes the space of all (complex-valued) continuous functions on X.

Remark 3. *When the measure μ is ergodic, it follows from the mean ergodic theorem that every Følner sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ admits a subsequence $\Phi' = (\Phi'_N)_{N\in\mathbb{N}}$ such that μ-almost every point in X is generic for μ along Φ'. Indeed, the mean ergodic theorem implies for each $f \in C(X)$ that* (1) *holds in L^2 norm, and therefore along a subsequence of Φ it holds for almost every $x \in X$. Since $C(X)$ has a countable dense set, a diagonal argument produces the desired subsequence.*

A nonzero function $f \in L^2(X, \mathscr{B}_X, \mu)$ is a *measurable eigenfunction* with *eigenvalue* $\theta \in \mathbb{C}$ if $f(Tx) = \theta f(x)$ holds for μ-almost every $x \in X$. A nonzero function $f \in C(X)$ is a *topological eigenfunction* if $f(Tx) = \theta f(x)$ for every $x \in X$. We emphasize that every topological eigenfunction is a measurable one, but not necessarily the other way around.

It is well known that the closed subspace of $L^2(X, \mathscr{B}_X, \mu)$ spanned by its measurable eigenfunctions is of the form $L^2(X, \mathscr{K}, \mu)$ for a sub-σ-algebra $\mathscr{K}$ of $\mathscr{B}_X$ (see [EW11, Theorem 6.10 and Theorem C.11]). Write $\mathbb{E}(f \mid \mathscr{K})$ for the orthogonal projection in $L^2(X, \mathscr{B}_X, \mu)$ of f on $L^2(X, \mathscr{K}, \mu)$. If $f \geq 0$ then $\mathbb{E}(f \mid \mathscr{K}) \geq 0$.

Remark 4. *Given a measurable set $E \subset X$, let $D = \{x \in X : \mathbb{E}(\mathbf{1}_E \mid \mathscr{K})(x) > 0\}$ denote the support of $\mathbb{E}(\mathbf{1}_E \mid \mathscr{K})$. Since $X \backslash D \in \mathscr{K}$ and $\mathbb{E}(\mathbf{1}_E \mid \mathscr{K}) \geq 0$, it follows that*

$$0 = \int_{X \backslash D} \mathbb{E}(\mathbf{1}_E \mid \mathscr{K}) \, \mathrm{d}\mu = \mu(E \backslash D).$$

We conclude that for μ-almost every $x \in E$ we have $\mathbb{E}(\mathbf{1}_E \mid \mathscr{K})(x) > 0$. We use this property in the proof of our main result.

We call $f \in L^2(X, \mathscr{B}_X, \mu)$ a *weak mixing* function if for all $g \in L^2(X, \mathscr{B}_X, \mu)$ and all Følner sequences $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ we have

$$\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\langle T^n f, g \rangle| = 0. \tag{2}$$

Remark 5. *Note that the limit in* (2) *is clearly zero whenever g is orthogonal to $\{T^n f : n \in \mathbb{N}\}$. This implies that a function is weak mixing if and only if*

$$\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\langle T^n f, f \rangle| = 0.$$

The following fundamental result is frequently used and is implicit in the work of Koopman and von Neumann [KN32] (cf. [Kre85, Theorem 2.3.4] or [HK18, Chapter 4, Proposition 19]).

Theorem 6. *For any $f \in L^2(X, \mathscr{B}_X, \mu)$ we have*

$$f \text{ is weak mixing} \iff \mathbb{E}(f \mid \mathscr{K}) = 0.$$

A crucial tool for translating a combinatorial problem into ergodic theory is Furstenberg's correspondence principle, introduced in his proof [Fur77] of Szemerédi's theorem (see also [Fur81]). We utilize the following enhanced variant, which allows us to assume that all eigenfunctions are topological eigenfunctions.

Theorem 7 (Correspondence principle)**.** *For any set $A \subset \mathbb{N}$ with positive upper Banach density there exist a Følner sequence $\Phi = (\Phi_N)_{N \in \mathbb{N}}$, an ergodic measure-preserving system (X, μ, T), a clopen set $E \subset X$, and a point $x \in X$ such that:*

1. *The point x is generic for μ along Φ.*
2. *The measure $\mu(E) > 0$.*
3. *$A = \{n \in \mathbb{N} : T^n x \in E\}$.*
4. *Every measurable eigenfunction is equal μ-almost everywhere to a topological eigenfunction.*

Theorem 7 is implicit in [KMRR24a] as a combination of [KMRR24a, Theorem 2.10] and [KMRR24a, Lemma 5.8]. The former establishes the first three properties as in Furstenberg's original correspondence principle. The latter is then used to produce a topological model of that system that additionally satisfies the fourth property.

3 Reformulation from combinatorics to ergodic theory

Using Furstenberg's correspondence principle, we reduce the existence of sumsets to the following dynamical result, versions of which have appeared in all of [KMRR24a, KMRR24b, KMRR25, KMRR26].

Theorem 8 (Main dynamical result)**.** *Let (X, μ, T) be an ergodic measure-preserving system and let $x \in X$ be a point that is generic for μ along a Følner sequence $\Phi = (\Phi_N)_{N \in \mathbb{N}}$. Assume every measurable eigenfunction of (X, μ, T) is equal μ-almost everywhere to a topological eigenfunction. Then for any clopen set $E \subset X$ with $\mu(E) > 0$ there exist sequences $b_1 < b_2 < \ldots \in \mathbb{N}$ and $c_1 < c_2 < \ldots \in \mathbb{N}$ such that*

$$\lim_{j\to\infty}\lim_{i\to\infty} T^{b_i+c_j}x \in E \qquad \textit{and} \qquad \lim_{i\to\infty}\lim_{j\to\infty} T^{b_i+c_j}x \in E. \tag{3}$$

Proof that Theorem 8 implies Theorem 1. Assume that $A \subset \mathbb{N}$ has positive upper Banach density and let Φ be the Følner sequence, (X, μ, T) the ergodic measure-preserving system, $E \subset X$ clopen, and $x \in X$ the point given by Theorem 7. We inductively apply the limits to construct subsequences $(b'_i)_{i\in\mathbb{N}}$ of $(b_i)_{i\in\mathbb{N}}$ and $(c'_j)_{j\in\mathbb{N}}$ of $(c_j)_{j\in\mathbb{N}}$ such that $b'_i + c'_j \in A$ for all $i, j \in \mathbb{N}$. First use the second iterated limit in (3) to choose $b'_1 \in (b_i)_{i\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} T^{b'_1+c_j}x \in E. \tag{4}$$

Then use both (4) and the first iterated limit in (3) to choose $c'_1 \in (c_j)_{j\in\mathbb{N}}$ such that

$$T^{b'_1+c'_1}x \in E \quad \text{and} \quad \lim_{i\to\infty} T^{b_i+c'_1}x \in E.$$

We next choose $b'_2 \in (b_i)_{i\in\mathbb{N}}$ sufficiently large such that

$$T^{b'_2+c'_1}x \in E \quad \text{and} \quad \lim_{j\to\infty} T^{b'_2+c_j}x \in E,$$

and then choose $c'_2 \in (c_j)_{j\in\mathbb{N}}$ sufficiently large such that

$$T^{b'_1+c'_2}x, T^{b'_2+c'_2}x \in E \quad \text{and} \quad \lim_{i\to\infty} T^{b_i+c'_2}x \in E.$$

We then continue inductively to obtain sequences $(b'_i)_{i\in\mathbb{N}}$ and $(c'_j)_{j\in\mathbb{N}}$ with $T^{b'_i+c'_j}x \in E$ for all $i, j \in \mathbb{N}$. Applying Theorem 7 to the set A in the hypothesis of Theorem 1, it is immediate from property (iii) that A contains the infinite sumset $\{b'_i+c'_j : i, j \in \mathbb{N}\}$. □

4 Proof of main dynamical result

Proof of Theorem 8. After replacing the Følner sequence Φ by a subsequence if necessary, there exists a point y in the support of μ that is generic for μ along Φ (cf. Remark 3). Further refining Φ if necessary, we can also assume that the pair (x, y) is generic along

Φ with respect to the product transformation $T \times T$ for a measure λ on the product space $X \times X$. Note that for any $f \in C(X)$,

$$\int_{X\times X} f(x_1)\,\mathsf{d}\lambda(x_1,x_2) = \int_{X\times X} f(x_2)\,\mathsf{d}\lambda(x_1,x_2) = \int_X f\,\mathsf{d}\mu.$$

Let $d\colon X \times X \to [0,\infty)$ denote a metric on X. The central ingredient of the proof is the following claim, which is iterated to produce the desired sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$.

Claim Let $Y \subset X \times X$ be a measurable set. If $\lambda\big((X \times E) \cap Y\big) > 0$, then for every $\varepsilon > 0$ there exist infinitely many $b \in \mathbb{N}$ such that

$$d(T^b x, y) < \varepsilon \qquad \text{and} \qquad \lambda\big((T^{-b}E \times X) \cap Y\big) > 0. \tag{5}$$

Proof of Section 4. We write $f_c = \mathbb{E}(\mathbf{1}_E \mid \mathscr{K})$ for the orthogonal projection of $\mathbf{1}_E$ on the space $L^2(X,\mathscr{K},\mu)$ and $f_{\mathrm{wm}} = \mathbf{1}_E - f_c$, so that f_{wm} is weak mixing by Theorem 6. As noted in Remark 4, for μ-almost every $x \in E$ we have $f_c(x) > 0$, and hence

$$\lambda\big((X \times E) \cap Y\big) > 0 \quad \Longrightarrow \quad \big\langle (1 \otimes f_c),\, \mathbf{1}_Y \big\rangle > 0.$$

Let $\eta = \langle (1 \otimes f_c),\, \mathbf{1}_Y\rangle$. To prove (5), it therefore suffices to find infinitely many $b \in \mathbb{N}$ such that

$$\underbrace{d(T^b x, y) < \varepsilon,}_{[1]} \quad \underbrace{\big\|(T^b f_{\mathrm{c}} \otimes 1) \,-\, (1 \otimes f_c)\big\|_{L^2(\lambda)} < \frac{\eta}{2},}_{[2]} \quad \text{and} \quad \underbrace{\big|\big\langle (T^b f_{\mathrm{wm}} \otimes 1),\, \mathbf{1}_Y \big\rangle\big| < \frac{\eta}{2}}_{[3]}.$$

Since f_{c} can be approximated by eigenfunctions, there exist $r \in \mathbb{N}$ and topological eigenfunctions $\chi_1,\ldots,\chi_r$ such that $f_{\mathrm{c}}' = \chi_1 + \ldots + \chi_r$ satisfies $\|f_{\mathrm{c}}' - f_{\mathrm{c}}\|_{L^2} < \eta/6$. The advantage of working with f_{c}' over f_{c} is that f_{c}' is continuous because each χ_i is continuous. Let $\delta > 0$ be sufficiently small such that for $1 \leqslant i \leqslant r$ we have $|\chi_i(y) - \chi_i(z)| < \eta/6r$ whenever $d(z,y) < \delta$. Since χ_i is also an eigenfunction, we additionally obtain $\sup_{n\in\mathbb{N}} |\chi_i(T^n y) - \chi_i(T^n z)| < \eta/6r$ whenever $d(z,y) < \delta$. This implies the following equicontinuity property for the function f_{c}':

$$\sup_{n\in\mathbb{N}} |f_{\mathrm{c}}'(T^n y) - f_{\mathrm{c}}'(T^n z)| < \frac{\eta}{6} \ \text{ whenever } \ d(z,y) < \delta. \tag{6}$$

Next, consider the set $\mathscr{B} = \{b \in \mathbb{N} : d(T^b x, y) < \min\{\delta,\varepsilon\}\}$. We make three observations about $\mathscr{B}$. First, note that [1] is satisfied for all $b \in \mathscr{B}$. Second, it follows from (6) that

$$\sup_{b\in\mathscr{B}} \sup_{n\in\mathbb{N}} |f_{\mathrm{c}}'(T^{n+b}x) - f_{\mathrm{c}}'(T^n y)| \leqslant \frac{\eta}{6}. \tag{7}$$

Third, since y belongs to the topological support of μ and x is generic for μ along Φ, it follows that the orbit of x visits every neighborhood of y with positive frequency along the Følner sequence Φ. In particular, we have

$$\limsup_{M\to\infty} \frac{|\mathcal{B}\cap\Phi_M|}{|\Phi_M|} > 0. \tag{8}$$

Using $\|f_c' - f_c\|_{L^2} < \eta/6$ and (7) we get

$$\begin{aligned}\sup_{b\in\mathcal{B}} \left\|(T^b f_c \otimes 1) - (1\otimes f_c)\right\|_{L^2} &< \sup_{b\in\mathcal{B}} \left\|(T^b f_c' \otimes 1) - (1\otimes f_c')\right\|_{L^2} + \frac{\eta}{3} \\ &= \sup_{b\in\mathcal{B}} \left(\lim_{N\to\infty} \frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} \left|f_c'(T^{n+b}x) - f_c'(T^n y)\right|^2\right)^{\frac{1}{2}} + \frac{\eta}{3} < \frac{\eta}{2}.\end{aligned}$$

This proves that [2] holds for all $b\in\mathcal{B}$.

Finally, since f_{wm} is a weak mixing function in $L^2(X,\mathscr{B}_X,\mu)$, Remark 5 implies $(f_{\mathrm{wm}}\otimes 1)$ is a weak mixing function in $L^2(X\times X,\mathscr{B}_{X\times X},\lambda)$. In particular,

$$\lim_{M\to\infty}\frac{1}{|\Phi_M|}\sum_{b\in\Phi_M}\left|\left\langle T^b f_{\mathrm{wm}}\otimes 1,\ F\right\rangle\right| = 0.$$

Combined with (8), this proves that there exist infinitely many $b\in\mathcal{B}$ such that [3] holds. This completes the proof of Section 4. △

The proof of Theorem 8 now follows from a simple iteration of Section 4. Noting that $\lambda(X\times E) = \mu(E) > 0$, we apply the claim with $Y = X\times E$ to find b_1 such that

$$d(T^{b_1}x, y) < \frac{1}{2} \qquad \text{and} \qquad \lambda\big(T^{-b_1}E\times E\big) > 0.$$

Applying the claim again, this time with $Y = T^{-b_1}E\times E$, we can find $b_2 > b_1$ such that

$$d(T^{b_2}x, y) < \frac{1}{4} \qquad \text{and} \qquad \lambda\big((T^{-b_1}E\cap T^{-b_2}E)\times E\big) > 0.$$

Next, we apply the claim with $Y = (T^{-b_1}E\cap T^{-b_2}E)\times E$, and so on. Continuing this procedure produces a sequence $b_1 < b_2 < \ldots \in\mathbb{N}$ such that $T^{b_i}x \to y$ as $i\to\infty$ and

$$\lambda\big((T^{-b_1}E\cap\ldots\cap T^{-b_j}E)\times E\big) > 0, \qquad \text{for all } j\in\mathbb{N}. \tag{9}$$

From the definition of λ, it follows that

$$\lambda\big((T^{-b_1}E\cap\ldots\cap T^{-b_i}E)\times E\big) = \lim_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\mathbf{1}_E(T^{n+b_1}x)\cdots\mathbf{1}_E(T^{n+b_i}x)\mathbf{1}_E(T^n y).$$

In particular, for every $j\in\mathbb{N}$ we can find c_j such that $T^{c_j}y\in E$ and $T^{c_j+b_i}x\in E$ for all $i\leqslant j$. After replacing (c_j) by a subsequence such that the limits $\lim_{j\to\infty}T^{c_j}x$ and $\lim_{j\to\infty}T^{c_j}y$ exist, and (b_i) by a subsequence such that $\lim_{i\to\infty}\lim_{j\to\infty}T^{b_i+c_j}x$ exists, we have that (3) is satisfied. □

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References

[EW11] M. Einsiedler and T. Ward, *Ergodic theory with a view towards number theory*, *Graduate Texts in Mathematics* **259**, Springer-Verlag London, Ltd., London, 2011. doi: 10.1007/978-0-85729-021-2.

[Fur77] H. Furstenberg, Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions, *J. Analyse Math.* **31** (1977), 204–256. doi: 10.1007/BF02813304.

[Fur81] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, N.J., 1981, M. B. Porter Lectures.

[Hos19] B. Host, A short proof of a conjecture of Erdős proved by Moreira, Richter and Robertson, *Discrete Anal.* (2019), Paper No. 19, 10. doi: 10.19086/da.

[HK18] B. Host and B. Kra, *Nilpotent structures in ergodic theory*, *Mathematical Surveys and Monographs* **236**, American Mathematical Society, Providence, RI, 2018.

[KN32] B. O. Koopman and J. V. Neumann, Dynamical systems of continuous spectra, *Proceedings of the National Academy of Sciences of the United States of America* **18** no. 3 (1932), 255–263. Available at http://www.jstor.org/stable/86259.

[KMRR24a] B. Kra, J. Moreira, F. K. Richter, and D. Robertson, Infinite sumsets in sets with positive density, *J. Amer. Math. Soc.* **37** no. 3 (2024), 637–682. doi: 10.1090/jams/1030.

[KMRR24b] B. Kra, J. Moreira, F. K. Richter, and D. Robertson, A proof of Erdős's $B + B + t$ conjecture, *Commun. Am. Math. Soc.* **4** (2024), 480–494. doi: 10.1090/cams/34.

[KMRR25] B. Kra, J. Moreira, F. K. Richter, and D. Robertson, Problems on infinite sumset configurations in the integers and beyond, *Bulletin of the American Mathematical Society* **62** (2025), 537–574. doi: 10.1090/bull/1868.

[KMRR26] B. Kra, J. Moreira, F. K. Richter, and D. Robertson, The density finite sums theorem, *Invent. Math.* **243** no. 1 (2026), 1–31. doi: 10.1007/s00222-025-01371-8.

[Kre85] U. Krengel, *Ergodic theorems*, *De Gruyter Studies in Mathematics* **6**, Walter de Gruyter & Co., Berlin, 1985, With a supplement by Antoine Brunel. doi: 10.1515/9783110844641.

[MRR19] J. Moreira, F. K. Richter, and D. Robertson, A proof of a sumset conjecture of Erdős, *Ann. of Math. (2)* **189** no. 2 (2019), 605–652. doi: 10.4007/annals.2019.189.2.4.