THE SPACETIME OF A SHIFT ENDOMORPHISM

VAN CYR, JOHN FRANKS, AND BRYNA KRA

Abstract. The automorphism group of a one dimensional shift space over a finite alphabet exhibits different types of behavior: for a large class with positive entropy, it contains a rich collection of subgroups, while for many shifts of zero entropy, there are strong constraints on the automorphism group. We view this from a different perspective, considering a single automorphism (and sometimes endomorphism) and studying the naturally associated two dimensional shift system. In particular, we describe the relation between nonexpansive subspaces in this two dimensional system and dynamical properties of an automorphism of the shift.

1. Introduction

Suppose Σ is a finite alphabet and $X \subset \Sigma^{\mathbb{Z}}$ is a closed set that is invariant under the left shift $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$. The collection of automorphisms $\mathrm{Aut}(X,\sigma)$, consisting of all homeomorphisms $\phi\colon X \to X$ that commute with σ, forms a group (under composition). A useful approach to understanding a countable group G is knowing if it has subgroups which are isomorphic to (or are homomorphic images of) simpler groups which are relatively well understood, such as matrix groups, and in particular, lattices in classical Lie groups. While the automorphism group of a shift is necessarily countable (as an immediate corollary of the Curtis-Hedlund-Lyndon Theorem [13], any automorphism $\phi\colon X \to X$ is given by a block code), there are numerous results in the literature showing that the automorphism group of the full shift, and more generally any mixing shift of finite type, contains isomorphic copies of many groups: this collection includes, for example, any finite group, the direct sum of countably many copies of $\mathbb{Z}$, the free group on any finite number of generators, and the fundamental group of any 2-manifold (see [13, 4, 15]). In light of these results, it is natural to ask if there is any finitely generated (or even countable) group which fails to embed in any such automorphism group, meaning any group of the form $\mathrm{Aut}(X,\sigma)$. A partial answer is given in [1], where it is shown that if (X,σ) is a subshift of finite type then any group that embeds in the automorphism group must be residually finite. At the other end of the complexity spectrum for (X,σ), there has been recent work showing that $\mathrm{Aut}(X,\sigma)$ is significantly more tame for a shift with very low complexity (see for example [6, 7, 9]).

Instead of viewing the entire group, we focus on the structure inherent in a single automorphism $\phi \in \mathrm{Aut}(X,\sigma)$, as studied for example in [13, 1, 15, 16]. Given an automorphism ϕ, there is an obvious way to associate a $\mathbb{Z}^2$-shift action, which we

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call the spacetime of ϕ (in a slightly different setting, this is called the complete history by Milnor [19] and is referred to as the spacetime diagram in the cellular automata literature). We make use of a particular subset of the spacetime, dubbed the light cone, that is closely related to the notion of causal cone discussed in [19]. We show that the light cone gives a characterization of a well studied structural feature of a $\mathbb{Z}^2$-shift, namely the boundary of a component of expansive subspaces (see [3] and [14]). In particular, in §4 we show that the edges of a light cone for ϕ are always nonexpansive subspaces in its spacetime (the precise statement is given in Theorem 4.4).

We also provide a complement to this result: for many $\mathbb{Z}^2$-subshifts with nonexpansive subspace L, the system is isomorphic to the space time of an automorphism ϕ by an isomorphism which carries L to an edge of the light cone of ϕ.

We then use these structural results to describe obstructions to embedding in the automorphism group of a shift. An important concept in the study of lattices is the idea of a distortion element, meaning an element whose powers have sublinear growth of their minimal word length in some (and hence any) set of generators. In §5, we introduce a notion of range distortion for automorphisms, meaning that the range (see Section 2.1 for the definitions) of the associated block codes of iterates of the automorphism grow sublinearly. An immediate observation is that if an automorphism is distorted in $\mathrm{Aut}(X)$ (in the group sense), then it is also range distorted. We also introduce a measure of non-distortion called the asymptotic spread $A(\phi)$ of an automorphism ϕ and show that the topological entropies of ϕ and σ satisfy the inequality

$$h_{\mathrm{top}}(\phi) \leq A(\phi) h_{\mathrm{top}}(\sigma).$$

This recovers an inequality of Tisseur [24]; his context is more restrictive, covering the full shift endowed with the uniform measure. We do not appeal to measure theoretic entropy and our statement applies to a wider class of shifts.

This inequality proves to be useful in providing obstructions to various groups embedding in the automorphism group. These ideas are further explored in [8].

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2. Background

2.1. **Shift systems and endomorphisms.** We assume throughout that Σ is a finite set (which we call the *alphabet*) endowed with the discrete topology and endow $\Sigma^{\mathbb{Z}}$ with the product topology. For $x \in \Sigma^{\mathbb{Z}}$, we write $x[n] \in \Sigma$ for the value of x at $n \in \mathbb{Z}$.

The *left shift* $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is defined by $(\sigma x)[n] = x[n+1]$, and is a homeomorphism from $\Sigma^{\mathbb{Z}}$ to itself. We say that (X, σ) is a *subshift*, or just a *shift* when the context is clear, if $X \subset \Sigma^{\mathbb{Z}}$ is a closed set that is invariant under the left shift $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$.

Standing assumption: Throughout this article, (X, σ) denotes a shift system and we assume that the alphabet Σ of X is finite and that the shift (X, σ) is infinite, meaning that $|X| = \infty$.

Definition 2.1. An *endomorphism* of the shift (X, σ) is a continuous surjection $\phi\colon X \to X$ such that $\phi \circ \sigma = \sigma \circ \phi$. An *endomorphism* which is invertible is called an *automorphism* The group of all automorphisms of (X, σ) is denoted $\mathrm{Aut}(X, \sigma)$, or simply $\mathrm{Aut}(X)$ when σ is clear from the context. The semigroup of all endomorphisms of X with operation composition is denoted $\mathrm{End}(X, \sigma)$, or simply $\mathrm{End}(X)$. We also observe that $\mathrm{End}(X, \sigma)/\langle\sigma\rangle$, the set of cosets of the subgroup $\langle\sigma\rangle$, is naturally a semigroup with multiplication $\phi\langle\sigma\rangle\psi\langle\sigma\rangle$ defined to be $\phi\psi\langle\sigma\rangle$.

For an interval $[n, n+1, \dots, n+k-1] \in \mathbb{Z}$ and $x \in X$, we let $x[n, \dots, n+k-1]$ denote the element a of Σ^k with $a_j = x[n+j]$ for $j = 0, 1, \dots, k-1$. Define the *words* $\mathcal{L}_k(X)$ *of length* k *in* X to be the collection of all $[a_1, \dots, a_k] \in \Sigma^k$ such that there exist $x \in X$ and $m \in \mathbb{Z}$ with $x[m+i] = a_i$ for $1 \le i \le k$. The length of a word $w \in \mathcal{L}(X)$ is denoted by $|w|$. The language $\mathcal{L}(X) = \bigcup_{k=1}^{\infty} \mathcal{L}_k(X)$ is defined to be the collection of all finite words.

The *complexity* of (X, σ) is the function $P_X\colon \mathbb{N} \to \mathbb{N}$ that counts the number of words of length n in the language of X. Thus

$$P_X(n) = \big|\mathcal{L}_n(X)\big|.$$

The exponential growth rate of the complexity is the *topological entropy* h_{top} of the shift σ. Thus

$$h_{\mathrm{top}}(\sigma) = \lim_{n\to\infty} \frac{\log(P_X(n))}{n}.$$

This is equivalent to the usual definition of topological entropy using (n, ε)-separated sets (see, for example [18]).

A map $\phi\colon X \to X$ is a *sliding block code* if there exists $R \in \mathbb{N}$ such that for any $x, y \in X$ with $x[i] = y[i]$ for $-R \le i \le R$, we have that $\phi(x)[0] = \phi(y)[0]$. The least R such that this holds is called the *range* of ϕ.

By the Curtis-Hedlund-Lyndon Theorem [13], any endomorphism $\phi\colon X \to X$ of a shift (X, σ) is a sliding block code. In particular, $\mathrm{End}(X)$ is always countable.

Definition 2.2. Suppose (X, σ) and (X', σ') are shifts and $\phi \in \mathrm{End}(X, \sigma)$ and $\phi' \in \mathrm{End}(X', \sigma')$ are endomorphisms. We say that ϕ and ϕ' are *conjugate endomorphisms* if there is a homeomorphism $h\colon X \to X'$ such that

$$h \circ \sigma = \sigma' \circ h \text{ and } h \circ \phi = \phi' \circ h.$$

A homeomorphism h satisfying these properties is a sliding block code. If ϕ and ϕ' both lie in $\mathrm{Aut}(X, \sigma)$, then ϕ and ϕ' are conjugate if and only if they are conjugate as elements of the group $\mathrm{Aut}(X, \sigma)$.

A shift X is *irreducible* if for all words $u, v \in \mathcal{L}(X)$, there exists $w \in \mathcal{L}(X)$ such that $uwv \in \mathcal{L}(X)$.

Definition 2.3. A shift (X, σ) is a *subshift of finite type* provided it is defined by a finite set of excluded words. In other words, there is a finite set $\mathcal{F} \subset \mathcal{L}(\Sigma^{\mathbb{Z}})$ such that $x \in X$ if and only if there are no $n \in \mathbb{Z}$ and $k > 0$ such that $x[n, \dots, n+k] \in \mathcal{F}$.

We make use of the following proposition due to Bowen [2]. A proof can be found in [18, Theorem 2.1.8].

Proposition 2.4. *A shift (X, σ) is a shift of finite type if and only if there exists $n_0 \geq 0$ such that whenever $uw, wv \in \mathcal{L}(X)$ and $|w| \geq n_0$, then also $uwv \in \mathcal{L}(X)$.*

2.2. **Higher dimensions.** More generally, one can consider a multidimensional shift $X \subset \Sigma^{\mathbb{Z}^d}$ for some $d \geq 1$, where X is a closed set (with respect to the product topology) that is invariant under the $\mathbb{Z}^d$ action $(T^u x)(v) = x(u+v)$ for $u \in \mathbb{Z}^d$. We refer to X with the $\mathbb{Z}^d$ action as a $\mathbb{Z}^d$-*subshift* and to $\eta \in X$ as an X-*coloring of* $\mathbb{Z}^d$.

We note that we have made a slight abuse of notation in passing to the multidimensional setting by denoting the entries of an element $x \in X$ by $x(u)$ (where $u \in \mathbb{Z}^d$), rather than $x[u]$ as we did for a one dimensional shift. This is done to avoid confusion with interval notation, as we frequently restrict ourselves to the two dimensional case, writing $x(i,j)$ rather than the possibly confusing $x[i,j]$.

Definition 2.5. Suppose $X \subset \Sigma^{\mathbb{Z}^d}$ is a $\mathbb{Z}^d$-subshift, endowed with the natural $\mathbb{Z}^d$-action by translations. If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and $\alpha \colon \mathcal{S} \to \Sigma$, define the *cylinder set*

$$[\mathcal{S}, \alpha] := \{\eta \in X \colon \text{the restriction of } \eta \text{ to } \mathcal{S} \text{ is } \alpha\}.$$

The set of all cylinder sets forms a basis for the topology of X. The *complexity function* for X is the map $P_X \colon \{\text{finite subsets of } \mathbb{Z}^d\} \to \mathbb{N}$ given by

$$P_X(\mathcal{S}) := \left|\{\alpha \in \Sigma^{\mathcal{S}} \colon [\mathcal{S}, \alpha] \neq \emptyset\}\right|$$

which counts the number of colorings of $\mathcal{S}$ which are restrictions of elements of X. If $\alpha \colon \mathcal{S} \to \Sigma$, is the restriction of an element of X we say it *extends uniquely* to an X-coloring if there is exactly one legal $\eta \in X$ whose restriction to $\mathcal{S}$ is α. Similarly, if $\mathcal{S} \subset \mathcal{T} \subset \mathbb{Z}^d$ and if $\alpha \colon \mathcal{S} \to \Sigma$ is such that $[\alpha, \mathcal{S}] \neq \emptyset$, then we say α *extends uniquely* to an X-coloring of $\mathcal{T}$ if there is a unique $\beta \colon \mathcal{T} \to \Sigma$ such that $[\beta, \mathcal{T}] \neq \emptyset$ and the restriction of β to $\mathcal{S}$ is α.

Note that as in the one dimensional setting, the complexity function is translation invariant, meaning that for any $v \in \mathbb{Z}^d$, we have

$$P_X(\mathcal{S}) = P_X(\mathcal{S} + v).$$

2.3. **Expansive subspaces.** An important concept in the study of higher dimensional systems is the notion of an expansive subspace (see Boyle and Lind [3] in particular). For our purposes it suffices to restrict to the case $d = 2$.

Definition 2.6. Suppose $X \subset \Sigma^{\mathbb{Z}^2}$ is a $\mathbb{Z}^2$-subshift and L is a one-dimensional subspace of $\mathbb{R}^2$. We consider $\mathbb{Z}^2 \subset \mathbb{R}^2$ in the standard way. For $r > 0$, define

$$L(r) = \{z \in \mathbb{Z}^2 \colon d(z, L) \leq r\}.$$

We say that the the line L is *expansive* if there exists $r > 0$ such that for any $\eta \in X$, the restriction $\eta|_{L(r)}$ extends uniquely to an X-coloring of $\mathbb{Z}^2$. We call the one-dimensional subspace L *nonexpansive* if it fails to be expansive.

It is also important for us to consider one-sided expansiveness for a subspace L. To define this we need to specify a particular side of a one-dimensional subspace. For this we require an orientation of $\mathbb{R}^2$ (or $\mathbb{Z}^2$) and an orientation of the subspace. We use the standard orientation of $\mathbb{R}^2$ given by the two form $\omega = dx \wedge dy$ or equivalently the orientation for which the standard ordered basis $\{(1,0),(0,1)\}$ is positively oriented.

If L is an *oriented* one-dimenstional subspace of $\mathbb{R}^2$ then the orientation determines a choice of one component L^+ of $L \setminus \{0\}$ which we call the *positive* subset of L. We then denote by $H^+(L)$ the open half space in $\mathbb{R}^2 \setminus L$ with the property that $\omega(v,w) > 0$ for all $v \in L^+$ and $w \in H^+(L)$. Alternatively, $H^+(L)$ is the set of all $w \in \mathbb{R}^2$ such that $\{v,w\}$ is a positively oriented basis of $\mathbb{R}^2$ whenever $v \in L^+$ and $w \in H^+(L)$. Equivalently

$$H^+(L) = \{w \in \mathbb{R}^2 \colon i_v\omega(w) > 0\}$$

whenever $v \in L^+$ and i_v is the interior product. The half space $H^-(L)$ is defined analogously or by $H^-(L) = -H^+(L)$.

Definition 2.7. Suppose L is an oriented one-dimenstional subspace of $\mathbb{R}^2$, i.e. it has a distinguished choice of one component L^+ of $L \setminus \{0\}$. Then L is *positively expansive* if there exists $r > 0$ such that for every $\eta \in X$, the restriction $\eta|_{L(r)}$ extends uniquely to the half space $H^+(L)$. Similarly L is *negatively expansive* if the restriction $\eta|_{L(r)}$ extends uniquely to the half space $H^-(L)$.

Proposition 2.8. *The oriented subspace L is positively expansive if for every $\eta \in X$, the restriction $\eta|_{H^-(L)}$ extends uniquely to an X-coloring of $\mathbb{Z}^2$. Equivalently L fails to be positively expansive if and only if there are colorings $\eta, \nu \in X$ such that $\eta \neq \nu$, but $\eta(i,j) = \nu(i,j)$ for all $(i,j) \in H^-(L)$.*

Proof. Suppose L is positively expansive and $\eta, \nu \in X$ are such that $\eta(i,j) = \nu(i,j)$ for all $(i,j) \in H^-(L)$. Find r such that for any $\xi \in X$, $\xi|_{L(r)}$ extends uniquely to the half-space $H^+(L)$. Let $v \in H^-(L)$ be such that the functions $\eta_v, \nu_v \in X$ defined by $\eta_v(x) = \eta(x+v)$ and $\nu_v(x) = \nu(x+v)$ have the same restriction to $L(r) \cup H^-(L)$. Then by positive expansiveness of L, η_v and ν_v coincide on $H^+(L)$ and hence on all of $\mathbb{Z}^2$. So $\eta_v = \nu_v$ and it follows that $\eta = \nu$. In other words, the restriction of η to $H^-(L)$ extends uniquely to an X-coloring of $\mathbb{Z}^2$.

Now suppose that for all $\eta \in X$ the restriction $\eta|_{H^-(L)}$ extends uniquely to an X-coloring of $\mathbb{Z}^2$. We claim that L is positively expansive. For contradiction, suppose that for all $r > 0$ there exist $\eta_r, \nu_r \in X$ such that $\eta_r|_{L(r)} = \nu_r|_{L(r)}$ but there exists $a_r \in H^+(L)$ such that $\eta_r(a_r) \neq \nu_r(a_r)$. Define

$$B_r = \{(i,j) \in H^+(L) \colon \eta_r(i,j) \neq \nu_r(i,j)\}.$$

Let H be the intersection of all closed half-planes (in $\mathbb{R}^2$) contained in $H^+(L)$ that contain B_r. Fix some $x \in B_r$. These half-planes are linearly ordered by inclusion, all of them are contained in $H^+(L)$, and all of them contain x. Thus their intersection is a closed half-plane (which might not have any integer points on its boundary). Therefore we can find a closed half-plane $J \subseteq H^+(L)$, with integer points on its boundary, that contains H and is such that for all $y \in J \cap \mathbb{Z}^2$ there exists $z \in H \cap \mathbb{Z}^2$ with $\|y - z\| \leq 1$. Choose an integer vector $w_r \in \mathbb{R}^2 \setminus J$ such that there exists $v_r \in B_r \cap \mathbb{Z}^2$ satisfying $\|w_r - v_r\| \leq 2$. Finally, define $\eta_{r,w_r}, \nu_{r,w_r} \in X$ by $\eta_{r,w_r}(y) = \eta_r(y+w_r)$ and $\nu_{r,w_r}(y) = \nu_r(y+w_r)$. Note that although vectors w_r are not bounded, we shift η

and ν so that w_r is moved to the origin. This shift is in the direction taking $H^-(L)$ into itself and thus preserves orientation in $\mathbb{R}^2$, ensuring that the shifted functions still agree on $H^-(L)$. The purpose of the shift is that the point at which the functions disagree now can be bound in a bounded set.) Then $\eta_{r,w_r}|_{H^-(L)} = \nu_{r,w_r}|_{H^-(L)}$ but there exists $t_r \in H^+(L) \cap ([-2,2] \times [-2,2])$ such that $\eta_{r,w_r}(t_r) \neq \nu_{r,w_r}(t_r)$. We pass to a subsequence $r_1 < r_2 < \cdots$ such that t_r is constant. By compactness of X, we can pass if needed to a further subsequence along which $\eta_{r_k,w_{r_k}}$ and $\nu_{r_k,w_{r_k}}$ both converge; call these limiting functions η_∞ and ν_∞. By construction $\eta_\infty(t_{r_1}) \neq \nu_\infty(t_{r_1})$, but $\eta_\infty|_{H^-(L)} = \nu_\infty|_{H^-(L)}$, a contradiction. □

Proposition 2.9. *Assume that $X \subset \Sigma^{\mathbb{Z}^2}$ is a $\mathbb{Z}^2$-subshift and L is a one-dimensional oriented subspace in the u,v-plane. Suppose there is a convex polygon $P \subset \mathbb{R}^2$ such that*

1. *There is a finite set $F \subset \mathbb{Z}^2$ such that P is the convex hull of F.*
2. *There is a unique $e \in F$ which is an extreme point of P and which lies in $H^+(L)$.*
3. *For any $\eta \in X$, the restriction of η to $F \setminus \{e\}$ extends uniquely to F.*

Then L is positively expansive.

Proof. For contradiction, suppose not. Let $\eta, \nu \in X$ be such that $\eta|_{H^-(L)} = \nu|_{H^-(L)}$, but $\eta \neq \nu$. Define $B = \{(i,j) \in H^+(L) \colon \eta(i,j) \neq \nu(i,j)\}$. For each $b \in B$, define $d(b,L)$ to be the distance from b to L and let

$$I = \inf\{d(b,L) \colon b \in B\}.$$

For each $f \in F \setminus \{e\}$, let $d(f,e)$ be the distance between lines L_e and L_f parallel to L that pass through e and f, respectively. Since $e \in H^+(L)$ and $f \notin H^+(L)$, for all $f \in F \setminus \{e\}$, we have $L_e \neq L_f$. Thus

$$\varepsilon := \min\{d(f,e) \colon f \in F \setminus \{e\}\} > 0.$$

If there exists $b \in B$ such that $d(b,L) = I$, then define $\tilde{\eta}, \tilde{\nu} \in X$ by $\tilde{\eta}(x) = \eta(x+b-e)$ and $\tilde{\nu}(x) = \nu(x+b-e)$. Then $\tilde{\eta}|_{H^-(L)} = \tilde{\nu}|_{H^-(L)}$ but $\tilde{\eta}(e) \neq \tilde{\nu}(e)$. This contradicts the fact that the restriction of η to $F \setminus \{e\}$ extends uniquely to an X-coloring of F.

If for all $b \in B$ we have $d(b,L) > I$, then there exists $b \in B$ such that $d(b,L) - I < \varepsilon/2$. Define $\tilde{\eta}, \tilde{\nu} \in X$ by $\tilde{\eta}(x) = \eta(x+b-e)$ and $\tilde{\nu}(x) = \nu(x+b-e)$. Then $\tilde{\eta}(e) \neq \tilde{\nu}(e)$, but $\tilde{\eta}|_{F\setminus\{e\}} = \tilde{\nu}|_{F\setminus\{e\}}$, again a contradiction. □

Examples 2.10.

1. Suppose (X,σ) is a shift and $\phi = \sigma^k$, $k \neq 0$. If L is the line $i = kj$ and $L^+ = L \cap \{(u,v) \colon v > 0\}$, then L is neither positively or negatively expansive, but all other lines are expansive.
2. (Ledrappier's three dot system [17]). With the alphabet $\Sigma = \{0,1\}$, consider the subset of $\Sigma^{\mathbb{Z}^2}$ defined by

$$x(i,j) + x(i+1,j) + x(i,j+1) = 0 \pmod 2$$

for all $i,j \in \mathbb{Z}$. Other than the horizontal axis, the vertical axis, and the reflected diagonal $y = -x$, every one-dimensional subspace is expansive. None of these three subspaces is expansive, but each of them is either positively or negatively expansive.

(3) (Algebraic examples; see [3, 10] for further background). With the alphabet $\Sigma = \{0, 1\}$, consider the subset of $\Sigma^{\mathbb{Z}^2}$ defined by

$$x(i, j) + x(i + 1, j + 1) + x(i - 1, j + 2) = 0 \pmod 2$$

for all $i, j \in \mathbb{Z}$. It is not difficult to see that the subspaces parallel to the sides of the triangle with vertices $(0, 0), (1, 1)$, and $(-1, 2)$ each fail to be one of positively or negatively expansive (but not both). All other one-dimensional subspaces are expansive.

3. The spacetime of an endomorphism

3.1. **ϕ-coding.** We continue to assume that (X, σ) is an infinite shift over the finite alphabet Σ.

Some of the results in this section overlap with results of Nasu [21], where he studies endomorphisms of subshifts that are resolving, which roughly speaking is a notion of being determined. While his language and terminology are different from ours, Lemma 3.4 and Proposition 3.5 correspond to results in Section 6 of [21] and the limiting objects given in Definition 3.12 and some of their properties (portions of Proposition 3.13) are described in Section 9 of [21].

Definition 3.1. If $\phi \in \mathrm{End}(X, \sigma)$ is an endomorphism we say that a subset $A \subset \mathbb{Z}$ *ϕ-codes* (or simply *codes* if ϕ is clear from context) a subset $B \subset \mathbb{Z}$ if for any $x, y \in X$ satisfying $x[a] = y[a]$ for all $a \in A$, it follows that $\phi(x)[b] = \phi(y)[b]$ for all $b \in B$.

We remark that if $\phi \in \mathrm{End}(X, \sigma)$ is an endomorphism then, as ϕ is determined by a block code of some range (say R), the ray $(-\infty, 0]$ ϕ-codes the ray $(-\infty, -R]$. Similarly the ray $[0, \infty)$ ϕ-codes the ray $[R, \infty)$. Of course, it could be the case that $(-\infty, 0]$ ϕ-codes a larger ray than $(-\infty, -R]$. This motivates the following definition:

Definition 3.2. If $\phi \in \mathrm{End}(X, \sigma)$ and $n \geq 0$, let $W^+(n, \phi)$ be the smallest element of $\mathbb{Z}$ such that the ray $[W^+(n, \phi), \infty)$ is ϕ^n-coded by $[0, \infty)$ meaning that if x and y agree on $[0, \infty)$, then necessarily $\phi^n(x)$ and $\phi^n(y)$ agree on $[W^+(n, \phi), \infty)$ and this is the largest ray with that property. Similarly $W^-(n, \phi)$ is the largest element of $\mathbb{Z}$ such that the ray $(-\infty, W^-(n, \phi)]$ is ϕ^n-coded by $(-\infty, 0]$. When ϕ is clear from the context, we omit it from the notation and denote $W^+(n, \phi)$ and $W^-(n, \phi)$ by $W^+(n)$ and $W^-(n)$, respectively.

Note that for $n \geq 1$ we have $W^+(n, \phi) = W^+(1, \phi^n)$ and $W^-(n, \phi) = W^-(1, \phi^n)$. These quantities have been studied in [22, 24] in order to define Lyapunov exponents for cellular automata, and then used to study the speed of propagation of perturbations with respect to a shift invariant measure. They use this to give bounds on the entropy of the measure in terms of these (left and right) Lyapunov exponents. We do not consider the role of an invariant measure in this article, but we give an estimate for topological entropy closely related to a result of [24] (see our Theorem 5.13 below).

The following result, known as Schwartzman's Theorem, has been frequently rediscovered. It was proven in Schwartzman's Thesis [23] and an account of the result seems to have first been published in [12] (see [3] for further references.)

Theorem 3.3. *If X is an infinite one-dimensional subshift, then there exist $x, y \in X$ such that $x[0,\infty) = y[0,\infty)$ but $x[-1] \neq y[-1]$, and there exist $w, z \in X$ such that $w(-\infty, 0] = z(-\infty, 0]$ but $w[1] \neq z[1]$.*

We check that $W^+(n,\phi)$ and $W^-(n,\phi)$ are well-defined:

Lemma 3.4. *If X is infinite, then $W^+(n,\phi) > -\infty$ and $W^-(n,\phi) < \infty$.*

Proof. This is an immediate consequence of Schwartzman's Theorem!3.3.. □

By Lemma 3.4, the function $\Theta_n^+ \colon \Sigma^{[0,\infty)} \to \Sigma^{[W^+(n,\phi),\infty)}$ defined by

$$\Theta_n^+(x[0,\infty)) = \phi^n(x)[W^+(n,\phi),\infty)$$

is well defined for all $n \geq 0$, as is the analogous function $\Theta_n^- \colon \Sigma^{(-\infty,0]} \to \Sigma^{(-\infty,W^-(n,\phi)]}$. These functions are continuous:

Proposition 3.5. *The functions Θ_n^+ and Θ_n^- are continuous. In particular, there exists $k = k(n,\phi) > 0$ such that $[0,k]$ ϕ^n-codes $\{W^+(n,\phi)\}$ and $[-k,0]$ ϕ^n-codes $\{W^-(n,\phi)\}$.*

Proof. Assume Θ_n^+ is not continuous. Then there exist x_j and y in X and $r \geq W^+(n,\phi)$ such that $x_j[0,m_j] = y[0,m_j]$, for a sequence $\{m_j\}$ with $\lim_{j\to\infty} m_j = \infty$, and such that $\phi(x_n)[r] \neq \phi(y)[r]$. By passing to a subsequence, w can assume that there exists $z \in X$ with $\lim_{n\to\infty} x_n = z$. Clearly $z[0,\infty) = y[0,\infty)$ and hence $\phi^n(z)[W^+(n,\phi),\infty) = \phi(y)[W^+(n,\phi),\infty)$. In particular, $\phi(z)[r] = \phi(y)[r]$, and so by continuity of ϕ we conclude that $\lim_{n\to\infty} \phi(x_n)[r] = \phi(z)[r] = \phi(y)[r]$. But since $\phi(x_n)[r] \neq \phi(y)[r]$, we also have that $\lim_{n\to\infty} \phi(x_n)[r] \neq \phi(y)[r]$, a contradiction. Thus Θ_n^+ is continuous, and a similar argument shows that Θ_n^- is continuous. □

3.2. **The spacetime of ϕ.**

Definition 3.6. If $\phi \in \mathrm{End}(X,\sigma)$ is an endomorphism, its *ϕ-spacetime* $\mathcal{U} = \mathcal{U}(\phi)$ is a $\mathbb{Z}^2$-subshift together with a preferred ordered basis for $\mathbb{Z}^2$ which defines what we call the "horizontal" and "vertical" directions. It is defined to be the closed subset of $x \in \Sigma^{\mathbb{Z}^2}$ such that for all $i \in \mathbb{Z}$ and $j \geq 0$ $\phi^j(x)[i] = x(i,j)$.

Thus the rows of $\mathcal{U}$ are elements of X with row n equal to ϕ of row $n-1$. There is an action of $\mathbb{Z}^2$ on $\mathcal{U}$ given by having (i,j) shift i times in the horizontal direction and j times in the vertical direction. A vertical shift by $j \geq 0$ can also be viewed as applying ϕ^j to each row of $\mathcal{U}$.

It follows immediately from the definition of expansiveness (Definition 2.6) that the horizontal axis in a spacetime $\mathcal{U}$ of an automorphism is always an expansive subspace for the $\mathbb{Z}^2$-subshift $\mathcal{U}$ with the $\mathbb{Z}^2$-action by translations. Also if L is the horizontal axis in the spacetime of an endomorphism and L^+ is the intersection of

L with the positive horizontal axis, then $H^+(L)$ is the upper half space and L is positively expansive.

Note that given a spacetime $\mathcal{U}$ (including the preferred basis of $\mathbb{Z}^2$), one can extract the shift (X, σ) by taking X to be the Σ-colorings of $\mathbb{Z}$ obtained by restricting the colorings in $\mathcal{U}$ to the i-axis ($j = 0$). Likewise, one can extract the endomorphism ϕ by using the fact that if $y \in \mathcal{U}$ and $x \in X$ is given by $x[i] = y[i, 0]$, then $\phi(x)[i] = y[i, 1]$.

A concept somewhat more general than our notion of spacetime is defined in Milnor [19] and referred to as the *complete history* of a cellular automaton. Our context is narrower, using the spacetime to study a single endomoprhism rather than the full system. However, there are analogs in our deveopment; Milnor defines an m-step forward cone, which corresponds to our interval $[W^-(m, \phi), W^+(m, \phi)]$, his definition of a limiting forward cone corresponds to our asymptotic light cone, and the case $n_0 = 0$ of Theorem 3.22 corresponds to results in Milnor.

We say that spacetimes $\mathcal{U}$ and $\mathcal{U}'$, which share the same alphabet Σ, are *isomorphic* if there is a homeomorphism $h \colon \mathcal{U} \to \mathcal{U}'$ such that

$$h(z)(i', j') = z(i, j),$$

where the isomorphism of $\mathbb{Z}^2$ for which $(i, j) \mapsto (i', j')$ is given by sending the preferred basis of $\mathbb{Z}^2$ for $\mathcal{U}$ to the preferred basis of $\mathcal{U}'$. (Note that the assumption that the spacetimes share the same alphabet is not necessary, but simplifies our notation.) It is straightforward to check that $\phi, \phi' \in \mathrm{Aut}(X)$ are conjugate automorphisms (see Definition 2.2) if and only if their respective spacetimes with the obvious preferred bases are isomorphic.

We extend definition 3.1 of coding to a spacetime:

Definition 3.7. If $\mathcal{U}$ is a $\mathbb{Z}^2$-subshift, we say that a subset $A \subset \mathbb{Z}^2$ *codes* a subset $B \subset \mathbb{Z}^2$ if for any $x, y \in \mathcal{U}$ satisfying $x(i, j) = y(i, j)$ for all $(i, j) \in A$, it follows that $x(i', j') = y(i', j')$ for all $(i', j') \in B$. Equivalently if x and y differ at some point of B, they also differ at some point of A.

Definition 3.8 (Light Cone). The *future light cone* $\mathcal{C}_f(\phi)$ *of* $\phi \in \mathrm{End}(X)$ is defined to be

$$\mathcal{C}_f(\phi) = \{(i, j) \in \mathbb{Z}^2 \colon W^-(j, \phi) \le i \le W^+(j, \phi),\ j \ge 0\}$$

The *past light cone* $\mathcal{C}_p(\phi)$ *of* ϕ is defined to be $\mathcal{C}_p(\phi) = -\mathcal{C}_f(\phi)$. The *full light cone* $\mathcal{C}(\phi)$ is defined to be $\mathcal{C}_f(\phi) \cup C_p(\phi)$.

The rationale for this terminology is that if $x \in X$ and $j > 0$, then a change in the value of $x(0)$ (and no other changes) can only cause a change in $\phi^j(x)[i]$, $j \ge 0$ if (i, j) lies in the future light cone of ϕ. Similarly if $\phi^j(y) = x$, $\jmath \ge 0$, then a change in $y[i]$ can only affect $x[0]$ if $(i, -j)$ lies in the past light cone of ϕ.

The light cone is naturally stratified into levels: define the n^{th} *level of* $\mathcal{C}(\phi)$ to be the set

$$\mathcal{I}(n, \phi) := \{i \in \mathbb{Z} \colon (i, n) \in \mathcal{C}(\phi)\}. \tag{3.1}$$

In Corollary 3.23 below, we show that if σ is a subshift of finite type and n is large, then the horizontal interval in the light cone at level $-n$ i.e., $\mathcal{I}(-n,\phi)$, is the unique minimal interval which ϕ^n-codes $\{0\}$, provided ϕ has infinite order in $\mathrm{End}(X,\sigma)/\langle\sigma\rangle$.

In general, it is not clear if $\phi \in \mathrm{Aut}(X)$, what the relationship, if any, between $\mathcal{C}(\phi)$ and $\mathcal{C}(\phi^{-1})$ is. However there are some restrictions given in Part (5) of Proposition 3.13.

Remark 3.9. A comment about notation is appropriate here. We are interested in subsets of the i,j-plane. Our convention is that i is the abscissa, or first coordinate, and we consider the i-axis to be horizontal. Likewise j is the ordinate, or second coordinate, and we consider the j-axis to be vertical. However some subsets of the plane we consider are naturally described as graphs of a function $i = f(j)$. For example, we frequently consider lines given by an equation like $i = \alpha j,\ j \in \mathbb{R}$, and think of α as a "slope" even in standard parlance it would be the reciprocal of the slope of the line $i = \alpha j$.

Our next goal is to study the asymptotic behavior of $W^+(j,\phi)$ and $W^-(j,\phi)$ for a fixed $\phi \in \mathrm{End}(X)$. We start by recalling Fekete's Lemma, which is then applied to the sequence $W^+(n) = W^+(n,\phi)$ for $n \geq 0$ which is shown to be subadditive.

Lemma 3.10 (Fekete's Lemma [11])**.** *If the sequence $a_n \in \mathbb{R},\ n \in \mathbb{N}$, is subadditive (meaning that $a_n + a_m \geq a_{m+n}$ for all $m,n \in \mathbb{N}$), then*
$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}.$$

We note a simple, but useful, consequence of this: if $s(n)$ is subadditive, and if $\lim\limits_{n\to\infty} \dfrac{s(n)}{n} \geq 0$, then $s(n) \geq 0$ for all $n \geq 1$ as otherwise $\inf\limits_{m\geq 1} \dfrac{s(m)}{m}$ would be negative.

Lemma 3.11. *If $\phi,\psi \in \mathrm{End}(X,\sigma)$ then $W^+(1,\phi\psi) \leq W^+(1,\phi) + W^+(1,\psi)$ and similarly $W^-(1,\phi\psi) \geq W^-(1,\phi) + W^-(1,\psi)$. In particular the sequences $\{W^+(n,\phi)\}$ and $\{-W^-(n,\phi)\},\ n \geq 0$, are subadditive.*

Proof. The ray $[0,\infty)$ ψ-codes $[W^+(1,\psi),\infty)$ and the ray $[W^+(1,\psi),\infty)$ ϕ-codes $[W^+(1,\phi) + W^+(1,\psi),\infty)$. Hence $[0,\infty)$ $\phi\psi$-codes $[W^+(1,\phi) + W^+(1,\psi),\infty)$ so $W^+(1,\phi\psi) \leq W^+(1,\phi) + W^+(1,\psi)$. This proves the first assertion.

Replacing ϕ by ϕ^m and ψ by ϕ^n in this inequality gives
$$W^+(1,\phi^{n+m}) \leq W^+(1,\phi^m) + W^+(1,\phi^n).$$
Since for $n \geq 1$ we have $W^+(n,\phi) = W^+(1,\phi^n)$ we conclude that $W^+(m+n,\phi) \leq W^+(m,\phi) + W^+(n,\phi)$, so $\{W^+(n,\phi)\}$ is subadditive. The proof for W^- is similar. ☐

We now want to consider two quantities which measure the asymptotic behavior of $W^\pm(n,\phi)$. These quantities (and other closely related ones) have been considered in [22, 24] in the context of measure preserving cellular automata and are referred to there as Lyapunov exponents of the automaton. If we fix ϕ and abbreviate $W^+(n,\phi)$ by $W^+(n)$ then Fekete's Lemma and Lemma 3.11, imply the limit $\lim\limits_{n\to\infty} \dfrac{W^+(n)}{n}$ exists.

Definition 3.12. We define

$$\alpha^+(\phi) := \lim_{n\to\infty} \frac{W^+(n)}{n}$$

and

$$\alpha^-(\phi) := \lim_{n\to\infty} \frac{W^-(n)}{n}.$$

Note that the limit $\alpha^+(\phi)$ is finite, since if $D \geq \text{range}(\phi)$, then for $j \geq 0$ we have $|W^+(j))| \leq Dj$ (and $|W^-(j))| \leq Dj$). As a consequence, we conclude that

$$W^+(n) = n\alpha^+(\phi) + \mathrm{o}(n). \tag{3.2}$$

This describes an important asymptotic property of the right light cone boundary function $W^+(n)$ used in the proof of Theorem 4.4 below. That theorem says that if $\alpha^+ = \alpha^+(\phi)$, then the line $i = \alpha^+ j$ is a nonexpansive subspace of $\mathbb{R}^2$ for the spacetime of ϕ.

Similarly, we can consider $W^-(n)$ and obtain a second nonexpansive subspace namely the line $x = \beta y$ where

$$\beta = \alpha^-(\phi) := \lim_{n\to\infty} \frac{W^-(n)}{n}.$$

As a consequence, we conclude that the left light cone boundary function satisfies

$$W^-(n) = n\alpha^-(\phi) + \mathrm{o}(n). \tag{3.3}$$

We list some elementary properties of the limits $\alpha^+(\phi)$ and $\alpha^-(\phi)$:

Proposition 3.13. *If $\phi \in \operatorname{End}(X,\sigma)$ then*

(1) *For all $k \in \mathbb{Z}$, $\alpha^-(\sigma^k\phi) = \alpha^-(\phi) + k$ and $\alpha^+(\sigma^k\phi) = \alpha^+(\phi) + k$.*
(2) *For all $m \in \mathbb{N}, \alpha^+(\phi^m) = m\alpha^+(\phi)$ and $\alpha^-(\phi^m) = m\alpha^-(\phi)$*
(3) *If X is infinite, then $\alpha^-(\phi) \leq \alpha^+(\phi)$*
(4) *If $\phi, \psi \in \operatorname{Aut}(X,\sigma)$ are commuting endomorphisms then*

$$\alpha^+(\phi\psi) \leq \alpha^+(\phi) + \alpha^+(\psi) \text{ and } \alpha^-(\phi\psi) \geq \alpha^-(\phi) + \alpha^-(\psi).$$

(5) *If ϕ is an automorphism and X is infinite, then*

$$\alpha^+(\phi) + \alpha^+(\phi^{-1}) \geq 0 \text{ and } \alpha^-(\phi) + \alpha^-(\phi^{-1}) \leq 0.$$

Proof. Since

$$W^+(n, (\sigma^k\phi)) = W^+(1, \sigma^{nk}\phi^n) = W^+(1, \phi^n) + nk = W^+(n, \phi) + nk,$$

property (1) follows. Since

$$\lim_{n\to\infty} \frac{W^+(mn,\phi)}{n} = m \lim_{n\to\infty} \frac{W^+(mn,\phi)}{mn} = m\alpha^+(\phi),$$

property (2) follows.

To show (3), we proceed by contradiction and assume that $\alpha^-(\phi) > \alpha^+(\phi)$. Since $W^+(n) = n\alpha^+(\phi) + \mathrm{o}(n)$ and $W^-(n) = n\alpha^-(\phi) + \mathrm{o}(n)$, the assumption that $\alpha^-(\phi) > \alpha^+(\phi)$ implies that $W^-(n) > W^+(n)$ for all sufficiently large n. By Proposition 3.5, Θ_n^+ is continuous and so for each fixed $n \in \mathbb{N}$, there exists $R > 0$ such that the interval $[0, R]$ ϕ^n-codes $\{W^+(n)\}$. Therefore the interval $[0, R{+}t]$ ϕ^n-codes the interval $[W^+(n), W^+(n)+t-1]$. Similarly there exists $R' > 0$ such that the interval $[-R', 0]$ ϕ^n codes $\{W^-(n)\}$ and so the interval $[0, R'+t]$ ϕ^n-codes the interval $[W^-(n)+R', W^-(n)+R'+t-1]$. Therefore for $t > W^+(n) - W^-(n) + R' - 1$, the interval $[0, R+t]$ ϕ^n-codes the interval $[W^+(n), W^-(n)+R+t]$ (in other words, the two intervals above overlap and so their union is an interval). For any such t, an interval of length $R{+}t{+}1$ ϕ^n-codes an interval of length $W^-(n){-}W^+(n){+}R{+}t{+}1 > R{+}t{+}1$. However this means that

$$P_X(R+t+1) \geq P_X(R+t+1+W^-(n)-W^+(n)),$$

as every word of length $R+t+1+W^-(n)-W^+(n)$ is ϕ^n-determined by some word of length $R+t+1$. Since $P_X(n)$ is a nondecreasing function, we have $P_X(R+t+1) = P_X(R+t+1+W^-(n)-W^+(n))$. Therefore any word in $\mathcal{L}(X)$ of length $R+t+1$ extends uniquely to the right to a word of greater length, namely of length $R+t+1+W^-(n)-W^+(n)$. Hence it extends uniquely to the infinite ray to the right. Similarly it extends uniquely to the infinite ray to the left. It follows that X contains at most $P_X(R+t+1)$ points, contradicting our standing assumption that X is infinite. This establishes (3).

To prove (4) we note that

$$\begin{aligned} W^+(n, \phi\psi) &= W^+(1, (\phi\psi)^n) = W^+(1, \phi^n\psi^n) \\ &\leq W^+(1, \phi^n) + W^+(1, \psi^n) = W^+(n, \phi) + W^+(n, \psi). \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \frac{W^+(n, \phi\psi)}{n} \leq \lim_{n\to\infty} \frac{W^+(n, \phi)}{n} + \lim_{n\to\infty} \frac{W^+(n, \psi)}{n}$$

giving the inequality of item (4). The result for α^- is similar.

Item (5) follows immediately from (4) if we replace ψ with ϕ^{-1}, since $\alpha^+(id) = \alpha^-(id) = 0$. $\square$

Other than the restriction that $\alpha^-(\phi) \leq \alpha^+(\phi)$, any rational values can be taken on for some automorphism of the full shift:

Example 3.14. We show that given rationals $r_1 \leq r_2$, there is a full shift (X, σ) with an automorphism ϕ such that $\alpha^-(\phi) = r_1$ and $\alpha^+(\phi) = r_2$.

Suppose that $r_2 = p_2/q_2 \geq 0$. Consider X_2 the Cartesian product of q_2 copies of the full two shift $\sigma\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$. Define an automorphism ϕ_0 by having it cyclically permuting the copies of $\{0,1\}^{\mathbb{Z}}$ and perform a shift on one of them. Then $\phi_0^{q_2} = \sigma_2\colon X_2 \to X_2$ is the shift (indeed a full shift on an alphabet of size 2^{q_2}). Since $\alpha^+(\sigma_2) = \alpha^-(\sigma_2) = 1$, it follows from parts (1) and (2) of Proposition 3.13 that $\alpha^+(\phi_0) = \alpha^-(\phi_0) = 1/q_2$. Setting $\phi_2 = \phi_0^{p_2}$, we have that $\alpha^+(\phi_2) = p_2\alpha^+(\phi_2) = p_2/q_2 = r_2$. Similarly $\alpha^-(\phi_2) = r_2$. If $r_2 = -p_2/q_2 < 0$ we can do the same construction, defining ϕ_0 to cyclically permute the copies of Σ_2 but use the inverse shift (instead of the shift) on one of the copies. Then $\phi^{q_2} = \sigma_2^{-1}\colon X_2 \to X_2$. In this way we still construct ϕ_2 with $\alpha^+(\phi_2) = \alpha^-(\phi_2) = r_2$.

By the same argument we can construct an automorphism ϕ_1 of (X_1, σ_1) such that $\alpha^+(\phi_1) = \alpha^-(\phi_1) = r_1$. Taking X to be the Cartesian product $X_1 \times X_2$ and considering the (full) shift $\sigma = \sigma_1 \times \sigma_2 \colon X \to X$, and the automorphism $\phi = \phi_1 \times \phi_2$, it is straightforward to check that $\alpha^+(\phi) = \alpha^+(\phi_2) = r_2$ and $\alpha^-(\phi) = \alpha^-(\phi_1) = r_1$.

In light of the work on Lyapunov exponents for cellular auomata, it is natural to ask for a general shift σ and $\phi \in Aut(X, \sigma)$ which conditions on ϕ and/or σ suffice for the existence of a σ-invariant ϕ-ergodic measure μ such that $\alpha^{\pm}(\phi)$ are Lyapunov exponents in the sense defined by [22, 24].

3.3. **Two dimensional coding.**

Remark 3.15. We thank Samuel Petite for suggesting the short proof of the following lemma (in an earlier version of this paper we had a longer proof of this lemma).

Lemma 3.16. *Let $\varphi \in \mathrm{End}(X)$ and suppose that there exists K such that* $\mathrm{range}(\varphi^n) \leq K$ *for infinitely many n. Then φ has finite order.*

Proof. There are only finitely many block maps of range $\leq K$ and so, by the pigeonhole principle, there exist $0 < m < n$ such that $\varphi^m = \varphi^n$. It follows that φ^{n-m} is the identity. □

Recall that the interval $\mathcal{I}(n, \phi)$ is defined in equation (3.1) to be $\{i \in \mathbb{Z} \colon (i, n) \in \mathcal{C}(\phi)\}$. Thus for $n \in \mathbb{N}$, we have $|\mathcal{I}(-n, \phi)| = W^+(n, \phi) - W^-(n, \phi) + 1$, is the width of the n^{th} level of the light cone for ϕ.

Lemma 3.17. *Suppose ϕ is an endomorphism of the shift (X, σ) and $n \geq 0$. If J is any interval in $\mathbb{Z}$ which ϕ^n-codes $\{0\}$, then $J \supset \mathcal{I}(-n, \phi)$.*

Proof. If the interval $J = [a, b]$ ϕ^n-codes $\{0\}$, then $[a, \infty)$ ϕ^n-codes $[0, \infty)$, and so $[0, \infty)$ ϕ^n-codes $[-a, \infty)$. It follows that $-a \geq W^+(n, \phi)$ and hence $a \leq -W^+(n, \phi)$. Similarly $b \geq -W^-(n, \phi)$ and so $\mathcal{I}(-n, \phi) \subset [a, b]$. □

Lemma 3.18. *Assume ϕ is an endomorphism of a shift of finite type (X, σ) and suppose that*

$$\lim_{n \to \infty} \big|\mathcal{I}(-n, \phi)\big| = +\infty.$$

Then there is n_0 such that whenever $n \geq n_0$, the interval $\mathcal{I}(-n, \phi)$ ϕ^n-codes $\{0\}$. Moreover, if σ is a full shift we can take n_0 to be 0 and the hypothesis $\lim_{n\to\infty} |\mathcal{I}(-n, \phi)| = \infty$ is unnecessary.

Proof. Suppose that ϕ is an endomorphism and σ is a subshift of finite type. Then by Proposition 2.4, there exists $m_0 \geq 0$ such that if w is a word of length at least m_0 and if $w_1^- w w_1^+$ and $w_2^- w w_2^+$ are elements of X for some semi-infinite words $w_i^{\pm}$, then both $w_1^- w w_2^+$ and $w_2^- w w_1^+$ are elements of X. Clearly $m_0 = 0$ suffices if σ is a full shift.

By hypothesis,

$$\lim_{n \to \infty} \big|\mathcal{I}(-n, \phi)\big| = \lim_{n \to \infty} \big|W^+(n, \phi) - W^-(n, \phi)\big| + 1 = +\infty,$$

and so we can choose n_0 such that the length of $\mathcal{I}(-n,\phi)$ is greater than m_0 when $n \geq n_0$. Suppose $n \geq n_0$ and that $x, y \in X$ agree on the interval $\mathcal{I}(-n,\phi)$. We show that $\phi^n(x)[0] = \phi^n(y)[0]$. Let $w = x[-W^+(n,\phi), -W^-(n,\phi)] = y[-W^+(n,\phi), -W^-(n,\phi)]$ and define $w_i^\pm$ by $x(-\infty,\infty) = w_1^- w w_1^+$ and $y(-\infty,\infty) = w_2^- w w_2^+$. Then $w_1^- w w_2^+$ is an element of X satisfying $x[i] = w_1^- w w_2^+[i]$ for all $i \leq -W^-(n,\phi)$ and $y[i] = w_1^- w w_2^+[i]$ for all $i \geq -W^+(n,\phi)$. It follows that $\phi^n(x)[0] = \phi^n(w_1^- w w_2^+)[0]$ and that $\phi^n(y)[0] = \phi^n(w_1^- w w_2^+)[0]$. Hence $\phi^n(x)[0] = \phi^n(y)[0]$ and $\{0\}$ is ϕ^n-coded by $[-W^+(n,\phi), -W^-(n,\phi)]$.

□

Definition 3.19. Let X be a subshift and let $\phi \in \mathrm{End}(X,\sigma)$. Define $r(n,\phi)$ to be the minimal width of an interval which ϕ^n-codes $\{0\}$.

Lemma 3.20. *Suppose (X,σ) is a subshift of finite type and $\phi \in \mathrm{End}(X,\sigma)$. Then there is a constant $C(\phi)$ such that $|\mathcal{I}(-n,\phi)| \leq r(n,\phi) \leq |\mathcal{I}(-n,\phi)| + C(\phi)$. If X is a full shift we can take $C(\phi) = 0$.*

Proof. The first inequality follows immediately from Lemma 3.17. We prove the second inequality by contradiction. Thus suppose that for any C, there exist infinitely many $n \in \mathbb{N}$ and points $x_{C,n} \neq y_{C,n}$ which agree on the interval $[-W^+(n,\phi), -W^-(n,\phi) + C]$ but are such that $\phi^n(x_{C,n})[0] \neq \phi^n(y_{C,n})[0]$.

Recall from Proposition 2.4 that there exists a constant n_0 (depending on the subshift X) such that if $x, y \in X$ agree for n_0 consecutive places, say $x[i] = y[i]$ for all $p \leq i < p + n_0$, then the $\mathbb{Z}$-coloring whose restriction to $(-\infty, p + n_0 - 1]$ coincides with that of x and whose restriction to $[p + n_0, \infty)$ coincides with that of y, is an element of X.

Choose $C > n_0$. By assumption, there exist infinitely many $n \in \mathbb{N}$ and points $x_n, y_n \in X$ which agree on $[-W^+(n,\phi), -W^-(n,\phi)+C]$ but are such that $\phi^n(x_n)[0] \neq \phi^n(y_n)[0]$. Let $z \in X$ be the $\mathbb{Z}$-coloring whose restriction to $(-\infty, -W^-(n,\phi) + C]$ coincides with x_n and whose restriction to $[-W^-(n,\phi)+C+1, \infty)$ coincides with y_n. Then since $C > n_0$, we have that $z \in X$. Since z agrees with y_n on $[-W^+(n,\phi), \infty)$, it follows that $(\phi^n z)[0] = (\phi^n y_n)[0]$. On the other hand, $(\phi^n z)[0] = (\phi^n x_n)[0]$, since z agrees with x_n on $(-\infty, -W^-(n,\phi)]$. But this contradicts the fact that $(\phi^n x_n)[0] \neq (\phi^n y_n)[0]$, and so $C(\phi)$ exists. □

Recall that in any semigroup G an element g has *finite order* if $\{g^n\}_{n \geq 1}$ is finite and otherwise g has *infinite order.*

Proposition 3.21. *Suppose X is a subshift of finite type and $\phi \in \mathrm{End}(X,\sigma)$. If*

$$\liminf_{n\to\infty} |\mathcal{I}(-n,\phi)| < \infty,$$

then ϕ has finite order in $\mathrm{End}(X,\sigma)/\langle\sigma\rangle$.

Proof. By hypothesis, there exists M such that $|\mathcal{I}(-n,\phi)| < M$ for infinitely many n. By Lemma 3.20, there is a constant $C(\phi)$ such that $r(n,\phi) < M + C(\phi)$ for infinitely many n. Let $n_1 < n_2 < \cdots$ be a subsequence along which $r(n_i,\phi) < M + C(\phi)$ is constant; define this constant to be R. Then for each $i = 1, 2, \ldots$ there is an interval

$[a_i, b_i]$ of length R which ϕ^{n_i}-codes $\{0\}$. Therefore the interval $[0, R]$ $(\sigma^{-a_i}\phi^{n_i})$-codes $\{0\}$ for all i. It follows that $\sigma^{-a_i}\phi^{n_i}$ is a block map of range R for all i. There are only finitely many block maps of range R, so there must exist $i_1 < i_2$ such that $\sigma^{-a_{i_1}}\phi^{n_{i_1}} = \sigma^{-a_{i_2}}\phi^{n_{i_2}}$ or simply

$$\phi^{n_{i_1}}(x) = \sigma^{a_{i_1}-a_{i_2}}\phi^{n_{i_2}}(x) = \sigma^{a_{i_1}-a_{i_2}}\phi^{n_{i_2}-n_{i_1}}(\phi^{n_{i_1}}(x))$$

for all $x \in X$. Since $\phi^{n_{i_1}}$ is a surjection, we have

$$y = \sigma^{a_{i_1}-a_{i_2}}\phi^{n_{i_2}-n_{i_1}}(y)$$

for all $y \in X$. In other words, $\phi^{n_{i_2}-n_{i_1}} = \sigma^{a_{i_2}-a_{i_1}}$. □

Theorem 3.22. *Assume that ϕ is an endomorphism of a shift of finite type (X, σ) and that ϕ has infinite order in $\mathrm{End}(X)/\langle\sigma\rangle$. Then there exists n_0 such that whenever $n \geq n_0$, the interval $\mathcal{I}(-n, \phi)$ ϕ^n-codes $\{0\}$. If σ is a full shift, we can take n_0 to be 0.*

Proof. If σ is a full shift, the result follows from Lemma 3.18. Otherwise, since ϕ has infinite order in $\mathrm{End}(X)/\langle\sigma\rangle$, Proposition 3.21 tells us that

$$\lim_{n\to\infty} |\mathcal{I}(-n, \phi)| = +\infty.$$

Thus we can apply Lemma 3.18 to conclude that $\mathcal{I}(-n, \phi)$ ϕ^n-codes $\{0\}$. □

Corollary 3.23. *If ϕ has infinite order in $\mathrm{End}(X)/\langle\sigma\rangle$, then for n sufficiently large, $\mathcal{I}(-n, \phi)$ is the unique minimal interval which ϕ^n-codes $\{0\}$.*

Proof. The fact that $\mathcal{I}(-n, \phi)$ ϕ^n-codes $\{0\}$ for large n follows from Theorem 3.22. Minimality and uniqueness follow from Lemma 3.17. □

Question 3.24. *Is the hypothesis that (X, σ) is an SFT necessary in Theorem 3.22?*

4. The light cone and nonexpansive subspaces

The main result of this section is Theorem 4.4: it states that the line $u = \alpha^+(\phi)v$ in the u, v-plane is a nonexpansive subspace of $\mathbb{R}^2$ for the spacetime of ϕ. The analogous statement holds in the other direction: the line $u = \alpha^-(\phi)v$ in the u, v-plane is a nonexpansive subspace.

4.1. **The deviation function.** We begin by investigating the properties of the function which measures the deviation of $W^+(n, \phi)$ from $\alpha^+(\phi)n$.

Definition 4.1. Suppose $\phi \in \mathrm{End}(X, \sigma)$. For $n \geq 0$ define the positive and negative *deviation functions* $\delta^+(n) = \delta^+(n, \phi)$ and $\delta^-(n) = \delta^-(n, \phi)$ by $\delta^+(n) = W^+(n) - n\alpha^+(\phi)$ and $\delta^-(n) = W^-(n) - n\alpha^-(\phi)$

Lemma 4.2. *Suppose $\delta^+(n)$ and $\delta^-(n)$ are the deviation functions associated to ϕ. Then*

(1) *The functions $\delta^+(n)$ and $-\delta^-(n)$ are subadditive.*
(2) *The deviation functions satisfy $\displaystyle\lim_{n\to\infty}\frac{\delta^+(n)}{n} = 0$ and $\displaystyle\lim_{n\to\infty}\frac{\delta^-(n)}{n} = 0$.*

(3) *For all* $n \geq 0$, *we have* $\delta^+(n) \geq 0$ *and* $\delta^-(n) \leq 0$.

Proof. Since $\delta^+(n)$ is the sum of the subadditive function $W^+(n) = W^+(\phi^n)$ and the additive function $-n\alpha$, part (1) follows. Since

$$\lim_{n\to\infty} \frac{\delta^+(n)}{n} = \lim_{n\to\infty} \frac{W^+(n) - n\alpha^+(\phi)}{n} = \lim_{n\to\infty} \frac{W^+(n)}{n} - \alpha^+(\phi) = 0,$$

part (2) follows.

To see part (3), observe that parts (1) and (2) together with Fekete's Lemma (Lemma 3.10) imply

$$\inf_{n\geq 1} \frac{\delta^+(n)}{n} = 0$$

and so $\delta^+(n) < 0$ is impossible. The analogous results for $\delta^-(n)$ are proved similarly. □

Lemma 4.3. *Let* $\mathcal{U}$ *be the* ϕ*-spacetime of* (X, σ) *for* $\phi \in \mathrm{End}(X)$ *and let* $\alpha = \alpha^+(\phi)$ *and* $\delta(n) = \delta^+(n, \phi)$. *Suppose that* $\alpha \geq 0$ *and the deviation* $\delta(n)$ *is unbounded for* $n \geq 0$. *Then there exist two sequences* $\{x_m\}_{m\geq 1}$ *and* $\{y_m\}_{m\geq 1}$ *in* $\mathcal{U}$ *such that*

(1) $x_m(i,j) = y_m(i,j)$ *for all* (i,j) *with* $-m \leq j \leq 0$ *and* $i \geq \alpha j$
(2) $x_m(i,j) = y_m(i,j)$ *for all* (i,j) *with* $j \geq 0$ *and* $i \geq (\alpha + \frac{1}{m})j$
(3) $x_m(-1,0) \neq y_m(-1,0)$ *for all* $m \in \mathbb{N}$.

The analogous result for $\alpha^-(\phi)$ *and* $\delta^-(n,\phi)$ *also holds.*

Proof. For notational simplicity, denote $W^+(n)$ by $W(n)$, so $\delta(n) = W(n) - n\alpha$.

We define a piecewise linear $F(t)$ from the set $\{t \in \mathbb{Z} : t \geq -m\}$ to $\mathbb{Z}$ and show $W(t) \leq F(t)$ for all $t \geq -m$. We then use this to define x_m, y_m satisfying the three properties.

Given $m \in \mathbb{N}$ and using the facts that $\lim_{k\to\infty} \frac{W(k)}{k} = \alpha$ and $\lim_{k\to\infty} \frac{\delta(k)}{k} = 0$, we can choose $n_0 = n_0(m) > m$ such that

$$\frac{\delta(k)}{k} < \frac{1}{m}$$

for all $k > n_0$. For the moment as m is fixed we suppress the dependence of n_0 on m. By hypothesis, $\delta(k)$ is unbounded above and so we can also choose n_0 so that

$$\delta(n_0) > \delta(j) \text{ for all } 0 \leq j < n_0. \tag{4.1}$$

Define a line $i = L(j)$ in the i,j-plane by

$$L(j) = \frac{1}{m}(j - n_0) + \delta(n_0).$$

We claim that the set of j with $\delta(j) \geq L(j)$ is finite. By Lemma 4.2,

$$\lim_{j\to\infty} \frac{\delta(j)}{j - n_0} = \lim_{j\to\infty} \frac{\delta(j)}{j} = 0$$

and so for sufficiently large j,

$$\delta(j) \leq \frac{1}{m}(j-n_0) < \frac{1}{m}(j-n_0) + \delta(n_0) = L(j),$$

since $\delta(n_0) \geq 0$ (by Lemma 4.2). This proves the claim.

Let J be the finite set $\{j : \ \delta(j) \geq L(j),\ j \geq 0\}$ and let $S = \{(\delta(j), j) : \ j \in J\}$. Note that $S \neq \emptyset$ since $(\delta(n_0), n_0) \in S$.

Let $t_0 = t_0(m) \in \mathbb{N}$ be the value of j with $j \geq n_0$ for which $\delta(j) - L(j)$ is maximal. Then $(\delta(t_0), t_0) \in S$. Since, for the moment m is fixed, we suppress the m and simply write t_0 for $t_0(m)$.

Suppose now that $j \in [n_0, t_0]$. Then since $\delta(t_0) - L(t_0) \geq \delta(j) - L(j)$, it follows that $\delta(t_0) \geq \delta(j) + L(t_0) - L(j) \geq \delta(j)$ since $j \in [n_0, t_0]$ and L is monotonic increasing. Thus we have

$$\delta(t_0) \geq \delta(j) \text{ for all } j \in [n_0, t_0]. \tag{4.2}$$

Let $\alpha_m = \alpha + \dfrac{1}{m}$ and consider the two lines

$$i = \mathcal{K}(j), \text{ where } \mathcal{K}(j) = \alpha(j - t_0) + W(t_0)$$

and

$$i = \mathcal{L}(j), \text{ where } \mathcal{L}(j) = \alpha_m(j - t_0) + W(t_0).$$

Both lines pass through $(W(t_0), t_0)$.

Define

$$F(j) = \begin{cases} \mathcal{K}(j), & \text{if } 0 \leq j \leq t_0 \\ \mathcal{L}(j), & \text{if } j \geq t_0. \end{cases} \tag{4.3}$$

We claim that for all $j \geq 0$

$$W(j) \leq F(j).$$

We prove this claim by considering two separate ranges of values for j, first $j \geq t_0$, then $0 \leq j \leq t_0$.

In the range $j \geq t_0$, by the choice of t_0 we have that $\delta(j) - L(j) \leq \delta(t_0) - L(t_0)$ if $j \in J$. But the same inequality holds for $j \notin J$ since then $\delta(j) - L(j) < 0$ and $\delta(t_0) - L(t_0) \geq 0$. Thus $\delta(j) \leq L(j) + \delta(t_0) - L(t_0)$ for all $j \geq t_0$. Therefore

$$\begin{aligned} W(j) &= \delta(j) + \alpha j \\ &\leq L(j) + \delta(t_0) - L(t_0) + \alpha j \\ &= \frac{1}{m}(j - n_0) - \frac{1}{m}(t_0 - n_0) + \delta(t_0) + \alpha j \\ &= \frac{1}{m}(j - t_0) + \delta(t_0) + \alpha t_0 + \alpha(j - t_0) \\ &= \alpha_m(j - t_0) + \delta(t_0) + \alpha t_0 \\ &= \alpha_m(j - t_0) + W(t_0) \\ &= \mathcal{L}(j). \end{aligned}$$

This proves the claim for the first range, i.e.,

$$W(j) \leq \mathcal{L}(j) \text{ for } j \geq t_0. \tag{4.4}$$

Next we consider the range $0 \leq j \leq t_0$. Note if $j \leq n_0$ then $W(j) = \delta(j) + \alpha j \leq \delta(n_0) + \alpha j$ by Equation (4.1), so $W(j) \leq \delta(t_0) + \alpha j$ since $\delta(t_0) \geq \delta(n_0)$ by Equation (4.2). But if $j \in [n_0, t_0]$ then $W(j) = \delta(j) + \alpha j \leq \delta(t_0) + \alpha j$ by Equation (4.2). So we conclude $W(j) \leq \delta(t_0) + \alpha j$ for all $0 \leq j \leq t_0$.

Hence in this range

$$\begin{aligned} W(j) &\leq \delta(t_0) + \alpha j \\ &= \delta(t_0) + \alpha t_0 + \alpha(j - t_0) \\ &= W(t_0) + \alpha(j - t_0) = \mathcal{K}(j). \end{aligned}$$

Thus we have

$$W(j) \leq \mathcal{K}(j) \text{ for } 0 \leq j \leq t_0 \tag{4.5}$$

Hence Equations (4.4), and (4.5) establish the claim, demonstrating that

$$F(j) \geq W(j) \text{ for all } j \geq 0, \tag{4.6}$$

where

$$F(j) = \begin{cases} \mathcal{K}(j) & \text{if } 0 \leq j \leq t_0 \\ \mathcal{L}(j) & \text{if } j \geq t_0. \end{cases}$$

We now use this to define the elements x_m and y_m. From the definition of $W^+(n, \phi)$ (which we are denoting $W(n)$), we know that whenever $j \geq 0$ and $u, v \in X$ have the rays $u[0, \infty)$ and $v[0, \infty)$ equal, it follows that the rays $\phi^j(u)[W(j), \infty) = \phi^j(v)[W(j), \infty)$. Equivalently if x and y are the elements in ϕ-spacetime which agree on the ray $\{(i, 0) \in \mathbb{Z}^2 \colon i \geq 0\}$, then

$$j \geq 0 \text{ and } i \geq W(j) \text{ implies } x(i, j) = y(i, j). \tag{4.7}$$

Moreover for each $j \geq 0$, there exist $u_j, v_j \in X$ such that $u_j[0, \infty) = v_j[0, \infty)$ but

$$\phi^j(u_j)(W(j) - 1) \neq \phi^j(v_j)(W(j) - 1).$$

In particular this means that for $m \in \mathbb{N}$ there exist elements $\hat{x}_m, \hat{y}_m \in \mathcal{U}$ which are equal on the ray $\{(i, 0) \in \mathbb{Z}^2 \colon i \geq 0\}$, but such that

$$\hat{x}_m(W(t_0(m)) - 1, t_0(m)) \neq \hat{y}_m(W(t_0(m)) - 1, t_0(m)). \tag{4.8}$$

(Note that the dependence of $t_0 = t_0(m)$ on m is now salient so we return to the more cumbersome notation.) We use translates of $\hat{x}_m$ and $\hat{y}_m$ by the vectors $(W(t_0(m)), t_0(m)) = (\delta(t_0(m)) + \alpha t_0(m), t_0(m))$ to define $x_m, y_m \in \mathcal{U}$. More precisely, define

$$x_m(i, j) = \hat{x}_m(i + W(t_0(m)), j + t_0(m))$$

and

$$y_m(i, j) = \hat{y}_m(i + W(t_0(m)), j + t_0(m)).$$

Note that x_m and y_m agree on the ray $\{(i, 0) \in \mathbb{Z}^2 \colon i \geq 0\}$.

We proceed to check properties (1), (2), and (3) of the lemma's conclusion.

From Equation (4.8) and the definition of x_m and y_m we have

$$x_m(-1,0) = \hat{x}_m(W(t_0(m)) - 1, t_0(m)) \neq \hat{y}_m(W(t_0(m)) - 1, t_0(m)) = y_m(-1,0),$$

and so (3) follows.

To check (1), suppose $-m \leq j \leq 0$ and $i \geq \alpha j$. Let $i' = i + W(t_0)$ and $j' = j + t_0$ and so $x_m(i,j) = \hat{x}_m(i',j')$ and $y_m(i,j) = \hat{y}_m(i',j')$. Hence if we show that $\hat{x}_m(i',j') = \hat{y}_m(i',j')$, then we have that $x_m(i,j) = y_m(i,j)$, which is the statement of (1). This in turn follows from Equation (4.7) if we show $j' \geq 0$ and $i' \geq W(j')$. We proceed to do so.

Note that since $-m \leq j \leq 0$ and since, by construction, $n_0(m)$ and $t_0(m)$ satisfy $m < n_0(m) < t_0(m))$, we have

$$0 \leq t_0(m) - m \leq t_0(m) + j = j'$$

To show $i' \geq W(j')$ observe that since $i \geq j\alpha$, it follows that $i' = i + W(t_0) \geq j\alpha + W(t_0) = (j' - t_0)\alpha + W(t_0) = \mathcal{K}(j')$. Since $j' = j + t_0(m) \leq t_0(m)$ the definition of F (equation 4.3) shows $\mathcal{K}(j') = F(j')$, and we may apply Equation (4.6) to conclude $i' \geq W(j')$. Then by Equation 4.7 applied to $\hat{x}_m$ and $\hat{y}_m$ at (i',j') we have $\hat{x}_m(i',j') = \hat{y}_m(i',j')$, so $x_m(i,j) = y_m(i,j)$. This completes the proof of property (1).

To check (2), we assume that $j \geq 0$ and $i \geq \alpha_m j$. Again we let $i' = i + W(t_0(m))$ and $j' = j + t_0(m)$ and so $j' \geq t_0(m)$. To show $x_m(i,j) = y_m(i,j)$, it suffices to show $\hat{x}_m(i',j') = \hat{y}_m(i',j')$ when

$$j' \geq t_0(m) \text{ and } i' \geq \alpha_m j + W(t_0(m))$$

But $\alpha_m j + W(t_0(m)) = \alpha_m(j' - t_0) + W(t_0) = \mathcal{L}(j')$, so we have $j' \geq t_0(m)$ and $i' \geq \mathcal{L}(j')$.

Since $j' \geq t_0(m)$ we conclude from the definition of F (Equation (4.3)) that $F(j') = \mathcal{L}(j')$. So $i' \geq F(j')$ and hence by Equation (4.6), $i' \geq F(j') \geq W(j')$. Since $i' \geq W(j')$ we have $x_m(i',j') = y_m(i',j')$ by Equation 4.7, completing the proof of (2).

The proof of the analogous result for $\alpha^-(\phi)$ and $\delta^-(n,\phi)$ is done similarly. □

4.2. **Nonexpansiveness of light cone edges.**

Theorem 4.4. *Suppose $\phi \in \operatorname{End}(X,\sigma)$ and $\alpha^+ = \alpha^+(\phi)$. In the spacetime $\mathcal{U}$ of ϕ orient the line $u = \alpha^+ v$ so that $\langle \alpha^+, 1\rangle$ is positive. Then this oriented line is not a positively expansive subspace. Similarly if $\alpha^- = \alpha^-(\phi)$, the line $u = \alpha^- v$ (oriented so that $\langle \alpha^-, 1\rangle$ is positive) is not a negatively expansive subspace.*

Proof. Let $\mathcal{U}$ be the ϕ-spacetime of (X,σ). Replacing ϕ with $\sigma^k\phi^m$ and using part (1) of Proposition 3.13, without loss of generality we can assume that $\alpha^+(\phi) \geq 0$.

Case 1: bounded deviation. As a first case we assume that the non-negative deviation function δ is bounded. Say $\delta(j) < D$ for some $D > 0$ and all $j \in \mathbb{N}$. Since $\delta(j) \geq 0$ and $\alpha^+ \geq 0$, we have $0 \leq W^+(j,\phi) - \alpha^+ j = \delta(j) < D$.

If we have two elements $x, y \in \mathcal{U}$ satisfying $x(k,0) = y(k,0)$ for $k \geq 0$ then whenever $j \geq 0$ and $i \geq D + \alpha^+ j$, we have $i > W^+(j)$. Hence

$$x(i,j) = y(i,j) \text{ for all } j \geq 0 \text{ and } i \geq D + \alpha^+ j \tag{4.9}$$

(see Equation (4.7)). Thus x and y agree in the part of the upper half space to the right of the line $i = D + \alpha^+ j$.

By the definition of $W^+(n) = W^+(n, \phi)$ for $n \in \mathbb{N}$ we may choose $\hat{x}_n, \hat{y}_n \in \mathcal{U}$ which agree on the ray $\{(i, 0) \in \mathbb{Z}^2 \colon i \geq 0\}$ such that $\hat{x}_n(W^+(n) - 1, n) \neq \hat{y}_n(W^+(n) - 1, n)$.

We want to create new colorings by translating $\hat{x}_n$ and $\hat{y}_n$ by the vector $(W^+(n), n)$. More precisely for $n \geq 0$ we define x_n and y_n by $x_n(i, j) = \hat{x}_n(i + W^+(n), j + n)$. Note that $x_n(-1, 0) \neq y_n(-1, 0)$, since $x_n(-1, 0) = \hat{x}_n(W^+(n) - 1, n) \neq \hat{y}_n(W^+(n) - 1, n) = y_n(-1, 0)$.

For all $j \geq -n$ and $i \geq D + \alpha^+ j$, we claim that

$$x_n(i, j) = y_n(i, j).$$

To see this define $i' = i + W^+(n)$ and $j' = j + n$ and so $x_n(i, j) = \hat{x}_n(i', j')$ and $y_n(i, j) = \hat{y}_n(i', j')$. Then

$$\begin{aligned}
i' &= i + W^+(n) \\
&\geq D + \alpha^+ j + W^+(n) \\
&= D + \alpha^+ j' + (W^+(n) - \alpha^+ n) \\
&= D + \alpha^+ j' + \delta(n) \\
&\geq D + \alpha^+ j'.
\end{aligned}$$

Hence $\hat{x}_n(i', j')$ and $\hat{y}_n(i', j')$ are equal by Equation 4.9 whenever $i \geq D + \alpha^+ j$ and $j \geq -n$ (since $j' \geq 0$ when $j \geq -n$). But $x_n(i, j) = \hat{x}_n(i', j')$ and $y_n(i, j) = \hat{y}_n(i', j')$ so $x_n(i, j) = y_n(i, j)$. Thus x_n and y_n agree at (i, j) if $i \geq D + \alpha^+ j$ and $j \geq -n$.

Since $\mathcal{U}$ is compact we can choose convergent subsequences (also denoted x_n and y_n). Say $\lim x_n = \hat{x}$ and $\lim y_n = \hat{y}$. Then clearly $\hat{x}(-1, 0) \neq \hat{y}(-1, 0)$ and $\hat{x}(i, j) = \hat{y}(i, j)$ for all $i > D + \alpha^+ j$. So $\hat{x}$ and $\hat{y}$ agree on the half space $H^+ = \{(i, j) \colon i > D + \alpha^+ j\}$. This implies the oriented line $u = \alpha^+ v$ is not positively expansive. The case of the line $u = \alpha^- v$ is handled similarly.

Case 2: unbounded deviation. We consider the elements x_m, y_m guaranteed by Lemma 4.3, and recall that they satisfy properties (1)-(3) of the lemma.

Since $\mathcal{U}$ is compact, by passing to subsequences, we can assume that both sequences converge in $\mathcal{U}$, say to $\hat{x}$ and $\hat{y}$. Clearly $\hat{x}(-1, 0) \neq \hat{y}(-1, 0)$. We claim the colorings $\hat{x}$ and $\hat{y}$ agree on the half space $H^+ = \{(i, j) \colon i > \alpha j\}$ of $\mathbb{Z}^2$. It then follows that the oriented line $u = \alpha v$ is not positively expansive (see Definition 2.6).

To prove the claim, note that if $(i, j) \in H^+$, $-m_0 \leq j \leq 0$ and $m \geq m_0$ then $x_m(i, j) = y_m(i, j)$. Hence the limits satisfy $\hat{x}(i, j) = \hat{y}(i, j)$ whenever $(i, j) \in H^+$ and $j \leq 0$. But also if $j > 0$ and $i > \alpha j$, then for some $n_0 > 0$ we have $i \geq (\alpha + \frac{1}{n_0}) j$ and it follows that $x_m(i, j) = y_m(i, j)$ whenever $m > n_0$. Hence the limits satisfy $\hat{x}(i, j) = \hat{y}(i, j)$.

The case of the line $u = \alpha^- v$ is handled similarly. $\square$

4.3. **Expansive subspaces.** We want to investigate which one-dimensional subspaces in a spacetime are expansive. Since the horizontal axis in a spacetime is always positively expansive for an endomorphism and expansive for an automorphisms, we restrict our attention to lines in $\mathbb{R}^2 = \{(u, v)\}$ given by $u = mv$ where $m \in \mathbb{R}$. (We write the abscissa as a function of the ordinate for convenient comparison with the edges of $\mathcal{A}(\phi)$ which are $u = \alpha^+ v$ and $u = \alpha^- v$.)

Proposition 4.5. *Suppose L is a line in $\mathbb{R}^2$ given by $u = mv$ and oriented so that $\langle m, 1\rangle$ is positive. Then:*

(1) *If $m > \alpha^+(\phi)$, then L is positively expansive.*
(2) *If $m < \alpha^-(\phi)$, then L is negatively expansive.*

Moreover if ϕ is an automorphism and if $m > \max\{\alpha^+(\phi), -\alpha^-(\phi^{-1})\}$ or if $m < \min\{\alpha^-(\phi), -\alpha^+(\phi^{-1})\}$, then L is expansive.

Proof. We first consider (1). We show that if $\mathcal{U}$ is the spacetime of ϕ and $x, y \in \mathcal{U}$ agree on the right side of $u = mv$, then they also agree on the left side. This implies that the oriented line L is positively expansive. Since $m > \alpha^+(\phi)$, the vector $\langle \alpha^+(\phi), 1\rangle$ is not parallel to L and points in the direction from the right side of L to the left side.

Let $W^+(n) = W^+(n, \phi)$ so

$$\lim_{n\to\infty} \frac{W^+(n)}{n} = \alpha^+(\phi)$$

(see Equation (3.12)) and hence

$$\lim_{n\to\infty} \frac{1}{n}\langle W^+(n), n\rangle = \langle \alpha^+(\phi), 1\rangle.$$

It follows that for sufficiently large n, the vector $\langle W^+(n), n\rangle$ is also not parallel to L and points in the direction from the right side of L to the left side. Hence, given any $(u_0, v_0) \in \mathbb{Z}^2$ on the left side of L, there exists $n_0 > 0$ such that if $u_1 = u_0 - W^+(n_0)$ and $v_1 = v_0 - n_0$, then (u_1, v_1) is on the right side of L. The ray $\{(t, v_1) \colon u_1 \le t\}$ in $\mathcal{U}$ lies entirely to the right of L and codes $\{(u_0, v_0)\}$.

It follows that if $x, y \in \mathcal{U}$ agree to the right of L, then they also agree at (u_0, v_0). Since (u_0, v_0) is an arbitrary point to the left of L, it follows that L is positively expansive. The proof of (2) is analogous.

To show the final statement, note the the reflection $R \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $R(u, v) = (u, -v)$ has the property that it switches the spacetimes $\mathcal{U}(\phi)$ and $\mathcal{U}(\phi^{-1})$, i.e., it induces a map $R^* \colon \mathcal{U}(\phi) \to \mathcal{U}(\phi^{-1})$ given by $R^*(\eta) = \eta \circ R$.

If L is the line $i = mj$, then our convention for the orientation of L was chosen so that

$$L^+ = \{\langle u, v\rangle \colon \langle u, v\rangle \in L \text{ and } v > 0\}.$$

Hence the convention implies that $R(L^+)$ is the set of *negative* vectors in $R(L)$ and the positive vectors in $R(L)$ are $R(L^-)$ where $L^- = -L^+$. Note that $H^+(L)$ consists of the vectors above the line L so $R(H^+(L))$ is the set of vectors below $R(L)$. (see Definition 2.7 and the paragraph preceding it). But since R reverses the orientation of L we have $H^+(R(L)) = R(H^+(L)$. It follows that L is positively (resp. negatively) expansive in $\mathcal{U}(\phi)$ if and only if $R(L)$ is positively (resp. negatively) expansive in $\mathcal{U}(\phi^{-1})$, i.e. R acting on non-vertical lines preserves positive expansiveness and negative expansiveness.

Now consider the line L given by $i = mj$ in $\mathcal{U}(\phi)$, and so $R(L)$ is the line $i = -mj$ in $\mathcal{U}(\phi^{-1})$. By part (2), if $-m < \alpha^-(\phi^{-1})$ (or equivalently if $m > -\alpha^-(\phi^{-1})$), then the line $R(L)$ is negatively expansive in $\mathcal{U}(\phi^{-1})$. Hence $m > -\alpha^-(\phi^{-1})$ implies that L is negatively expansive in $\mathcal{U}(\phi)$. If we also have $m > \alpha^+(\phi)$ then by part (1), the line L is also positively expansive and thus it is, in fact, expansive. The case that $m < \min\{\alpha^-(\phi), -\alpha^+(\phi^{-1})\}$ is handled similarly. $\square$

5. Asymptotic behavior

5.1. **The asymptotic light cone.** The edges of the light cone $\mathcal{C}(\phi)$ are given by the graphs of the functions $i = W^+(j,\phi)$ $i = W^-(j,\phi)$. Since these functions have nice asymptotic properties, so does the cone they determine, which motivates the following definition:

Definition 5.1. The *asymptotic light cone* of ϕ is defined to be

$$\mathcal{A}(\phi) = \{(u,v) \in \mathbb{R}^2 \colon \alpha^-(\phi)v \le u \le \alpha^+(\phi)v\}.$$

This means $\mathcal{A}(\phi)$ is the cone in $\mathbb{R}^2$ which does not contain the i-axis and which is bounded by the lines $u = \alpha^+(\phi)v$ and $u = \alpha^-(\phi)v$. We view $\mathcal{A}(\phi)$ as a subset of $\mathbb{R}^2$ rather than of $\mathbb{Z}^2$, as we want to consider lines with irrational slope that may lie in $\mathcal{A}(\phi)$ but would intersect $\mathcal{C}(\phi)$ only in $\{0\}$.

We begin by investigating the deviation of the function $W^+(n,\phi)$ from the linear function $n\alpha^+(\phi)$. Observe that the asymptotic light cone $\mathcal{A}(\phi)$ is a subset of the light cone $\mathcal{C}(\phi)$, as an immediate corollary of part (3) of Lemma 4.2.

Corollary 5.2. *The set of integer points in the asymptotic light cone $\mathcal{A}(\phi)$ is a subset of the light cone $\mathcal{C}(\phi)$.*

If $\phi \in \mathrm{Aut}(X)$ it is natural to consider the relationship between $\mathcal{C}(\phi)$ and $\mathcal{C}(\phi^{-1})$, or between $\mathcal{A}(\phi)$ and $\mathcal{A}(\phi^{-1})$. The spacetime $\mathcal{U}(\phi)$ of ϕ is not the same as the spacetime $\mathcal{U}(\phi^{-1})$ of ϕ^{-1}, but there is a natural identification of $\mathcal{U}(\phi)$ with the reflection of $\mathcal{U}(\phi^{-1})$ about the horizontal axis $j = 0$. In general, it is not true that $\mathcal{A}(\phi^{-1})$ is the reflection of $\mathcal{A}(\phi)$ about the u-axis (Example 2.10 is one where this fails). On the other hand, if (X,σ) is a subshift, there is at least one line in the intersection of $\mathcal{A}(\phi^{-1})$ with the reflection of $\mathcal{A}(\phi)$ about the u-axis.

To see this, note that the cone $\mathcal{A}(\phi^{-1})$ has edges which are the lines

$$u = \alpha^+(\phi^{-1})v \text{ and } u = \alpha^-(\phi^{-1})v, \tag{5.1}$$

while the cone obtained by reflecting $\mathcal{A}(\phi)$ about the u-axis has edges given by

$$u = -\alpha^-(\phi)v \text{ and } u = -\alpha^+(\phi)v. \tag{5.2}$$

Hence the line $u = mv$ lies in the intersection $\mathcal{A}(\phi^{-1})$ and the reflection of $\mathcal{A}(\phi)$ in the line u-axis if

$$m \in [\alpha^-(\phi^{-1}), \alpha^+(\phi^{-1})] \cap [-\alpha^+(\phi), -\alpha^-(\phi)].$$

If these two intervals are disjoint, then either

$$\alpha^+(\phi^{-1}) < -\alpha^+(\phi) \text{ or } -\alpha^-(\phi) < \alpha^-(\phi^{-1}).$$

Either of these inequalities contradict part (5) of Proposition 3.13.

In a different vein, the cone $\mathcal{A}(\phi)$ is a conjugacy invariant:

Proposition 5.3. *Suppose (X_i, σ_i) is a shift for $i = 1, 2$ and $\phi_i \in \mathrm{End}(X_i)$. Suppose further that $h\colon X_1 \to X_2$ is a topological conjugacy from σ_1 to σ_2. If*

$$\phi_2 = h \circ \phi_1 \circ h^{-1},$$

then $\mathcal{A}(\phi_1) = \mathcal{A}(\phi_2)$.

Proof. Since h is a block code, there is a constant $D > 0$, depending only on h, such that for any $n \in \mathbb{Z}$ the ray $[n, \infty)$ h-codes $[n + D, \infty)$ and the ray $(-\infty, n]$ h-codes $(-\infty, n - D]$. It follows that $W^+(m, \phi_1) \leq W^+(m, \phi_2) + 2D$. Switching the roles of ϕ_1 and ϕ_2 and considering h^{-1}, for which there is $D' > 0$ with properties analogous to those of D, we see that $W^+(m, \phi_2) \leq W^+(m, \phi_1) + 2D'$. By the definition of α^+ (see Equation 3.12),

$$\alpha^+(\phi_1) = \lim_{n\to\infty} \frac{W^+(n, \phi_1)}{n} = \lim_{n\to\infty} \frac{W^+(n, \phi_2)}{n} = \alpha^+(\phi_1).$$

The proof that $\alpha^-(\phi_1) = \alpha^-(\phi_2)$ is similar, and thus the asymptotic light cones of ϕ_1 and ϕ_2 are identical. □

5.2. **A complement to Theorem 4.4.** In Theorem 4.4 we showed that lines in the spacetime of an endomorphism ϕ which form the boundary of its asymptotic light cone $\mathcal{A}(\phi)$ are nonexpansive subspaces. In this section we want to show that in many instances, given an arbitrary $\mathbb{Z}^2$-subshift Y and a nonexpansive subspace $L \subset \mathbb{R}^2$ for Y, there is a $\mathbb{Z}^2$-subshift isomorphism Ψ from Y to the spacetime $\mathcal{U}$ of an automorphism $\phi \in \mathrm{Aut}(X, \sigma)$ for some shift (X, σ) such that $\Psi(L)$ is an edge of the asymptotic light cone $\mathcal{A}(\phi)$. In particular this holds if Y has finitely many nonexpansive subspaces. Hence in that case every nonexpansive subspace in Y is (up to isomorphism) an edge of an asymptotic light cone for some automorphism.

To do this it is useful to introduce the notion of *expansive ray* which incorporates both the subspace and its orientation

By a *ray* in $\mathbb{R}^2$ we mean a set $\rho \subset \mathbb{R}^2$ such that there exists $w \neq 0 \in \mathbb{R}^2$ with

$$\rho = \rho(w) = \{tw\colon t \in [0, \infty)\}.$$

The space of all rays in $\mathbb{R}^2$ is naturally homeomorphic to the set of unit vectors in $\mathbb{R}^2$, which is the circle S^1.

Definition 5.4. Let Y be a $\mathbb{Z}^2$-subshift. We say ρ is an *expansive ray* for Y if the line L containing ρ with orientation given by $L^+ = \rho \cap (L \setminus \{0\})$ is positively expansive (see Definition 2.7 and the paragraph preceding it).

The concept of expansive ray is essentially the same as that of *oriented expansive line* introduced in §3.1 of [5]. We emphasize that this concept is defining *one-sided expansiveness* for the line L containing ρ. Which side of L codes the other is determined by the orientation of ρ and the orientation of $\mathbb{Z}^2$.

To relate this to our earlier notions of expansive (Definition 2.6) observe that if L is the subspace containing ρ, then L is expansive if and only if both ρ and $-\rho$ are expansive rays. In this terminology, Theorem 4.4 says that the rays $\rho^+(\phi) := \{\langle \alpha^+ v, v\rangle : v \geq 0\}$ and $\rho^-(\phi) := \{\langle \alpha^- v, v\rangle : v \leq 0\}$ are nonexpansive rays. We note that it is not in general the case that $-\rho^+(\phi)$ and $-\rho^-(\phi)$ are nonexpansive rays.

The following lemma is essentially contained in [3], but differs from results there in that we consider one-sided expansiveness. In particular note the following result implies that being positively expansive is an open condition for oriented one-dimensional subspaces of the $\mathbb{R}^2$ associated to a $\mathbb{Z}^2$-subshift. Similarly being negatively expansive is an open condition.

Lemma 5.5. *If $\mathcal{E} \subset S^1$ is the set of expansive rays for a $\mathbb{Z}^2$-subshift Y, then $\mathcal{E}$ is open.*

Proof. We show that the set $\mathcal{N}$ of nonexpansive rays is closed. Suppose that $\rho_n = \{tw_n : t \geq 0\}_{n=1}^{\infty}$ is a sequence of rays in $\mathbb{R}^2$ with $\lim_{n\to\infty} w_n = w_0 \neq 0$ so that ρ_0 is the limit of the rays $\rho_n, n \geq 1$. If the rays ρ_n are nonexpansive we must show that ρ_0 is nonexpansive.

Let L_n be the line containing w_n with the orientation such that $w_n \in L_n^+$ and let $H^+(L_n)$ be the component of $\mathbb{R}^2 \setminus L_n$ such that for all $w' \in H^+(L_n)$ the ordered basis $\{w_n, w'\}$ is positively oriented and let $H^-(L_n)$ be the other component of $\mathbb{R}^2 \setminus L_n$. Define the linear function $f_n : \mathbb{R}^2 \to \mathbb{R}$ by $f_n(u) = u \cdot v_n$ where v_n is a unit vector in $H^+(L_n)$ which is orthogonal to w_n. Then we have the following:

- $L_n = \ker(f_n)$
- A vector u is in $H^+(L_n)$ if and only if $f_n(u) > 0$ and in $H^-(L_n)$ if and only if $f_n(u) < 0$.
- $\lim_{n\to\infty} f_n(v_0) = f_0(v_0) = 1$.

By Proposition 2.8 we know there exist $\eta_n, \eta_n' \in Y$ and $z_n \in \mathbb{Z}^2$ such that $\eta_n(v) = \eta_n'(v)$ for all $v \in H^-(L_n)$ but $\eta_n(z_n) \neq \eta_n'(z_n)$. By shifting η_n and η_n' we may assume lengths $|z_n|$ are bounded. Choosing a subsequence we may assume $\{z_n\}$ is constant, say, $z_n = z_0 \in Z^2$. Since Y is compact we may further choose subsequences $\{\eta_n\}_{n=1}^{\infty}$ and $\{\eta_n'\}_{n=1}^{\infty}$ which converge, say to η_0 and η_0' respectively. Clearly $\eta_0(z_0) \neq \eta_0'(z_0)$. Now if $y \in H^-(L_0) \cap \mathbb{Z}^2$ then $f_0(y) < 0$ so $f_n(y) < 0$ for sufficiently large n and hence $y \in H^-(L_n) \cap \mathbb{Z}^2$. It follows that $\eta_0(y) = \eta_0'(y)$.

Since η_0 and η_0' agree on $H^-(L_n) \cap \mathbb{Z}^2$ but disagree at z_0 we conclude that $H^-(L_n) \cap \mathbb{Z}^2$ does not code $H^+(L_n) \cap \mathbb{Z}^2$ so ρ_0 is a nonexpansive ray. □

Proposition 5.6. *Suppose Y is a $\mathbb{Z}^2$-subshift and $\mathcal{E}$ is the set of expansive rays for Y (thought of as a subset of S^1). Suppose C is a component of $\mathcal{E}$ and ρ_1, ρ_2 are the endpoints of the open interval C. Then there exists a shift (X, σ) with automorphism $\phi \in \mathrm{Aut}(X)$ and an isomorphism $\Psi : Y \to \mathcal{U}(\phi)$ from Y to the spacetime of ϕ such* *that the lines* $L_1 := \operatorname{span}(\Psi(\rho_1))$ *and* $L_2 := \operatorname{span}(\Psi(\rho_2))$ *are the two edges of the asymptotic light cone* $\mathcal{A}(\phi)$ *of* ϕ.

Proof. We consider C as an open interval (ρ_1, ρ_2) in the circle S^1 of rays in $\mathbb{R}^2$. There is a $\mathbb{Z}^2$-subshift isomorphism $\Psi_0 \colon Y \to Y_0$, where Y_0 is a $\mathbb{Z}^2$-subshift with $\langle 1, 0\rangle \in \Psi_0(C)$. Thus the horizontal axis is an expansive subspace for the $\mathbb{Z}^2$-subshift Y_0. We may recode Y_0 to Y_1 by an isomorphism $\Psi_1 \colon Y_0 \to Y_1$ such that the horizontal axis H_0 in $\mathbb{Z}^2$ codes the positive half space $\{\langle i, j\rangle \in \mathbb{Z}^2 \colon j > 0\}$ (this follows from Lemma 3.2 in [3] where we recode Y_0 such that "symbols" in Y_1 are vertically stacked arrays of symbols from Y_0 of an appropriate height). We let $\Psi \colon Y \to Y_1$ be the composition $\Psi_1 \circ \Psi_0$.

Let X denote the set of colorings of $\mathbb{Z}$ obtained by restricting elements $\eta \in Y_1$ to H_0. We could equally well describe X as the colorings of $\mathbb{Z}$ obtained by restricting elements of Y to the horizontal row $H_{-1} := \{\langle i, j\rangle \in \mathbb{Z}^2 \colon j = -1\}$ and define $\phi \colon X \to X$ by $\phi(x) = x'$ if there is $\eta \in Y_1$ such that $x = \eta|_{H_0}$ and $x' = \eta|_{H_{-1}}$. Then clearly $\phi \in \operatorname{Aut}(X)$ and Y_1 is $\mathcal{U}(\phi)$, the spacetime of ϕ.

Note that the ray $\rho^+(\phi) := \{\langle \alpha^+ v, v\rangle \colon v \geq 0\}$ lies in the light cone $\mathcal{A}(\phi)$ of ϕ (and in the upper half space of $\mathbb{R}^2$). If $m > \alpha^+(\phi)$ and ρ_m is the ray $\rho_m := \{\langle mv, v\rangle \colon v \geq 0\}$, then by Proposition 4.5 ρ_m is an expansive ray. Since by Theorem 4.4 $\rho^+(\phi)$ is not an expansive ray, it follows that $\Psi(\rho_2) = \rho^+(\phi)$.

Letting $\rho^-(\phi) := \{\langle \alpha^- v, v\rangle \colon v \leq 0\}$, a similar proof shows that $\Psi(\rho_1) = \rho^-(\phi)$. Hence the lines L_1 and L_2 form the edges of the asymptotic light cone $\mathcal{A}(\phi)$. $\square$

We are not able to show which lines can arise as the edges of the asymptotic light cone:

Question 5.7. *Does there exist a subshift of finite type* X *and an automorphism* $\phi \in \operatorname{Aut}(X)$ *such that some edge of the asymptotic light cone of* ϕ *has irrational slope? If so, what set of angles is achievable? More generally, for a subshift of finite type* X *or for a general shift* X*, what are all of the components of the expansive subspaces?*

Hochman [14] points out that, as there are only countably many shifts of finite type, this set must be countable (and, in particular, cannot contain all irrational slopes). If X is not required to be a subshift of finite type, then Hochman's results show that for the first question, the only constraint on the light cone for an automorphism (of an infinite subshift) comes from $-\infty < \alpha^- \leq \alpha^+ < \infty$.

5.3. **Asymptotic spread.** Let $\ell(n, \phi)$ be the minimal length of an interval $J \subset \mathbb{Z}$ which contains 0 and ϕ^n-codes $\{0\}$ and let $\mathcal{L}(\phi^n)$ be the minimal length of an interval $J_0 \subset \mathbb{Z}$ which is symmetric about 0 and ϕ^n-codes $\{0\}$. It is straightforward to see that both $\ell(n, \phi)$ and $\mathcal{L}(\phi^n)$ are subadditive sequences.

Definition 5.8. Define the *asymptotic spread* $A(\phi)$ of $\phi \in \operatorname{End}(X)$ to be

$$A(\phi) = \lim_{n \to \infty} \frac{\ell(n, \phi)}{n}. \tag{5.3}$$

We say ϕ is *range distorted* if $A(\phi) = 0$.

Note that since the sequence $\ell(n, \phi)$ is subadditive, Fekete's Lemma implies that the limit in (5.3) exists.

The asymptotic spread is a measure of both the width of the asymptotic light cone, as well as how that cone deviates from the vertical.

Remark 5.9. Since the function $\mathcal{L}(\phi^n)$ is a subadditive function of $n \geq 0$, by Fekete's Lemma, the limit

$$\rho(\phi) = \lim_{n\to\infty} \frac{\mathcal{L}(\phi^n)}{n}$$

exists. Clearly $\mathcal{L}(\phi^n) \leq \ell(n, \phi) \leq 2\mathcal{L}(\phi^n) + 1$ and so

$$\rho(\phi) \leq A(\phi) \leq 2\rho(\phi).$$

In particular, ϕ is range distorted if and only if

$$\lim_{n\to\infty} \frac{\mathcal{L}(\phi^n)}{n} = 0$$

Proposition 5.10. *If $\phi \in \mathrm{Aut}(X)$ and $\alpha^+(\phi) = \alpha^-(\phi) = \alpha^+(\phi^{-1}) = \alpha^-(\phi^{-1})$, then the line $u = \alpha^+(\phi)v$ is the unique nonexpansive one-dimensional subspace. In particular, if $\phi, \phi^{-1} \in \mathrm{Aut}(X)$ are both range distorted, then the vertical axis ($u = 0$) is the unique nonexpansive subspace*

Proof. The first statement follows immediately from Theorem 4.4 and Proposition 4.5. The second statement follows from the first, since ϕ and ϕ^{-1} are both range distorted if and only if $\alpha^+(\phi) = \alpha^-(\phi) = \alpha^+(\phi^{-1}) = \alpha^-(\phi^{-1}) = 0$. □

It was shown by M. Hochman [14] that if L is any 1-dimensional subspace of $\mathbb{R}^2$, then there exists a subshift X_L and an automorphism $\phi_L \in \mathrm{Aut}(X_L)$ such that L is the unique nonexpansive subspace for the spacetime of ϕ_L. Moreover, the automorphisms ϕ_L in his examples always have infinite order (in particular, when L is vertical, ϕ_L is range distorted and has infinite order). However, the space X_L he constructs lacks many natural properties one might assume about a subshift; for example, it is not a subshift of finite type and it is not transitive. He asks the following natural question:

Question 5.11 (Hochman [14, Problem 1.2])**.** *Does every nonempty closed set of one-dimensional subspaces of $\mathbb{R}^2$ arise as the nonexpansive subspaces of a $\mathbb{Z}^2$-action that is transitive (or even minimal) and supports a global ergodic measure?*

We do not answer this question, but recall it here as, in particular, we do not know whether a transitive subshift can have a range distorted automorphism of infinite order. We mention further that, in the special case that L is vertical, Hochman shows that his example (X_L, ϕ_L) is logarithmically distorted.

Proposition 5.12. *If ϕ is an endomorphism of a subshift of finite type (X, σ), then $A(\phi)$ is determined by the light cone of ϕ and is, in fact, the length of the smallest interval containing $0, \alpha^-(\phi)$ and $\alpha^+(\phi)$.*

Proof. It follows from Proposition 3.18 that if σ is a subshift of finite type, then for all $x \in X$ and all sufficiently large $n > 0$, the interval $[W^-(n), W^+(n)]$ is an interval which codes $\phi^n(x)[0]$ and which is contained in any interval which contains 0 and codes $\phi^n(x)[0]$. It follows that if J_n is the smallest interval containing $0, W^-(n)$ and $W^+(n)$, then

$$A(\phi) = \lim_{n\to\infty} \frac{|J_n|}{n}.$$

Hence $A(\phi)$ is the length of the smallest interval containing $0, \alpha^-(\phi)$ and $\alpha^+(\phi)$. $\square$

The following result is essentially the same as Proposition 5.3 of Tisseur's paper [24], except that we consider an arbitrary $\phi \in \mathrm{Aut}(X, \sigma)$ with σ an arbitrary shift while he considers a cellular automaton defined on the full shift and preserving the uniform measure on that shift. Our proof is quite short and makes no use of measure. It makes explicit the connection between the topological entropy of a shift and the topological entropy of an automorphism of that shift.

Theorem 5.13. *If $\phi \in \mathrm{End}(X)$, then*

$$h_{\mathrm{top}}(\phi) \le A(\phi) h_{\mathrm{top}}(\sigma),$$

where $A(\phi)$ is the asymptotic spread of ϕ. In particular, if ϕ is range distorted then $h_{\mathrm{top}}(\phi) = 0$.

Proof. Let $\mathcal{U}$ be the spactime of ϕ. For $z \in \mathcal{U}$, let $R_{m,n} = \{(i,j) \in \mathbb{Z}^2 \colon 0 \le i < m, 0 \le j < n\}$ and let $z|_{R_{m,n}}$ denote the restriction of z to $R_{m,n}$. Recall that $P_{\mathcal{U}}$ denotes the two dimensional complexity function $P_{\mathcal{U}}$ (see Definition 2.5). Then

$$h_{\mathrm{top}}(\phi) = \lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{n} \log(P_{\mathcal{U}}(R_{m,n})).$$

Since $A(\phi)$ is the length of the smallest interval containing $0, \alpha^-(\phi)$ and $\alpha^+(\phi)$, for a fixed m there is an interval J in $\mathbb{Z}$ with length $A(\phi)n + \mathrm{o}(n) + m$ that ϕ^j-codes the block $[0, m]$ for all $0 \le j \le n$. In other words, the interval $J \times \{0\} \subset \mathcal{U}$ codes $R_{m,n}$. Therefore, for any $\varepsilon > 0$, and m and n sufficiently large,

$$P_{\mathcal{U}}(R_{m,n}) \le P_X(A(\phi)n + \mathrm{o}(n) + m) \le (\exp(h_\sigma + \varepsilon))^{A(\phi)n+m}.$$

Hence $\log(P_{\mathcal{U}}(R_{m,n})) \le (A(\phi)n + m)(h_\sigma + \varepsilon)$ and

$$h_{\mathrm{top}}(\phi) = \lim_{m\to\infty} \lim_{n\to\infty} \frac{\log(P_{\mathcal{U}}(R_{m,n}))}{n} \le \lim_{m\to\infty} \lim_{n\to\infty} \frac{(A(\phi)n + m)(h_\sigma + \varepsilon)}{n} = A(\phi)(h_\sigma+\varepsilon).$$

Since this holds for all $\varepsilon > 0$, the desired inequality follows.

By definition ϕ is range distorted if and only if $A(\phi) = 0$, and so the last two assertions of the proposition are immediate. $\square$

5.4. **Distortion and inert automorphisms.** Recall that if (Σ_A, σ) is a subshift of finite type, there is a dimension group representation $\Psi \colon \mathrm{Aut}(\Sigma_A) \to \mathrm{Aut}(D_A)$ mapping automorphisms of the shift to automorphisms of its dimension group D_A (see [18], [25], and [1] for definitions). A particularly important subgroup of $\mathrm{Aut}(\Sigma_A)$ is $\mathrm{Inert}(\Sigma_A)$, defined to be the kernel of Ψ. An automorphism $\phi \in \mathrm{Aut}(\Sigma_A)$ is called *inert* if $\Psi(\phi) = \mathrm{Id}$.

There is one special case when Ψ can be thought of as a homomorphism from $\mathrm{Aut}(\Sigma_A)$ to the group of positive reals under multiplication. This occurs when Σ_A is an irreducible subshift of finite type and $\det(I - At)$ is an irreducible polynomial. In this setting, one can associate to each $\phi \in \mathrm{Aut}(\Sigma_A)$ an element $\lambda_\phi = \Psi_0(\phi)$ in $(0, \infty)$ such that Ψ_0 is a homomorphism and $\lambda_\phi = 1$ if and only if ϕ is inert.

To investigate the relationship between being inert and being distorted, we quote the following important result of Boyle and Krieger:

Theorem 5.14 (Boyle and Krieger [1, Theorem 2.17])**.** *Suppose (Σ_A, σ) is an irreducible subshift of finite type and $\det(I - At)$ is an irreducible polynomial. Then if $\phi \in \mathrm{Aut}(\Sigma_A)$ and m is sufficiently large, $\sigma^m\phi$ is conjugate to a subshift of finite type and*

$$h_{\mathrm{top}}(\sigma^m\phi) = \log(\lambda_\phi) + m h_{\mathrm{top}}(\sigma).$$

Theorem 5.15. *Suppose (Σ_A, σ) is an irreducible subshift of finite type such that $\det(I - At)$ is an irreducible polynomial, and let $\phi \in \mathrm{Aut}(\Sigma_A)$. If ϕ and ϕ^{-1} are range distorted, then ϕ is inert.*

Proof. Let $\lambda_\phi = \Psi(\phi)$ and note that by replacing ϕ with ϕ^{-1} if necessary, we can assume that $\lambda_\phi \geq 1$. Suppose ϕ is range distorted and so $\alpha^+(\phi) = \alpha^-(\phi) = 0$; we show that ϕ is inert. From parts (1) and (2) of Proposition 3.13, we conclude that $\alpha^+(\sigma^k\phi) = \alpha^-(\sigma^k\phi) = k$. By Proposition 5.12, it follows that $A_{\sigma^k\phi} = |k|$. Hence by Theorem 5.13, we have $h_{\mathrm{top}}(\sigma^k\phi) \leq |k| h_{\mathrm{top}}(\sigma)$. Combining this with the fact from Theorem 5.14 which says for large k we have $h_{\mathrm{top}}(\sigma^k\phi) = \log(\lambda_\phi) + k h_{\mathrm{top}}(\sigma)$, we conclude that $\log(\lambda_\phi) \leq 0$ or $\lambda_\phi \leq 1$. Since we also have $\lambda_\phi \geq 1$, we conclude that $\lambda_\phi = 1$ and ϕ is inert. $\square$

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Bucknell University, Lewisburg, PA 17837 USA
E-mail address: `van.cyr@bucknell.edu`

Northwestern University, Evanston, IL 60208 USA
E-mail address: `j-franks@northwestern.edu`

Northwestern University, Evanston, IL 60208 USA
E-mail address: `kra@math.northwestern.edu`