CHAPTER I

BASIC NOTIONS

1.1. (a) $86.66\ldots$ and $88.33\ldots$.

(b) $a_1 = 0.6, a_2 = 0.4$ will work in the first case, but there are no possible such weightings to produce the second case, since Student 1 and Student 3 have to end up with the same score.

1.2. (a) $x = 2, y = -1/3$. (b) $x = 1, y = 2, z = 2$. (c) This system does not have a solution since by adding the first two equations, we obtain $x + 2y + z = 7$ and that contradicts the third equation. (d) Subtracting the second equation from the first yields $x + y = 0$ or $x = -y$. This system has infinitely many solutions since x and y can be arbitrary as long as they satisfy this relation.

2.1.

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \quad 3\mathbf{x} - 5\mathbf{y} + \mathbf{z} = \begin{bmatrix} 14 \\ 1 \\ -25 \end{bmatrix}.$$

2.2.

$$A\mathbf{x} = \begin{bmatrix} -15 \\ -10 \\ 4 \\ -10 \end{bmatrix}, \quad A\mathbf{y} = \begin{bmatrix} -2 \\ -2 \\ 14 \\ -20, \end{bmatrix} \quad A\mathbf{x} + A\mathbf{y} = A(\mathbf{x} + \mathbf{y}) = \begin{bmatrix} -17 \\ -12 \\ 18 \\ -30 \end{bmatrix}.$$

2.3. $A + 3B, C + 2D, DC$ are not defined.

$$A + C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad AB = \begin{bmatrix} -9 & 8 \\ -8 & 4 \end{bmatrix},$$

$$BA = \begin{bmatrix} 1 & -5 & 7 \\ 1 & -1 & 3 \\ -3 & -1 & -5 \end{bmatrix}, \quad CD = \begin{bmatrix} -4 & -3 & 13 \\ -2 & -6 & 10 \end{bmatrix}.$$

2.4. We have for the first components of these two products

$$a_{11} + a_{12}2 = 3$$
$$a_{11}2 + a_{12} = 6$$

This is a system of 2 equations in 2 unknowns, and you can solve it by the usual methods of high school algebra to obtain $a_{11} = 3, a_{12} = 0$. A similar argument applied to the second components yields $a_{2,1} = 7/3, a_{22} = -2/3$. Hence,

$$A = \begin{bmatrix} 3 & 0 \\ 7/3 & -2/3 \end{bmatrix}.$$

2.5. For example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11}+0+0 \\ a_{21}+0+0 \\ a_{31}+0+0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}.$$

2.6.

(a)

$$\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 2 & -3 \\ -4+2 & \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

(c)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

2.7. (a) $|\mathbf{u}| = \sqrt{10}, |\mathbf{v}| = \sqrt{2}, |\mathbf{w}| = \sqrt{8}$. (b) Each is perpendicular to the other two. Just take the dot products. (c) Multiply each vector by the reciprocal of its length: $\dfrac{1}{\sqrt{10}}\mathbf{u}, \dfrac{1}{\sqrt{2}}\mathbf{v}, \dfrac{1}{\sqrt{8}}\mathbf{w}$.

2.8. (b) Let $\mathbf{u}$ be the $\mathbf{n} \times 1$ column vector all of whose entries are 1, and let $\mathbf{v}$ the the corresponding $1 \times n$ row vector. The conditions are $A\mathbf{u} = c\mathbf{u}$ and $\mathbf{v}A = c\mathbf{v}$ for the same c.

2.9. We need to determine the relative number of individuals in each age group after 10 years has elapsed. Notice however that the individuals in any given age group become (less those who die) the individuals in the next age group and that new individuals appear in the $0 \dots 9$ age group.

$$A = \begin{bmatrix} 0 & .01 & .04 & .03 & .01 & .001 & 0 & 0 & 0 & 0 \\ .99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .99 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .98 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .97 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .96 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .90 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .70 & 0 \end{bmatrix}$$

Note that this model is not meant to be realistic.

3.1. (a) Every power of I is just I. (b) $J^2 = I$, the 2×2 identity matrix.

3.2. There are lots of answers. Here is one

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3.3. By the distributive law,

$$A(a\mathbf{x} + b\mathbf{y}) = A(a\mathbf{x}) + A(b\mathbf{y}).$$

However, one of the rules says we may move scalars around at will in a matrix product, so the above becomes

$$a(A\mathbf{x}) + b(A\mathbf{y}).$$

3.4. This is an exercise in the proper use of subscripts. The i, r entry of $(AB)C = DC$ is

$$\sum_{k=1}^{p} d_{ik}c_{kr} = \sum_{k=1}^{p}\sum_{j=1}^{n} a_{ij}b_{jk}c_{kr}.$$

Similarly, the i, r entry of $A(BC) = AE$ is

$$\sum_{j=1}^{n} a_{ij}e_{jr} = \sum_{j=1}^{n}\sum_{k=1}^{p} a_{ij}b_{jk}c_{kr}.$$

These are the same since the double sums amount to the same thing.

4.1. (a) $x_1 = -3/2, x_2 = 1/2, x_3 = 3/2$.
(b) No solutions.
(c) $x_1 = -27, x_2 = 9, x_3 = 27, x_4 = 27$. In vector form,

$$\mathbf{x} = \begin{bmatrix} -27 \\ 9 \\ 27 \\ 27 \end{bmatrix}.$$

4.2.

$$X = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

4.3. (a) Row reduction yields

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

Since the last row consists of zeroes to the left of the separator and does not consist of zeroes to the right, the system is inconsistent and does not have a solution.
(b) The solution is

$$X = \begin{bmatrix} 3/2 & 0 \\ 1/2 & 1 \\ -1/2 & 0 \end{bmatrix}.$$

4.4. The effect is to add a times the first *column* to the second *column*. The general rule is that if you multiply a matrix on the right by the matrix with an a in the i, j-position ($i \neq j$) and ones on the diagonal, the effect is to add a times the ith column to the jth column.

4.5.

$$\begin{bmatrix} 11 & 13 & 15 \\ -2 & -1 & 0 \\ 7 & 8 & 9 \end{bmatrix}$$

5.1.

$$\text{(a)} \begin{bmatrix} 0 & 1 & -1/2 \\ 1 & -3 & 5/2 \\ -1 & 2 & -3/2 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} -5 & -1 & 7 \\ 1 & 0 & -1 \\ 2 & 1 & -3 \end{bmatrix}$$

$$\text{(c) not invertible,} \quad \text{(d)} \begin{bmatrix} -4 & -2 & -3 & 5 \\ 2 & 1 & 1 & -2 \\ 6 & 2 & 4 & -7 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

5.2.

5.3. $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$

5.4. The basic argument does work except that you should start with the second column instead. If that consists of zeroes, go on to the third column, etc. The matrix obtained at the end of the Gauss-Jordan reduction will have as many columns at the beginning which consist only of zeroes as did the original matrix. For example,

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 4 \end{bmatrix} \to \cdots \to \begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.5. (b) The coefficient matrix is almost singular. Replacing 1.0001 by 1.0000 would make it singular.

5.6. The answer in part (a) is way off but the answer in part (b) is pretty good. This exercise shows you some of the numerical problems which can arise if the entries in the coefficient matrix differ greatly is size. One way to avoid such problems is always to use the largest pivot available in a given column. This is called *partial pivoting.*

5.7. The LU decomposition is

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The solution to the system is

$$\mathbf{x} = \begin{bmatrix} 1/2 \\ -1/2 \\ 3/2 \end{bmatrix}.$$

6.1.

$$\text{(a) } \mathbf{x} = \begin{bmatrix} -3/5 \\ 2/5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}, \quad \text{(b) } \mathbf{x} = \begin{bmatrix} 3/5 \\ 1/5 \end{bmatrix},$$

$$\text{(c) } \mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

6.2. Only the Gaussian part of the reduction was done. The Jordan part of the reduction was not done. In particular, there is a pivot in the $2,2$ position with a non-zero entry above it. As a result, the separation into bound and free variables is faulty.

The correct solution to this problem is $x_1 = 1, x_2 = -x_3$ with x_3 free.

6.3.

$$\text{(a) } \mathbf{x} = x_4 \begin{bmatrix} 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \quad \text{(b) } \mathbf{x} = x_3 \begin{bmatrix} -10 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

6.4. We have $\mathbf{u} \cdot \mathbf{v} = -1$, $|\mathbf{u}| = \sqrt{2}$, and $|\mathbf{v}| = \sqrt{3}$. Hence, $\cos\theta = \frac{-1}{\sqrt{6}}$. Hence, $\theta = \cos^{-1} \frac{-1}{\sqrt{6}} \approx 1.99$ radians or about 114 degrees.

6.5. (a) has rank 3 and (b) has rank 3.

6.6. The ranks are 2, 1, and 1.

6.7. (a) is always true because the rank can't be larger than the number of rows. Similarly, (b) and (d) are never true. (c) and (e) are each sometimes true. (f) is true just for the zero matrix.

6.8. In case (a), after reduction, there won't be a row of zeroes to the left of the 'bar' in the augmented matrix. Hence, it won't matter what is to the right of the 'bar'. In case (b), there will be at least one row of zeroes to the left of the 'bar', so we can always arrange for a contradictory system by making sure that there is something non-zero in such a row to the right of the 'bar'.

6.9. The rank of AB is always less than or equal to the rank of A.

6.10. A right pseudo-inverse is

$$\begin{bmatrix} -1 & 1 \\ 2 & -1 \\ 0 & 0 \end{bmatrix}.$$

There are no left pseudo-inverses for A. For if B were a left pseudo-inverse of A, A would be a right pseudo-inverse of B, and B has more 3 rows and 2 columns. According to the the text, a matrix with more rows than columns never has a right pseudo-inverse.

6.11. Suppose $m < n$ and A has a left pseudo-inverse A' such that $A'A = I$. It would follow that A' is an $n \times m$ matrix with $n > m$ (more rows that columns) and A' has a right pseudo-inverse, namely A. But we already know that is impossible.

7.1. The augmented matrix has one row $[1 \quad -2 \quad 1 \quad | \quad 4]$. It is already in Gauss–Jordan reduced form with the first entry being the single pivot. The general solution is $x_1 = 4 + 2x_2 - x_3$ with x_2, x_3 free. The general solution vector is

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The second two terms form a general solution of the homogeneous equation.

7.2. (a) is a subspace, since it is a plane in $\mathbf{R}^3$ through the origin.

(b) is not a subspace since it is a plane in $\mathbf{R}^3$ not through the origin. One can also see that it it doesn't satisfy the defining condition that it be closed under forming linear combinations. Suppose for example that $\mathbf{u}$ and $\mathbf{v}$ are vectors whose components satisfy this equation, and s and t are scalars. Then

$$\begin{aligned} u_1 - u_2 + 4u_3 &= 3 \\ v_1 - v_2 + 4v_3 &= 3 \end{aligned}$$

Multiply the first equation by s and the second by t and add. You get

$$(su_1 + tv_1) - (su_2 + tv_2) + 4(su_3 + tv_3) = 3(s + t).$$

This is the equation satisfied by the components of $s\mathbf{u} + t\mathbf{v}$. Only in the special circumstances that $s + t = 1$ will this again satisfy the same condition. Hence, most linear combinations will not end up in the same subset. A much shorter but less instructive argument is to notice that the components of the zero vector $\mathbf{0}$ don't satisfy the condition.

(c) is not a subspace because it is a curved surface in $\mathbf{R}^3$. Also, with some effort, you can see that it is not closed under forming linear combinations. Probably, the easiest thing to notice is that the components of the zero vector don't satisfy the condition.

(d) is not a subspace because the components give a parametric representation for a line in $\mathbf{R}^3$ which doesn't pass through the origin. If it did, from the first component you could conclude that $t = -1/2$, but this would give non-zero values for the second and third components. Here is a longer argument which shows that if you add two such vectors, you get a vector not of the same form.

$$\begin{bmatrix} 1 + 2t_1 \\ -3t_1 \\ 2t_1 \end{bmatrix} + \begin{bmatrix} 1 + 2t_2 \\ -3t_2 \\ 2t_2 \end{bmatrix} = \begin{bmatrix} 2 + 2(t_1 + t_2) \\ -3(t_1 + t_2) \\ 2(t_1 + t_2) \end{bmatrix}$$

The second and third components have the right form with $t = t_1 + t_2$, but the first component does not have the right form because of the '2'.

(e) is a subspace. In fact it is the plane spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}.$$

This is a special case of a subspace spanned by a finite set of vectors. Here is a detailed proof showing that the set satisfies the required condition.

$$\begin{bmatrix} s_1 + 2t_1 \\ 2s_1 - 3t_1 \\ s_1 + 2t_1 \end{bmatrix} + \begin{bmatrix} s_2 + 2t_2 \\ 2s_2 - 3t_2 \\ s_2 + 2t_2 \end{bmatrix} = \begin{bmatrix} s_1 + s_2 + 2(t_1 + t_2) \\ 2(s_1 + s_2) - 3(t_1 + t_2) \\ s_1 + s_2 + 2(t_1 + t_2) \end{bmatrix}.$$

$$c \begin{bmatrix} s + 2t \\ 2s - 3t \\ s + 2t \end{bmatrix} = \begin{bmatrix} cs + 2(ct) \\ 2(cs) - 3(ct) \\ cs + 2(ct) \end{bmatrix}.$$

What this shows is that any sum is of the same form and also any scalar multiple is of the same form. However, an arbitrary linear combination can always be obtained by combining the process of addition and scalar multiplication in some order.

Note that in cases (b), (c), (d), the simplest way to see that the set is not a subspace is to notice that the zero vector is not in the set.

7.3. No. Pick $\mathbf{v}_1$ a vector in L_1 and $\mathbf{v}_2$ a vector in L_2. If s and t are scalars, the only possible way in which $s\mathbf{v}_1 + t\mathbf{v}_2$ can point along one or the other of the lines is if s or t is zero. Hence, it is not true that every linear combination of vectors in the set S is again in the set S.

7.4. It is a plane through the origin. Hence it has an equation of the form $a_1x_1 + a_2x_2 + a_3x_3 = 0$. The given data show that

$$\begin{aligned} a_1 + a_2 &= 0 \\ a_2 + 3a_3 &= 0 \end{aligned}$$

We can treat these as homogeneous equations in the unknowns a_1, a_2, a_3. The general solution is

$$\begin{aligned} a_1 &= 3a_3 \\ a_2 &= -3a_3 \end{aligned}$$

with a_3 free. Taking $a_3 = 1$ yields the specific solution $a_1 = 3, a_2 = -3, a_3 = 1$ or the equation $3x_1 - 3x_2 + x_3 = 0$ for the desired plane. Any other non-zero choice of a_3 will yield an equation with coefficients proportion to these, hence it will have the same locus.

Another way to find the equation is to use the fact that $\mathbf{u}_1 \times \mathbf{u}_2$ is perpendicular to the desired plane. This cross product ends up being the vector with components $\langle 3, -3, 1 \rangle$.

7.5. (a) The third vector is the sum of the other two. The subspace is the plane through the origin spanned by the first two vectors. In fact, it is the plane through the origin spanned by any two of the three vectors. A normal vector to this plane may be obtained by forming the vector product of any two of the three vectors.

(b) This is actually the same plane as in part (a).

7.6. (a) A spanning set is given by

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Take dot products to check perpendicularity.

(b) A spanning set is given by

$$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

8.1. (a) No. $\mathbf{v}_1 = \mathbf{v}_2 + \mathbf{v}_3$. See also Section 9 which provides a more systematic way to answer such questions. (b) Yes. Look at the pattern of ones and zeroes. It is clear that none of these vectors can be expressed as a linear combination of the others.

8.2.

$$\left\{ \begin{bmatrix} -1/3 \\ 2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ -2/3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

8.3. (a) One. (b) Two.

8.4. No. $\mathbf{0}$ can always be expressed a linear combination of other vectors simply by taking the coefficients to be zero. One has to quibble about the set which has only one element, namely $\mathbf{0}$. Then there aren't any other vectors for it to be a linear combination of. However, in this case, we have avoided the issue by defining the set to be linearly dependent. (Alternately, one could ask if the zero vector is a linear combination of the other vectors in the set, i.e., the empty set. However, by convention, any empty sum is defined to be zero, so the criterion also works in this case.)

8.5. Suppose first that the set is linearly independent. If there were such a relation without all the coefficients c_1, c_2, c_3 zero, then one of the coefficients, say it was c_2 would not be zero. Then we could divide by that coefficient and solve for $\mathbf{v}_2$ to get

$$\mathbf{v}_1 = -\frac{c_1}{c_2}\mathbf{v}_1 - \frac{c_3}{c_2}\mathbf{v}_3,$$

i.e., $\mathbf{v}_2$ would be a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$. A similar argument would apply if c_1 or c_3 were non-zero. That contradicts the assumption of linear independence.

Suppose conversely that there is no such relation. Suppose we could express $\mathbf{v}_1$ in terms of the other vectors

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

This could be rewritten

$$-v_1 + c_2v_2 + c_3v_3 = 0.$$

which would be a relation of the form

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

with $c_1 = -1 \neq 0$. By assumption there are no such relations. A similar argument shows that neither of the other vectors could be expressed as a linear combination of the others.

A similar argument works for any number of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

8.6. (a) $\mathbf{v}_1 \times \mathbf{v}_2 \cdot \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2 \neq 0$, so $\mathbf{v}_3$ is not perpendicular to $\mathbf{v}_1 \times \mathbf{v}_2$. Similarly, calculate $\mathbf{v}_1 \times \mathbf{v}_3 \cdot \mathbf{v}_2$ and $\mathbf{v}_2 \times \mathbf{v}_3 \cdot \mathbf{v}_1$.

(b) The subspace spanned by these vectors has dimension 3. Hence, it must be all of $\mathbf{R}^3$.

(c) Solve the system

$$\mathbf{v}_1s_1 + \mathbf{v}_2s_2 + \mathbf{v}_3s_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

for s_1, s_2, s_3. The solution is $s_1 = 0, s_2 = 1, s_3 = 1$.

8.7. It is clear that the vectors form a linearly independent pair since neither is a multiple of the other. To find the coordinates of $\mathbf{e}_1$ with respect to this new basis, solve

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The solution is $x_1 = x_2 = 1/2$. Hence, the coordinates are given by

$$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

Similarly, solving

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

yields the following coordinates for $\mathbf{e}_2$.

$$\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}.$$

One could have found both sets of coordinates simultaneously by solving

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which amounts to finding the inverse of the matrix $[\, \mathbf{u}_1 \quad \mathbf{u}_2 \,]$.

8.8. (a) The set is linearly independent since neither vector is a multiple of the other. Hence, it is a basis for W.

(b) We can answer both questions by trying to solve

$$\mathbf{v}_1c_1 + \mathbf{v}_2c_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

for c_1, c_2. If there is no solution, the vector is not in subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$. If there is a solution, it provides the coordinates. In this case, there is the unique solution $c_1 = 1, c_2 = -2$.

8.9. (a) You can see you can't have a non-trivial linear relation among these vectors because of the pattern of zeroes and ones. Each has a one where the others are zero.

(b) This set of vectors does not span $\mathbf{R}^\infty$. For example, the 'vector'

$$(1, 1, 1, \dots, 1, \dots)$$

with all entries 1 cannot be written a linear combination of *finitely many* of the $\mathbf{e}_i$. Generally, the only vectors you can get as such finite linear combinations are the ones which have all components zero past a certain point.

9.1. Gauss–Jordan reduction of the matrix with these columns yields

$$\begin{bmatrix} 1 & 0 & 3/2 & -1/2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the first two vectors in the set form a basis for the subspace spanned by the set.

9.2. (a) Gauss-Jordan reduction yields

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 5 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis.

(b) A basis for the row space is

$$\{[\,1 \quad 0 \quad 2 \quad 1 \quad 1\,], [\,0 \quad 1 \quad 5 \quad 1 \quad 2\,]\}.$$

Note that neither of these has any obvious connection to the solution space which has basis

$$\left\{ \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

9.3. Reduce

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -2 & 2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

to get

$$\begin{bmatrix} 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}.$$

Picking out the first, second, and fourth columns shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_2\}$ is a basis for $\mathbf{R}^3$ containing $\mathbf{v}_1$ and $\mathbf{v}_2$.

9.5. (a) Gaussian reduction shows that A has rank 2 with pivots in the first and third columns. Hence,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}$$

is a basis for its column space.

(b) Solve the system

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 3 & 6 & 7 & 10 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

It does have solutions, so the vector on the right is in the column space.

9.6. (a) Every such system is solvable. For, the column space of A must be all of $\mathbf{R}^7$ since it is a subspace of $\mathbf{R}^7$ and has dimension 7.

(b) There are definitely such systems which don't have solutions. For, the dimension of the column space is the rank of A, which is at most 7 in any case. Hence, the column space of A must be a proper subspace of $\mathbf{R}^{12}$.

10.2. (a) The rank of A turns out to be 2, so the dimension of its nullspace is $4-2=3$. (b) The dimension of the column space is the rank, which is 2. (c) These add up to the number of columns of A which is 5.

10.3. The formula is correct if the order of the terms on the right is reversed. Since matrix multiplication is not generally commutative, we can't generally conclude that $A^{-1}B^{-1} = B^{-1}A^{-1}$.

10.4. (a) will be true if the rank of A is 15. Otherwise, there will be vectors $\mathbf{b}$ in $\mathbf{R}^{15}$ for which there is no solution.

(b) is always true since there are more unknowns that equations. In more detail, the rank is at most 15, and the number of free variables is 23 less the rank, so there are at least $23 - 15 = 8$ free variables which may assume any possible values.

10.5. (a) The Gauss–Jordan reduction is $\begin{bmatrix} 1 & 0 & 10 & -3 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. The rank is 3, and the free variables are x_3 and x_4. A basis for the nullspace is

$$\left\{ \begin{bmatrix} -10 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Whenever you do a problem of this kind, make sure you go all the way to Jordan reduced form! Also, make sure the number of free variables is the total number of unknowns less the rank.

(b) The dimension of the null space is the number of free variables which is 2. The dimension of the column space of A is the rank of A, which in this case is 3.

(c) In this case, the column space is a subspace of $\mathbf{R}^3$ with dimension 3, so it is all of $\mathbf{R}^3$. Hence, the system $A\mathbf{x} = \mathbf{b}$ has a solution for *any* possible $\mathbf{b}$. If the rank of A had been smaller than the number of rows of A (usually called m), you would have had to try to solve $A\mathbf{x} = \mathbf{b}$ for the given $\mathbf{b}$ to answer the question.

10.6. $A^{-1} = \begin{bmatrix} 4 & -3/2 & 0 \\ 1 & -1/2 & 2 \\ -1 & 1/2 & 0 \end{bmatrix}$. You should be able to check your answer yourself. Just multiply it by A and see if you get I.

10.7. (a) Reduction yields the matrix $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. x_2 and x_4 are the free variables. A basis for the solution space is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(b) Pick out the columns of the original matrix for which we have pivots in the reduced matrix. A basis is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}.$$

Of course, any other linearly independent pair of columns would also work.

(c) The columns do not form a linearly independent set since the matrix does not have rank 4.

(d) Solve the system

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 3 & 6 & 7 & 10 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

You should discover that it doesn't have a solution. The last row of the reduced augmented matrix is $[0 \quad 0 \quad 0 \quad 0 \quad | \quad -2]$. Hence, the vector is not in the column space.

10.8. (a) is a vector subspace because it is a plane through the origin. (b) is not because it is a curved surface. Also, any vector subspace contains the element $\mathbf{0}$, but this does not lie on the sphere.

10.9. (a) The rank is 2.

(b) The dimension of the solution space is the number of variables less the rank, which in this case is $5 - 2 = 3$.

10.10. (a) Yes, the set is linearly independent. The easiest way to see this is as follows. Form the 4×4 matrix with these vectors as columns, but in the opposite order to that in which they are given. That matrix is upper triangular with non-zero entries on the diagonal, so its rank is 4.

(b) Yes, it is a basis for $\mathbf{R}^4$. The subspace spanned by this set has a basis with 4 elements, so its dimension is 4. The only 4 dimensional subspace of $\mathbf{R}^4$ is the whole space itself.