CHAPTER II

DETERMINANTS AND EIGENVALUES

1.1. The first two components of $\mathbf{u}\times\mathbf{v}$ are zero and the third component is the given determinant, which might be negative.

1.2. (a) (i) 1, (ii) -1, (iii) 1.

(b) In case (ii), the orientation is reversed, so the sign changes. In case (iii), the two parallelograms can be viewed as having the same base and same height—one of the sides is shifted—so they have the same area.

1.3. (b) $(\mathbf{v}\times\mathbf{u})\cdot\mathbf{w} = (-\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$, so the sign changes. A similar argument shows the sign changes if the second and third columns are interchanged. The last determinant can be obtained by the two switches

$$[\,\mathbf{u}\quad\mathbf{v}\quad\mathbf{w}\,]\to[\,\mathbf{u}\quad\mathbf{w}\quad\mathbf{v}\,]\to[\,\mathbf{w}\quad\mathbf{u}\quad\mathbf{v}\,]$$

each of which changes the sign, so the net result is no change.

(c) $(\mathbf{u}+\mathbf{v})\times\mathbf{v} = \mathbf{u}\times\mathbf{u}+\mathbf{u}\times\mathbf{v} = \mathbf{u}\times\mathbf{v}$, so $((\mathbf{u}+\mathbf{v})\times\mathbf{v})\cdot\mathbf{w} = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$.

(d) The determinant is multiplied by -3.

1.4. We have

$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} e \\ f \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix} \\ &= \frac{1}{ad-bc}\begin{bmatrix} de-bf \\ -ce+af \end{bmatrix}.\end{aligned}$$

2.1. (a) -16. (b) 40. (c) 3. (d) 0.

2.2. This is a lot of algebra, which I leave to you.

If you have verified rules (i) and (ii) only for the first row, and you have also verified rule (iii), then you can verify rules (i) and (ii) for the second row as follows. Interchange the two rows. The second row is now the first row, but the sign has changed. Use rules (i) and (ii) on the new first row, then exchange rows again. The sign changes back and the rules are verified for the second row.

2.3. I leave the algebra to you. The corresponding rule for the second row follows by exchanging rows, applying the new rule to both sides of the equation and then exchanging back.

2.4. The matrix is singular if and only if its determinant is zero.

$$\det \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix} = 1 - z^2 = 0$$

yields $z = \pm 1$.

2.5. $\det A = -\lambda^3 + 2\lambda = 0$ yields $\lambda = 0, \pm\sqrt{2}$.

2.6. The relevant point is that the determinant of any matrix which has a column consisting of zeroes is zero. For example, in the present case, if we write out the formula for the determinant of the above 5×5 matrix, each term will involve the determinant of a 4×4 matrix with a column of zeroes. Similarly, in the formula for the determinant of such a 4×4 matrix, each term will involve the determinant of a 3×3 matrix with a column of zeroes. Continuing this way, we eventually get to determinants of 2×2 matrices, each with a column of zeroes. However, it is easy to see that the determinant of such a 2×3 matrix is zero.

Note that we will see later that a formula like that used to define the determinant works for *any* column or indeed any row. Hence, if a column (or row) consists of zeroes, the coefficients in that formula would all be zero, and the net result would be zero. However, it would be premature to use such a formula at this point.

2.7. $\det(cA) = c^n \det A$. For, multiplying one row of A by c multiplies its determinant by c, and in cA, all n rows are multiplied by c.

2.8. By a previous exercise, we have $\det(-A) = (-1)^6 \det A = \det A$. The only way we could have $\det A = -\det A$ is if $\det A = 0$, in which case A would be singular.

2.9. Almost anything you come up, with the exception of a few special cases, should work. For example, suppose $\det A \neq 0$ and $B = -A$. Then, since A is 2×2, it follows that $\det(-A) = (-1)^2 \det A = \det A$. Hence, $\det A + \det B = 2 \det A \neq \det 0 = 0$.

2.10. (a) The recursive formula for $n = 7$ uses seven 6×6 subdeterminants. Each of these requires $N(6) = 876$ multiplications. Since there are 7 of these, this requires $7 * 876 = 6132$ multiplications. However, in addition, once these 7 subdeterminants have been calculated, each must be multiplied by the appropriate entry, and this adds 7 additional multiplications. Hence, the total $N(7) = 6132 + 7 = 6139$.

(b) The recursive rule is $N(n) = nN(n-1) + n$.

3.1. The first matrix has determinant 31, and the second matrix has determinant 1. The product matrix is

$$\begin{bmatrix} 6 & 5 & -3 \\ 7 & 9 & 2 \\ -4 & -6 & -1 \end{bmatrix}$$

which has determinant 31.

3.2. If A and B both have rank n, they are both non-singular. Hence, $\det A$ and $\det B$ are nonzero. Hence, by the product rule, $\det(AB) = \det A \det B \neq 0$. Hence, AB is also non-singular and has rank n.

3.3. The determinant of any lower triangular matrix is the product of its diagonal entries. For example, you could just use the transpose rule.

3.4. (a) If A is invertible, then $AA^{-1} = I$. Hence, $\det(AA^{-1}) = 1$. Using the product rule yields $\det A \det(A^{-1} = 1$. Hence, $\det A \neq 0$, and dividing both sides by it yields $\det(A^{-1}) = \dfrac{1}{\det A}$.

(b) $\det(PAP^{-1}) = \det P \det A \det(P^{-1})$. But $\det(P^{-1}) = \dfrac{1}{\det P}$. $\det P$ and $\dfrac{1}{\det P}$ cancel, so the net result is $\det A$ as claimed.

3.5. Cramer's rule has $\det A$ in the denominator. Hence, the formula is meaningless if A is singular since in that case $\det A = 0$.

3.6. The determinant of the coefficient matrix is 1. The solution by Cramer's rule or by Gauss-Jordan reduction is $x_1 = -2, x_2 = 1, x_3 = 4, x_4 = 2$.

4.1. In each case we give the eigenvalues and for each eigenvalue a basis consisting of one or more basic eigenvectors for that eigenvalue. (a)

$$\lambda = 2, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda = 3, \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

(b)

$$\lambda = 2, \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} \qquad \lambda = 1, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad \lambda = -1, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

(c)

$$\lambda = 2, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad \lambda = 1, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

(d)

$$\lambda = 3, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

4.2. Compute

$$A\mathbf{v} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$

You see that $A\mathbf{v}$ is *not* a scalar multiple of $\mathbf{v}$, so *by definition*, it is not an eigenvector for A. Note that trying to find the eigenvalues and eigenvectors of A would be much more time consuming. In this particular case, the eigenvalues turn out to be $\lambda = -2, -2 + \sqrt{2}, -2 - \sqrt{2}$, and the radicals make it a bit complicated to find the eigenvectors.

4.3. You say that an eigenvector can't be zero, so there had to be a mistake somewhere in the calculation. Either the characteristic equation was not solved correctly to find the eigenvalue λ or the solution of the system $(A - \lambda I)\mathbf{v} = \mathbf{0}$ was not done properly to find the eigenspace.

4.4. The eigenvalues of A are the roots of the equation $\det(A - \lambda I) = 0$. $\lambda = 0$ is a root of this equation if and only if $\det(A - 0\,I) = 0$, i.e., $\det A = 0$. Hence, A would have to be singular.

4.5. Multiply $A\mathbf{v} = \lambda\mathbf{v}$ by A. We get

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

In general, $A^n\mathbf{v} = \lambda^n\mathbf{v}$.

4.6. $A\mathbf{v} = \lambda\mathbf{v}$ implies that

$$\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}(\lambda\mathbf{v}) = \lambda A^{-1}\mathbf{v}.$$

Since A is non-singular, $\lambda \neq 0$ by a problem above, so we may divide through by λ to obtain $\lambda^{-1}\mathbf{v} = A^{-1}\mathbf{v}$. This just says λ^{-1} is an eigenvalue for A^{-1}.

4.7. (a) and (b) are done by expanding the determinants

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \qquad \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix}.$$

I leave the details to you.

(c) The coefficient of λ^n is $(-1)^n$, i.e., it is 1 if n is even and -1 if n is odd.

5.1. (a) $\lambda = 3, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda = -1, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. However, other answers are possible, depending on how you did the problem.

5.2. (a)

$$\lambda = -3, \mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \lambda = 6, \mathbf{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis, but other answers are possible, depending on how you went about doing the problem,

5.3. (a) $\lambda = 2, \mathbf{v}_1 = \mathbf{e}_1$ and $\lambda = 1, \mathbf{v}_2 = \mathbf{e}_3$. (b) For $\lambda = 2$, the dimension of the eigenspace is strictly less than the multiplicity. For $\lambda = 1$, the number of basic eigenvectors does equal the multiplicity; they are both one. A is not diagonalizable because equality does not hold for at least one of the eigenvalues.

5.4. (a)

$$\lambda = 1, \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \lambda = 4, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

However, other answers are possible.

(b) This depends on your answer for part (a). For example,

$$P = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3\,] = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

will work.

5.5. Note that if $m_1 + m_2 + m_3 \neq 5$, that means that the characteristic equation has complex roots which are not considered candidates for real eigenvalues.

(a) The dimension of the eigenspace equals the multiplicity for each eigenvalue and the multiplicities add up to five. Hence, the matrix is diagonalizable.

(b) $d_1 < m_1$ so the matrix is not diagonalizable.

(c) $d_1 > m_1$, which is never possible. No such matrix exists.

(d) $m_1 + m_2 + m_3 = 3 < 5$. Hence, there are necessarily some complex roots of the characteristic equation. The matrix is not diagonalizable (in the purely real theory).

5.6. (a) The characteristic equation is $\lambda^2 - 13\lambda + 36 = 0$. Its roots $\lambda = 4, 9$ are distinct, so the matrix is diagonalizable. In Chapter III, we will learn a simpler more direct way to see that a matrix of this type is diagonalizable.

(b) The characteristic equation is $(\lambda - 1)^2 = 0$ so the only eigenvalue is $\lambda = 1$ and it has multiplicity two. That, in itself, is not enough to conclude the matrix isn't diagonalizable. However,

$$\begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which has rank 1. Hence, the eigenspace has dimension $2 - 1 = 1$ which is less than the multiplicity of the eigenvalue. Hence, the matrix is not diagonalizable.

(c) The characteristic equation is $\lambda^2 + 1 = 0$. Since its roots are non-real complex numbers, this matrix is not diagonalizable in our sense, since we restrict attention to real scalars.

6.1. (a) For any non-negative integer n, we have

$$A^n = \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix},$$

so

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \begin{bmatrix} \sum_{n=0}^{\infty} \lambda^n t^n / n! & 0 \\ 0 & \sum_{n=0}^{\infty} \mu^n t^n / n! \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}.$$

(b) We have

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

6.2. (a)

$$e^{Nt} = I + t\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

This used the fact that in this case $N^k = 0$ for $k \geq 2$.

(b) In this case $N^k = 0$ for $k \geq 3$.

$$e^{Nt} = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2/2 & t & 1 \end{bmatrix}.$$

(c) The smallest such k is n. e^{Nt} has the form suggested by the answers in parts (a) and (b). It is an $n \times n$ matrix with '1's on the diagonal, 't's just below the diagonal, '$t^2/2$'s just below that, etc. In the lower left hand corner there is a '$t^{n-1}/(n-1)!$'.

6.3. (a)

$$e^{At} = e^{\lambda t}\left\{I + t\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\} = e^{\lambda t}\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

This used the fact that in this case $(A - \lambda I)^k = 0$ for $k \geq 2$.

(b) In this case $(A - \lambda I)^k = 0$ for $k \geq 3$.

$$e^{At} = e^{\lambda t}\begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2/2 & t & 1 \end{bmatrix}.$$

(c) The smallest such k is n. e^{At} has the form suggested by the answers in parts (a) and (b). There is a scalar factor of $e^{\lambda t}$ followed by a lower triangular matrix with '1's on the diagonal, 't's just below the diagonal, '$t^2/2$'s just below that, etc. In the lower left hand corner there is a '$t^{n-1}/(n-1)!$'.

6.4. In general $PA^nP^{-1} = (PAP^{-1})^n$. Hence,

$$P(\sum_{n=0}^{\infty} t^n/n!A^n)P^{-1} = \sum_{n=0}^{\infty} t^n/n!PA^nP^{-1} = \sum_{n=0}^{\infty} t^n/n!(PAP^{-1})^n = e^{PAP^{-1}t}.$$

6.5.

$$\begin{aligned} e^{B+C} &= \sum_{n=0}^{\infty} \frac{1}{n!}(B+C)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i+j=n} \frac{n!}{i!j!} B^i C^j \\ &= \sum n = 0^{\infty} \sum_{i+j=n} \frac{1}{i!} B^i \frac{1}{j!} C^j \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} B^i \sum_{j=0}^{\infty} \frac{1}{j!} C^j = e^B e^C. \end{aligned}$$

6.6. (c) We have

$$B + C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

so by Example 2,

$$e^{B+C} = \begin{bmatrix} \cos 1 & sin 1 \\ -\sin 1 & \cos 1 \end{bmatrix}.$$

On the other hand,

$$e^B e^C = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},$$

7.1. This is *not* an upper or lower triangular matrix. However, after interchanging the first and third rows, it becomes an upper triangular matrix with determinant equal to the product of its diagonal entries. The determinant is -6 because we have to change the sign due to the interchange.

7.2. (a) and (c) are true. (b) is false. The correct rule is $\det(cA) = c^n \det A$. (d) is true. One way to see this is to notice that $\det A^t = \det A \neq 0$.

7.3. The characteristic equation is $-(\lambda - 2)(\lambda + 1)^2 = 0$. The eigenvalues are $\lambda = 2$ and $\lambda = -1$ which is a double root. For $\lambda = 2$,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis for the eigenspace. For $\lambda = -1$,

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the eigenspace.

7.4. Use Gauss Jordan reduction to get an upper triangular matrix. You might speed things up also by using selected column operations. The answer is -23.

7.5. (a) This is never true. It is invertible if and only if its determinant is not zero.

(b) This condition is the definition of 'diagonalizable matrix'. There are many non-diagonalizable matrices. This will happen for example when the dimension of an eigenspace is less than the multiplicity of the corresponding eigenvalue. (It can also happen if the characteristic equation has non-real complex roots.)

(c) The statement is only true for square matrices.

7.6. No. $A\mathbf{v}$ is not a scalar multiple of $\mathbf{v}$.

7.7. (a) It is not diagonalizable since the dimension of the eigenspace for $\lambda = 3$ is one and the multiplicity of the eigenvalue is two.

(b) There are three distinct eigenvalues, so the matrix is diagonalizable.

7.8. Take $\mathbf{v}$ to be the element of $\mathbf{R}^n$ with all its entries equal to one. Then the ith component of $A\mathbf{v}$ is just the sum of the entries in the ith row of A. Since these are all equal to a, it follows that $A\mathbf{v} = a\mathbf{v}$, so $\mathbf{v}$ is an eigenvector with corresponding eigenvalue a.