Solutions to Problems

Solutions for Chapter 1, Section 1.

1.1 (a) $86.66\ldots$ and $88.33\ldots$.

(b) $a_1 = 0.6, a_2 = 0.4$ will work in the first case, but there are no possible such weightings to produce the second case, since Student 1 and Student 3 have to end up with the same score.

1.2 (a) $x = 2, y = -1/3$ (b) $x = 1, y = 2, z = 2$ (c) This system does not have a solution since by adding the first two equations, we obtain $x + 2y + z = 7$ and that contradicts the third equation. (d) Subtracting the second equation from the first yields $x + y = 0$ or $x = -y$ This system has infinitely many solutions since x and y can be arbitrary as long as they satisfy this relation.

Solutions for Chapter 1, Section 2.

2.1

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \quad 3\mathbf{x} - 5\mathbf{y} + \mathbf{z} = \begin{bmatrix} 14 \\ 1 \\ -25 \end{bmatrix}.$$

2.2

$$A\mathbf{x} = \begin{bmatrix} -15 \\ -10 \\ 4 \\ -10 \end{bmatrix}, \quad A\mathbf{y} = \begin{bmatrix} -2 \\ -2 \\ 14 \\ -20, \end{bmatrix} \quad A\mathbf{x} + A\mathbf{y} = A(\mathbf{x} + \mathbf{y}) = \begin{bmatrix} -17 \\ -12 \\ 18 \\ -30 \end{bmatrix}.$$

2.3 $A + 3B, C + 2D, DC$ are not defined.

$$A + C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad AB = \begin{bmatrix} -9 & 8 \\ -8 & 4 \end{bmatrix},$$

$$BA = \begin{bmatrix} 1 & -5 & 7 \\ 1 & -1 & 3 \\ -3 & -1 & -5 \end{bmatrix}, \quad CD = \begin{bmatrix} -4 & -3 & 13 \\ -2 & -6 & 10 \end{bmatrix}.$$

2.4 We have for the first components of these two products

$$a_{11} + a_{12}2 = 3$$
$$a_{11}2 + a_{12} = 6$$

This is a system of 2 equations in 2 unknowns, and you can solve it by the usual methods of high school algebra to obtain $a_{11} = 3, a_{12} = 0$ A similar argument applied to the second components yields $a_{2,1} = 7/3, a_{22} = -2/3$ Hence,

$$A = \begin{bmatrix} 3 & 0 \\ 7/3 & -2/3 \end{bmatrix}.$$

2.5 For example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} + 0 + 0 \\ a_{21} + 0 + 0 \\ a_{31} + 0 + 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}.$$

2.6 (a)

$$\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 2 & -3 \\ -4+2 & \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

(c)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

2.7 (a) $|\mathbf{u}| = \sqrt{10}, |\mathbf{v}| = \sqrt{2}, |\mathbf{w}| = \sqrt{8}$. (b) Each is perpendicular to the other two. Just take the dot products. (c) Multiply each vector by the reciprocal of its length: $\dfrac{1}{\sqrt{10}}\mathbf{u}$, $\dfrac{1}{\sqrt{2}}\mathbf{v}$, $\dfrac{1}{\sqrt{8}}\mathbf{w}$.

2.8 (b) Let $\mathbf{u}$ be the $\mathbf{n} \times 1$ column vector all of whose entries are 1, and let $\mathbf{v}$ the the corresponding $1 \times n$ row vector. The conditions are $A\mathbf{u} = c\mathbf{u}$ and $\mathbf{v}A = c\mathbf{v}$ for the same c.

2.9 We need to determine the relative number of individuals in each age group after 10 years has elapsed. Notice however that the individuals in any given age group become (less those who die) the individuals in the next age group and that new individuals appear in the $0 \ldots 9$ age group.

$$A = \begin{bmatrix} 0 & .01 & .04 & .03 & .01 & .001 & 0 & 0 & 0 & 0 \\ .99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .99 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .98 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .97 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .96 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .90 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .70 & 0 \end{bmatrix}$$

Note that this model is not meant to be realistic.

Solutions for Chapter 1, Section 3.

3.1 (a) Every power of I is just I. (b) $J^2 = I$, the 2×2 identity matrix.

3.2 There are lots of answers. Here is one

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3.3 By the distributive law,

$$A(a\mathbf{x} + b\mathbf{y}) = A(a\mathbf{x}) + A(b\mathbf{y}).$$

However, one of the rules says we may move scalars around at will in a matrix product, so the above becomes

$$a(A\mathbf{x}) + b(A\mathbf{y}).$$

3.4 This is an exercise in the proper use of subscripts. The i, r entry of $(AB)C = DC$ is

$$\sum_{k=1}^{p} d_{ik}c_{kr} = \sum_{k=1}^{p}\sum_{j=1}^{n} a_{ij}b_{jk}c_{kr}.$$

Similarly, the i, r entry of $A(BC) = AE$ is

$$\sum_{j=1}^{n} a_{ij}e_{jr} = \sum_{j=1}^{n}\sum_{k=1}^{p} a_{ij}b_{jk}c_{kr}.$$

These are the same since the double sums amount to the same thing.

Solutions for Chapter 1, Section 4.

4.1 (a) $x_1 = -3/2, x_2 = 1/2, x_3 = 3/2$.
(b) No solutions.
(c) $x_1 = -27, x_2 = 9, x_3 = 27, x_4 = 27$. In vector form,

$$\mathbf{x} = \begin{bmatrix} -27 \\ 9 \\ 27 \\ 27 \end{bmatrix}.$$

4.2

$$X = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

4.3 (a) Row reduction yields

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

Since the last row consists of zeroes to the left of the separator and does not consist of zeroes to the right, the system is inconsistent and does not have a solution.
(b) The solution is

$$X = \begin{bmatrix} 3/2 & 0 \\ 1/2 & 1 \\ -1/2 & 0 \end{bmatrix}.$$

4.4 The effect is to add a times the first *column* to the second *column*. The general rule is that if you multiply a matrix on the right by the matrix with an a in the i, j-position $(i \neq j)$ and ones on the diagonal, the effect is to add a times the ith column to the jth column.

4.5

$$\begin{bmatrix} 11 & 13 & 15 \\ -2 & -1 & 0 \\ 7 & 8 & 9 \end{bmatrix}$$

Solutions for Chapter 1, Section 5.

5.1

$$\text{(a)}\ \begin{bmatrix} 0 & 1 & -1/2 \\ 1 & -3 & 5/2 \\ -1 & 2 & -3/2 \end{bmatrix}, \quad \text{(b)}\ \begin{bmatrix} -5 & -1 & 7 \\ 1 & 0 & -1 \\ 2 & 1 & -3 \end{bmatrix}$$

$$\text{(c) not invertible,} \quad \text{(d)}\ \begin{bmatrix} -4 & -2 & -3 & 5 \\ 2 & 1 & 1 & -2 \\ 6 & 2 & 4 & -7 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

5.2 Just compute AA^{-1} and see that you get I.

Note that this formula is probably the fastest way to find the inverse of a 2×2 matrix. In words, you do the following: interchange the diagonal entries, change the signs of the off diagonal entries, and divide by the determinant $ad - bc$. Unfortunately, there no rule for $n \times n$ matrices, even for $n = 3$, which is quite so simple.

5.3 $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$.

5.4 The basic argument does work except that you should start with the second column instead. If that consists of zeroes, go on to the third column, etc. The matrix obtained at the end of the Gauss-Jordan reduction will have as many columns at the beginning which consist only of zeroes as did the original matrix. For example,

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 4 \end{bmatrix} \to \cdots \to \begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.5 (b) The coefficient matrix is almost singular. Replacing 1.0001 by 1.0000 would make it singular.

5.6 The answer in part (a) is way off but the answer in part (b) is pretty good. This exercise shows you some of the numerical problems which can arise if the entries in the coefficient matrix differ greatly is size. One way to avoid such problems is always to use the largest pivot available in a given column. This is called *partial pivoting.*

5.7 The LU decomposition is

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The solution to the system is

$$\mathbf{x} = \begin{bmatrix} 1/2 \\ -1/2 \\ 3/2 \end{bmatrix}.$$

Solutions for Chapter 1, Section 6.

6.1

$$\text{(a)}\ \mathbf{x} = \begin{bmatrix} -3/5 \\ 2/5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}, \quad \text{(b)}\ \mathbf{x} = \begin{bmatrix} 3/5 \\ 1/5 \end{bmatrix},$$

$$\text{(c) } \mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

6.2 Only the Gaussian part of the reduction was done. The Jordan part of the reduction was not done. In particular, there is a pivot in the $2,2$ position with a non-zero entry above it. As a result, the separation into bound and free variables is faulty.

The correct solution to this problem is $x_1 = 1, x_2 = -x_3$ with x_3 free.

6.3

$$\text{(a) } \mathbf{x} = x_4 \begin{bmatrix} 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \quad \text{(b) } \mathbf{x} = x_3 \begin{bmatrix} 4-10 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

6.4 We have $\mathbf{u} \cdot \mathbf{v} = -1$, $|\mathbf{u}| = \sqrt{2}$, and $|\mathbf{v}| = \sqrt{3}$. Hence, $\cos\theta = \dfrac{-1}{\sqrt{6}}$. Hence, $\theta = \cos^{-1} \dfrac{-1}{\sqrt{6}} \approx 1.99$ radians or about 114 degrees.

6.5 (a) has rank 3 and (b) has rank 3.

6.6 The ranks are 2, 1, and 1.

6.7 (a) is always true because the rank can't be larger than the number of rows. Similarly, (b) and (d) are never true. (c) and (e) are each sometimes true. (f) is true just for the zero matrix.

6.8 In case (a), after reduction, there won't be a row of zeroes to the left of the 'bar' in the augmented matrix. Hence, it won't matter what is to the right of the 'bar'. In case (b), there will be at least one row of zeroes to the left of the 'bar', so we can always arrange for a contradictory system by making sure that there is something non-zero in such a row to the right of the 'bar'.

6.9 The rank of AB is always less than or equal to the rank of A.

6.10 A right pseudo-inverse is

$$\begin{bmatrix} -1 & 1 \\ 2 & -1 \\ 0 & 0 \end{bmatrix}.$$

There are no left pseudo-inverses for A. For if B were a left pseudo-inverse of A, A would be a right pseudo-inverse of B, and B has more 3 rows and 2 columns. According to the the text, a matrix with more rows than columns never has a right pseudo-inverse.

6.11 Suppose $m < n$ and A has a left pseudo-inverse A' such that $A'A = I$. It would follow that A' is an $n \times m$ matrix with $n > m$ (more rows that columns) and A' has a right pseudo-inverse, namely A. But we already know that is impossible.

Solutions for Chapter 1, Section 7.

7.1 The augmented matrix has one row $\begin{bmatrix} 1 & -2 & 1 & | & 4 \end{bmatrix}$. It is already in Gauss–Jordan reduced form with the first entry being the single pivot. The general solution is $x_1 = 4 + 2x_2 - x_3$ with x_2, x_3 free. The general solution vector is

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The second two terms form a general solution of the homogeneous equation.

7.2 (a) is a subspace, since it is a plane in $\mathbf{R}^3$ through the origin.

(b) is not a subspace since it is a plane in $\mathbf{R}^3$ not through the origin. One can also see that it it doesn't satisfy the defining condition that it be closed under forming linear combinations. Suppose for example that $\mathbf{u}$ and $\mathbf{v}$ are vectors whose components satisfy this equation, and s and t are scalars. Then

$$\begin{aligned} u_1 - u_2 + 4u_3 &= 3 \\ v_1 - v_2 + 4v_3 &= 3 \end{aligned}$$

Multiply the first equation by s and the second by t and add. You get

$$(su_1 + tv_1) - (su_2 + tv_2) + 4(su_3 + tv_3) = 3(s + t).$$

This is the equation satisfied by the components of $s\mathbf{u} + t\mathbf{v}$. Only in the special circumstances that $s+t = 1$ will this again satisfy the same condition. Hence, most linear combinations will not end up in the same subset. A much shorter but less instructive argument is to notice that the components of the zero vector $\mathbf{0}$ don't satisfy the condition.

(c) is not a subspace because it is a curved surface in $\mathbf{R}^3$. Also, with some effort, you can see that it is not closed under forming linear combinations. Probably, the easiest thing to notice is that the components of the zero vector don't satisfy the condition.

(d) is not a subspace because the components give a parametric representation for a line in $\mathbf{R}^3$ which doesn't pass through the origin. If it did, from the first component you could conclude that $t = -1/2$, but this would give non-zero values for the second and third components. Here is a longer argument which shows that if you add two such vectors, you get a vector not of the same form.

$$\begin{bmatrix} 1 + 2t_1 \\ -3t_1 \\ 2t_1 \end{bmatrix} + \begin{bmatrix} 1 + 2t_2 \\ -3t_2 \\ 2t_2 \end{bmatrix} = \begin{bmatrix} 2 + 2(t_1 + t_2) \\ -3(t_1 + t_2) \\ 2(t_1 + t_2) \end{bmatrix}$$

The second and third components have the right form with $t = t_1 + t_2$, but the first component does not have the right form because of the '2'.

(e) is a subspace. In fact it is the plane spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}.$$

This is a special case of a subspace spanned by a finite set of vectors. Here is a detailed proof showing that the set satisfies the required condition.

$$\begin{bmatrix} s_1+2t_1 \\ 2s_1-3t_1 \\ s_1+2t_1 \end{bmatrix} + \begin{bmatrix} s_2+2t_2 \\ 2s_2-3t_2 \\ s_2+2t_2 \end{bmatrix} = \begin{bmatrix} s_1+s_2+2(t_1+t_2) \\ 2(s_1+s_2)-3(t_1+t_2) \\ s_1+s_2+2(t_1+t_2) \end{bmatrix}. \tag{58}$$

$$c\begin{bmatrix} s+2t \\ 2s-3t \\ s+2t \end{bmatrix} = \begin{bmatrix} cs+2(ct) \\ 2(cs)-3(ct) \\ cs+2(ct) \end{bmatrix}. \tag{59}$$

What this shows is that any sum is of the same form and also any scalar multiple is of the same form. However, an arbitrary linear combination can always be obtained by combining the process of addition and scalar multiplication in some order.

Note that in cases (b), (c), (d), the simplest way to see that the set is not a subspace is to notice that the zero vector is not in the set.

7.3 No. Pick $\mathbf{v}_1$ a vector in L_1 and $\mathbf{v}_2$ a vector in L_2. If s and t are scalars, the only possible way in which $s\mathbf{v}_1 + t\mathbf{v}_2$ can point along one or the other of the lines is if s or t is zero. Hence, it is not true that every linear combination of vectors in the set S is again in the set S.

7.4 It is a plane through the origin. Hence it has an equation of the form $a_1x_1 + a_2x_2 + a_3x_3 = 0$. The given data show that

$$\begin{aligned} a_1 + a_2 &= 0 \\ a_2 + 3a_3 &= 0 \end{aligned}$$

We can treat these as homogeneous equations in the unknowns a_1, a_2, a_3. The general solution is

$$\begin{aligned} a_1 &= 3a_3 \\ a_2 &= -3a_3 \end{aligned}$$

with a_3 free. Taking $a_3 = 1$ yields the specific solution $a_1 = 3, a_2 = -3, a_3 = 1$ or the equation $3x_1 - 3x_2 + x_3 = 0$ for the desired plane. Any other non-zero choice of a_3 will yield an equation with coefficients proportion to these, hence it will have the same locus.

Another way to find the equation is to use the fact that $\mathbf{u}_1 \times \mathbf{u}_2$ is perpendicular to the desired plane. This cross product ends up being the vector with components $\langle 3, -3, 1\rangle$.

7.5 (a) The third vector is the sum of the other two. The subspace is the plane through the origin spanned by the first two vectors. In fact, it is the plane through the origin spanned by any two of the three vectors. A normal vector to this plane may be obtained by forming the vector product of any two of the three vectors.

(b) This is actually the same plane as in part (a).

7.6 (a) A spanning set is given by

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Take dot products to check perpendicularity.

(b) A spanning set is given by

$$\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}.$$

Solutions for Chapter 1, Section 8.

8.1 (a) No. $\mathbf{v}_1 = \mathbf{v}_2 + \mathbf{v}_3$. See also Section 9 which provides a more systematic way to answer such questions. (b) Yes. Look at the pattern of ones and zeroes. It is clear that none of these vectors can be expressed as a linear combination of the others.

8.2

$$\left\{\begin{bmatrix} -1/3 \\ 2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ -2/3 \\ 0 \\ 1 \end{bmatrix}\right\}$$

8.3 (a) One. (b) Two.

8.4 No. $\mathbf{0}$ can always be expressed a linear combination of other vectors simply by taking the coefficients to be zero. One has to quibble about the set which has only one element, namely $\mathbf{0}$. Then there aren't any other vectors for it to be a linear combination of. However, in this case, we have avoided the issue by defining the set to be linearly dependent. (Alternately, one could ask if the zero vector is a linear combination of the other vectors in the set, i.e., the empty set. However, by convention, any empty sum is defined to be zero, so the criterion also works in this case.)

8.5 Suppose first that the set is linearly independent. If there were such a relation without all the coefficients c_1, c_2, c_3 zero, then one of the coefficients, say it was c_2 would not be zero. Then we could divide by that coefficient and solve for $\mathbf{v}_2$ to get

$$\mathbf{v}_1 = -\frac{c_1}{c_2}\mathbf{v}_1 - \frac{c_3}{c_2}\mathbf{v}_3,$$

i.e., $\mathbf{v}_2$ would be a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$. A similar argument would apply if c_1 or c_3 were non-zero. That contradicts the assumption of linear independence.

Suppose conversely that there is no such relation. Suppose we could express $\mathbf{v}_1$ in terms of the other vectors

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

This could be rewritten

$$-v_1 + c_2v_2 + c_3v_3 = 0.$$

which would be a relation of the form

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

with $c_1 = -1 \neq 0$. By assumption there are no such relations. A similar argument shows that neither of the other vectors could be expressed as a linear combination of the others.

A similar argument works for any number of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

8.6 (a) $\mathbf{v}_1 \times \mathbf{v}_2 \cdot \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2 \neq 0$, so $\mathbf{v}_3$ is not perpendicular to $\mathbf{v}_1 \times \mathbf{v}_2$. Similarly, calculate $\mathbf{v}_1 \times \mathbf{v}_3 \cdot \mathbf{v}_2$ and $\mathbf{v}_2 \times \mathbf{v}_3 \cdot \mathbf{v}_1$.

(b) The subspace spanned by these vectors has dimension 3. Hence, it must be all of $\mathbf{R}^3$.

(c) Solve the system

$$\mathbf{v}_1 s_1 + \mathbf{v}_2 s_2 + \mathbf{v}_3 s_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

for s_1, s_2, s_3. The solution is $s_1 = 0, s_2 = 1, s_3 = 1$.

8.7 It is clear that the vectors form a linearly independent pair since neither is a multiple of the other. To find the coordinates of $\mathbf{e}_1$ with respect to this new basis, solve

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The solution is $x_1 = x_2 = 1/2$. Hence, the coordinates are given by

$$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

Similarly, solving

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

yields the following coordinates for $\mathbf{e}_2$.

$$\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}.$$

One could have found both sets of coordinates simultaneously by solving

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which amounts to finding the inverse of the matrix $\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$.

8.8 (a) The set is linearly independent since neither vector is a multiple of the other. Hence, it is a basis for W.

(b) We can answer both questions by trying to solve

$$\mathbf{v}_1 c_1 + \mathbf{v}_2 c_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

for c_1, c_2. If there is no solution, the vector is not in subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$. If there is a solution, it provides the coordinates. In this case, there is the unique solution $c_1 = 1, c_2 = -2$.

8.9 (a) You can see you can't have a non-trivial linear relation among these vectors because of the pattern of zeroes and ones. Each has a one where the others are zero.

(b) This set of vectors does not span $\mathbf{R}^\infty$. For example, the 'vector'

$$(1, 1, 1, \dots, 1, \dots)$$

with all entries 1 cannot be written a linear combination of *finitely many* of the $\mathbf{e}_i$. Generally, the only vectors you can get as such finite linear combinations are the ones which have all components zero past a certain point.

Solutions for Chapter 1, Section 9.

9.1 Gauss–Jordan reduction of the matrix with these columns yields

$$\begin{bmatrix} 1 & 0 & 3/2 & -1/2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the first two vectors in the set form a basis for the subspace spanned by the set.

9.2 (a) Gauss-Jordan reduction yields

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 5 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis.

(b) A basis for the row space is

$$\{\begin{bmatrix} 1 & 0 & 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 5 & 1 & 2 \end{bmatrix}\}.$$

Note that neither of these has any obvious connection to the solution space which has basis

$$\left\{ \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

9.3 Reduce

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -2 & 2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

to get

$$\begin{bmatrix} 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}.$$

Picking out the first, second, and fourth columns shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_2\}$ is a basis for $\mathbf{R}^3$ containing $\mathbf{v}_1$ and $\mathbf{v}_2$.

9.5 (a) Gaussian reduction shows that A has rank 2 with pivots in the first and third columns. Hence,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}$$

is a basis for its column space.

(b) Solve the system

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 3 & 6 & 7 & 10 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

It does have solutions, so the vector on the right is in the column space.

9.6 (a) Every such system is solvable. For, the column space of A must be all of $\mathbf{R}^7$ since it is a subspace of $\mathbf{R}^7$ and has dimension 7.

(b) There are definitely such systems which don't have solutions. For, the dimension of the column space is the rank of A, which is at most 7 in any case. Hence, the column space of A must be a proper subspace of $\mathbf{R}^{12}$.

Solutions for Chapter 1, Section 10.

10.2 (a) The rank of A turns out to be 2, so the dimension of its nullspace is $4-2=3$. (b) The dimension of the column space is the rank, which is 2. (c) These add up to the number of columns of A which is 5.

10.3 The formula is correct if the order of the terms on the right is reversed. Since matrix multiplication is not generally commutative, we can't generally conclude that $A^{-1}B^{-1} = B^{-1}A^{-1}$.

10.4 (a) will be true if the rank of A is 15. Otherwise, there will be vectors $\mathbf{b}$ in $\mathbf{R}^{15}$ for which there is no solution.

(b) is always true since there are more unknowns that equations. In more detail, the rank is at most 15, and the number of free variables is 23 less the rank, so there are at least $23-15=8$ free variables which may assume any possible values.

10.5 (a) The Gauss–Jordan reduction is $\begin{bmatrix} 1 & 0 & 10 & -3 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. The rank is 3, and the free variables are x_3 and x_4. A basis for the nullspace is

$$\left\{ \begin{bmatrix} -10 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Whenever you do a problem of this kind, make sure you go all the way to Jordan reduced form! Also, make sure the number of free variables is the total number of unknowns less the rank.

(b) The dimension of the null space is the number of free variables which is 2. The dimension of the column space of A is the rank of A, which in this case is 3.

(c) In this case, the column space is a subspace of $\mathbf{R}^3$ with dimension 3, so it is all of $\mathbf{R}^3$. Hence, the system $A\mathbf{x} = \mathbf{b}$ has a solution for *any* possible $\mathbf{b}$. If the rank of A had been smaller than the number of rows of A (usually called m), you would have had to try to solve $A\mathbf{x} = \mathbf{b}$ for the given $\mathbf{b}$ to answer the question.

10.6 $A^{-1} = \begin{bmatrix} 4 & -3/2 & 0 \\ 1 & -1/2 & 2 \\ -1 & 1/2 & 0 \end{bmatrix}$. You should be able to check your answer yourself. Just multiply it by A and see if you get I.

10.7 (a) Reduction yields the matrix $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. x_2 and x_4 are the free variables. A basis for the solution space is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(b) Pick out the columns of the original matrix for which we have pivots in the reduced matrix. A basis is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}.$$

Of course, any other linearly independent pair of columns would also work.

(c) The columns do not form a linearly independent set since the matrix does not have rank 4.

(d) Solve the system

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 3 & 6 & 7 & 10 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

You should discover that it doesn't have a solution. The last row of the reduced augmented matrix is $\begin{bmatrix} 0 & 0 & 0 & 0 & | & -2 \end{bmatrix}$. Hence, the vector is not in the column space.

10.8 (a) is a vector subspace because it is a plane through the origin. (b) is not because it is a curved surface. Also, any vector subspace contains the element $\mathbf{0}$, but this does not lie on the sphere.

10.9 (a) The rank is 2.

(b) The dimension of the solution space is the number of variables less the rank, which in this case is $5 - 2 = 3$.

10.10 (a) Yes, the set is linearly independent. The easiest way to see this is as follows. Form the 4×4 matrix with these vectors as columns, but in the opposite order to that in which they are given. That matrix is upper triangular with non-zero entries on the diagonal, so its rank is 4.

(b) Yes, it is a basis for $\mathbf{R}^4$. The subspace spanned by this set has a basis with 4 elements, so its dimension is 4. The only 4 dimensional subspace of $\mathbf{R}^4$ is the whole space itself.

Solutions for Chapter 2, Section 1.

1.1 The first two components of $\mathbf{u} \times \mathbf{v}$ are zero and the third component is the given determinant, which might be negative.

1.2 (a) (i) 1, (ii) -1, (iii) 1.

(b) In case (ii), the orientation is reversed, so the sign changes. In case (iii), the two parallelograms can be viewed as having the same base and same height—one of the sides is shifted—so they have the same area.

1.3 (b) $(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w} = (-\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$, so the sign changes. A similar argument shows the sign changes if the second and third columns are interchanged. The last determinant can be obtained by the two switches

$$\begin{bmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{bmatrix} \to \begin{bmatrix}\mathbf{u} & \mathbf{w} & \mathbf{v}\end{bmatrix} \to \begin{bmatrix}\mathbf{w} & \mathbf{u} & \mathbf{v}\end{bmatrix}$$

each of which changes the sign, so the net result is no change.

(c) $(\mathbf{u}+\mathbf{v}) \times \mathbf{v} = \mathbf{u} \times \mathbf{u} + \mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{v}$, so $((\mathbf{u}+\mathbf{v}) \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

(d) The determinant is multiplied by -3.

1.4 We have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}$$
$$= \frac{1}{ad-bc} \begin{bmatrix} de - bf \\ -ce + af \end{bmatrix}.$$

Solutions for Chapter 2, Section 2.

2.1 (a) -16. (b) 40. (c) 3. (d) 0.

2.2 This is a lot of algebra, which I leave to you.

If you have verified rules (i) and (ii) only for the first row, and you have also verified rule (iii), then you can verify rules (i) and (ii) for the second row as follows. Interchange the two rows. The second row is now the first row, but the sign has changed. Use rules (i) and (ii) on the new first row, then exchange rows again. The sign changes back and the rules are verified for the second row.

2.3 I leave the algebra to you. The corresponding rule for the second row follows by exchanging rows, applying the new rule to both sides of the equation and then exchanging back.

2.4 The matrix is singular if and only if its determinant is zero.

$$\det \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix} = 1 - z^2 = 0$$

yields $z = \pm 1$.

2.5 $\det A = -\lambda^3 + 2\lambda = 0$ yields $\lambda = 0, \pm\sqrt{2}$.

2.6 The relevant point is that the determinant of any matrix which has a column consisting of zeroes is zero. For example, in the present case, if we write out the formula for the determinant of the above 5×5 matrix, each term will involve the determinant of a 4×4 matrix with a column of zeroes. Similarly, in the formula for the determinant of such a 4×4 matrix, each term will involve the determinant of a 3×3 matrix with a column of zeroes. Continuing this way, we eventually get to determinants of 2×2 matrices, each with a column of zeroes. However, it is easy to see that the determinant of such a 2×3 matrix is zero.

Note that we will see later that a formula like that used to define the determinant works for *any* column or indeed any row. Hence, if a column (or row) consists of zeroes, the coefficients in that formula would all be zero, and the net result would be zero. However, it would be premature to use such a formula at this point.

2.7 $\det(cA) = c^n \det A$. For, multiplying one row of A by c multiplies its determinant by c, and in cA, all n rows are multiplied by c.

2.8 By a previous exercise, we have $\det(-A) = (-1)^6 \det A = \det A$. The only way we could have $\det A = -\det A$ is if $\det A = 0$, in which case A would be singular.

2.9 Almost anything you come up, with the exception of a few special cases, should work. For example, suppose $\det A \neq 0$ and $B = -A$. Then, since A is 2×2, it follows that $\det(-A) = (-1)^2 \det A = \det A$. Hence, $\det A + \det B = 2 \det A \neq \det 0 = 0$.

2.10 (a) The recursive formula for $n = 7$ uses seven 6×6 subdeterminants. Each of these requires $N(6) = 876$ multiplications. Since there are 7 of these, this requires $7 * 876 = 6132$ multiplications. However, in addition, once these 7 subdeterminants have been calculated, each must be multiplied by the appropriate entry, and this adds 7 additional multiplications. Hence, the total $N(7) = 6132 + 7 = 6139$.

(b) The recursive rule is $N(n) = nN(n-1) + n$.

Solutions for Chapter 2, Section 3.

3.1 The first matrix has determinant 31, and the second matrix has determinant 1. The product matrix is

$$\begin{bmatrix} 6 & 5 & -3 \\ 7 & 9 & 2 \\ -4 & -6 & -1 \end{bmatrix}$$

which has determinant 31.

3.2 If A and B both have rank n, they are both non-singular. Hence, $\det A$ and $\det B$ are nonzero. Hence, by the product rule, $\det(AB) = \det A \det B \neq 0$. Hence, AB is also non-singular and has rank n.

3.3 The determinant of any lower triangular matrix is the product of its diagonal entries. For example, you could just use the transpose rule.

3.4 (a) If A is invertible, then $AA^{-1} = I$. Hence, $\det(AA^{-1}) = 1$. Using the product rule yields $\det A \det(A^{-1} = 1$. Hence, $\det A \neq 0$, and dividing both sides by it yields $\det(A^{-1}) = \dfrac{1}{\det A}$.

(b) $\det(PAP^{-1}) = \det P \det A \det(P^{-1})$. But $\det(P^{-1}) = \dfrac{1}{\det P}$. $\det P$ and $\dfrac{1}{\det P}$ cancel, so the net result is $\det A$ as claimed.

3.5 Cramer's rule has $\det A$ in the denominator. Hence, the formula is meaningless if A is singular since in that case $\det A = 0$.

3.6 The determinant of the coefficient matrix is 1. The solution by Cramer's rule or by Gauss-Jordan reduction is $x_1 = -2, x_2 = 1, x_3 = 4, x_4 = 2$.

Solutions for Chapter 2, Section 4.

4.1 In each case we give the eigenvalues and for each eigenvalue a basis consisting of one or more basic eigenvectors for that eigenvalue. (a)

$$\lambda = 2, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda = 3, \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

(b)

$$\lambda = 2, \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} \qquad \lambda = 1, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad \lambda = -1, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

(c)

$$\lambda = 2, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad \lambda = 1, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

(d)

$$\lambda = 3, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

4.2 Compute

$$A\mathbf{v} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$

You see that $A\mathbf{v}$ is *not* a scalar multiple of $\mathbf{v}$, so *by definition*, it is not an eigenvector for A. Note that trying to find the eigenvalues and eigenvectors of A would be much more time consuming. In this particular case, the eigenvalues turn out to be $\lambda = -2, -2+\sqrt{2}, -2-\sqrt{2}$, and the radicals make it a bit complicated to find the eigenvectors.

4.3 You say that an eigenvector can't be zero, so there had to be a mistake somewhere in the calculation. Either the characteristic equation was not solved correctly to find the eigenvalue λ or the solution of the system $(A - \lambda I)\mathbf{v} = \mathbf{0}$ was not done properly to find the eigenspace.

4.4 The eigenvalues of A are the roots of the equation $\det(A - \lambda I) = 0$. $\lambda = 0$ is a root of this equation if and only if $\det(A - 0\, I) = 0$, i.e., $\det A = 0$. Hence, A would have to be singular.

4.5 Multiply $A\mathbf{v} = \lambda\mathbf{v}$ by A. We get

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

In general, $A^n\mathbf{v} = \lambda^n\mathbf{v}$.

4.6 $A\mathbf{v} = \lambda\mathbf{v}$ implies that

$$\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}(\lambda\mathbf{v}) = \lambda A^{-1}\mathbf{v}.$$

Since A is non-singular, $\lambda \neq 0$ by a problem above, so we may divide through by λ to obtain $\lambda^{-1}\mathbf{v} = A^{-1}\mathbf{v}$. This just says λ^{-1} is an eigenvalue for A^{-1}.

4.7 (a) and (b) are done by expanding the determinants

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \qquad \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix}.$$

I leave the details to you.

(c) The coefficient of λ^n is $(-1)^n$, i.e., it is 1 if n is even and -1 if n is odd.

Solutions for Chapter 2, Section 5.

5.1 (a) $\lambda = 3, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda = -1, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. However, other answers are possible, depending on how you did the problem.

5.2 (a)

$$\lambda = -3, \mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \lambda = 6, \mathbf{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis, but other answers are possible, depending on how you went about doing the problem,

5.3 (a) $\lambda = 2, \mathbf{v}_1 = \mathbf{e}_1$ and $\lambda = 1, \mathbf{v}_2 = \mathbf{e}_3$. (b) For $\lambda = 2$, the dimension of the eigenspace is strictly less than the multiplicity. For $\lambda = 1$, the number of basic eigenvectors does equal the multiplicity; they are both one. A is not diagonalizable because equality does not hold for at least one of the eigenvalues.

5.4 (a)

$$\lambda = 1, \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \lambda = 4, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

However, other answers are possible.

(b) This depends on your answer for part (a). For example,

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

will work.

5.5 Note that if $m_1 + m_2 + m_3 \neq 5$, that means that the characteristic equation has complex roots which are not considered candidates for real eigenvalues.

(a) The dimension of the eigenspace equals the multiplicity for each eigenvalue and the multiplicities add up to five. Hence, the matrix is diagonalizable.

(b) $d_1 < m_1$ so the matrix is not diagonalizable.

(c) $d_1 > m_1$, which is never possible. No such matrix exists.

(d) $m_1 + m_2 + m_3 = 3 < 5$. Hence, there are necessarily some complex roots of the characteristic equation. The matrix is not diagonalizable (in the purely real theory).

5.6 (a) The characteristic equation is $\lambda^2 - 13\lambda + 36 = 0$. Its roots $\lambda = 4, 9$ are distinct, so the matrix is diagonalizable. In Chapter 3,we will learn a simpler more direct way to see that a matrix of this type is diagonalizable.

(b) The characteristic equation is $(\lambda - 1)^2 = 0$ so the only eigenvalue is $\lambda = 1$ and it has multiplicity two. That, in itself, is not enough to conclude the matrix isn't diagonalizable. However,

$$\begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which has rank 1. Hence, the eigenspace has dimension $2 - 1 = 1$ which is less than the multiplicity of the eigenvalue. Hence, the matrix is not diagonalizable.

(c) The characteristic equation is $\lambda^2 + 1 = 0$. Since its roots are non-real complex numbers, this matrix is not diagonalizable in our sense, since we restrict attention to real scalars.

Solutions for Chapter 2, Section 6.

6.1 (a) For any non-negative integer n, we have

$$A^n = \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix},$$

so

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \begin{bmatrix} \sum_{n=0}^{\infty} \lambda^n t^n / n! & 0 \\ 0 & \sum_{n=0}^{\infty} \mu^n t^n / n! \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}.$$

(b) We have

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

6.2 (a)

$$e^{Nt} = I + t \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

This used the fact that in this case $N^k = 0$ for $k \geq 2$.

(b) In this case $N^k = 0$ for $k \geq 3$.

$$e^{Nt} = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2/2 & t & 1 \end{bmatrix}.$$

(c) The smallest such k is n. e^{Nt} has the form suggested by the answers in parts (a) and (b). It is an $n \times n$ matrix with '1's on the diagonal, 't's just below the diagonal, '$t^2/2$'s just below that, etc. In the lower left hand corner there is a '$t^{n-1}/(n-1)!$'.

6.3 (a)

$$e^{At} = e^{\lambda t} \left\{ I + t \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} = e^{\lambda t} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

This used the fact that in this case $(A - \lambda I)^k = 0$ for $k \geq 2$.

(b) In this case $(A - \lambda I)^k = 0$ for $k \geq 3$.

$$e^{At} = e^{\lambda t} \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2/2 & t & 1 \end{bmatrix}.$$

(c) The smallest such k is n. e^{At} has the form suggested by the answers in parts (a) and (b). There is a scalar factor of $e^{\lambda t}$ followed by a lower triangular matrix with '1's on the diagonal, 't's just below the diagonal, '$t^2/2$'s just below that, etc. In the lower left hand corner there is a '$t^{n-1}/(n-1)!$'.

6.4 In general $PA^nP^{-1} = (PAP^{-1})^n$. Hence,

$$P(\sum_{n=0}^{\infty} t^n/n!A^n)P^{-1} = \sum_{n=0}^{\infty} t^n/n!PA^nP^{-1} = \sum_{n=0}^{\infty} t^n/n!(PAP^{-1})^n = e^{PAP^{-1}t}.$$

6.5

$$\begin{aligned} e^{B+C} &= \sum_{n=0}^{\infty} \frac{1}{n!}(B+C)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i+j=n} \frac{n!}{i!j!} B^i C^j \\ &= \sum n = 0^{\infty} \sum_{i+j=n} \frac{1}{i!} B^i \frac{1}{j!} C^j \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} B^i \sum_{j=0}^{\infty} \frac{1}{j!} C^j = e^B e^C. \end{aligned}$$

6.6 (c) We have

$$B + C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

so by Example 6.1,

$$e^{B+C} = \begin{bmatrix} \cos 1 & sin1 \\ -\sin 1 & \cos 1 \end{bmatrix}.$$

On the other hand,

$$e^B e^C = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},$$

Solutions for Chapter 2, Section 7.

7.1 This is *not* an upper or lower triangular matrix. However, after interchanging the first and third rows, it becomes an upper triangular matrix with determinant equal to the product of its diagonal entries. The determinant is -6 because we have to change the sign due to the interchange.

7.2 (a) and (c) are true. (b) is false. The correct rule is $\det(cA) = c^n \det A$. (d) is true. One way to see this is to notice that $\det A^t = \det A \neq 0$.

7.3 The characteristic equation is $-(\lambda - 2)(\lambda + 1)^2 = 0$. The eigenvalues are $\lambda = 2$ and $\lambda = -1$ which is a double root. For $\lambda = 2$,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis for the eigenspace. For $\lambda = -1$,

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the eigenspace.

7.4 Use Gauss Jordan reduction to get an upper triangular matrix. You might speed things up also by using selected column operations. The answer is -23.

7.5 (a) This is never true. It is invertible if and only if its determinant is not zero.

(b) This condition is the definition of 'diagonalizable matrix'. There are many non-diagonalizable matrices. This will happen for example when the dimension of an eigenspace is less than the multiplicity of the corresponding eigenvalue. (It can also happen if the characteristic equation has non-real complex roots.)

(c) The statement is only true for square matrices.

7.6 No. $A\mathbf{v}$ is not a scalar multiple of $\mathbf{v}$.

7.7 (a) It is not diagonalizable since the dimension of the eigenspace for $\lambda = 3$ is one and the multiplicity of the eigenvalue is two.

(b) There are three distinct eigenvalues, so the matrix is diagonalizable.

7.8 Take $\mathbf{v}$ to be the element of $\mathbf{R}^n$ with all its entries equal to one. Then the ith component of $A\mathbf{v}$ is just the sum of the entries in the ith row of A. Since these are all equal to a, it follows that $A\mathbf{v} = a\mathbf{v}$, so $\mathbf{v}$ is an eigenvector with corresponding eigenvalue a.

Solutions for Chapter 3, Section 1.

1.1 The eigenvalues are $\lambda = 5, -5$. An orthonormal basis of eigenvectors consists of

$$\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}, \quad \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\end{bmatrix}.$$

1.2 The eigenvalues are $\lambda = 5, -5$. A basis of eigenvectors consists of

$$\begin{bmatrix}1\\4\end{bmatrix}, \quad \begin{bmatrix}-1\\1\end{bmatrix}$$

which are not perpendicular. However, the matrix is not symmetric, so there is no special reason to expect that the eigenvectors will be perpendicular.

1.3 The eigenvalues are $0, 1, 2$. An orthonormal basis is

$$\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}.$$

1.4 The columns of the matrix

$$\begin{bmatrix} -\frac{5}{\sqrt{2}} & 0 & \frac{5}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} & 3 & \frac{4}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & -4 & \frac{3}{\sqrt{2}} \end{bmatrix}$$

form an orthonormal basis of eigenvectors corresponding to the eigenvalues $-4, 1, 6$.

1.5 $(P^tAP)^t = P^tA^t(P^t)^t = P^tAP$.

Solutions for Chapter 3, Section 3.

3.1 (a) $\begin{bmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{bmatrix}$.

3.2 Replacing θ by $-\theta$ doesn't change the cosine entries and changes the signs of the sine entries.

3.3 The 'P' matrix is

$$\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}.$$

Its inverse is its transpose, so the components of $-g\mathbf{j}$ in the new coordinate system are given by

$$\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}0\\-g\end{bmatrix} = -g\begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix}.$$

3.4 Such a matrix is

$$\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}.$$

The diagonal entries are $-1, 3$.

3.5 Such a matrix is

$$\begin{bmatrix} -\frac{5}{5\sqrt{2}} & 0 & \frac{5}{5\sqrt{2}} \\ \frac{4}{5\sqrt{2}} & \frac{3}{5} & \frac{4}{5\sqrt{2}} \\ \frac{3}{5\sqrt{2}} & \frac{-4}{5} & \frac{3}{5\sqrt{2}} \end{bmatrix}$$

The diagonal entries are $-4, 1, 6$.

3.6 Let A, B be orthogonal, i.e., they are invertible and $A^t = A^{-1}, B^t = B^{-1}$. Then AB is invertible and

$$(AB)^t = B^t A^t = B^{-1} A^{-1} = (AB)^{-1}.$$

The inverse of an orthogonal matrix is orthogonal. For, if A^t is the inverse of A, then A is the inverse of A^t. But $(A^t)^t = A$, so A^t has the property that its transpose is its inverse.

3.7 The rows are also mutually perpendicular unit vectors. The reason is that another way to characterize an orthogonal matrix P is to say that the P^t is the inverse of P, i.e., $P^t P = P P^t = I$. However, it is easy to see from this that P^t is also orthogonal. (Its transpose $(P^t)^t = P$ is also its inverse.) Hence, the columns of P^t are mutually perpendicular unit vectors. But these are the rows of P.

Solutions for Chapter 3, Section 4.

4.1 The principal axes are given by the basis vectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In the new coordinates system the equation is

$$(x')^2 + 3(y')^2 = 2$$

which is the equation of an ellipse.

4.2 The eigenvalues are $-1, 3$. The conic is a hyperbola. The principal axes may be specified by the unit vectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The equation in the new coordinates is $-(x')^2 + 3(y')^2 = 4$. The points closest to the origin are at $x' = 0, y' = \pm\dfrac{2}{\sqrt{3}}$. In the original coordinates, these points are $\pm(\sqrt{2/3}, \sqrt{2/3})$.

4.3 The principal axes are those along the unit vectors

$$\mathbf{u}_1 = \frac{1}{5} \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

The equation in the new coordinate system is

$$-25(x')^2 + 50(y')^2 = 50.$$

The curve is a hyperbola. The points closest to the origin are given by $x' = 0, y' = \pm 1$. In the original coordinates these are the points $\pm\dfrac{1}{5}(4, 3)$. There is no upper bound on the distance of points to the origin for a hyperbola.

4.4 The principal axes are along the unit vectors given by

$$\mathbf{u}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

The equation in the new coordinates system is

$$-3(x')^2 + (y')^2 + 3(z')^2 = 1.$$

This is an elliptic hyperboloid of one sheet centered on the x'-axis.

Solutions for Chapter 3, Section 5.

5.1 Compare this with Exercise 1 in Section 7.

The equations are

$$\begin{aligned}2x &= \lambda(2x+y)\\ 2y &= \lambda(x+2y)\\ x^2+xy+y^2 &= 1\end{aligned}$$

This in effect says that (x, y) give the components of an eigenvector for the matrix

$$\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

with $\dfrac{2}{\lambda}$ as eigenvalue. However, since we already classified the conic in the aforementioned exercise, it is easier to use that information here. In the new coordinates the equation is $(x')^2 + 3(y')^2 = 2$. The maximum distance to the origin is at $x' = \pm\sqrt{2}, y' = 0$ and the minimum distance to the origin is at $x' = 0, y' = \pm\sqrt{2/3}$. Since the change of coordinates is orthogonal, we may still measure distance by $\sqrt{(x')^2 + (y')^2}$. Hence, the maximum square distance to the origin is 2 and the minimum square distance to the origin is $2/3$.

Note that the problem did not ask for the locations in the original coordinates where the maximum and minimum are attained.

5.2 (Look in the previous section for an analogous problem.) The equations are

$$\begin{bmatrix}1 & -2 & 0\\ -2 & -1 & -2\\ 0 & -2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \lambda\begin{bmatrix}x\\ y\\ z\end{bmatrix}, \quad x^2+y^2+z^2 = 1.$$

Thus we need to find eigenvectors of length 1 for the given matrix. However, we already did this in the aforementioned exercise. The answers are

$$\pm\frac{1}{\sqrt{6}}(1,2,1), \quad \pm\frac{1}{\sqrt{2}}(-1,0,1), \quad \pm\frac{1}{\sqrt{3}}(1,-1,1).$$

The values of the function at these three points are respectively $-3, 1, 3$. Since a continuous function on a closed bounded set must attain both a maximum and minimum values, the maximum is 3 at the third point and the minimum is -3 at the first point.

5.3 The Lagrange multiplier condition yields the equations

$$\begin{aligned}x &= \lambda x\\ y &= \lambda y\\ z &= \lambda z\\ x^2+y^2 &= z^2.\end{aligned}$$

If $\lambda \neq 1$, then the first two equations show that $x = y = 0$. From this, the last equation shows that $z = 0$. Hence, $(0, 0, 0)$ is one possible maximum point. If $\lambda = 1$, then the third equation shows that $z = 0$, and then the last equation shows that $x = y = 0$. Hence, that gives the same point. Finally, we have to consider all points where $\nabla g = \langle 2x, 2y, -2z\rangle = \mathbf{0}$. Again, $(0, 0, 0)$ is the only such point. Since $x^2 + y^2 + z^2 \geq 0$ and it does attain the value 0 on the given surface, it follows that 0 is its minimum value.

Note that in this example $\nabla g = \mathbf{0}$ didn't give us any other candidates to examine, but in general that might not have been the case.

5.4 On the circle $x^2+y^2=1$, $f(x,y)=1+4xy$, so maximizing $f(x,y)$ is the same as maximizing xy. The level sets $xy=c$ are hyperbolas. Some of these intersect the circle $x^2+y^2=1$ and some don't intersect. The boundary between the two classes are the level curves $xy=1$ and $xy=-1$. The first is tangent to the circle in the first and third quadrants and the second is tangent in the second and fourth quadrants. As you move on the circle toward one of these points of tangency, you cross level curves with either successively higher values of c or successively lower values of c. Hence, the points of tangency are either maximum or minimum points for xy. The maximum points occur in the first and third quadrants with xy attaining the value 1 at those points.

Note also that the points of tangency are exactly where the normal to the circle and the normal to the level curve are parallel.

Solutions for Chapter 3, Section 6.

6.1 The characteristic polynomial of

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

is

$$\begin{aligned} \det \begin{bmatrix} -1-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} &= (-1-\lambda)((-2-\lambda)(-1-\lambda)-1) - 1(-1-\lambda) \\ &= -(\lambda+1)(\lambda^2+3\lambda) = -(\lambda+1)(\lambda+3)\lambda. \end{aligned}$$

Hence, the eigenvalues are $-\omega^2(m/k) = \lambda = -1, -3$, and 0. As indicated in the problem statement, the eigenvalue 0 corresponds to a non-oscillatory solution in which the system moves freely at constant velocity.

For $\lambda = -1$, we have $\omega = \sqrt{k/m}$ and

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ yielding basic eigenvector } \mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

This corresponds to both end particles moving with equal displacements in opposite directions and the middle particle staying still.

For $\lambda = -3$, we have $\omega = \sqrt{3k/m}$ and

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ yielding basic eigenvector } \mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

This corresponds to both end particles moving together with equal displacements in the same direction and the middle particle moving with twice that displacement in the opposite direction.

6.2 The characteristic polynomial of

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

is

$$\det \begin{bmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{bmatrix} = (-2-\lambda)((-2-\lambda)(-2-\lambda)-1)-1(-2-\lambda)$$
$$= -(\lambda+2)(\lambda^2+4\lambda+2).$$

Hence, the eigenvalues are $-\omega^2(m/k) = \lambda = -2, -2+\sqrt{2}$, and $= -2+\sqrt{2}$.

For $\lambda = -2$, we have $\omega = \sqrt{2k/m}$ and

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ yielding basic eigenvector } \mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

This corresponds to both end particles moving with equal displacements in opposite directions and the middle particle staying still.

For $\lambda = -2-\sqrt{2}$, we have $\omega = \sqrt{(2+\sqrt{2})k/m}$ and

$$\begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{yielding basic eigenvector } \mathbf{u} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}.$$

This corresponds to both end particles moving together with equal displacements in the same direction and the middle particle moving with $\sqrt{2}$ times that displacement in the opposite direction.

For $\lambda = -2+\sqrt{2}$, we have $\omega = \sqrt{(2-\sqrt{2})k/m}$ and

$$\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \text{ yielding basic eigenvector } \mathbf{u} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}.$$

This corresponds to both end particles moving together with equal displacements in the same direction and the middle particle moving with $\sqrt{2}$ times that displacement in the same direction. Notice that the intuitive significance of this last normal mode is not so clear.

6.3 The given information about the first normal mode tells us that a corresponding basic eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Any basic eigenvector for the second normal mode must be perpendicular to $\mathbf{v}_1$, so we can take $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Hence, the relation between the displacements for the second normal mode is $x_2 = -2x_1$.

6.4 For one normal mode, $\omega = \sqrt{\dfrac{k}{m}}$, and the relative motions of the particles satisfy $x_2 = 2x_1$. For the other normal mode, $\omega = \sqrt{\dfrac{6k}{m}}$ and the relative motions of the particles satisfy $x_1 = 2x_2$.

6.5 For one normal mode, $\lambda = -3 + \sqrt{5}$ and $\omega = \sqrt{(3 - \sqrt{5})\frac{k}{m}}$. The eigenspace is obtained by reducing the matrix

$$\begin{bmatrix} -1 - \sqrt{5} & 2 \\ 2 & 1 - \sqrt{5} \end{bmatrix}.$$

Note that this matrix *must be singular* so that the first row must be a multiple of the second row. (The multiple is in fact $\frac{-1 - \sqrt{5}}{2}$. Check it!) Hence, the reduced matrix is

$$\begin{bmatrix} 1 & (1 - \sqrt{5})/2 \\ 0 & 0 \end{bmatrix}.$$

The relative motions of the particles satisfy $x_1 = \frac{\sqrt{5} - 1}{2} x_2$. A similar analysis show that for the other normal mode, $\omega = \sqrt{(3 + \sqrt{5})\frac{k}{m}}$, and the relative motions of the particles satisfy $x_1 = -\frac{\sqrt{5} + 1}{2} x_2$.

6.6 The information given tells us that two of the eigenvectors are

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Any basic eigenvector for the third normal mode must be perpendicular to this. If its components are $\begin{bmatrix} v_1 \\ v_2 \\ v_2 \end{bmatrix}$, then we must have

$$v_1 + v_2 + v_3 = 0 \qquad \text{and} \qquad v_2 - 2v_2 + v_3 = 0.$$

By the usual method, we find that a basis for the null space of this system is given by

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

whence we conclude that the relative motions satisfy $x_1 = -x_3, x_2 = 0$.

Solutions for Chapter 3, Section 7.

7.1 The set is not linearly independent.

7.2 (a) For $\lambda = 2$, a basis for the corresponding eigenspace is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

For $\lambda = -1$, a basis for the corresponding eigenspace is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) An orthonormal basis of eigenvectors is

$$\left\{ \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}, \quad \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\-1\\2\end{bmatrix} \right\}.$$

(c) We have

$$P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}, \quad P^tAP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The answers would be different if the eigenvalues or eigenvectors were chosen in some other order.

7.3 The columns of P must *also* be unit vectors.

7.4 (a) For $\lambda = 6$, $\frac{1}{\sqrt{5}}\begin{bmatrix}-1\\2\end{bmatrix}$ constitutes a basis for the corresponding eigenspace. For $\lambda = 1$, $\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}$ constitutes a basis for the corresponding eigenspace.

(b) $P = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$. The equation is $6u^2 + v^2 = 24$.

(c) The conic is an ellipse with principal axes along the axes determined by the two basic eigenvectors.

7.5 First find an orthonormal basis of eigenvectors for $\begin{bmatrix}2 & 0 & 0\\0 & 1 & 2\\0 & 2 & 1\end{bmatrix}$. The eigenvalues are $\lambda = 3, 2, -1$. The corresponding basis is

$$\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix} \right\}.$$

Picking new coordinates with these as unit vectors on the new axes, the equation is $3(x')^2 + 2(y')^2 - (z')^2 = 6$. This is a hyperboloid of one sheet. Its axis is the z'-axis and this points along the third vector in the above list.

If the eigenvalues had been given in another order, you would have listed the basic eigenvectors in that order also. You might also have conceivably picked a negative of one of those eigenvectors for a basic eigenvector. The new axes would be the same except that they would be labeled differently and some might have directions reversed. The graph would be the same but its equation would look different because of the labeling of the coordinates.

Note that just to identify the graph, all you needed was the eigenvalues. The orthonormal basis of eigenvectors is only necessary if you want to sketch the surface relative to the *original* coordinate axes.

7.6 One normal mode $\mathbf{x} = \mathbf{v}\cos(\omega t)$ has $\lambda = -1$,

$$\omega = \sqrt{\frac{k}{m}}, \quad \mathbf{v} = \begin{bmatrix}1\\1\end{bmatrix}.$$

In this mode, $x_1 = x_2$, so the particles move in the same direction with equal displacements.

A second normal mode $\mathbf{x} = \mathbf{v}\cos(\omega t)$ has $\lambda = -5$,

$$\omega = \sqrt{\frac{5k}{m}}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

In this mode, $x_1 = -x_2$, so the particles move in the opposite directions with equal displacements.

7.7 (a) Not diagonalizable. 1 is a triple root of the characteristic equation, but $A-I$ has rank 2, which shows that the corresponding eigenspace has dimension 1.

(b) If you subtract 2 from each diagonal entry, and compute the rank of the resulting matrix, you find it is one. So the eigenspace corresponding to $\lambda = 2$ has dimension $3 - 1 = 2$. The other eigenspace necessarily has dimension one. Hence, the matrix is diagonalizable.

(c) The matrix is real and symmetric so it is diagonalizable.