CHAPTER III

APPLICATIONS

1. Real Symmetric Matrices

The most common matrices we meet in applications are *symmetric*, that is, they are square matrices which are equal to their transposes. In symbols, $A^t = A$.

Examples.

$$\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

are symmetric, but

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

are not.

Symmetric matrices are in many ways much simpler to deal with than general matrices.

First, as we noted previously, it is not generally true that the roots of the characteristic equation of a matrix are necessarily real numbers, even if the matrix has only real entries. However,

if A is a symmetric matrix with real entries, then the roots of its characteristic equation are all real.

Example 1. The characteristic equations of

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

are

$$\lambda^2 - 1 = 0 \qquad \text{and} \qquad \lambda^2 + 1 = 0$$

respectively. Notice the dramatic effect of a simple change of sign.

The reason for the reality of the roots (for a real symmetric matrix) is a bit subtle, and we will come back to it later sections.

The second important property of real symmetric matrices is that *they are always diagonalizable*, that is, there is always a basis for $\mathbf{R}^n$ consisting of eigenvectors for the matrix.

Example 2. We previously found a basis for $\mathbf{R}^2$ consisting of eigenvectors for the 2×2 symmetric matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = 3, \lambda_2 = 1$, and the basis of eigenvectors is

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

If you look carefully, you will note that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ not only form a basis, but they are perpendicular to one another, i.e., $\mathbf{v}_1 \cdot \mathbf{v}_2 = 1(-1) + 1(1) = 0$.

The perpendicularity of the eigenvectors is no accident. It is always the case for a symmetric matrix by the following reasoning.

First, recall that the dot product of two column vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$ can be written as a row by column product

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^t \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \dots u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} u_i v_i.$$

Suppose now that $A\mathbf{u} = \lambda \mathbf{u}$ and $A\mathbf{v} = \mu \mathbf{v}$, i.e., $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors for A with corresponding eigenvalues λ and μ. Assume $\lambda \neq \mu$. Then

$$\mathbf{u} \cdot (A\mathbf{v}) = \mathbf{u}^t (A\mathbf{v}) = \mathbf{u}^t (\mu \mathbf{v}) = \mu (\mathbf{u}^t \mathbf{v}) = \mu (\mathbf{u} \cdot \mathbf{v}). \tag{1}$$

On the other hand,

$$(A\mathbf{u}) \cdot \mathbf{v} = (A\mathbf{u})^t \mathbf{v} = (\lambda \mathbf{u})^t \mathbf{v} = \lambda (\mathbf{u}^t \mathbf{v}) = \lambda (\mathbf{u} \cdot \mathbf{v}). \tag{2}$$

However, since the matrix is symmetric, $A^t = A$, and

$$(A\mathbf{u})^t \mathbf{v} = (\mathbf{u}^t A^t) \mathbf{v} = (\mathbf{u}^t A) \mathbf{v} = \mathbf{u}^t (A\mathbf{v}).$$

The first of these expressions is what was calculated in (2) and the last was calculated in (1), so the two are equal, i.e.,

$$\mu (\mathbf{u} \cdot \mathbf{v}) = \lambda (\mathbf{u} \cdot \mathbf{v}).$$

If $\mathbf{u} \cdot \mathbf{v} \neq 0$, we can cancel the common factor to conclude that $\mu = \lambda$, which is contrary to our assumption, so it must be true that $\mathbf{u} \cdot \mathbf{v} = 0$, i.e., $\mathbf{u} \perp \mathbf{v}$.

We summarize this as follows.

Eigenvectors for a real symmetric matrix which belong to different eigenvalues are necessarily perpendicular.

This fact has important consequences. Assume first that the eigenvalues of A are distinct and that it is real and symmetric. Then not only is there a basis consisting of eigenvectors, but the basis elements are also mutually perpendicular.

This is reminiscent of the familiar situation in $\mathbf{R}^2$ and $\mathbf{R}^3$, where coordinate axes are almost always assumed to be mutually perpendicular. For arbitrary matrices, we may have to face the prospect of using 'skew' axes, but the above remark tells us we can avoid this possibility in the symmetric case.

In two or three dimensions, we usually require our basis vectors to be unit vectors. There is no problem with that here. Namely, if $\mathbf{u}$ is not a unit vector, we can always obtain a unit vector *by dividing* $\mathbf{u}$ *by its length* $|\mathbf{u}|$. Moreover, if $\mathbf{u}$ is an eigenvector for A with eigenvalue λ, then any nonzero multiple of $\mathbf{u}$ is also such an eigenvector, in particular, the unit vector $\dfrac{1}{|\mathbf{u}|}\mathbf{u}$ is.

Example 2, revisited. The eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ both have length $\sqrt{2}$. So we replace them by the corresponding unit vectors

$$\frac{1}{\sqrt{2}}\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \frac{1}{\sqrt{2}}\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

which also constitute a basis for $\mathbf{R}^2$.

There is some special terminology which is commonly used in linear algebra for the familiar concepts discussed above. Two vectors are said to be *orthogonal* if they are perpendicular. A unit vector is said to be *normalized*. The idea is that if we started with a non-unit vector, we would produce an equivalent unit vector by dividing it by its length. The latter process is called *normalization*. Finally, a basis for $\mathbf{R}^n$ consisting of mutually perpendicular unit vectors is called an *orthonormal basis*.

Exercises for Section 1.

1. (a) Find a basis of eigenvectors for $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$.

(b) Check that the basis vectors are orthogonal, and normalize them to yield an orthonormal basis.

2. (a) Find a basis of eigenvectors for $A = \begin{bmatrix} -3 & 2 \\ 8 & 3 \end{bmatrix}$.

(b) Are the basis vectors orthogonal to one another? If not what might be the problem?

3. (a) Find a basis of eigenvectors for $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(b) Check that the basis vectors are orthogonal, and normalize them to yield an orthonormal basis.

4. Let $A = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$. Find an orthonormal basis of eigenvectors.

5. Let A be a symmetric $n \times n$ matrix, and let P be any $n \times n$ matrix. Show that $P^t AP$ is also symmetric.

2. Repeated Eigenvalues, The Gram–Schmidt Process

We now consider the case in which one or more eigenvalues of a real symmetric matrix A is a repeated root of the characteristic equation. It turns out that we can still find an orthonormal basis of eigenvectors, but it is a bit more complicated.

Example 1. Consider

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

The characteristic equation is

$$\begin{aligned} \det &\begin{bmatrix} -1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & 1 & -1-\lambda \end{bmatrix} \\ &= -(1+\lambda)((1+\lambda)^2-1) - 1(-1-\lambda-1) + 1(1+1+\lambda) \\ &= -(1+\lambda)(\lambda^2+2\lambda) + 2(\lambda+2) \\ &= -(\lambda^3+3\lambda^2-4) = 0. \end{aligned}$$

Using the method suggested in Chapter 2, we may find the roots of this equation by trying the factors of the constant term. The roots are $\lambda = 1$, which has multiplicity 1, and $\lambda = -2$, which has multiplicity 2.

For $\lambda = 1$, we need to reduce

$$A - I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The general solution is $v_1 = v_3, v_2 = v_3$ with v_3 free. A basic eigenvector is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

but we should normalize this by dividing it by $|\mathbf{v}_1| = \sqrt{3}$. This gives

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For $\lambda = -2$, the situation is more complicated. Reduce

$$A + 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which yields the general solution $v_1 = -v_2 - v_3$ with v_2, v_3 free. This gives basic eigenvectors

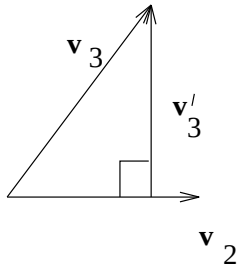
$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Note that, as the general theory predicts, $\mathbf{v}_1$ is perpendicular to both $\mathbf{v}_2$ and $\mathbf{v}_3$. (The eigenvalues are different). Unfortunately, $\mathbf{v}_2$ and $\mathbf{v}_3$ are *not perpendicular to each other*. However, with a little effort, this is easy to remedy. All we have to do is pick *another basis* for the subspace spanned by $\{\mathbf{v}_2, \mathbf{v}_3\}$. The eigenvectors with eigenvalue -2 are exactly the non-zero vectors in this subspace, so any basis will do as well. Hence, we arrange to pick a basis consisting of mutually perpendicular vectors.

It is easy to construct the new basis. Indeed we need only replace one of the two vectors. Keep $\mathbf{v}_2$, and let $\mathbf{v}_3' = \mathbf{v}_3 - c\mathbf{v}_2$ where c is chosen so that

$$\mathbf{v}_2 \cdot \mathbf{v}_3' = \mathbf{v}_2 \cdot \mathbf{v}_3 - c\mathbf{v}_2 \cdot \mathbf{v}_2 = 0,$$

i.e., take $c = \dfrac{\mathbf{v}_2 \cdot \mathbf{v}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}$. (See the diagram to get some idea of the geometry behind this calculation.)



We have

$$\frac{\mathbf{v}_2 \cdot \mathbf{v}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2} = \frac{1}{2}$$

$$\mathbf{v}_3' = \mathbf{v}_3 - \frac{1}{2}\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

We should also normalize this basis by choosing

$$\mathbf{u}_2 = \frac{1}{|\mathbf{v}_2|}\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_3 = \frac{1}{|\mathbf{v}_3'|}\mathbf{v}_3' = \sqrt{\frac{2}{3}}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Putting this all together, we see that

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_3 = \sqrt{\frac{2}{3}}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

form an orthonormal basis for $\mathbf{R}^3$ consisting of eigenvectors for A.

The Gram–Schmidt Process. In Example 1, we used a special case of a more general algorithm in order to construct an orthonormal basis of eigenvectors. The algorithm, called the *Gram–Schmidt Process* works as follows. Suppose

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$$

is a *linearly independent set* spanning a certain subspace W of $\mathbf{R}^n$. We construct an orthonormal basis for W as follows. Let

$$\begin{aligned}
\mathbf{v}_1' &= \mathbf{v}_1 \\
\mathbf{v}_2' &= \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1'}{\mathbf{v}_1' \cdot \mathbf{v}_1'}\mathbf{v}_1' \\
\mathbf{v}_3' &= \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{v}_1'}{\mathbf{v}_1' \cdot \mathbf{v}_1'}\mathbf{v}_1' - \frac{\mathbf{v}_3 \cdot \mathbf{v}_2'}{\mathbf{v}_2' \cdot \mathbf{v}_2'}\mathbf{v}_2' \\
&\vdots \\
\mathbf{v}_k' &= \mathbf{v}_k - \sum_{j=1}^{k-1} \frac{\mathbf{v}_k \cdot \mathbf{v}_j'}{\mathbf{v}_j' \cdot \mathbf{v}_j'}\mathbf{v}_j'.
\end{aligned}$$

It is not hard to see that each new $\mathbf{v}_k'$ is perpendicular to those constructed before it. For example,

$$\mathbf{v}_1' \cdot \mathbf{v}_3' = \mathbf{v}_1' \cdot \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{v}_1'}{\mathbf{v}_1' \cdot \mathbf{v}_1'}\mathbf{v}_1' \cdot \mathbf{v}_1' - \frac{\mathbf{v}_3 \cdot \mathbf{v}_2'}{\mathbf{v}_2' \cdot \mathbf{v}_2'}\mathbf{v}_1' \cdot \mathbf{v}_2'.$$

However, we may suppose that we already know that $\mathbf{v}_1' \cdot \mathbf{v}_2' = 0$ (from the previous stage of the construction), so the above becomes

$$\mathbf{v}_1' \cdot \mathbf{v}_3' = \mathbf{v}_1' \cdot \mathbf{v}_3 - \mathbf{v}_3 \cdot \mathbf{v}_1' = 0.$$

The same argument works at each stage.

It is also not hard to see that at each stage, replacing $\mathbf{v}_j$ by $\mathbf{v}_j'$ in

$$\{\mathbf{v}_1', \mathbf{v}_2', \ldots, \mathbf{v}_{j-1}', \mathbf{v}_j\}$$

does not change the subspace spanned by the set. Hence, for $j = k$, we conclude that $\{\mathbf{v}_1', \mathbf{v}_2', \ldots, \mathbf{v}_k'\}$ is a basis for W consisting of mutually perpendicular vectors. Finally, to complete the process simply divide each $\mathbf{v}_j'$ by its length

$$\mathbf{u}_j = \frac{1}{|\mathbf{v}_j'|}\mathbf{v}_j'.$$

Then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal basis for W.

Example 2. Consider the subspace of $\mathbf{R}^4$ spanned by

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then

$$\mathbf{v}_1' = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2' = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \\ -\frac{2}{3} \end{bmatrix}$$

$$\mathbf{v}_3' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{0}{3}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{-1}{\frac{15}{9}}\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \\ \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}.$$

Normalizing, we get

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{3}{\sqrt{15}}\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \\ -\frac{2}{3} \end{bmatrix} = \frac{1}{\sqrt{15}}\begin{bmatrix} -1 \\ 1 \\ 3 \\ -2 \end{bmatrix}$$

$$\mathbf{u}_3 = \frac{5}{\sqrt{35}}\begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \\ \frac{3}{5} \\ \frac{3}{5} \end{bmatrix} = \frac{1}{\sqrt{35}}\begin{bmatrix} 4 \\ 1 \\ 3 \\ 3 \end{bmatrix}.$$

The Principal Axis Theorem. The *Principal Axis Theorem* asserts that the process outlined above for finding mutually perpendicular eigenvectors always works.

If A is a real symmetric $n \times n$ matrix, there is always an orthonormal basis for $\mathbf{R}^n$ consisting of eigenvectors for A.

Here is a summary of the method. If the roots of the characteristic equation are all different, then all we need to do is find an eigenvector for each eigenvalue and if necessary normalize it by dividing by its length. If there are repeated roots, then it will usually be necessary to apply the Gram–Schmidt process to the set of basic eigenvectors obtained for each repeated eigenvalue.

Exercises for Section 2.

1. Apply the Gram–Schmidt Process to each of the following sets of vectors.

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} \right\}$.

2. Find an orthonormal basis of eigenvectors for $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

3. Find an orthonormal basis of eigenvectors for

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

Hint: 3 is an eigenvalue.

4. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly independent set. Suppose $\{\mathbf{v}_1', \mathbf{v}_2', \mathbf{v}_3'\}$ is the set obtained (before normalizing) by the Gram-Schmidt Process. (a) Explain why $\mathbf{v}_2'$ is not zero. (b) Explain why $\mathbf{v}_3'$ is not zero.

The generalization of this to an arbitrary linearly independent set is one reason the Gram-Schmidt Process works. The vectors produced by that process are mutually perpendicular *provided they are non-zero*, and so they form a linearly independent set. Since they are in the subspace W spanned by the original set of vectors and there are just enough of them, they must form a basis a basis for W.

3. Change of Coordinates

As we have noted previously, it is probably a good idea to use a special basis like an orthonormal basis of eigenvectors. Any problem associated with the matrix A is likely to take a particularly simple form when expressed relative to such a basis.

To study this in greater detail, we need to talk a bit more about changes of coordinates. Although the theory is quite general, we shall concentrate on some simple examples.

In $\mathbf{R}^n$, the entries in a column vector $\mathbf{x}$ may be thought of as the coordinates $x_1, x_2, \ldots, x_n$ of the vector with respect to the standard basis. To simplify the algebra, let's concentrate on one specific n, say $n = 3$. In that case, we may make the usual identifications $\mathbf{e}_1 = \mathbf{i}, \mathbf{e}_2 = \mathbf{j}, \mathbf{e}_3 = \mathbf{k}$ for the elements of the standard basis. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is another basis. The coordinates of $\mathbf{x}$ with respect to the new basis—call them x_1', x_2', x_3'—are defined by the relation

$$\mathbf{x} = \mathbf{v}_1 x_1' + \mathbf{v}_2 x_2' + \mathbf{v}_3 x_3' = [\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \,] \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = [\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \,]\, \mathbf{x}'. \tag{1}$$

One way to view this relation is as a system of equations in which the old coordinates

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

are given, and we want to solve for the new coordinates

$$\mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}.$$

The coefficient matrix of this system

$$P = [\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \,]$$

is called the *change of basis matrix. It's columns are the old coordinates of the new basis vectors.*

The relation (1) may be rewritten

$$\mathbf{x} = P\mathbf{x}' \tag{2}$$

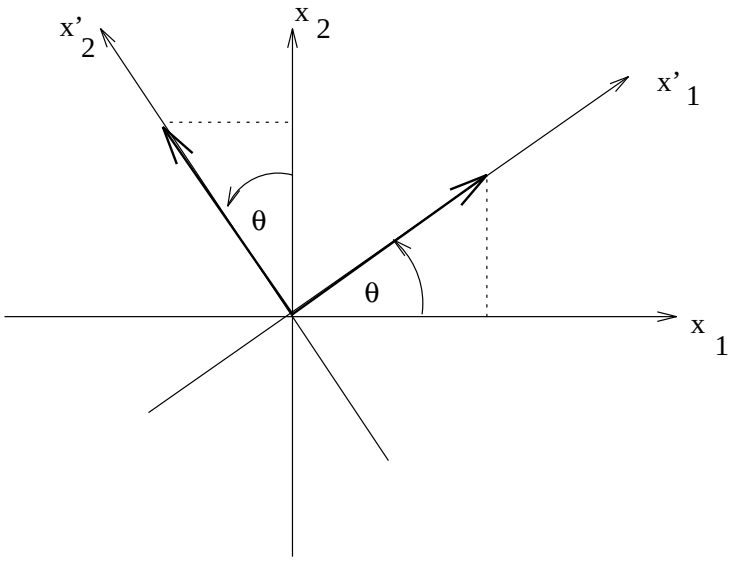
and it may also be interpreted as expressing the 'old' coordinates of a vector in terms of its 'new' coordinates. This seems backwards, but it is easy to turn it around. Since the columns of P are linearly independent, P is invertible and we may write instead

$$\mathbf{x}' = P^{-1}\mathbf{x} \tag{3}$$

where we express the 'new' coordinates in terms of the 'old' coordinates.

Example 1. Suppose in $\mathbf{R}^2$ we pick a new set of coordinate axes by rotating each of the old axes through angle θ in the counterclockwise direction. Call the old coordinates (x_1, x_2) and the new coordinates (x_1', x_2'). According to the above discussion, the columns of the change of basis matrix P come from the old coordinates of the new basis vectors, i.e., of unit vectors along the new axes. From the diagram, these are

$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$



Hence,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}.$$

The change of basis matrix is easy to invert in this case. (Use the special rule which applies to 2×2 matrices.)

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

(You could also have obtained this by using the matrix for rotation through angle $-\theta$.) Hence, we may express the 'new' coordinates in terms of the 'old' coordinates through the relation

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

For example, suppose $\theta = \pi/6$. The new coordinates of the point with original coordinates $(2, 6)$ are given by

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} \sqrt{3}+3 \\ -1+3\sqrt{3} \end{bmatrix}.$$

So with respect to the rotated axes, the coordinates are $(3+\sqrt{3}, 3\sqrt{3}-1)$.

Orthogonal Matrices. You may have noticed that the matrix P obtained in Example 1 has the property $P^{-1} = P^t$. This is no accident. It is a consequence of the fact that its columns are mutually perpendicular unit vectors. Indeed,

The columns of an $n \times n$ matrix form an orthonormal basis for $\mathbf{R}^n$ if and only if its inverse is its transpose.

An $n \times n$ real matrix with this property is called *orthogonal*.

Example 2. Let

$$P = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

The columns of P are

$$\mathbf{u}_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix},$$

and it is easy to check that these are mutually perpendicular unit vectors in $\mathbf{R}^2$. To see that $P^{-1} = P^t$, it suffices to show that

$$P^t P = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Of course, it is easy to see that this true by direct calculation, but it may be more informative to write it out as follows

$$P^t P = \begin{bmatrix} (\mathbf{u}_1)^t \\ (\mathbf{u}_2)^t \end{bmatrix} [\, \mathbf{u}_1 \quad \mathbf{u}_2 \,] = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \mathbf{u}_1 \cdot \mathbf{u}_2 \\ \mathbf{u}_2 \cdot \mathbf{u}_1 & \mathbf{u}_2 \cdot \mathbf{u}_2 \end{bmatrix}$$

where the entries in the product are exhibited as row by column dot products. The off diagonal entries are zero because the vectors are perpendicular, and the diagonal entries are ones because the vectors are unit vectors.

The argument for $n \times n$ matrices is exactly the same except that there are more entries.

Note. The terminology is very confusing. The definition of an orthogonal matrix requires that the columns be mutually perpendicular and also that they be unit vectors. Unfortunately, the terminology reminds us of the former condition but not of the latter condition. It would have been better if such matrices had been named 'orthonormal' matrices rather than 'orthogonal' matrices, but that is not how it happened, and we don't have the option of changing the terminology at this late date.

The Principal Axis Theorem Again. As we have seen, given a real symmetric $n \times n$ matrix A, the Principal Axis Theorem assures us that we can always find an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for $\mathbf{R}^n$ consisting of eigenvectors for A. Let

$$P = [\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \ldots \quad \mathbf{v}_n \,]$$

be the corresponding change of basis matrix. As in Chapter II, Section 5, we have

$$A\mathbf{v}_1 = \mathbf{v}_1\lambda_1 = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n\,] \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A\mathbf{v}_2 = \mathbf{v}_2\lambda_2 = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n\,] \begin{bmatrix} 0 \\ \lambda_2 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vdots$$

$$A\mathbf{v}_n = \mathbf{v}_n\lambda_n = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n\,] \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \lambda_n \end{bmatrix}$$

where some eigenvalues λ_j for different eigenvectors might be repeated. These equations can be written in a single matrix equation

$$A\,[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n\,] = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n\,] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

or

$$AP = PD$$

where D is a diagonal matrix with eigenvalues (possibly repeated) on the diagonal. This may also be written

$$P^{-1}AP = D. \tag{4}$$

Since we have insisted that the basic eigenvectors form an orthonormal basis, the change of basis matrix P is orthogonal, and we have $P^{-1} = P^t$. Hence, (4) can be written in the alternate form

$$P^tAP = D \qquad \text{with } P \text{ orthogonal.} \tag{5}$$

Example 3. Let $A = \begin{bmatrix} -7 & 14 \\ 14 & 7 \end{bmatrix}$. The characteristic equation of A turns out to be $\lambda^2 - 625 = 0$, so the eigenvalues are $\lambda = \pm 25$. Calculation shows that an orthonormal basis of eigenvectors is formed by

$$\mathbf{u}_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \quad \text{for } \lambda = 25 \qquad \text{and} \qquad \mathbf{u}_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} \quad \text{for } \lambda = -25.$$

Hence, we may take P to be the orthogonal matrix

$$\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

The reader should check in this case that

$$P^tAP = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \cdot \begin{bmatrix} -7 & 14 \\ 14 & 7 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & -25 \end{bmatrix}.$$

Appendix. A Proof of the Principal Axis Theorem.

The following section outlines how the Principal Axis Theorem is proved for the very few of you who may insist on seeing it. It is not necessary for what follows.

In view of the previous discussions, we can establish the Principal Axis Theorem by showing that there is an orthogonal $n \times n$ matrix P such that

$$AP = PD \qquad \text{or equivalently} \qquad P^t AP = D \tag{6}$$

where D is a diagonal matrix with the eigenvalues of A (possibly repeated) on its diagonal.

The method is to proceed by induction on n.

If $n = 1$ there really isn't anything to prove. (Take $P = [\,1\,]$.) Suppose the theorem has been proved for $(n-1) \times (n-1)$ matrices. Let $\mathbf{u}_1$ be a unit eigenvector for A with eigenvalue λ_1. Consider the subspace W consisting of all vectors perpendicular to $\mathbf{u}_1$. It is not hard to see that W is an $n-1$ dimensional subspace. Choose (by the Gram–Schmidt Process) an orthonormal basis $\{\mathbf{w}_2, \mathbf{w}_2 \ldots, \mathbf{w}_n\}$ for W. Then $\{\mathbf{u}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is an orthonormal basis for $\mathbf{R}^n$, and

$$A\mathbf{u}_1 = \mathbf{u}_1\lambda_1 = \underbrace{\begin{bmatrix}\mathbf{u}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_n\end{bmatrix}}_{P_1} \begin{bmatrix}\lambda_1 \\ 0 \\ \vdots \\ 0\end{bmatrix}.$$

This gives the first column of AP_1, and we want to say something about its remaining columns

$$A\mathbf{w}_2, \quad A\mathbf{w}_2, \ldots, A\mathbf{w}_n.$$

To this end, note that if $\mathbf{w}$ is any vector in W, then $A\mathbf{w}$ is also a vector in W. For, we have

$$\mathbf{u}_1 \cdot (A\mathbf{w}) = (\mathbf{u}_1)^t A\mathbf{w} = (\mathbf{u}_1)^t A^t \mathbf{w} = (A\mathbf{u}_1)^t \mathbf{w} = \lambda_1 (\mathbf{u}_1)^t \mathbf{w}) = \lambda_1 (\mathbf{u}_1 \cdot \mathbf{w}) = 0,$$

which is to say, $A\mathbf{w}$ is perpendicular to $\mathbf{u}_1$ if $\mathbf{w}$ is perpendicular to $\mathbf{u}_1$. It follows that each $A\mathbf{w}_j$ is a linear combination just of $\mathbf{w}_2, \mathbf{w}_3, \ldots, \mathbf{w}_n$, i.e.,

$$A\mathbf{w}_j = \begin{bmatrix}\mathbf{u}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_n\end{bmatrix} \begin{bmatrix}0 \\ * \\ \vdots \\ *\end{bmatrix}$$

where '$*$' denotes some unspecified entry. Putting this all together, we see that

$$AP_1 = P_1 \underbrace{\begin{bmatrix}\lambda_1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{bmatrix}}_{A_1}$$

where A' is an $(n-1)\times(n-1)$ matrix. P_1 is orthogonal (since its columns form an orthonormal basis) so

$$P_1{}^t A P_1 = A_1,$$

and it is not hard to derive from this the fact that A_1 is symmetric. Because of the structure of A_1, this implies that A' is symmetric. Hence, by induction we may assume there is an $(n-1)\times(n-1)$ orthogonal matrix P' such that $A'P' = P'D'$ with D' diagonal. It follows that

$$A_1 \underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & P' & \\ 0 & & & \end{bmatrix}}_{P_2} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & P' & \\ 0 & & & \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & A'P' & \\ 0 & & & \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & P'D' & \\ 0 & & & \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & P' & \\ 0 & & & \end{bmatrix}}_{P_2} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & D' & \\ 0 & & & \end{bmatrix}}_{D} = P_2 D.$$

Note that P_2 is orthogonal and D is diagonal. Thus,

$$A \underbrace{P_1 P_2}_{P} = P_1 A_1 P_2 = \underbrace{P_1 P_2}_{P} D$$

$$\text{or} \qquad AP = PD.$$

However, a product of orthogonal matrices is orthogonal—see the Exercises—so P is orthogonal as required.

This completes the proof.

There is one subtle point involved in the above proof. We have to know that a real symmetric $n\times n$ matrix has at least one real eigenvalue. This follows from the fact, alluded to earlier, that the roots of the characteristic equation for such a matrix are necessarily real. Since the equation does have a root, that root is the desired eigenvalue.

Exercises for Section 3.

1. Find the change of basis matrix for a rotation through (a) 30 degrees in the counterclockwise direction and (b) 30 degrees in the clockwise direction

2. Let $P(\theta)$ be the matrix for rotation of axes through θ. Show that $P(-\theta) = P(\theta)^t = P(\theta)^{-1}$.

3. An inclined plane makes an angle of 30 degrees with the horizontal. Change to a coordinate system with x_1' axis parallel to the inclined plane and x_2' axis perpendicular to it. Use the change of variables formula derived in the section to find the components of the gravitational acceleration vector $-g\mathbf{j}$ in the new coordinate system. Compare this with what you would get by direct geometric reasoning.

4. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Find a 2×2 orthogonal matrix P such that P^tAP is diagonal. What are the diagonal entries?

5. Let $A = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$. Find a 3×3 orthogonal matrix P such that P^tAP is diagonal. What are the diagonal entries?

6. Show that the product of two orthogonal matrices is orthogonal. How about the inverse of an orthogonal matrix?

7. The columns of an orthogonal matrix are mutually perpendicular unit vectors. Is the same thing true of the rows? Explain.

4. Classification of Conics and Quadrics

The Principal Axis Theorem derives its name from its relation to classifying conics, quadric surfaces, and their higher dimensional analogues.

The general quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey = f$$

(with enough of the coefficients non-zero) defines a curve in the plane. Such a curve is generally an ellipse, a hyperbola, a parabola, all of which are called conics, or two lines, which is considered a degenerate case. (See the Appendix to this section for a review.)

Examples.

$$x^2 + \frac{y^2}{4} = 1$$
$$x^2 - y^2 = 1$$
$$x^2 - 2xy + 2y^2 = 1$$
$$x^2 + 2xy - y^2 + 3x - 5y = 10$$

If the linear terms are not present ($d = e = 0$ and $f \neq 0$), we call the curve a *central conic*. It turns out to be an ellipse or hyperbola (but its axes of symmetry may not be the coordinate axes) or a pair of lines in the degenerate case. Parabolas can't be obtained this way.

In this section, we shall show how to use linear algebra to classify such central conics and higher dimensional analogues such as quadric surfaces in $\mathbf{R}^3$. Once you understand the central case, it is fairly easy to reduce the general case to that. (You just use completion of squares to get rid of the linear terms in the same way that you identify a circle with center not at the origin from its equation.)

In order to apply the linear algebra we have studied, we adopt a more systematic notation, using subscripted variables x_1, x_2 instead of x, y.

Consider the central conic defined by the equation

$$f(\mathbf{x}) = a_{11}{x_1}^2 + 2a_{12}x_1x_2 + a_{22}{x_2}^2 = C$$

(The reason for the 2 will be clear shortly.)

It is more useful to express the function f as follows.

$$\begin{aligned} f(\mathbf{x}) &= (x_1a_{11} + x_2a_{21})x_1 + (x_1a_{12} + x_2a_{22})x_2 \\ &= x_1(a_{11}x_1 + a_{12}x_2) + x_2(a_{21}x_1 + a_{22}x_2), \end{aligned}$$

where we have introduced $a_{21} = a_{12}$. The above expression may also be written in matrix form

$$f(\mathbf{x}) = \sum_{j,k=1}^{2} x_j a_{jk} x_k = \mathbf{x}^t A \mathbf{x}$$

where A is the symmetric matrix of coefficients.

Note what has happened to the coefficients. The coefficients of the squares appear on the diagonal of A, while the coefficient of the cross term $2bx_1x_2$ is divided into two equal parts. Half of it appears as b in the $1,2$ position (corresponding to the product x_1x_2) while the other half appears as b in the $2,1$ position (corresponding to the product x_2x_1 which of course equals x_1x_2). So it is clear why the matrix is symmetric.

This may be generalized to $n > 2$ in a rather obvious manner. Let A be a real symmetric $n \times n$ matrix, and define

$$f(\mathbf{x}) = \sum_{j,k=1}^{n} x_j a_{jk} x_k = \mathbf{x}^t A \mathbf{x}.$$

For $n = 3$ this may be written explicitly

$$\begin{aligned} f(\mathbf{x}) &= (x_1a_{11} + x_2a_{21} + x_3a_{31})x_1 \\ &\quad + (x_1a_{12} + x_2a_{22} + x_3a_{32})x_2 \\ &\quad + (x_1a_{13} + x_2a_{23} + x_3a_{33})x_3 \\ &= a_{11}{x_1}^2 + a_{22}{x_2}^2 + a_{33}{x_3}^2 \\ &\quad + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3. \end{aligned}$$

The rule for forming the matrix A from the equation for $f(\mathbf{x})$ is the same as in the 2×2 case. The coefficients of the squares are put on the diagonal. The

coefficient of a cross term involving $x_i x_j$ is split in half, with one half put in the i, j position, and the other half is put in the j, i position.

The level set defined by

$$f(\mathbf{x}) = C$$

is called a *central hyperquadric*. It should be visualized as an $n-1$ dimensional curved object in $\mathbf{R}^n$. For $n = 3$ it will be an ellipsoid or a hyperboloid (of one or two sheets) or perhaps a degenerate 'quadric' like a cone. (As in the case of conics, we must also allow linear terms to encompass paraboloids.)

If the above descriptions are accurate, we should expect the locus of the equation $f(\mathbf{x}) = C$ to have certain axes of symmetry which we shall call its *principal axes*. It turns out that these axes are determined by an *orthonormal basis of eigenvectors* for the coefficient matrix A. To see this, suppose $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is such a basis and $P = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \ldots \quad \mathbf{u}_n\,]$ is the corresponding orthogonal matrix. By the Principal Axis Theorem, $P^t A P = D$ is diagonal with the eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, of A appearing on the diagonal. Make the change of coordinates $\mathbf{x} = P\mathbf{x}'$ where $\mathbf{x}$ represents the 'old' coordinates and $\mathbf{x}'$ represents the 'new' coordinates. Then

$$f(\mathbf{x}) = \mathbf{x}^t A \mathbf{x} = (P\mathbf{x}')^t A (P\mathbf{x}') = (\mathbf{x}')^t P^t A P \mathbf{x}' = (\mathbf{x}')^t D \mathbf{x}'.$$

Since D is diagonal, the quadratic expression on the right has no cross terms, i.e.

$$\begin{aligned}(\mathbf{x}')^t D \mathbf{x}' &= [\,x_1' \quad x_2' \quad \cdots x_n'\,] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} \\ &= \lambda_1 (x_1')^2 + \lambda_2 (x_2')^2 + \cdots + \lambda_n (x_n')^2.\end{aligned}$$

In the new coordinates, the equation takes the form

$$\lambda_1 (x_1')^2 + \lambda_2 (x_2')^2 + \cdots + \lambda_n (x_n')^2 = C$$

and its graph is usually quite easy to describe.

Example 1. We shall investigate the conic $f(x, y) = x^2 + 4xy + y^2 = 1$. First rewrite the equation

$$[\,x \quad y\,] \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1.$$

(Note how the 4 was split into two symmetrically placed 2s.) Next, find the eigenvalues of the coefficient matrix by solving

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = 0.$$

This equation is easy to factor, and the roots are $\lambda = 3, \lambda = -1$.

For $\lambda = 3$, to find the eigenvectors, we need to solve

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

Reduction of the coefficient matrix yields

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

with the general solution $v_1 = v_2$, v_2 free. A basic *normalized* eigenvector is

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda = -1$, a similar calculation (which you should make) yields the basic normalized eigenvector

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(Note that $\mathbf{u}_1 \perp \mathbf{u}_2$ as expected.)

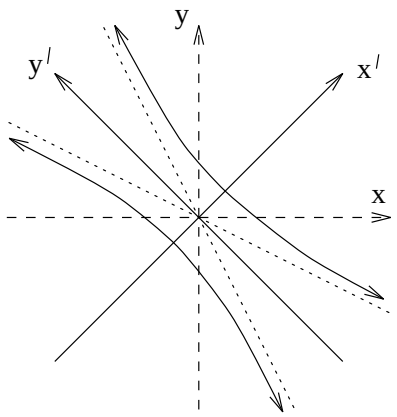
From this we can form the corresponding orthogonal matrix P and make the change of coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x' \\ y' \end{bmatrix},$$

and, according to the above analysis, the equation of the conic in the new coordinate system is

$$3(x')^2 - (y')^2 = 1.$$

It is clear that this is a hyperbola with principal axes pointing along the new axes.



Example 2. Consider the quadric surface defined by

$$x_1{}^2 + x_2{}^2 + x_3{}^2 - 2x_1x_3 = 1.$$

We take

$$f(\mathbf{x}) = x_1{}^2 + x_2{}^2 + x_3{}^2 - 2x_1x_3 = [\, x_1 \quad x_2 \quad x_3 \,] \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Note how the coefficient in $-2x_1x_3$ was split into two equal parts, a -1 in the $1,3$-position and a -1 in the $3,1$-position. The coefficients of the other cross terms were zero. As usual, the coefficients of the squares were put on the diagonal.

The characteristic equation of the coefficient matrix is

$$\det\begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^3 - (1-\lambda) = -(\lambda-2)(\lambda-1)\lambda = 0$$

Thus, the eigenvalues are $\lambda = 2, 1, 0$.

For $\lambda = 2$, reduce

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

to obtain $v_1 = -v_3, v_2 = 0$ with v_3 free. Thus,

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

is a basic eigenvector for $\lambda = 2$, and

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

is a basic unit eigenvector.

Similarly, for $\lambda = 1$ reduce

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which yields $v_1 = v_3 = 0$ with v_2 free. Thus a basic unit eigenvector for $\lambda = 1$ is

$$\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Finally, for $\lambda = 0$, reduce

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This yields $v_1 = x_3, v_2 = 0$ with v_3 free. Thus, a basic unit eigenvector for $\lambda = 0$ is

$$\mathbf{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

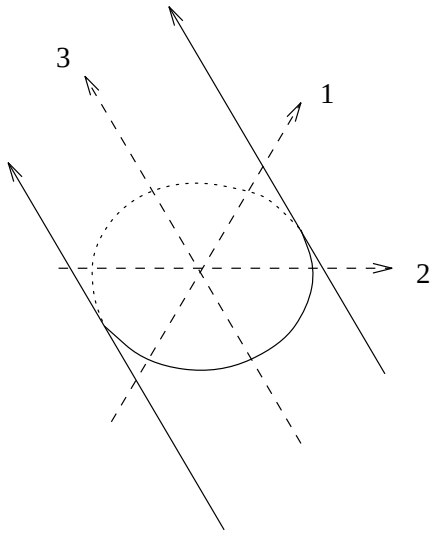
The corresponding orthogonal change of basis matrix is

$$P = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3\,] = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Moreover, putting $\mathbf{x} = P\mathbf{x}'$, we can express the equation of the quadric surface in the new coordinate system

$$2{x_1'}^2 + 1{x_2'}^2 + 0{x_3'}^2 = 2{x_1'}^2 + {x_2'}^2 = 1. \tag{1}$$

Thus it is easy to see what this quadric surface is: an elliptical cylinder perpendicular to the x_1', x_2' plane. (This is one of the degenerate cases.) The three 'principal axes' in this case are the two axes of the ellipse in the x_1', x_2' plane and the x_3' axis, which is the central axis of the cylinder.



Tilted cylinder relative to the new axes. The new axes are labelled.

Representing the graph in the new coordinates makes it easy to understand its geometry. Suppose, for example, that we want to find the points on the graph which are closest to the origin. These are the points at which the x_1'-axis intersects the surface. These are the points with new coordinates $x_1' = \pm\dfrac{1}{\sqrt{2}}, x_2' = x_3' = 0$. If you want the coordinates of these points in the original coordinate system, use the change of coordinates formula

$$\mathbf{x} = P\mathbf{x}'.$$

Thus, the old coordinates of the minimum point with new coordinates $(1/\sqrt{2}, 0, 0)$ are given by

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}.$$

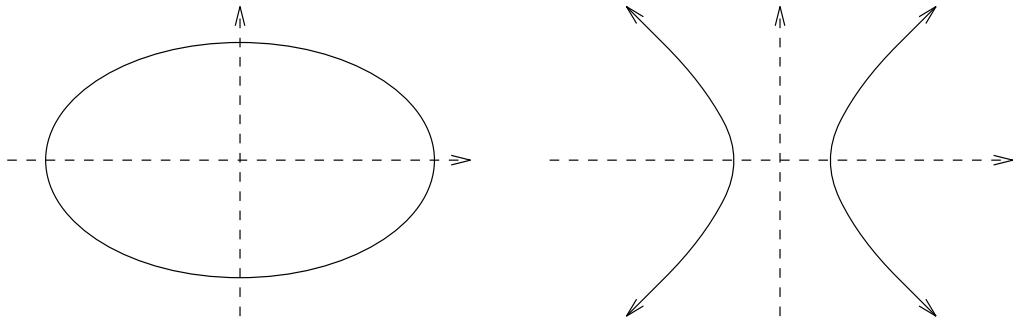
Appendix. A review of conics and quadrics.

You are probably familiar with certain graphs when they arise in standard configurations.

In two dimensions, the central conics have equations of the form

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1.$$

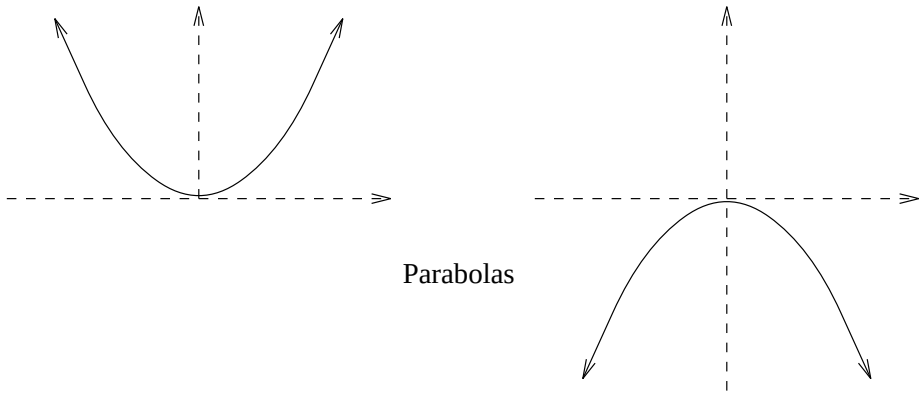
If both signs are +, the conic is an ellipse. If one sign is + and one is −, then the conic is a hyperbola. The + goes with the axis which crosses the hyperbola. Some examples are sketched below.



Ellipse Hyperbola

Two − signs result in an *empty graph*, i.e., there are no points satisfying the equation.

Parabolas arise from equations of the form $y = px^2$ with $p \neq 0$.



Parabolas

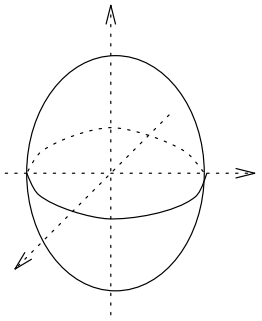
For $x = py^2$, the parabola opens along the positive or negative y-axis.

There are also some degenerate cases. For example, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ defines two lines which intersect at the origin.

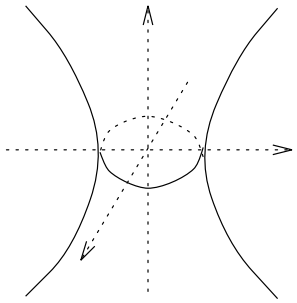
In three dimensions, the central quadrics have equations of the form

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1.$$

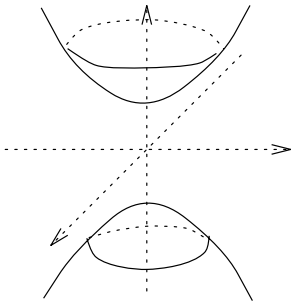
If all three signs are +, the quadric is an ellipsoid. If two of the three signs are + and one is −, the quadric is a hyperboloid of one sheet. If one of the two signs is + and the other two are −, the quadric is a hyperboloid of two sheets. Notice that the number of sheets is the same as the number of − signs. It is not hard to figure out how the quadric is oriented, depending on how the signs are arranged. The 'axis' of a hyperboloid is labeled by the variable whose sign in the equation is in the minority, i.e., the − sign in the one sheet case and the + sign in the two sheet case.



Ellipsoid



Hyperboloid of one sheet

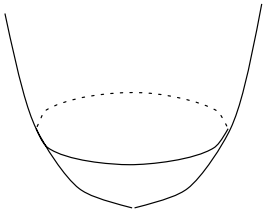


Hyperboloid of two sheets

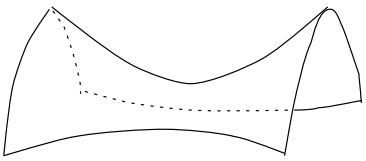
If all three signs are −, we get an empty graph.
Paraboloids arise from equations of the form

$$z = \pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2},$$

or similar equations with x, y, z rearranged. If both signs are + or both are −, then the quadric is an elliptic paraboloid or 'bowl'. The bowl opens up along the axis of the variable appearing on the left of the equation if the signs are + and it opens along the negative axis of that variable if the signs are −. If one sign is + and the other is −, the surface is a hyperbolic paraboloid or 'saddle'. Equations of the form $z = cxy, c \neq 0$ also describe saddles.



Elliptic paraboloid



Hyperbolic paraboloid

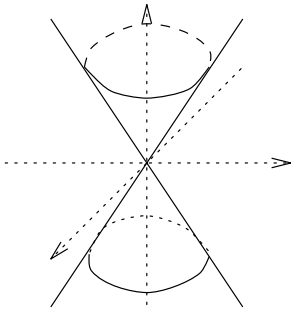
There are many degenerate cases. One example would be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

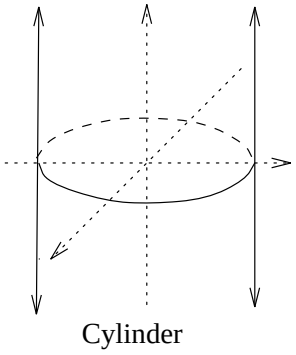
Its graph is a double cone with elliptical cross sections. Another would be

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

with at least one + sign. Its graph is a 'cylinder' perpendicular to the x, y-plane. The cross sections are ellipses or hyperbolas, depending on the combination of signs.



Cone



Cylinder

Exercises for Section 4.

1. Find the principal axes and classify the central conic $x^2 + xy + y^2 = 1$.

2. Identify the conic defined by $x^2 + 4xy + y^2 = 4$. Find its principal axes, and find the points closest and furthest (if any) from the origin.

3. Identify the conic defined by $2x^2 + 72xy + 23y^2 = 50$. Find its principal axes, and find the points closest and furthest (if any) from the origin.

4. Find the principal axes and classify the central quadric defined by

$$x^2 - y^2 + z^2 - 4xy - 4yz = 1.$$

5. (Optional) Classify the surface defined by

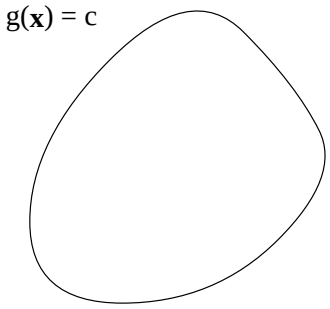
$$x^2 + 2y^2 + z^2 + 2xy + 2yz - z = 0.$$

Hint: This is not a central quadric. To classify it, first apply the methods of the section to the quadratic expression $x^2+2y^2+z^2+2xy+2yz$ to find a new coordinate system in which this expression has the form $\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2$. Use the change of coordinates formula to express z in terms of x', y', and z' and then complete squares to eliminate all linear terms. At this point, it should be clear what the surface is.

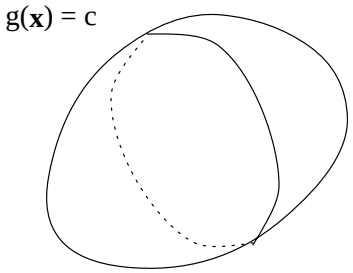
5. Conics and the Method of Lagrange Multipliers

There is another approach to finding the principal axes of a conic, quadric, or hyperquadric. Consider for an example an ellipse in $\mathbf{R}^2$ centered at the origin. One of the principal axes intersects the conic in the two points at greatest distance from the origin, and the other intersects it in the two points at least distance from the origin. Similarly, two of the three principal axes of a central ellipsoid in $\mathbf{R}^3$ may be obtained in this way. Thus, if we didn't know about eigenvalues and eigenvectors, we might try to find the principal axes by maximizing (or minimizing) the function giving the distance to the origin *subject to* the quadratic equation defining the conic or quadric. In other words, we need to minimize a function given a *constraint* among the variables. Such problems are solved by the method of Lagrange multipliers, which you learned in your multidimensional calculus course.

Here is a review of the method. Suppose we want to maximize (minimize) the real valued function $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ subject to the constraint $g(\mathbf{x}) = g(x_1, x_2, \ldots, x_1) = c$. For $n = 2$, this has a simple geometric interpretation. The locus of the equation $g(x_1, x_2) = c$ is a level curve of the function g, and we want to maximize (minimize) the function f *on that curve.* Similarly, for $n = 3$, the level set $g(x_1, x_2.x_3) = c$ is a surface in $\mathbf{R}^3$, and we want to maximize (minimize) f *on that surface.*



n = 2. Level curve in the plane.



n = 3. Level surface in space.

Examples. Maximize $f(x, y) = x^2 + y^2$ on the ellipse $g(x, y) = x^2 + 4y^2 = 3$. (This is easy if you draw the picture.)

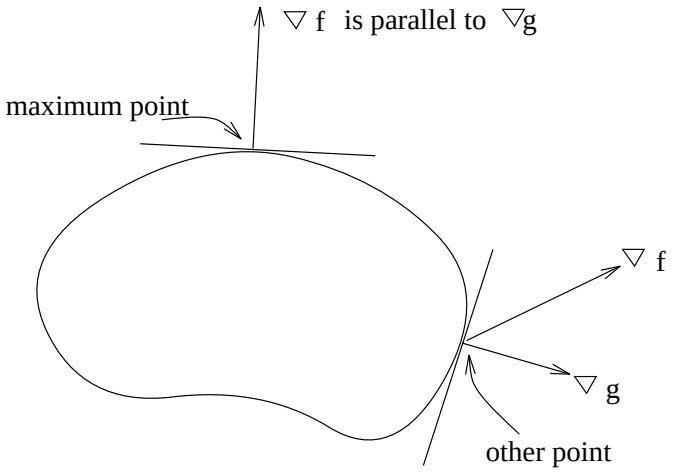
Minimize $f(x, y, z) = 2x^2 + 3xy + y^2 + xz - 4z^2$ on the sphere $g(x, y, z) = x^2 + y^2 + z^2 = 1$.

Minimize $f(x, y, z, t) = x^2 + y^2 + z^2 - t^2$ on the 'hypersphere' $g(x, y, z, t) = x^2 + y^2 + z^2 + t^2 = 1$.

We shall concentrate on the case of $n = 3$ variables, but the reasoning for any n is similar. We want to maximize (or minimize) $f(\mathbf{x})$ on a level surface $g(\mathbf{x}) = c$ in $\mathbf{R}^3$, where as usual we abbreviate $\mathbf{x} = (x_1, x_2, x_3)$. At any point $\mathbf{x}$ on the level surface at which such an extreme value is obtained, we must have

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \tag{1}$$

for some scalar λ.



(1) is a *necessary condition* which must hold at the relevant points. (It doesn't by itself guarantee that there is a maximum or a minimum at the point. There could be no extreme value at all at the point.) In deriving this condition, we assume implicitly that the level surface is smooth and has a well defined normal vector $\nabla g \neq \mathbf{0}$, and that the function f is also smooth. If these conditions are violated at some point, that point could also be a candidate for a maximum or minimum.

Taking components, we obtain 3 scalar equations for the 4 variables x_1, x_2, x_3, λ. We would not expect, even in the best of circumstances to get a unique solution from this, but the defining equation for the level surface

$$g(\mathbf{x}) = c$$

provides a 4th equation. We still won't generally get a unique solution, but we will usually get at most a finite number of possible solutions. Each of these can be examined further to see if f attains a maximum (or minimum) at that point in the level set. Notice that the variable λ plays an auxiliary role since we really only want the coordinates of the point $\mathbf{x}$. (In some applications, λ has some significance beyond that.) λ is called a *Lagrange multiplier*.

The method of Lagrange multipliers often leads to a set of equations which is difficult to solve. However, in the case of quadratic functions f, there is a typical pattern which emerges.

Example 1. Suppose we want to minimize the function $f(x, y) = x^2 + 4xy + y^2$ on the circle $x^2 + y^2 = 1$. For this problem $n = 2$, and the level set is a curve. Take $g(x, y) = x^2 + y^2$. Then $\nabla f = \langle 2x + 4y, 4x + 2y \rangle$, $\nabla g = \langle 2x, 2y \rangle$, and $\nabla f = \lambda \nabla g$ yields the equations

$$\begin{aligned} 2x + 4y &= \lambda(2x) \\ 4x + 2y &= \lambda(2y) \end{aligned}$$

to which we add

$$x^2 + y^2 = 1.$$

After canceling a common factor of 2, the first two equations may be written in matrix form

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

which says that

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

is an *eigenvector* for the eigenvalue λ, and the equation $x^2 + y^2 = 1$ says it is a *unit eigenvector*. You should know how to solve such problems, and we leave it to you to make the required calculations. (See also Example 1 in the previous section where we made these calculations in another context.) The eigenvalues are $\lambda = 3$ and $\lambda = -1$. For $\lambda = 3$, a basic unit eigenvector is

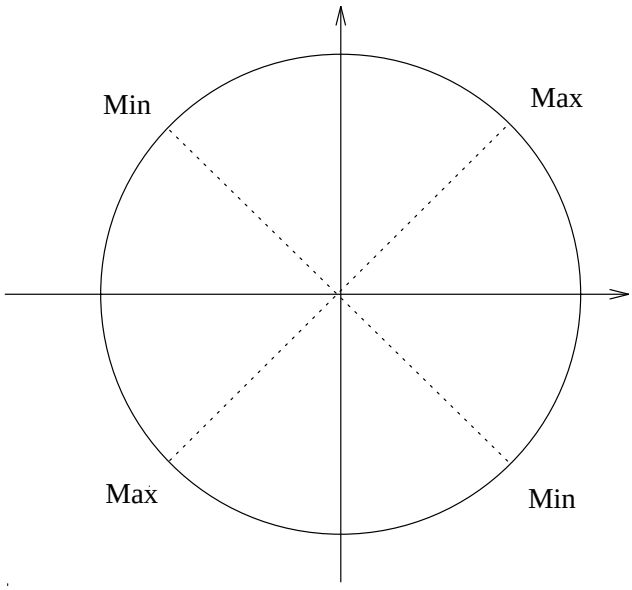
$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and every other eigenvector is of the form $c\mathbf{u}_1$. The latter will be a *unit* vector if and only $|c| = 1$, i.e., $c = \pm 1$. We conclude that $\lambda = 3$ yields two solutions of the Lagrange multiplier problem: $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. At each of these points $f(x, y) = x^2 + 4xy + y^2 = 3$.

For $\lambda = -1$, we obtain the basic unit eigenvector

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

and a similar analysis (which you should do) yields the two points: $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$. At each of these points $f(x, y) = x^2 + 4xy + y^2 = -1$.



Hence, the function attains its maximum value at the first two points and its minimum value at the second two.

Example 2. Suppose we want to minimize the function $g(x, y) = x^2+y^2$ (which is the square of the distance to the origin) on the conic $f(x, y) = x^2+4xy+y^2 = 1$. Note that this is basically the same as the previous example except that the roles of the two functions are reversed. The Lagrange multiplier condition $\nabla g = \lambda \nabla f$ is the same as the condition $\nabla f = (1/\lambda)\nabla g$ provided $\lambda \neq 0$. ($\lambda \neq 0$ in this case since otherwise $\nabla g = 0$, which yields $x = y = 0$. However, $(0, 0)$ is not a point on the conic.) We just solved that problem and found eigenvalues $1/\lambda = 3$ or $1/\lambda = -1$. In this case, we don't need unit eigenvectors, so to avoid square roots we choose basic eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

corresponding respectively to $\lambda = 3$ and $\lambda = -1$. The endpoint of $\mathbf{v}_1$ does not lie on the conic, but any other eigenvector for $\lambda = 3$ is of the form $c\mathbf{v}_1$, so all we need to do is adjust c so that the point satisfies the equation $f(x, y) = x^2+4xy+y^2 = 1$. Substituting $(x, y) = (c, c)$ yields $6c^2 = 1$ or $c = \pm 1/\sqrt{6}$. Thus, we obtain the two points $(1/\sqrt{6}, 1/\sqrt{6})$ and $(-1/\sqrt{6}, -1/\sqrt{6})$. For $\lambda = -1$, substituting $(x, y) = (-c, c)$ in the equation yields $-2c^2 = 1$ which has no solutions.

Thus, the only candidates for a minimum (or maximum) are the first pair of points: $(1/\sqrt{6}, 1/\sqrt{6})$ and $(-1/\sqrt{6}, -1/\sqrt{6})$. A simple calculation shows these are both $1/\sqrt{3}$ units from the origin, but without further analysis, we can't tell if this is the maximum, the minimum, or neither. However, it is not hard to classify this conic—see the previous section—and discover that it is a hyperbola. Hence, the two points are minimum points.

The Rayleigh-Ritz Method. Example 1 above is typical of a certain class of Lagrange multiplier problems. Let A be a real symmetric $n \times n$ matrix, and consider the problem of maximizing (minimizing) the quadratic function $f(\mathbf{x}) = \mathbf{x}^t A\mathbf{x}$ subject to the constraint $g(\mathbf{x}) = |\mathbf{x}|^2 = 1$. This is called the *Rayleigh–Ritz problem*. For $n = 2$ or $n = 3$, the level set $|\mathbf{x}|^2 = 1$ is a circle or sphere, and for $n > 3$, it is called a *hypersphere*.

Alternately, we could reverse the roles of the functions f and g, i.e., we could try to maximize (minimize) the square of the distance to the origin $g(\mathbf{x}) = |\mathbf{x}|^2$ on the level set $f(\mathbf{x}) = 1$. Because the Lagrange multiplier condition in either case asserts that the two gradients ∇f and ∇g are parallel, these two problems are very closely related. The latter problem—finding the points on a conic, quadric, or hyperquadric furthest from (closest to) the origin—is easier to visualize, but the former problem—maximizing or minimizing the quadratic function f on the hypersphere $|\mathbf{x}| = 1$ —is easier to compute with.

Let's go about applying the Lagrange Multiplier method to the Rayleigh–Ritz problem. The components of ∇g are easy:

$$\frac{\partial g}{\partial x_i} = 2x_i, \qquad i = 1, 2, \ldots n.$$

The calculation of ∇f is harder. First write

$$f(\mathbf{x}) = \sum_{j=1}^{n} x_j \Big(\sum_{k=1}^{n} a_{jk} x_k\Big)$$

and then carefully apply the product rule together with $a_{jk} = a_{kj}$. The result is

$$\frac{\partial f}{\partial x_i} = 2\sum_{j=1}^{n} a_{ij}x_j \qquad i = 1, 2, \ldots, n.$$

(Work this out explicitly in the cases $n = 2$ and $n = 3$ if you don't believe it.) Thus, the Lagrange multiplier condition $\nabla f = \lambda \nabla g$ yields the equations

$$2\sum_{j=1}^{n} a_{ij}x_j = \lambda(2x_i) \qquad i = 1, 2, \ldots, n$$

which may be rewritten in matrix form (after canceling the 2's)

$$A\mathbf{x} = \lambda \mathbf{x}. \tag{3}$$

To this we must add the equation of the level set

$$g(\mathbf{x}) = |\mathbf{x}|^2 = 1.$$

Thus, any potential solution $\mathbf{x}$ is a *unit* eigenvector for the matrix A with eigenvalue λ. Note also that for such a unit eigenvector, we have

$$f(\mathbf{x}) = \mathbf{x}^t A\mathbf{x} = \mathbf{x}^t(\lambda \mathbf{x}) = \lambda \mathbf{x}^t\mathbf{x} = \lambda|\mathbf{x}|^2 = \lambda.$$

Thus the eigenvalue is the extreme value of the quadratic function at the point on the (hyper)sphere given by the unit eigenvector.

The upshot of this discussion is that for a real symmetric matrix A, the Rayleigh–Ritz problem is equivalent to the problem of finding an orthonormal basis of eigenvectors for A.

The Rayleigh–Ritz method may be used to show that the characteristic equation of a real symmetric matrix only has real eigenvalues. This was an issue left unresolved in our earlier discussions. Here is an outline of the argument. The hypersphere $g(\mathbf{x}) = |\mathbf{x}|^2 = 1$ is a closed bounded set in $\mathbf{R}^n$ for any n. It follows from a basic theorem in analysis that any continuous function, in particular the quadratic function $f(\mathbf{x})$, must attain both maximum and minimum values on the hypersphere. Hence, the Lagrange multiplier problem always has solutions, which by the above algebra amounts to the assertion that the real symmetric matrix A must have at least one eigenvalue. This suggests a general procedure for showing that all the eigenvalues are real. First find the largest eigenvalue by maximizing the quadratic function $\mathbf{f}(\mathbf{x})$ on the set $|\mathbf{x}|^2 = 1$. Let $\mathbf{x} = \mathbf{u}_1$ be the corresponding eigenvector. Change coordinates by choosing an orthonormal basis starting with $\mathbf{u}_1$. Then the additional basis elements will span the subspace perpendicular to $\mathbf{u}_1$ and we may obtain a lower dimensional quadratic function by restricting f to that subspace. We can now repeat the process to find the next smaller real eigenvalue. Continuing in this way, we will obtain an orthonormal basis of eigenvectors for A and each of the corresponding eigenvalues will be real.

Exercises for Section 5.

1. Find the maximum and minimum values of the function $f(x, y) = x^2+y^2$ given the constraint $x^2 + xy + y^2 = 1$.

2. Find the maximum and/or minimum value of $f(x, y, z) = x^2-y^2+z^2-4xy-4yz$ subject to $x^2 + y^2 + z^2 = 1$.

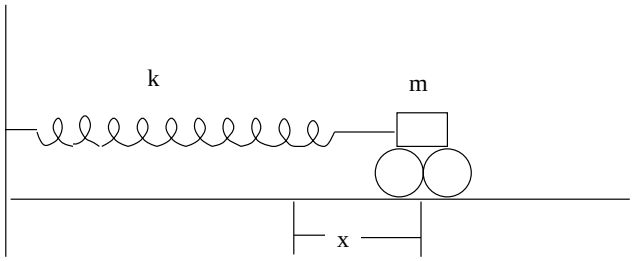
3. (Optional) The derivation of the Lagrange multiplier condition $\nabla f = \lambda\nabla g$ assumes that the $\nabla g \neq 0$, so there is a well defined tangent 'plane' at the potential maximum or minimum point. However, a maximum or minimum could occur at a point where $\nabla g = 0$, so all such points should also be checked. (Similarly, either f or g might fail to be smooth at a maximum or minimum point.) With these remarks in mind, find where $f(x, y, z) = x^2 + y^2 + z^2$ attains its minimum value subject to the constraint $g(x, y, z) = x^2 + y^2 - z^2 = 0$.

4. Consider as in Example 2 the problem of maximizing $f(x, y) = x^2 + 4xy + y^2$ given the constraint $x^2 + y^2 = 1$. This is equivalent to maximizing $F(x, y) = xy$ on the circle $x^2+y^2 = 1$. (Why?) Draw a diagram showing the circle and selected level curves $F(x, y) = c$ of the function F. Can you see why $F(x, y)$ attains its maximum at $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$ without using any calculus? Hint: consider how the level curves of F intersect the circle and decide from that where F is increasing, and where it is decreasing on the circle.

6. Normal Modes

Eigenvalues and eigenvectors are an essential tool in solving systems of linear differential equations. We leave an extended treatment of this subject for a course in differential equations, but it is instructive to consider an interesting class of vibration problems that have many important scientific and engineering applications.

We start with some elementary physics you may have encountered in a physics class. Imagine an experiment in which a small car is placed on a track and connected to a wall though a stiff spring. With the spring in its rest position, the car will just sit there forever, but if the car is pulled away from the wall a small distance and then released, it will oscillate back and forth about its rest position. If we assume the track is so well greased that we can ignore friction, this oscillation will in principle continue forever.



We want to describe this situation symbolically. Let x denote the displacement of the car from equilibrium, and suppose the car has mass m. Hooke's Law tells

us that there is a restoring force of the form $F = -kx$ where k is a constant called the spring constant. Newton's second law relating force and acceleration tells us

$$m\frac{d^2x}{dt^2} = -kx. \tag{1}$$

This is also commonly written $\dfrac{d^2x}{dt^2} + \dfrac{k}{m}x = 0$. You may have learned how to solve this differential equation in a previous course, but in this particular case, it is not really necessary. From the physical characteristics of the solution, we can pretty much guess what it should look like.

$$x = A\cos(\omega t) \tag{2}$$

where A is the amplitude of the oscillation and ω is determined by the frequency or rapidity of the oscillation. It is usually called the *angular frequency* and it is related to the actual frequency f by the equation

$$\omega = 2\pi f.$$

A is determined by the size of the initial displacement. It gives the maximum displacement attained as the car oscillates. ω however is determined by the spring constant. To see how, just substitute (2) in (1). We get

$$m(-\omega^2 A\cos(\omega t)) = -kA\cos(\omega t)$$

which after canceling common factors yields

$$m\omega^2 = k$$

$$\text{or} \qquad \omega = \sqrt{\frac{k}{m}}.$$

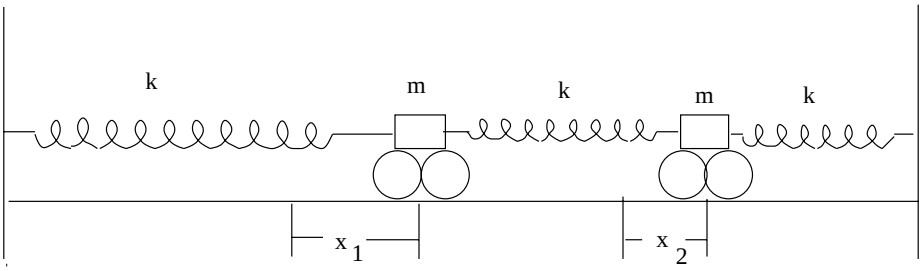
The above discussion is a bit simplified. We could not only have initially displaced the car from rest, but we could also have given it an initial shove or velocity. In that case, the maximal displacement would be shifted in time. The way to describe this symbolically is

$$x = A\cos(\omega t + \delta)$$

where δ is called the phase shift. This complication does not change the basic character of the problem since it is usually the fundamental vibration of the system that we are interested in, and that turns out to be the same if we include a possible phase shift.

We now want to generalize this to more than one mass connected by several springs. This may seem a bit bizarre, but it is just a model for situations commonly met in scientific applications. For example, in chemistry, one often needs to determine the basic vibrations of a complex molecule. The molecule consists of atoms 'connected' by interatomic forces. As a first approximation, we may treat the atoms as point masses and the forces between them as linear restoring forces from equilibrium positions. Thus the mass-spring model may tell us something useful about real problems.

Example 1. Consider the the configuration of masses and springs indicated below, where m is the common mass of the two particles and k is the common spring constant of the three springs.



Look at the first mass. When it is displaced a distance x_1 to the right from equilibrium, it will be acted upon by two forces. Extension of the spring on the left will pull it back with force $-kx_1$. At the same time, the spring in the middle will push or pull it depending on whether it is compressed or stretched. If x_2 is the displacement of the second mass from equilibrium, the change in length of the second spring will be $x_1 - x_2$, so the force on the first mass will be $-k(x_1 - x_2)$. This yields a total force of

$$-kx_1 - k(x_1 - x_2) = -2kx_1 + kx_2.$$

A similar analysis works for the second mass. Thus, we obtain the system of differential equations

$$\begin{aligned} m\frac{d^2x_1}{dt^2} &= -2kx_1 + kx_2 \\ m\frac{d^2x_2}{dt^2} &= kx_1 - 2kx_2. \end{aligned}$$

The system may also be rewritten in matrix form

$$m\frac{d^2\mathbf{x}}{dt^2} = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}\mathbf{x} \qquad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{3}$$

Note that the matrix on the right is a symmetric matrix. This is always the case in such problems. It is an indirect consequence of Newton's third law which asserts that the forces exerted by two masses on each other must be equal and opposite.

To solve this, we look for solutions of the form

$$\begin{aligned} x_1 &= v_1\cos(\omega t) \\ x_2 &= v_2\cos(\omega t) \end{aligned} \tag{4}$$

In such a solution, the two particles oscillate with the same frequency but with possibly different amplitudes v_1 and v_2. Such a solution is called a *normal mode*. General motions of the system can be quite a bit more complicated. First of all, we have to worry about possible phase shifts. More important, we also have to

allow for linear combinations of the normal modes in which there is a mixture of different frequencies. In this way the situation is similar to that of a musical instrument which may produce a complex sound which can be analyzed in terms of basic frequencies or harmonics. We leave such complications for another course. Here we content ourselves at doing the first step, which is to find the fundamental oscillations or normal modes.

(4) may be rewritten in matrix form

$$\mathbf{x} = \mathbf{v}\cos(\omega t) \tag{5}$$

where ω and $\mathbf{v} \neq 0$ are to be determined. Then

$$\frac{d^2\mathbf{x}}{dt^2} = -\omega^2 \mathbf{v}\cos(\omega t)$$

Hence, putting (5) in (3) yields

$$m(-\omega^2 \mathbf{v}\cos(\omega t)) = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix} \mathbf{v}\cos(\omega t).$$

Now factor out the common scalar factor $\cos(\omega t)$ to obtain

$$-\omega^2 m\mathbf{v} = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix} \mathbf{v}.$$

Note that the 'amplitude' $\mathbf{v}$ is a vector in this case, so we cannot cancel it as we did in the case of a single particle. The above equation may now be rewritten

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{v} = -\omega^2 \frac{m}{k}\mathbf{v}.$$

This is a trifle messy, but if we put abbreviate $\lambda = -\omega^2 \frac{m}{k}$ for the scalar on the right, we can write it

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{v} = \lambda \mathbf{v}.$$

This equation should look familiar. It says that $\mathbf{v}$ is an eigenvector for the matrix on the left, and that $\lambda = -\omega^2 \dfrac{m}{k}$ is the corresponding eigenvalue. However, we know how to solve such problems. First we find the eigenvalues by solving the characteristic equation. For each eigenvalue, we can find the corresponding frequency ω from $\omega = \sqrt{\dfrac{\lambda m}{k}}$. Next, for each eigenvalue, we can determine basic eigenvectors as before.

In this example, the characteristic equation is

$$\begin{aligned} \det \begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} &= (-2-\lambda)^2 - 1 \\ &= \lambda^2 + 4\lambda + 4 - 1 \\ &= \lambda^2 + 4\lambda + 3 \\ &= (\lambda+1)(\lambda+3) = 0. \end{aligned}$$

Hence, the roots are $\lambda = -1$ ($\omega = \sqrt{k/m}$) and $\lambda = -3$ ($\omega = \sqrt{3k/m}$).

For $\lambda = -1$ ($\omega = \sqrt{k/m}$), finding the eigenvectors results in reducing the matrix

$$\begin{bmatrix} -2+1 & 1 \\ 1 & -2+1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Hence, the solution is $v_1 = v_2$ with v_2 free. A basic solution vector for the subspace of solutions is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The corresponding normal mode has the form

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\sqrt{k/m}\, t).$$

Note that $x_1(t) = x_2(t)$ for all t, so the two particles move together in tandem with the same angular frequency $\sqrt{k/m}$. This behavior of the particles is a consequence of the fact that the components of the basic vector $\mathbf{v}_1$ are equal.

Similarly, for $\lambda = -3$ ($\omega = \sqrt{3k/m}$), we have

$$\begin{bmatrix} -2+3 & 1 \\ 1 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

The solution is $v_1 = -v_2$ with v_2 free, and a basic solution vector for the system is

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The corresponding normal mode is is

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos(\sqrt{3k/m}\, t).$$

Note that $x_1(t) = -x_2(t)$ for all t, so the two masses move opposite to one another with the same amplitude and angular frequency $\sqrt{\dfrac{3k}{m}}$.

Note that in the above example, we could have determined the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ by inspection. As noted, the first corresponds to motion in which the particles move in tandem and the spring between them experiences no net change in length. The second corresponds to motion in which the particles move back and forth equal amounts in opposite directions but with the same frequency. This would have simplified the problem quite a lot. For, if you know an eigenvector of a matrix, it is fairly simple to find the corresponding eigenvalue, and hence the angular frequency. In fact, it is often true that careful consideration of the physical arrangement of the particles, with particular attention to any symmetries that may be present, may suggest possible normal modes with little or no calculation.

Relation to the Principal Axis Theorem. As noted above normal mode problems typically result in systems of the form

$$\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x} \tag{7}$$

where A is a real symmetric matrix. (In the case that all the particles have the same mass, $A = \dfrac{1}{m}K$, where K is a symmetric matrix of 'spring constants'. If the masses are different, the situation is a bit more complicated, but the problem may still be restated in the above form.)

If P is a matrix with columns the elements of a basis of eigenvectors for A, then we saw earlier that

$$AP = PD$$

where D is a diagonal matrix with the eigenvalues on the diagonal. Assume we make the change of coordinates

$$\mathbf{x} = P\mathbf{x}'.$$

Then

$$\begin{aligned}
\frac{d^2P\mathbf{x}'}{dt^2} &= AP\mathbf{x}' \\
P\frac{d^2\mathbf{x}'}{dt^2} &= AP\mathbf{x}' \\
\frac{d^2\mathbf{x}'}{dt^2} &= P^{-1}AP\mathbf{x}' = D\mathbf{x}'.
\end{aligned}$$

However, since D is diagonal, this last equation may be written as n scalar equations

$$\frac{d^2x_j'}{dt^2} = \lambda_j x_j' \qquad j = 1, 2, \ldots, n.$$

In the original coordinates, the motions of the particles are 'coupled' since the motion of each particle may affect the motion of the other particles. In the new coordinate system, these motions are 'decoupled'. The new coordinates are called *normal* coordinates. Each x_j' may be thought of as the displacement of one of n fictitious particles, each of which oscillates independently of the others in one of n mutually perpendicular directions. The physical significance in terms of the original particles of each normal coordinate is a but murky, but they presumably represent underlying structure of some importance.

Example 1, revisited.

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{k}{m}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{x}.$$

A basis of eigenvectors for the coefficient matrix is as before

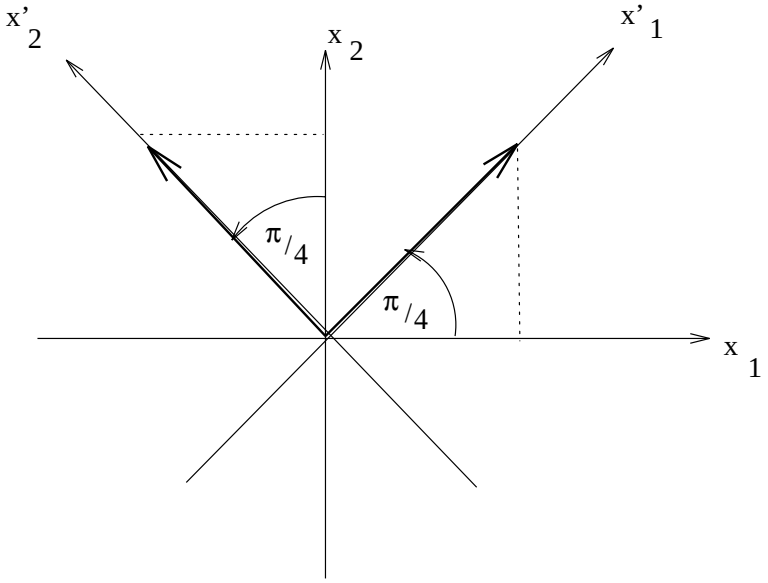
$$\left\{\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}.$$

If we divide the vectors by their lengths, we obtain the orthonormal basis

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

This in turn leads to the change of coordinates matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

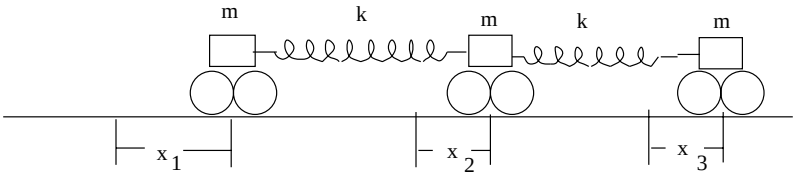


If you look carefully, you will see this represents a rotation of the original x_1, x_2-axes through an angle $\pi/4$. However, this has nothing to do with the original geometry of the problem. x_1 and x_2 stand for displacements of two different particles along the same one dimensional axis. The x_1, x_2 plane is a fictitious configuration space in which a single point represents a pair of particles. It is not absolutely clear what a rotation of axes means for this plane, but the new normal coordinates x_1', x_2' obtained thereby give us a formalism in which the normal modes appear as decoupled oscillations.

Exercises for Section 6.

1. Determine the normal modes, including frequencies and relative motions for the system

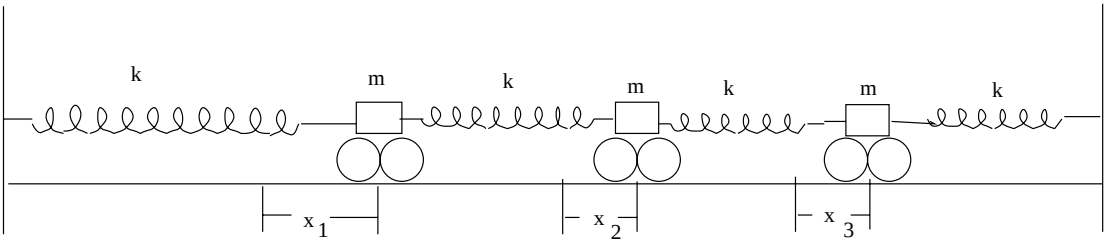
$$\begin{aligned} m\frac{d^2x_1}{dt^2} &= k(x_2 - x_1) = -kx_1 + kx_2 \\ m\frac{d^2x_2}{dt^2} &= k(x_1 - x_2) + k(x_3 - x_2) = kx_1 - 2kx_2 + kx_3 \\ m\frac{d^2x_3}{dt^2} &= k(x_2 - x_3) = kx_2 - kx_3 \end{aligned}$$



Note that since the masses are not fixed to any wall, one possibility is that they will together move freely at constant velocity without oscillating. This is reflected in the linear algebra by one *zero* eigenvalue which does not actually give an oscillatory solution. Ignore that eigenvalue and the corresponding eigenvector.

2. Determine the normal modes, including frequencies and relative motions for the system

$$m\frac{d^2x_1}{dt^2} = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$$
$$m\frac{d^2x_2}{dt^2} = k(x_1 - x_2) + k(x_3 - x_2) = kx_1 - 2kx_2 + kx_3$$
$$m\frac{d^2x_3}{dt^2} = k(x_2 - x_3) - kx_3 = kx_2 - 2kx_3$$



3. Suppose a normal mode problem involving two particles has one normal mode in which the displacements satisfy $x_1 = 2x_2$ for all time. What relation do the displacements have for the other normal mode?

4. A system of two particles is similar to the example in the text except that one end is free. It is described by the system

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{k}{m}\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\mathbf{x} \qquad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Find the normal modes.

5. A system of two particles is as in the example in the text except that one end is free. It is described by the system

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{k}{m}\begin{bmatrix} -4 & 2 \\ 2 & -2 \end{bmatrix}\mathbf{x} \qquad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Find the normal modes.

6. A certain molecule has three normal modes. One is degenerate and corresponds to the eigenvalue $\lambda = 0$. The eigenvector for this degenerate mode is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The relative motions for another normal mode satisfy $x_1 = x_3, x_2 = -2x_3$. What relations do the relative motions for the third normal mode satisfy?

7. Review

Exercises for Section 7.

1. The Gram-Schmidt process fails when applied to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$$

in $\mathbf{R}^4$. Explain why.

2. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.
   (a) Find the eigenvalues and eigenvectors of A.
   (b) Find an orthonormal basis for $\mathbf{R}^3$ consisting of eigenvectors for A.
   (c) Find an orthogonal matrix P such that P^tAP is diagonal. What is P^tAP?

3. What is wrong with the following statement? If the columns of an $n \times n$ matrix P are mutually perpendicular, then P is orthogonal.

4. Consider the matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$ which has eigenvalues $\lambda = 6, 1$.
   (a) Find the eigenvectors of A.
   (b) Consider the conic section $2x^2 - 4xy + 5y^2 = 24$. Find an orthogonal matrix P such that the coordinate change $\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} u \\ v \end{bmatrix}$ transforms the equation of the conic into the form $\alpha u^2 + \beta v^2 = 24$ (that is, into an equation with zero cross term).
   (c) Sketch the conic section of part (b). Include in the same sketch the xy axes and the uv axes.

5. Use the methods introduced in this course to sketch the graph of the equation

$$2x^2 + y^2 + z^2 + 4yz = 6.$$

6. A system of two particles with displacements x_1 and x_2 satisfies the system of differential equations

$$m\frac{d^2x_1}{dt^2} = -3kx_1 + 2kx_2$$
$$m\frac{d^2x_2}{dt^2} = 2kx_1 - 3kx_2$$

Find the normal modes. Include the 'angular frequencies' ω and the initial displacements (u_1, u_2) for each normal mode.

7. Determine whether or not each of the following matrices may be diagonalized. In each case, explain your answer. Using general principles may help you avoid difficult computations.

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(b) $B = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$. *Note:* The characteristic polynomial of B is $-(\lambda - 2)^2(\lambda - 4)$.

(c) $C = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 4 & 0 \\ 1 & 0 & 4 & 0 & 5 \\ 1 & 1 & 0 & 5 & 0 \end{bmatrix}$.