LOCALILAZATION

1. Local Rings

Let A be a ring. We call it a *local* ring if the *complement* J of the set of left invertible elements of A is a left ideal. (The reason for the name will become apparent later when we discuss the process of "localization" which always results in local rings.) In principle we would have to distinguish between "left" local and "right" local, but fortunately the following facts make that distinction unnecessary.

THEOREM. *Let A be a local ring, and let J be the complement of the set of all left invertible elements. Then*

(i) J is the Jacobson radical of A.

(ii) J contains all left ideals and it contains all right ideals. Hence, J is the unique maximal proper left ideal of A, the unique maximal proper right ideal of A, and the unique maximal proper 2-sided ideal of A.

(iii) J is also the complement of the set of all right invertible elements.

(iv) J is the complement of $U(A)$.

PROOF.

First we show that every proper left ideal $L \subseteq J$. Indeed, if L contains an element $x \notin J$, then x is left invertible from which it follows that $1 = ux \in L$ (where $u \in A$ is a left inverse of x). Hence J is the unique maximal left ideal of A and so it is trivially the intersection of all maximal left ideals, i.e., (i) $J = \mathrm{rad}(A)$.

We show next that $x \in A$ is right invertible if and only if $x \notin J$, i.e., x is left invertible. Suppose first that x is right invertible. Then $x \notin J$ since $xy = 1, x \in J \Rightarrow 1 \in J$ because J is a 2-sided ideal. Conversely, assume x is left invertible, i.e., $yx = 1$ for some $y \in A$. Then, $yxy = y$ or $y(xy - 1) = 0$. Consider the set L of all $z \in A$ such that $z(xy - 1) = 0$. It is easy to see that it is a left ideal of A, and $y \in L$. By what we first proved, if it is a proper left ideal, it is contained in J. However, since $y \in L$, and since y is *left invertible*, $y \notin J$. Hence, $L = A$, so $1 \in L$ and $1(xy - 1) = 0$ or $xy = 1$. Hence, x is also right invertible. This establishes both (iii) and (iv).

Since we now know that J is the complement of the set of right invertible elements, we may show as above that it is the maximal proper right ideal (ii).

Note: A crucial point in the above proof is that J is a 2-sided ideal. Can you prove that without using the theory of the Jacobson radical by a direct argument?

Of course, if A is commutative, all the arguments are quite a bit simpler. You should go through and check the statments and proof of the above theorem in the commutative case.

COROLLARY. *If A is a local ring and J is its unique maximal ideal, then A/J is a division ring. In the commutative case it is a field.*

PROOF.

A/J has no left ideal and no right ideals except $\{0\}$. It is easy to see from this that every non-zero element is both left and right invertible.

Examples of local rings:

0. Of course any field (or more generally any division ring) is trivially a local ring with $\{0\}$ the maximal ideal.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

1. Let k be a field and let X be an indeterminate. $k[X]$ is not a local ring. For, $U(k[X]) = k^*$, and its complement is the set of polynomials of degree > 0, and that is not an ideal. Let A be the *subring* of the *rational function field* $k(X)$ of all elements of the form $f(X)/g(X)$ where $g(0) \neq 0$ (i.e., the constant term of g(X) is nonzero.) It is easy to see that this is in fact a ring. Clearly, $U(A)$ consists of all elements of the form $f(X)/g(X)$ where $f(0) \neq 0$ and $g(0) \neq 0$. Hence, its complement is the set of all $f(X)/g(X)$ where $f(0) = 0$ but $g(0) \neq 0$. This is an ideal in A so A is a local ring. (We shall see later that A is the "localization of $k[X]$ at 0".)

2. Let k be a field and let X be an indeterminate; denote by $k[[X]]$ the set of all "formal expressions"

$$\sum_{n=0}^{\infty} a_n X^n, \qquad a_n \in k.$$

More precisely, $k[[X]]$ as a k-vector space is the direct *product* of a denumerable number of copies of k indexed by the set of monomials $X^n, n \geq 0$. $k[[X]]$ is made into a ring by defining addition and multiplication exactly as for polynomials except there is no restriction that the result must have all but a finite number of coefficients 0. In the formula for the product

$$\left(\sum_i a_i X^i\right)\left(\sum_j b_j X^j\right) = \sum_n \left(\sum_{i+j=n} a_i b_j\right) X^n$$

the coefficient of X^n is a sum with only a finite number of nonzero terms in any case, so the product is well defined.

$U(k[[X]])$ is the set of $\sum_n a_n X^n$ with $a_0 \in k^*$. For suppose $\left(\sum_i a_i X^i\right)\left(\sum_j b_j X^j\right) = 1$. Using the above formula yields

$$a_0 b_0 = 1$$

$$\sum_{i+j=n} a_i b_j = 0 \qquad \text{for } n > 0.$$

The first equation may be solved for b_0 if and only if a_0 is a unit in k. In that case, the remaining equations may then be solved recursively

$$b_n = -a_0^{-1}(a_1 b_{n-1} + \cdots + a_n b_0)$$

and the resulting formal series $\sum_n b_n X^n$ is easily seen to be the inverse of $\sum_n a_n X^n$. It follows that the complement of $U(k[[X]])$ is the set of $\sum_n a_n X^n$ with $a_n = 0$, and that is an ideal, the ideal generated by X. Hence, k[[X]] is a local ring with unique maximal ideal the ideal generated by X. $k[[X]]$ is called the ring of *formal power series* in the indeterminate X.

3. Let k be a field of characteristic p where $p > 0$, and let G be a group. Let A be the group ring $k[G]$, and define $\epsilon : k[G] \to k$ by $\epsilon(\sum_{g \in G} a_g g) = (\sum_{g \in G} a_g) \in k$. It is not hard to check that ϵ is a ring epimorphism. Moreover, in the case that G is a finite p-group, it is possible to show that up to isomorphism the only simple $k[G]$ module is the field k itself where the action of $k[G]$ on k is given by $ra = \epsilon(r)a$ for $r \in k[G], a \in k$. (Note that each element of G acts as the identity on k.) This is called the *trivial $k[G]$-module*. It follows that $I = \text{Ker}\,\epsilon$ is the only annihilator of a simple module, so $\text{rad}(A) = I$. Since, $k[G]/I \cong k$, it is clear that I is maximal. Using these facts, it is not hard to see that $k[G]$ is a local ring. [The details of this example are the subject of an exercise.] If G is not a p-group, then $k[G]$ will not in general be a local ring—as we shall see later. Of course, $k[G]$ is non-commutative if G is non-abelian.

THEOREM. *Let A be a local ring with unique maximal ideal J, and let M be a finitely generated left A-module. Suppose that as an A/J-module, M/JM is generated by $\{x_1 + JM, x_2 + JM, \ldots, x_n + JM\}$. Then M is generated over A by $\{x_1, x_2, \ldots, x_n\}$.*

PROOF. Let $N = Ax_1 + Ax_2 + \cdots + Ax_n$. Then clearly $JM + N = M$. Hence, by Nakayama's Lemma, $M = N$.

Exercises.

1. Restate and prove the first Theorem in the section under the assumption that A is commutative. Both the statement and proof should be considerably simpler.

2. Let p be a prime, G a finite p-group, and k a field of characteristic p. Let kG denote the group algebra of G over k.

(a) Let V be a finitely generated kG-module. Show that

$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\} \neq \{0\}.$$

Hint: Proceed by induction on the order of $|G|$. Use the fact that the center $Z(G)$ is nontrivial and show that $V^{Z(G)}$ is a $k(G/Z(G))$-module. To start the induction, assume G is cyclic of order p and note that if z is a generator, then $(z-1)^p = z^p - 1 = 0$.

(b) Using (a), show that up to isomorphism the only simple kG-module is k with trivial action. ($gv = v$ for all $g \in G, v \in V$.)

(c) Show that the augmentation ideal

$$I = \{\sum_g a_g g \mid \sum_g ag = 0\}$$

is the Jacobson radical of kG. Conclude kG is a local ring.

3. Show that in any commutative local domain, the Jacobson radical is not equal to the nil radical.

Exercises about composita.

4. Let E be a finite separable extension of k, and let Ω be any field extension of k. Show that $A = E \otimes_k \Omega$ is isomorphic to a direct product of fields. Hint: Let $E = k[x]$ where x has minimal polynomial $f(X) \in k[X]$, and show $A \cong \Omega[X]/f(X)$. Then factor $f(X)$ in $\Omega[X]$ and use the Chinese Remainder Theorem.

5. Assume as in the previous problem that E is a separable extension of k and Ω is any field extension of k.

(a) Show that every compositum $E\Omega$ formed in any field containing both E and Ω is an epimorphic image of $E \otimes_k \Omega$, and conversely every field which is such an epimorphic image is a compositum.

(b) Conclude that the constituents of the product of fields $E \otimes_k \Omega$ give all the composita which may be formed from E and Ω up to k-isomorphism.

6. Let k be a field of characteristic p and let $E = k[x]$ where the minimal polynomial $f(X) \in k[X]$ of x factors $f(X) = (X - a)^q$ in $E[X]$. (Thus, E is a purely inseparable extension of k and q is a power of p.) Show that $E \otimes_k E$ is an artinian local ring which is not a field.

2. Localization

We start with an example.

The field $\mathbf{Q}$ is obtained from $\mathbf{Z}$ by adjoining sufficiently many additional numbers such that every non-zero element of $\mathbf{Z}$ becomes invertible in $\mathbf{Q}$. In many cases we might not want to go quite so far. For example, if we are specially interested in a single prime number p, we might simply try to invert all numbers not divisible by p. Thus, we would consider the subring of $\mathbf{Q}$ consisting of all rational numbers a/b (with $\gcd(a, b) = 1$) such that b is relatively prime to p. In this ring, the only kind of divisibility that counts is divisibility by p or powers of p. This process of "inverting" a specified class of elements is called localization.

Suppose now that A is any *commutative* ring and that S is a subset of A which forms a monoid under multiplication, i.e., $1 \in S$ and $a, b \in S \Rightarrow ab \in S$. Such a set is called a *multiplicative* subset of A. Consider the set of pairs (a, s) with $a \in A$ and $s \in S$, and define an equivalence relation on this set by

$$(a, s) \sim (b, t) \Leftrightarrow \exists u \in S \text{ such that } u(at - bs) = 0.$$

We leave it to the student to check that this does in fact yield an equivalence relation. The definition seems a bit strange. However, notice that in case A is a domain and 0 is not in S the condition simply amounts to

$at - bs = 0$ which is the same condition used in defining the field of fractions of A—except the "denominators" are restricted to S. If A is not a domain, the condtion has to be weakened somewhat because of the existence of zero-divisors.

Denote by a/s the equivalence class of the pair (a, s), and denote the set of equivalence classes by $S^{-1}A$ (or sometimes $A[S^{-1}]$.) We may make $S^{-1}A$ into a ring by defining

$$a/s + b/t = (at + bs)/st$$
$$(a/s)(b/t) = (ab)/(st).$$

It is not hard to check that these operations are in fact well defined on equivalence classes. For example, suppose $a/s = a'/s'$ and $b/t = b'/t'$. Then $\exists u, v \in S$ such that $u(as' - a's) = 0$ and $v(bt' - b't) = 0$. However,

$$(at + bs)(s't') - (a't' + b's')(st)$$
$$= ats't' + bss't' - a't'st - b's'st$$
$$= (as' - a's)tt' + (bt' - b't)ss'$$

so uv kills it, and $(at + bs)/(st) = (a't' + b's')/(s't')$. A similar argument shows the product is well defined. It is not hard to see that with these operations $S^{-1}A$ becomes a commutative ring. The identity is $1/1 = s/s$ for every $s \in S$, and the zero element is $0/1 = 0/s$ for every $s \in S$. Note that if $0 \in S$, then there is only one equivalence class and $S^{-1}A$ is the trivial ring in which $1 = 0$.

Define $\phi : A \to S^{-1}A$ by $\phi(a) = a/1$. It is easy to check that ϕ is a ring homomorphism so that $S^{-1}A$ may be viewed as an A-algebra in this natural way. ϕ need not be a monomorphism. In fact, $\operatorname{Ker} \phi = \{x \in A \mid ux = 0 \text{ for some } u \in S\}$, and this will not be trivial if some elements of S are zero divisors in A. (If A is a domain, and if $0 \notin S$, then it is easy to see that ϕ is a monomorphism.) In any case, $A \to S^{-1}A$ does have a universal mappping property which makes it similar to an imbedding.

PROPOSITION. *Let A be a commutative ring and let S be a multiplicative subset of A. If $g : A \to B$ is a ring homomorphism into a commutative ring B such that $g(S) \subseteq U(B)$, then there is a unique ring homomorphism $\psi : S^{-1}A \to B$ such that*

$$A \xrightarrow{\phi} S^{-1}A$$
$$g \searrow \quad \downarrow \psi$$
$$B$$

commutes.

PROOF. If there is such a ring homomorphism, we must have

$$\psi(a/s) = \psi((a/1)(1/s)) = \psi(a/1)\psi(1/s).$$

However, $\psi(a/1) = g(a)$, and since $(s/1)(1/s) = 1$, it follows that $\psi((s/1)(1/s)) = g(s)\psi(1/s) = 1$, so $\psi(1/s) = g(s)^{-1}$. Hence,

$$\psi(a/s) = g(a)g(s)^{-1}.$$

This shows that there is at most one such ψ. On the other hand, since $g(S) \subseteq U(B)$, the above formula makes sense in any case, and it is easy to see that it is well defined on equivalence classes and that it defines a ring homomorphism such that $\psi \circ \phi = g$.

Note that if A is a domain and 0 is not in S, then as mentioned above, we may view A as a subring of $S^{-1}A$, and in this case the latter ring may also be viewed as a subring of the field of fractions K of A. In fact it will consist of all fractions $a/s \in K$ such that the denominators are in S.

There are two important classes of examples which arise in applications.

1. Suppose $S = \{f^n \mid n \geq 0\}$ consists of the nonnegative powers of some element $f \in A$.

2. Let $\mathfrak{p}$ be a prime ideal of A. Then the complement S of $\mathfrak{p}$ is a multiplicative set. For, if a and b are *not* in $\mathfrak{p}$, then the product ab can not be in $\mathfrak{p}$ by the defintion of primality. In this case we write $A_{\mathfrak{p}} = S^{-1}A$ and we call it the *localization* of A at the prime ideal $\mathfrak{p}$.

PROPOSITION. *Let A be a commutative ring and let $\mathfrak{p}$ be a prime ideal in A. Then the localization $A_{\mathfrak{p}}$ is a local ring with unique maximal ideal $S^{-1}\mathfrak{p} = \{x/s \mid x \in \mathfrak{p}, s \notin \mathfrak{p}\}$.*

PROOF. $a/s \in A_{\mathfrak{p}}$ is a unit if and only if there exist $b \in A, t, u \in S$ such that $u(ab - st) = 0$ (i.e., $ab/st = 1/1$). Thus, in this case $aw \in S$ for some $w \in A$. If a were not in S, it would be in $\mathfrak{p} = A - S$, so since $\mathfrak{p}$ is an ideal we would also have $aw \in \mathfrak{p}$ —which is a contradiction. Hence, $a \in S$. Thus every unit in $A_{\mathfrak{p}}$ is of the form a/s with $a, s \in S$; conversely, it is trivial that every such element is a unit. Hence, the non-units in $A_{\mathfrak{p}}$ consist precisely of the elements of the form x/s where $x \in \mathfrak{p}$ (i.e., x is *not* in S) and $s \in S$. It is easy to check that these elements form an ideal so $A_{\mathfrak{p}}$ is a local ring.

Exercises.

1. (a) Show that the relation $(a, s) \sim (b, t)$ defined in the text is an equivalence relation.
    (b) Using the notation in the text, show that

$$(a/s) + (b/t) = (at + bs)/(st)$$

is a well defined operation on such equivalence classes.
    (c) Assuming the operation

$$(a/s)(b/t) = (ab)/(st)$$

is well defined, check the distributive law.

2. Let A be a commutative ring and let S be a multiplicative subset of A. Define a one-to-one correspondence between the prime ideals $\mathfrak{p}'$ of $S^{-1}A$ and the prime ideals $\mathfrak{p}$ of A such that $\mathfrak{p} \cap S = \emptyset$. If $\mathfrak{p}$ is a prime ideal of A, define a one-to-one correspondence between the prime ideals of A contained in $\mathfrak{p}$ and the prime ideals of $A_{\mathfrak{p}}$.

3. Localization of modules, flatness

Let $\phi : A \to B$ be a ring homomorphism. Then we may view B as a right A-module by $ba = b\phi(a)$ for $a \in A, b \in B$. If M is any left A-module, treating B as a right A-module, we may form $B \otimes_A M$, and we may view this as a left B-module by defining $b(b' \otimes m) = (bb') \otimes m$ for $b, b' \in B$ and $m \in M$. $B \otimes_A M$ is sometimes called the module *induced* by the change of ring homomorphism $A \to B$. If $f : M \to M'$ is an A-module homomorphism, then it is easy to see that $B \otimes f : B \otimes_A M \to B \otimes_A M'$ is a B-module homomorphism. In fact, $B \otimes_A (-)$ is a functor from left A-modules to left B-modules. (What should you do for right A-modules?)

Let A be a commutative ring, let S be a multiplicative set in A, and apply the above construction to the ring homomorphism $\phi : A \to S^{-1}A$: for each A-module M, consider the $S^{-1}A$-module $S^{-1}A \otimes_A M$. By definition, each element of this module is a sum of the form

$$\sum_i (a_i/s_i) \otimes m_i.$$

Let $s = s_1 s_2 \ldots s_n$ (where n is the number of terms in the sum), and let

$$t_i = s_1 \ldots s_{i-1} s_{i+1} \ldots s_n.$$

Then in $S^{-1}A$, we have $a_i/s_i = a_i t_i/s$, so

$$\sum_i (a_i/s_i) \otimes m_i = \sum_i (a_i t_i/s) \otimes m_i = \sum_i (1/s) \otimes (a_i t_i m_i) = (1/s) \otimes m$$

where $m = \sum_i a_i t_i m_i$. Hence, we conclude that every element of $S^{-1}A \otimes M$ is expressible in the form $(1/s) \otimes m$ where $s \in S$ and $m \in M$.

We may approach this concept from another point of view. Suppose as above that S is a multiplicative subset of the commutative ring A, and suppose M is an A-module. Consider the collection of pairs (m, s) where $m \in M$ and $s \in S$. As previously, define an equivalence relation on the set of the pairs by

$$(m, s) \sim (n, t) \qquad \Leftrightarrow \qquad \exists u \in S \text{ such that } u(tm - sn) = 0.$$

Let $S^{-1}M$ denote the set of equivalence classes and denote the equivalence class of (m, s) by m/s. As previously, define operations

$$(m/s) + (n/t) = (tm + sn)/st \qquad m, n \in M, s, t \in S$$
$$(a/s)(m/t) = (am)/(st) \qquad a \in A, m \in M, s, t \in S.$$

One checks as earlier that these operations are well defined and they make $S^{-1}M$ into an $S^{-1}A$-module.

PROPOSITION. *Let A be a commutative ring and S a multiplicative subset of A, and let M be an A-module. Then there is an $S^{-1}A$-module isomorphism $S^{-1}A \otimes_A M \cong S^{-1}M$, and in fact these isomorphims provide an isomorphism of functors.*

PROOF. Define $\psi_M : S^{-1}A \otimes_A M \to S^{-1}M$ by $(a/s) \otimes m \mapsto (am)/s$. Of course, one has to show that this is well defined (i.e., does not depend on the representative of a/s) and also that it in fact defines a morphism of the tensor product (i.e., that the underlying map on the direct product is bilinear.) Also, one has to show that it is an $S^{-1}A$-module homomorphism, and finally that the collection ψ_M constitutes a natural transformation of functors. We leave all this to the student. ψ_M is clearly an epimorphism. We shall show that it is an monomorphism, hence an isomorphism. To this end, suppose $\psi((1/s) \otimes m) = m/s = 0$. Then $um = 0$ for some $u \in S$. Hence,

$$(1/s) \otimes m = (u/us) \otimes m = (1/us) \otimes (um) = (1/us) \otimes 0 = 0.$$

Note that if $f : M \to M'$ is an A-module homomorphism, then $S^{-1}f : S^{-1}M \to S^{-1}M'$ is given by $(S^{-1}f)(m/s) = f(m)/s$.

PROPOSITION. *Let A be a commutative ring, and let S be a multiplicative subset of A. The functor $S^{-1}A \otimes (-)$ is exact. That is, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then*

$$0 \to S^{-1}A \otimes_A M \to S^{-1}A \otimes_A M \to S^{-1}A \otimes_A M'' \to 0$$

is an exact sequence of $S^{-1}A$-modules.

PROOF. We already know from our earlier discussion of the tensor product that it is right exact, i.e., the induced sequence is exact except possibly on the left. It remains to prove that if $f : M' \to M$ is a monomorphism, then $S^{-1}A \otimes f$ is a monomorphism. Using the previous proposition, consider instead $S^{-1}f : S^{-1}M' \to S^{-1}M$. We have

$$S^{-1}f(m'/s) = f(m')/s = 0 \Leftrightarrow uf(m') = 0 \text{ for some } u \in S$$
$$\Leftrightarrow f(um') = 0 \Leftrightarrow um' = 0 \text{ (since } f \text{ is a monomorphism)}$$
$$\Rightarrow m'/s = 0.$$

In the future we shall use the single notation $S^{-1}M$, and we shall only make use of the fact that it is isomorphic to the above tensor product when that fact will be useful. Note that if $\mathfrak{p}$ is an ideal of A, that is a submodule of A, then by the above proposition $S^{-1}\mathfrak{p}$ is a submodule, that is an ideal of $S^{-1}A$.

In general, if a module M has the property that the functor $M \otimes_A (-)$ is exact, then we say that M is *flat*. As above, that amounts to asserting that it preserves monomorphisms. Hence, $S^{-1}A$ is a flat A-module.

PROPOSITION. *Let M and N be A-submodules of the A-module L. Then*

$$S^{-1}(M + N) = S^{-1}M + S^{-1}N$$
$$S^{-1}(M \cap N) = S^{-1}M \cap S^{-1}N.$$

PROOF. Exercise. The formulas may be derived directly from the definitions or they may be derived from the exactness of the functor $S^{-1}(-)$ if one is sufficiently clever.

PROPOSITION. *Suppose A is a commutative ring, S a multiplicative set in A and M and N are A-modules. Then*

$$S^{-1}(M \otimes_A N) \cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N.$$

PROOF. We have

$$S^{-1}(M \otimes_A N) \cong S^{-1}A \otimes_A (M \otimes_A N) \cong (S^{-1}A \otimes_A M) \otimes_{S^{-1}A} S^{-1}A \otimes_A N$$
$$\cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N.$$

If $\mathfrak{p}$ is a prime ideal of A, and $S = A - \mathfrak{p}$, then we use the notation $M_\mathfrak{p}$ for $S^{-1}M$.

PROPOSITION. *Let A be a commutative ring and let M be an A-module. Then the following are equivalent:*
(i) $M = 0$.
(ii) $M_\mathfrak{p} = 0$ for every prime ideal $\mathfrak{p}$ in A.
(iii) $M_\mathfrak{m} = 0$ for every maximal ideal $\mathfrak{m}$ in A.

PROOF. Since every maximal ideal is prime, (ii) $\Rightarrow$ (iii).

Suppose that $M \neq 0$. Let $x \neq 0 \in M$, and let $\mathfrak{m}$ be a maximal ideal of A containing $\text{Ann}_A(x)$ (which is a proper ideal since $x \neq 0$). We have $ux \neq 0$ for every $u \in S = A - \mathfrak{m}$ since no such u is in $\text{Ann}_A(x)$. Hence $x/1 \neq 0 \in S^{-1}M = M_\mathfrak{m}$. It follows that (iii) $\Rightarrow$ (i).

Since (i) $\Rightarrow$ (ii) is clear, we are done.

The ideas introduced above play a natural role in studying interesting rings of functions defined on a topological space with some additional structure. For example one could consider the ring of all real valued differentiable functions on a differentiable manifold, the ring of all holomorphic functions on a Riemann surface, etc. An important intuition first exploited in functional analysis is that in many cases one can reconstruct the space (with its appropriate structure) purely from the structure of the associated ring of functions. Even more generally, for any appropriate ring, one can construct an associated "space" and then interpret the ring as a ring of functions on that space.

Example. Let $A = k[X]$ be a polynomial ring in an indeterminate X over an algebraically closed field k. We may view A as a ring of functions on the "line" k. In this case, all prime ideals are maximal since $k[X]$ is a PID. For each point $t \in k$, the deal $\mathfrak{m} = A(X - t)$ is maximal; in fact, $\mathfrak{m}$ may be thought of as the kernel of the homomorphism $A \to k$ obtained by evaluating a polynomial at t. Conversely, every maximal ideal is of this form since k is algebraically closed and every monic irreducible polynomial is of the form $X - t$ for $t \in k$. Thus the maximal ideals $\mathfrak{m}$ of A are in one-to-one correspondence with the points t of the line k. Moreover, it is not hard to see that the local ring $A_\mathfrak{m}$ may be identified with the subring of the rational function field $k(X)$ consisting of all $f(X)/g(X)$ where $f(X), g(X) \in k[X]$ and $g(t) \neq 0$. We may think of these as the rational functions on k which are well defined in some neighborhood of the point t.

Let A be any commutative ring. With the above discussion in mind, we call the set of all maximal ideals of A the *maximal ideal spectrum* of A and we denote it $\text{Max}(A)$. Given $f \in A$, we define $f(\mathfrak{m}) = f \bmod \mathfrak{m} \in A/\mathfrak{m}$. Unfortunately, this does not quite give us a *function* on the set $\text{Max}(A)$ because the "function values" $f(\mathfrak{m})$ are taken in different rings. However, in many ways it is appropriate to view $f(\mathfrak{m})$ as a function. Similarly, there is a way to view each localization $A_\mathfrak{m}$ as a kind of limit of rings of functions defined locally in some neighborhood of $\mathfrak{m}$.

It turns out that for the purposes of algebraic geometry, the maximal ideal spectrum is usually not large enough. (The reasons for this are a bit mysterious and difficult to explain without developing quite a lot of algebraic geometry.) We call the set of all prime ideals of A the *prime ideal spectrum*, and we denote it $\mathrm{spec}(A)$. Note that $\mathrm{Max}(A)$ is a subset of $\mathrm{spec}(A)$. As above, we may think of $A_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ as structures defined locally at the point $\mathfrak{p} \in \mathrm{spec}(A)$. With this suggestive terminology in mind, we may restate the above proposition by saying that a module is trivial if and only if it is trivial at each point in $\mathrm{spec}(A)$ or if and only if it is trivial at each point of $\mathrm{Max}(A)$.

COROLLARY. *Let A be a commutative ring and let $f : M \to N$ be an A-module homomorphism. Then f is a monomorphism (epimorphism) if and only if $f_{\mathfrak{p}}$ is a monomorphism (epimorphism) for each $\mathfrak{p} \in \mathrm{spec}(A)$ which in turn holds if and only if $f_{\mathfrak{m}}$ is a a monomorphism (epimorphism) for each $\mathfrak{m} \in \mathrm{Max}(A)$.*

PROOF. Use the fact that localization is exact, and apply the previous propostion to the sequences

$$0 \to \mathrm{Ker}\, f \to M \to \mathrm{Im}\, f \to 0$$

and

$$0 \to \mathrm{Im}\, f \to N \to \mathrm{Coker}\, f \to 0.$$

PROPOSITION. *Let A be a commutative ring. The nil radical $N(A)$ of A is the intersection of all prime ideals of A.*

PROOF. Suppose $x \in A$ is nilpotent (i.e., $x \in N(A)$). Then $x^k = 0$ for some $k > 0$ so clearly $x^k \in \mathfrak{p}$ for every prime ideal $\mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows easily (by induction) that $x \in \mathfrak{p}$. Hence the nil radical is contained in $\cap \mathfrak{p}$.

Suppose on the other hand that x is not nilpotent (i.e., it is not in $N(A)$). Then $S = \{x^k \mid k = 1, 2, \dots\}$ does not contain 0 so it follows that $A' = S^{-1}A \neq \{0\}$. Let $\mathfrak{m}'$ be a maximal ideal of A' and let $\mathfrak{m} = \phi^{-1}(\mathfrak{m})$ where $\phi : A \to S^{-1}A$ is the homomorphism defined previously. The inverse image of a prime ideal is always prime for any ring homomorphism so $\mathfrak{m}$ is at least prime. $x \notin \mathfrak{m}$ since if it were then $x/1$ would be in $\phi(\mathfrak{m}) \subseteq \mathfrak{m}$. Since $x/1$ is a unit in $S^{-1}A$, this can't happen. Thus there is at least one prime ideal not containing x if x isn't nilpotent.

Note that one way to interpret the above proposition is to say that the nil radical is the set of "functions" in A which "vanish" at all points of $\mathrm{spec}(A)$. Similarly, the Jacobson radical is the set of all "functions" which "vanish" at all points of $\mathrm{Max}(A)$.

PROPOSITION. *Let A be a commutative ring and let S be a multiplicative subset of A. If A is noetherian (artinian), then $S^{-1}A$ is noetherian (artinian).*

PROOF. Let J' be an ideal in $S^{-1}A$, and let J be the inverse image of J' in A, i.e.,

$$J = \{a \in A \mid a/1 \in J'\}.$$

It is not hard to see that $J' = S^{-1}J$. (Exercise.) Given any chain

$$J'_1 \subseteq J'_2 \subseteq \cdots \subseteq J'_k \subseteq \dots$$

of ideals in $S^{-1}A$, the corresponding chain

$$J_1 \subseteq J_2 \subseteq \cdots \subseteq J_k \subseteq \dots$$

in A stablilizes; hence the original chain in $S^{-1}A$ stabilizes.

Exercises.

1.   Choose *one* of the many assertions, unproved in the text, which are necessary to prove the isomorphism of functors
$$S^{-1}A \otimes_A M \cong S^{-1}M$$

and prove it.

2.   Prove the formulas in the text

$$S^{-1}(M + N) = S^{-1}M + S^{-1}N$$
$$S^{-1}(M \cap N) = S^{-1}M \cap S^{-1}N.$$

3.   Let k be a commutative ring. Show that in the polynomial ring $k[X]$ the Jacobson radical is the same as the nil radical.