CHAPTER III

NORMAL SERIES

1. Normal Series

A group is called *simple* if it has no nontrivial, proper, normal subgroups. The only abelian simple groups are cyclic groups of prime order, but some authors exclude these by requiring simple groups to be non-abelian. A_n is a simple non-abelian group for $n > 4$.

Let G be a group. A sequence of subgroups

$$\{1\} = G_s \lhd \ldots \lhd G_2 \lhd G_1 \lhd G_0 = G$$

where each subgroup is normal in the next one as indicated is called a *normal series* for the group G. The length s of a normal series is the number of *factors* G_i/G_{i+1} rather than the number of subgroups. In general, a normal subgroup of a normal subgroup need not be normal. (Can you find an example in any of the previous pages?) So the terms in a normal series need not be normal in G.

Example:

$$\{1\} \lhd A_n \lhd S_n \text{ is a normal series for } S_n.$$

For $n = 4$, A_4 has the normal series $\{1\} \lhd V \lhd A_4$ where $V = \{\mathrm{id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ is easily seen to be a normal subgroup isomorphic to the Klein 4-group (the direct product of two cyclic subgroups of order 2.) In fact, in this case V is normal in S_4.

A *composition* series for G is a normal series such that each factor is simple, i.e., each factor is either cyclic of prime order or a simple nonabelian group.

Clearly, a finite group always has a composition series since we can keep inserting normal subgroups into the factors until all the factors have no proper nontrivial normal subgroups. On the other hand, an infinite group may or may not have a composition series. (**Z** does not have a composition series since the last nontrivial subgroup would be isomorphic to **Z** again.)

Let

$$\{1\} = G_s \lhd \ldots \lhd G_2 \lhd G_1 \lhd G_0 = G$$

be a normal series for G. A *refinement* of this series is another normal series such that every subgroup G_i in the first series appears as a term in the second series. In this case, we can describe the situation notationally as follows. For each layer $G_i \geq G_{i+1}$ in the first series, we can insert additional factors

$$G_{i+1} = H_{i,t} \lhd \ldots \lhd H_{i,1} \lhd H_{i,0} = G_i$$

and putting these together yields the refined series.

Note that a composition series does not have any proper refinements, i.e., the only way to refine a composition series is to add trivial layers. (That is, all the intermediate layers $H_{i,j+1} \lhd H_{i,j}$ are in fact equalities.)

Two normal series are said to be *equivalent* if they have the same length and the factors are the same up to isomorphism except for the order in which they occur. Note that two different groups could have equivalent normal series without being isomorphic. For example, both a cyclic group of order 4 and the Klein 4-group have composition series in which both factors are cyclic of order 2.

THEOREM. (Schreier) *Any two normal series for the same group have equivalent refinements.*

We shall prove this theorem below.

COROLLARY. (Jordan-Hölder) *If a group has a composition series, then all composition series for that group are equivalent.*

PROOF. The only way a refinement of a composition series can differ from the original composition series is by the insertion of trivial factors. Hence, using Schreier's theorem we see that there must be a one-to-one correspondence between the (isomorphic) nontrivial factors in the refinements, and these factors must be the same as the original factors in the two composition series. □

Proof of Schreier's Theorem.

We introduce some terminology. Let $L \leq M$ be subgroups of G with M normal in L. Then L/M is called a *section* of G. Suppose H is another subgroup of G. Put $H^* = (H \cap L)M$. Then H^*/M is a subgroup of the section L/M. Note that if $K \lhd H$ then $K^* \lhd H^*$. (For, $K \cap L \lhd H \cap L$ and it is not hard to see that multiplying by M will preserve normality.) H^*/M is in some sense the largest subgroup of the section L/M that we can "construct" from H.

LEMMA. (Zassenhaus) *Let G be a group with sections H/K and L/M. Then $(H \cap L)M/(K \cap L)M \cong (L \cap H)K/(M \cap H)K$. (This can be written $H^*/K^* \cong L^*/M^*$ if we stipulate that the $*$ operation on the left is relative to the section L/M and that on the right is relative to the section H/K.)*

$$\begin{array}{ccc} & G & \\ / & & \backslash \\ H & & L \\ | & & | \\ L^* & & H^* \\ | & & | \\ M^* & & K^* \\ | & & | \\ K & & M \end{array}$$

PROOF. Under the canonical epimorphism $L \to L/M$, the subgroup $H \cap L$ of L gets carried onto the factor group $(H \cap L)M/M$, and the kernel of the restriction of this canonical epimorphism to $H \cap L$ is

$$(H \cap L) \cap M = H \cap (L \cap M) = H \cap M.$$

Thus, using the canonical maps as in the second isomorphism theorem, we have

(*) $$(H \cap L)/(H \cap M) \cong (H \cap L)M/M.$$

Similarly, under the epimorphism $\phi : H \cap L \to (H \cap L)M/M$, the subgroup $K \cap L$ gets carried onto $(K \cap L)M/M$. On the other, hand the largest subgroup of $H \cap L$ with this property is $(K \cap L)(\mathrm{Ker}\phi) = (K \cap L)(H \cap M)$. Hence, the subgroup of $(H \cap L)/(H \cap M)$ on the left of (*) corresponding to $(K \cap L)M/M$ on the right is $(K \cap L)(H \cap M)/(H \cap M)$.

$$\begin{array}{ccc} (H \cap L)/(H \cap M) & \longrightarrow & (H \cap L)M/M \\ | & & | \\ (K \cap L)(H \cap M)/H \cap M & \rightarrow & (K \cap L)M/M \\ | & & | \\ 1 & & 1 \end{array}$$

From the third isomorphism theorem, we have

$$(H \cap L)/(K \cap L)(H \cap M) \cong (H \cap L)M/(K \cap L)M.$$

However, the expression on the left is does not change if the pairs (H, K) and (L, M) are switched. Hence, the result follows. □

To complete the proof of Schreier's Theorem, suppose we have two normal series

$$1 = G_s \lhd \dots \lhd G_i \lhd \dots \lhd G_0 = G$$

and

$$1 = H_t \lhd \dots \lhd H_j \lhd \dots \lhd H_0 = G.$$

Between G_{i+1} and G_i insert the subgroups $(H_j \cap G_i)G_{i+1}$ where $j = 0, 1, \dots, t$. Similarly, between H_{j+1} and H_j insert the subgroups $(G_i \cap H_j)H_{j+1}$ where $i = 0, 1, \dots, s$. The refined normal series have the same length st. Also, by Zassenhaus's Lemma, we have isomorphisms of factors

$$\frac{(H_j \cap G_i)G_{i+1}}{(H_{j+1} \cap G_i)G_{i+1}} \cong \frac{(G_i \cap H_j)H_{j+1}}{(G_{i+1} \cap H_j)H_{j+1}}$$

so that shows there is a one-to-one correspondence of isomorphic factors. □

As we noted earlier, a finite group necessarily has a composition series but an infinite group need not have one. In particular, $\mathbf{Z}$ has many infinite descending chains such that each factor is cyclic of prime order. Similarly, $\mathbf{Q}/\mathbf{Z}$ has many infinite ascending chains in which each factor is cyclic of prime order.

Important Note. In many circumstances, one wishes to consider normal series or composition series not of arbitrary subgroups but of subgroups meeting some further condition. As long as the basic isomorphism theorems remain true, then conclusions such at the Jordan–Hölder Theorem remain true. For example, the group G could be the additive group of a vector space, and we might restrict attention is subgroups which are subspaces. Provided the reasoning about homomorphisms and isomorphisms remains valid we can still derive the analogue of the Jordan–Hölder Theorem. Of course, in the definition of composition series, we would have to require the factors be irreducible in the new sense—i.e. have no proper, nontrivial subspaces.

Exercises.

1. Find all subgroups of S_4. Exhibit them in a lattice diagram Determine all normality relations and exhibit all composition series for S_4.

2. Solvability

Let G be a group. For $x, y \in G$, define

$$[x, y] = xyx^{-1}y^{-1} = ({}^x y)y^{-1}.$$

$[x, y]$ is called the *commutator* of x and y. (Group theorists often define $[x, y]$ to be $x^{-1}y^{-1}xy$ instead.) The commutator satisfies many important relations, one of which is

$$[x, yz] = [x, y]({}^y[x, z]) = [x, y][x, z]([y, [z, x]])^{-1}.$$

(See the Exercises.) We define the *commutator subgroup* $[G, G]$ to be the subgroup generated by all $[x, y]$ with $x, y \in G$. The commutator subgroup is also denoted G', and it is also called the *derived* subgroup.

Note that G/G' is abelian because modulo G' we have $xy \equiv yx$. Conversely, if N is any normal subgroup of G such that G/N is abelian, then $xy \equiv yx \bmod N$ holds for all $x, y \in G$. It follows that each $[x, y] = xyx^{-1}y^{-1} \in N$, i.e. $G' \le N$. It follows that G' is the intersection of all normal subgroups N of G for which G/N is abelian.

We can repeat the process of finding the commutator subgroup. Namely, define G'' to be $[G', G']$. It is not hard to see that G'' is even normal in G. (See the Exercises.) Continue in this way and define $G_1 = G, G_2 = G', \dots, G_{i+1} = [G_i, G_i], \dots$. This provides a descending chain of subgroups each normal in the whole group. It is called the *derived* series for the group. Notice that it could go on 'forever'. Even for a finite group, it is quite possible that the terms of the series could stabilize on a single non-trivial subgroup which repeats, so the series would go on forever without terminating with $\{1\}$,

A group G is called *solvable* if its derived series has $G_i = \{1\}$ for some i. Of course, every abelian group is solvable. Any p-group G is also solvable. (See the Exercises.) S_n is solvable for $n = 2, 3$, and 4, but it is not solvable if $n > 4$. (See the Exercises.)

Another criterion for solvability which is simpler to apply in specific examples is derived from the following proposition.

PROPOSITION. *Assume G has a sequence of subgroups*

$$\dots H_{i+1} \lhd H_i \lhd \dots H_2 \lhd H_1 = G$$

such that each factor H_i/H_{i+1} is abelian. Then for each i, the ith term in the derived series $G_i \leq H_i$. In particular, G is solvable if and only if it has a finite normal series in which each factor is abelian.

PROOF. Clear by induction and the definition of G_i as the commutator subgroup of G_{i-1}. □

PROPOSITION. *Let G have a composition series. Then G is solvable if and only if the factors in its composition series are cyclic of prime order.*

PROOF. Assume G is solvable. By Schreier's Theorem, we can find equivalent refinements of its derived series and some composition series. That means that the factors in the composition series must be sections of the groups G_i/G_{i+1} which are abelian. That excludes the possibility of simple nonabelian factors in the composition series so they are all cyclic of prime order.

Assume G has a composition series with cyclic factors. Then it has a normal series with abelian factors, and by the above proposition, the derived series must terminate. □

PROPOSITION. *Let $N \lhd G$. Then G is solvable if and only if both N and G/N are solvable. More generally, any subgroup and any factor group of a solvable group are solvable.*

PROOF. Assume first that G is solvable and choose a normal series for G with abelian factors. If H is any subgroup, then the intersection of this normal series with H will produce a normal series for H, and it is not hard to see the factors will be abelian. (Proof? Relate these factors somehow to the factors in the original normal series.) Similarly, if G/N is a factor group of G, we may project the normal series for G onto a normal series for G/N, and again it is easy to see that the factors in the second series will be abelian.

Conversely, assume G/N and N are solvable. Choose a normal series for G/N with abelian factors and pull it back to a series for G with abelian factors which terminates in N. (Use the third isomorphism theorem.) Since N is solvable, we can continue with a normal series for N with abelian factors, and the result is such a normal series for G. □

Exercises.

1. Derive the commutator inequality

$$[x, yz] = [x, y](^y[x, z]) = [x, y][x, z]([y, [z, x]])^{-1}.$$

2. Show that each term of the derived series is normal in the whole group.

3. Show that every group of order pq, where $p < q$ are primes, is solvable.

4. Show that every group of order smaller than 60 is solvable. Show in particular that S_n is solvable for $n = 2, 3, 4$, but it is not solvable for $n = 5$.

5. Give an example of a solvable group which does not have a composition series.

3. Nilpotent Groups

Let G be a group, and let K and H be subgroups of G. Define $[K, H]$ to be the subgroup generated by all $[k, h]$ where $k \in K$ and $h \in H$. If $H \lhd G$ and $K \lhd G$ then $[K, H] \lhd G$. For,

$$g[k, h]g^{-1} = [gkg^{-1}, ghg^{-1}] \in [K, H] \text{ for each } k \in K, h \in H.$$

Define $\Gamma_1(G) = G, \Gamma_2(G) = [G, \Gamma_1(G)], \dots, \Gamma_{i+1}(G) = [G, \Gamma_i(G)], \dots$. These groups form a descending chain of subgroups each normal in the whole group G.

The chain is called the *descending central series* of the group G. A group is called *nilpotent* if its descending central series stops with the trivial subgroup.

Every nilpotent group is solvable since we have $G_1 = \Gamma_1, G_2 = [G, G] = \Gamma_2, \dots, G_{i+1} = [G_i, G_i] \leq [G, G_i] \leq [G, \Gamma_i] = \Gamma_{i+1}$ by induction.

Proposition. *A finite p-group is nilpotent.*

Proof. Every abelian group is certainly nilpotent. Proceeding by induction, we may assume that $G/Z(G)$ is nilpotent. ($|G/Z(G)| < |G|$ since $Z(G)$ is nontrivial.) For $x, y \in G, z \in Z(G)$ we have

$$[xz, yz] = [x, y].$$

(This is easily verified by direct computation.) It follows that in G/Z, $[xZ, yZ] = [x, y]Z$. From this, it follows that

$$\Gamma_i(G/Z) = \Gamma_i(G)Z/Z \cong \Gamma_i(G)/\Gamma_i(G) \cap Z.$$

Assume $\Gamma_n(G/Z) = \{1\}$. Then $\Gamma_n(G) \subseteq Z(G)$. Hence, $\Gamma_{n+1}(G) \subseteq [G, Z] = \{1\}$ so G is also nilpotent. □

Note that a solvable group need not be nilpotent. For example, direct calculations in S_3 show that all $\Gamma_i = A_3$ for $i \geq 2$. In fact, it can be shown that a finite group is nilpotent if and only if it is isomorphic to the direct product of its p-Sylow subgroups for the primes p dividing its order. (See the Exercises.)

Exercises.

1. (a) Show that the center of any nilpotent group is non-trivial.

(b) Conversely, suppose G has a normal series

$$\{1\} = Z_0 \leq Z_1 \leq \cdots \leq Z_{k-1} \leq Z_k = G$$

such that each Z_i is normal in G and Z_{i+1}/Z_i is in the center of G/Z_i for each $i = 0, \dots, k-1$. Show that G is nilpotent.

2. Let G be a finite group.

(a) Assume G is nilpotent. Show that for every prime p dividing $|G|$, any p-Sylow subgroup G_p is normal. (Hence, there is only one Sylow subgroup for each prime.) Conclude from this that G is isomorphic to the direct product of its p-Sylow subgroups for p dividing $|G|$.

(b) Assume conversely that G is a direct product of p-groups for distinct primes p. Show that G is nilpotent.