2.3. Relations

2.3.1. Relations. Assume that we have a set of men M and a set of women W, some of whom are married. We want to express which men in M are married to which women in W. One way to do that is by listing the set of pairs (m, w) such that m is a man, w is a woman, and m is married to w. So, the relation "married to" can be represented by a subset of the Cartesian product $M \times W$. In general, a *relation* $\mathcal{R}$ from a set A to a set B will be understood as a subset of the Cartesian product $A \times B$, i.e., $\mathcal{R} \subseteq A \times B$. If an element $a \in A$ is related to an element $b \in B$, we often write $a \,\mathcal{R}\, b$ instead of $(a, b) \in \mathcal{R}$.

The set

$$\{a \in A \mid a \,\mathcal{R}\, b \text{ for some } b \in B\}$$

is called the *domain* of $\mathcal{R}$. The set

$$\{b \in B \mid a \,\mathcal{R}\, b \text{ for some } a \in A\}$$

is called the *range* of $\mathcal{R}$. For instance, in the relation "married to" above, the domain is the set of married men, and the range is the set of married women.

If A and B are the same set, then any subset of $A \times A$ will be a *binary relation* in A. For instance, assume $A = \{1, 2, 3, 4\}$. Then the binary relation "less than" in A will be:

$$\begin{aligned} <_A = \{(x, y) \in A \times A \mid x < y\} \\ = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}\,. \end{aligned}$$

Notation: A set A with a binary relation $\mathcal{R}$ is sometimes represented by the pair $(A, \mathcal{R})$. So, for instance, $(\mathbb{Z}, \leq)$ means the set of integers together with the relation of non-strict inequality.

2.3.2. Representations of Relations.

Arrow diagrams. Venn diagrams and arrows can be used for representing relations between given sets. As an example, figure 2.8 represents the relation from $A = \{a, b, c, d\}$ to $B = \{1, 2, 3, 4\}$ given by $\mathcal{R} = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$. In the diagram an arrow from x to y means that x is related to y. This kind of graph is called *directed graph* or *digraph.*

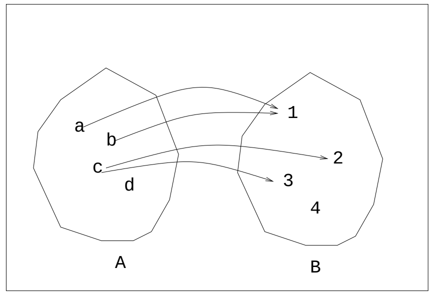

FIGURE 2.8. Relation.

Another example is given in diagram 2.9, which represents the divisibility relation on the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

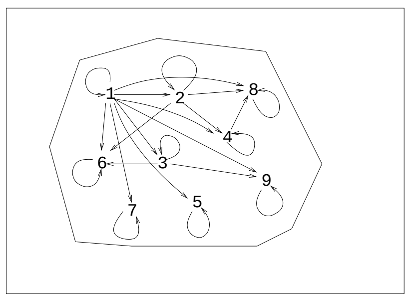

FIGURE 2.9. Binary relation of divisibility.

Matrix of a Relation. Another way of representing a relation $\mathcal{R}$ from A to B is with a matrix. Its rows are labeled with the elements of A, and its columns are labeled with the elements of B. If $a \in A$ and $b \in B$ then we write 1 in row a column b if $a \,\mathcal{R}\, b$, otherwise we write 0. For instance the relation $\mathcal{R} = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$ from $A = \{a, b, c, d\}$ to $B = \{1, 2, 3, 4\}$ has the following matrix:

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

2.3.3. Inverse Relation. Given a relation $\mathcal{R}$ from A to B, the inverse of $\mathcal{R}$, denoted $\mathcal{R}^{-1}$, is the relation from B to A defined as

$$b \,\mathcal{R}^{-1}\, a \Leftrightarrow a \,\mathcal{R}\, b\,.$$

For instance, if $\mathcal{R}$ is the relation "being a son or daughter of", then $\mathcal{R}^{-1}$ is the relation "being a parent of".

2.3.4. Composition of Relations. Let A, B and C be three sets. Given a relation $\mathcal{R}$ from A to B and a relation $\mathcal{S}$ from B to C, then the composition $\mathcal{S} \circ \mathcal{R}$ of relations $\mathcal{R}$ and $\mathcal{S}$ is a relation from A to C defined by:

$$a\,(\mathcal{S} \circ \mathcal{R})\,c \Leftrightarrow \text{there exists some } b \in B \text{ such that } a\,\mathcal{R}\,b \text{ and } b\,\mathcal{S}\,c\,.$$

For instance, if $\mathcal{R}$ is the relation "to be the father of", and $\mathcal{S}$ is the relation "to be married to", then $\mathcal{S} \circ \mathcal{R}$ is the relation "to be the father in law of".

2.3.5. Properties of Binary Relations. A binary relation $\mathcal{R}$ on A is called:

1. *Reflexive* if for all $x \in A$, $x\,\mathcal{R}\,x$. For instance on $\mathbb{Z}$ the relation "equal to" $(=)$ is reflexive.

2. *Transitive* if for all $x, y, z \in A$, $x\,\mathcal{R}\,y$ and $y\,\mathcal{R}\,z$ implies $x\,\mathcal{R}\,z$. For instance equality $(=)$ and inequality $(<)$ on $\mathbb{Z}$ are transitive relations.

3. *Symmetric* if for all $x, y \in A$, $x\,\mathcal{R}\,y \Rightarrow y\,\mathcal{R}\,x$. For instance on $\mathbb{Z}$, equality $(=)$ is symmetric, but strict inequality $(<)$ is not.

4. *Antisymmetric* if for all $x, y \in A$, $x\,\mathcal{R}\,y$ and $y\,\mathcal{R}\,x$ implies $x = y$. For instance, non-strict inequality $(\leq)$ on $\mathbb{Z}$ is antisymmetric.

2.3.6. Partial Orders. A *partial order*, or simply, an *order* on a set A is a binary relation "$\preccurlyeq$" on A with the following properties:

1. *Reflexive*: for all $x \in A$, $x \preccurlyeq x$.
2. *Antisymmetric*: $(x \preccurlyeq y) \wedge (y \preccurlyeq x) \Rightarrow x = y$.
3. *Transitive*: $(x \preccurlyeq y) \wedge (y \preccurlyeq z) \Rightarrow x \preccurlyeq z$.

Examples:

1. The non-strict inequality $(\leq)$ in $\mathbb{Z}$.

2. Relation of divisibility on $\mathbb{Z}^+$: $a|b \Leftrightarrow \exists t,\, b = at$.

3. Set inclusion ($\subseteq$) on $\mathcal{P}(A)$ (the collection of subsets of a given set A).

Exercise: prove that the aforementioned relations are in fact partial orders. As an example we prove that integer divisibility is a partial order:

1. Reflexive: $a = a\,1 \Rightarrow a|a$.
2. Antisymmetric: $a|b \Rightarrow b = at$ for some t and $b|a \Rightarrow a = bt'$ for some t'. Hence $a = att'$, which implies $tt' = 1 \Rightarrow t' = t^{-1}$. The only invertible positive integer is 1, so $t = t' = 1 \Rightarrow a = b$.
3. Transitive: $a|b$ and $b|c$ implies $b = at$ for some t and $c = bt'$ for some t', hence $c = att'$, i.e., $a|c$.

Question: is the strict inequality ($<$) a partial order on $\mathbb{Z}$?

Two elements $a, b \in A$ are said to be *comparable* if either $x \preccurlyeq y$ or $y \preccurlyeq x$, otherwise they are said to be *non comparable*. The order is called *total* or *linear* when every pair of elements $x, y \in A$ are comparable. For instance $(\mathbb{Z}, \leq)$ is totally ordered, but $(\mathbb{Z}^+, |)$, where "|" represents integer divisibility, is not. A totally ordered subset of a partially ordered set is called a *chain*; for instance the set $\{1, 2, 4, 8, 16, \dots\}$ is a chain in $(\mathbb{Z}^+, |)$.

2.3.7. Hasse diagrams. A Hasse diagram is a graphical representation of a partially ordered set in which each element is represented by a dot (node or vertex of the diagram). Its immediate successors are placed above the node and connected to it by straight line segments. As an example, figure 2.10 represents the Hasse diagram for the relation of divisibility on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Question: How does the Hasse diagram look for a totally ordered set?

2.3.8. Equivalence Relations. An *equivalence relation* on a set A is a binary relation "$\sim$" on A with the following properties:

1. *Reflexive*: for all $x \in A$, $x \sim x$.
2. *Symmetric*: $x \sim y \Rightarrow y \sim x$.
3. *Transitive*: $(x \sim y) \wedge (y \sim z) \Rightarrow x \sim z$.

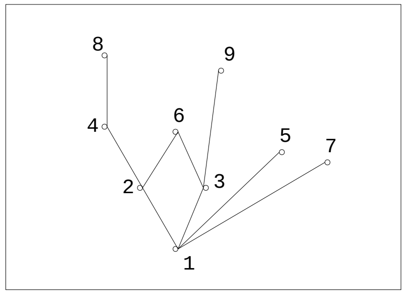

FIGURE 2.10. Hasse diagram for divisibility.

For instance, on $\mathbb{Z}$, the equality ($=$) is an equivalence relation.

Another example, also on $\mathbb{Z}$, is the following: $x \equiv y \pmod{2}$ ("x is congruent to y modulo 2") iff $x-y$ is even. For instance, $6 \equiv 2 \pmod{2}$ because $6-2=4$ is even, but $7 \not\equiv 4 \pmod{2}$, because $7-4=3$ is not even. Congruence modulo 2 is in fact an equivalence relation:

1. Reflexive: for every integer x, $x-x=0$ is indeed even, so $x \equiv x \pmod{2}$.
2. Symmetric: if $x \equiv y \pmod{2}$ then $x-y=t$ is even, but $y-x=-t$ is also even, hence $y \equiv x \pmod{2}$.
3. Transitive: assume $x \equiv y \pmod{2}$ and $y \equiv z \pmod{2}$. Then $x-y=t$ and $y-z=u$ are even. From here, $x-z=(x-y)+(y-z)=t+u$ is also even, hence $x \equiv z \pmod{2}$.

2.3.9. Equivalence Classes, Quotient Set, Partitions. Given an equivalence relation $\sim$ on a set A, and an element $x \in A$, the set of elements of A related to x are called the *equivalence class* of x, represented $[x]=\{y \in A \mid y \sim x\}$. Element x is said to be a *representative* of class

$[x]$. The collection of equivalence classes, represented $A/\!\sim\, = \{[x] \mid x \in A\}$, is called *quotient set* of A by $\sim$.

Exercise: Find the equivalence classes on $\mathbb{Z}$ with the relation of congruence modulo 2.

One of the main properties of an equivalence relation on a set A is that the quotient set, i.e. the collection of equivalence classes, is a partition of A. Recall that a *partition* of a set A is a collection of non-empty subsets $A_1, A_2, A_3, \ldots$ of A which are pairwise disjoint and whose union equals A:

1. $A_i \cap A_j = \emptyset$ for $i \neq j$,
2. $\bigcup_n A_n = A$.

Example: in $\mathbb{Z}$ with the relation of congruence modulo 2 (call it "$\sim_2$"), there are two equivalence classes: the set $\mathbb{E}$ of even integers and the set $\mathbb{O}$ of odd integers. The quotient set of $\mathbb{Z}$ by the relation "$\sim_2$" of congruence modulo 2 is $\mathbb{Z}/\sim_2 = \{\mathbb{E}, \mathbb{O}\}$. We see that it is in fact a partition of $\mathbb{Z}$, because $\mathbb{E} \cap \mathbb{O} = \emptyset$, and $\mathbb{Z} = \mathbb{E} \cup \mathbb{O}$.

Exercise: Let m be an integer greater than or equal to 2. On $\mathbb{Z}$ we define the relation $x \equiv y \pmod{m} \Leftrightarrow m|(y-x)$ (i.e., m divides exactly $y-x$). Prove that it is an equivalence relation. What are the equivalence classes? How many are there?

Exercise: On the Cartesian product $\mathbb{Z} \times \mathbb{Z}^*$ we define the relation $(a, b) \mathcal{R} (c, d) \Leftrightarrow ad = bc$. Prove that $\mathcal{R}$ is an equivalence relation. Would it still be an equivalence relation if we extend it to $\mathbb{Z} \times \mathbb{Z}$?